Review: Sequences and Series

Course Title: Real Analysis 2 **Course Code:** MTH322
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A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a finite list is not called a sequence; a sequence must continue without interruption. Formally it is defined as follows:

Sequence

A sequence is a function whose domain of definition is the set of natural numbers.

Or it can also be defined as an ordered set.

Notation:

An infinite sequence is denoted as

$\{s_n\}_{n=1}^{\infty}$ or $\{s_n : n\in \mathbb{N}\}$ or $\{s_1, s_2, s_3, ...\}$ or simply as $\{s_n\}$ or by (x_n).

The values s_n are called the *terms* or the *elements* of the sequence $\{s_n\}$.

e.g. i) $\{n\} = \{1, 2, 3, \ldots\}$.

ii) $\left\{\dfrac{1}{n}\right\} = \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\right\}$.

iii) $\left\{(-1)^{n+1}\right\} = \{1, -1, 1, -1, \ldots\}$.

iv) $\{2, 3, 5, 7, 11, ...\}$, a sequence of positive prime numbers.

Subsequence

It is a sequence whose terms are contained in given sequence. A subsequence of $\{s_n\}$ is usually written as $\{s_{n_k}\}$.

Increasing Sequence

A sequence $\{s_n\}$ is said to be an increasing sequence if

$$s_{n+1} \geq s_n \quad \forall\ n \geq 1.$$

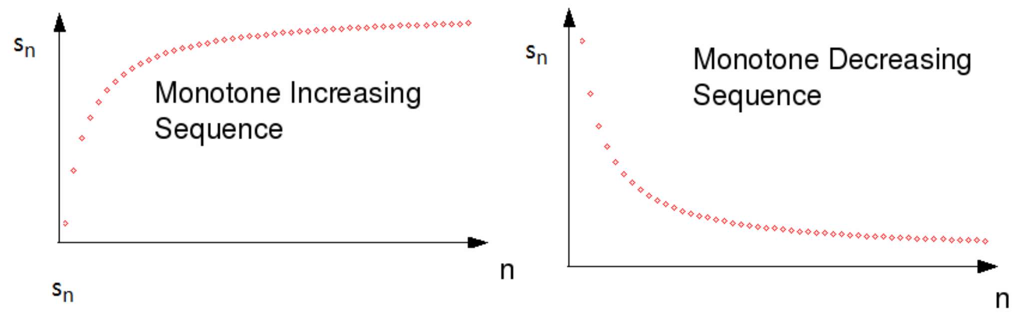
Decreasing Sequence

A sequence $\{s_n\}$ is said to be an decreasing sequence if

$$s_{n+1} \leq s_n \quad \forall\ n \geq 1.$$

Monotonic Sequence

A sequence $\{s_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.



Examples:

- $\{n\} = \{1, 2, 3, \ldots\}$ is an increasing sequence.
- $\left\{\frac{1}{n}\right\}$ is a decreasing sequence.
- $\{\cos n\pi\} = \{-1, 1, -1, 1, \ldots\}$ is neither increasing nor decreasing.

Bounded Sequence

A sequence is said to be bounded if its range is a bounded set.

Definition

A sequence $\{s_n\}$ is said to be bounded if there is a number λ so that

$$|s_n| < \lambda \quad \forall \; n \in \mathbb{N}.$$

Examples

a) $\{u_n\} = \left\{\frac{(-1)^n}{n}\right\}$ is a bounded sequence

b) $\{v_n\} = \{\sin nx\}$ is also bounded sequence. Its supremum is 1 and infimum is -1.

c) The geometric sequence $\{ar^{n-1}\}$, $r > 1$ is an unbounded above sequence. It is bounded below by *a*.

d) $\left\{\tan\frac{n\pi}{2}\right\}$ is an unbounded sequence.

Convergence of the sequence

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

is getting closer and closer to the number 0. We say that this sequence converges to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined?

The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like

> *A sequence* $\{s_n\}$ *converges to a number L if the terms of the sequence get closer and closer to L.*

However this is too vague and too weak to serve as definition but a rough guide for the intuition, this is misleading in other respects. What about the sequence

0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, 0.00001, 0.00002, ...?

Surely this should converge to 0 but the terms do not get steadily "closer and closer" but back off a bit at each second step.

The definition that captured the idea in the best way was given by Augustin Cauchy in the 1820s. He found a formulation that expressed the idea of "arbitrarily close" using inequalities.

