Chapter 2 – Sequences

Course Title: Real Analysis 1 **Course Code:** MTH321
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Course URL: *www.mathcity.org/atiq/fa18-mth321*

Sequences form an important component of Mathematical Analysis and arise in many situations. The first rigorous treatment of sequences was made by George Cantor (1845-1918) and A. Cauchy (1789-1857). A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a finite list is not called a sequence (some authors considered it finite sequence); a sequence must continue without interruption. Formally it is defined as follows:

Sequence

A sequence is a function whose domain of definition is the set of natural numbers.

Notation:

An infinite sequence is usually denoted as

$\{s_n\}_{n=1}^{\infty}$ or $\{s_n : n \in \mathbb{N}\}$ or $\{s_1, s_2, s_3, ...\}$ or simply as $\{s_