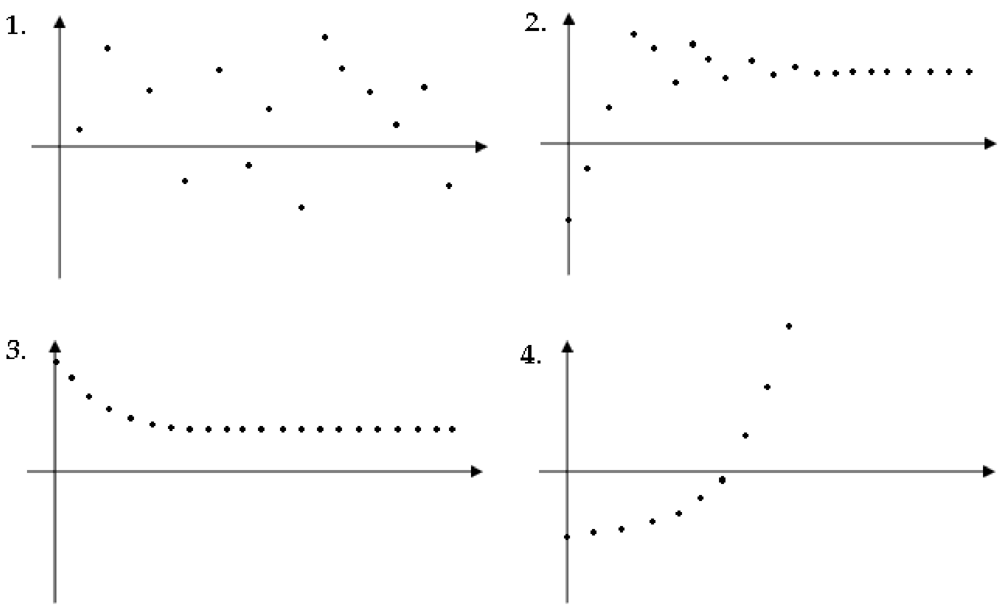
Definition

A sequence $\{s_n\}$ of real numbers is said to convergent to limit 's' as $n \to \infty$, if for every real number $\varepsilon > 0$, there exists a positive integer n_0, depending on ε, so that

$$|s_n - s| < \varepsilon \quad \text{whenever } n > n_0.$$

A sequence that converges is said to be *convergent*. A sequence that fails to converge is said to *divergent*.

We will try to understand it by graph of some sequence. Graph of any four sequences is drawn in the picture below.



Theorem

A convergent sequence of real number has one and only one limit (i.e. limit of the sequence is unique.)

Theorem (Sandwich Theorem or Squeeze Theorem)

Suppose that $\{s_n\}$ and $\{t_n\}$ be two convergent sequences such that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = s$. If $s_n < u_n < t_n \quad \forall\ n \geq n_0$, then the sequence $\{u_n\}$ also converges to *s*.

Cauchy Sequence

A sequence $\{s_n\}$ of real number is said to be a *Cauchy sequence* if for given number $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$|s_n - s_m| < \varepsilon \qquad \forall\ m,n > n_0$$

Theorem

A Cauchy sequence of real numbers is bounded.

Theorem

Let *a* and *b* be fixed real numbers if $\{s_n\}$ and $\{t_n\}$ converge to *s* and *t* respectively, then

(i) $\{as_n + bt_n\}$ converges to *as* + *bt*.

(ii) $\{s_n t_n\}$ converges to *st*.

(iii) $\left\{\dfrac{s_n}{t_n}\right\}$ converges to $\dfrac{s}{t}$, provided $t_n \neq 0 \ \forall\ n$ and $t \neq 0$.

Theorem

For each irrational number *x*, there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim_{n\to\infty} r_n = x$.

Theorem

Let a sequence $\{s_n\}$ be a bounded sequence.

(i) If $\{s_n\}$ is monotonically increasing then it converges to its supremum.

(ii) If $\{s_n\}$ is monotonically decreasing then it converges to its infimum.

Remark:

Let $\{s_n\}$ be a sequence and $\lim_{n\to\infty} s_n = s$. Then $\lim_{n\to\infty} s_{n+1} = s$.

Recurrence Relation

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

Exercises:

- Let $\{t_n\}$ be a positive term sequence. Find the limit of the sequence if $4t_{n+1} = \frac{2}{5} - 3t_n$ for all $n \geq 1$.
- Let $\{u_n\}$ be a sequence of positive numbers. Then find the limit of the sequence if $u_{n+1} = \frac{1}{u_n} + \frac{1}{4}u_{n-1}$ for $n \geq 1$.
- The Fibonacci numbers are: $F_1 = F_2 = 1$, and for every $n \geq 3$, F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Find the $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ (this limit is known as golden number)

Theorem

Every Cauchy sequence of real numbers has a convergent subsequence.

Theorem (Cauchy's General Principle for Convergence)

A sequence of real number is convergent if and only if it is a Cauchy sequence.

Limit Inferior of the sequence

Suppose $\{s_n\}$ is bounded below then we define limit inferior of $\{s_n\}$ as follow

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} u_n, \quad \text{where} \quad u_n = \inf\{s_k : k \geq n\}.$$

If s_n is not bounded below then

$$\liminf_{n\to\infty} s_n = -\infty.$$

Limit Superior of the sequence

Suppose $\{s_n\}$ is bounded above then we define limit superior of $\{s_n\}$ as follow

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} v_n, \quad \text{where} \quad v_n = \sup\{s_k : k \geq n\}$$

If s_n is not bounded above then we have

$$\limsup_{n\to\infty} s_n = +\infty.$$

Theorem

If $\{s_n\}$ is a convergent sequence then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(\inf s_n) = \lim_{n\to\infty}(\sup s_n).$$

Infinite Series

Given a sequence $\{a_n\}$, we use the notation $\sum_{i=1}^{\infty} a_n$ or simply $\sum a_n$ to denotes the sum $a_1 + a_2 + a_3 + \dots$ and called a infinite series or just series.

The numbers $s_n = \sum_{k=1}^{n} a_k$ are called the partial sum of the series.

If the sequence $\{s_n\}$ converges to *s*, we say that the series converges and write $\sum_{n=1}^{\infty} a_n = s$, the number *s* is called the sum of the series but it should be clearly understood that the '*s*' is the limit of the sequence of sums and is not obtained simply by addition.

If the sequence $\{s_n\}$ diverges then the series is said to be diverge.

Note:

The behaviors of the series remain unchanged by addition or deletion of the certain terms

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Note:

The converse of the above theorem is false. For example the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, although $\lim_{n\to\infty} a_n = 0$.

This implies that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent (It is known as basic divergent test).

Theorem (General Principle of Convergence)

A series $\sum a_n$ is convergent if and only if for any real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left| \sum_{i=m+1}^{n} a_i \right| < \varepsilon \qquad \forall \; n > m > n_0 .$$

Theorem

Let $\sum a_n$ be an infinite series of non-negative terms and let $\{s_n\}$ be a sequence of its partial sums then $\sum a_n$ is convergent if $\{s_n\}$ is bounded and it diverges if $\{s_n\}$ is unbounded.

Theorem (Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are infinite series such that $a_n > 0$, $b_n > 0$ $\forall\ n$. Also suppose that for a fixed positive number λ and positive integer k, $a_n < \lambda b_n \quad \forall\ n \geq k$.

(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Example

The series $\sum \frac{1}{n^\alpha}$ is convergent if $\alpha > 1$ and diverges if $\alpha \leq 1$.

Theorem

Let $a_n > 0$, $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lambda \neq 0$ then the series $\sum a_n$ and $\sum b_n$ behave alike.

Theorem (Cauchy Condensation Test)

Let $a_n \geq 0$, $a_n > a_{n+1}$ $\forall\ n \geq 1$. Then the series $\sum a_n$ and $\sum 2^{n-1} a_{2^{n-1}}$ converges or diverges together.

Alternating Series

A series in which successive terms have opposite signs is called an alternating series.

e.g. $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is an alternating series.

Theorem (Alternating Series Test or Leibniz Test)

Let $\{a_n\}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$ then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots \text{ converges.}$$

Absolute Convergence

$\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

Theorem

An absolutely convergent series is convergent.

Note

The converse of the above theorem does not hold.

e.g. $\sum \frac{(-1)^{n+1}}{n}$ is convergent but $\sum \frac{1}{n}$ is divergent.

Theorem (Dirichlet test for infinite series)

Suppose that $\{s_n\}$, $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ is bounded. Let $\{a_n\}$ be positive term decreasing sequence such that $\lim_{n\to\infty} a_n = 0$ and $\{s_n\}$, $s_n = \sum_{k=1}^{n} b_k$ is bounded, then $\sum a_n b_n$ is convergent.

Theorem (Abel's test for infinite series)

If $\{a_n\}$ is monotonic convergent sequence and $\sum b_n$ is convergent then $\sum a_n b_n$ is also convergent.

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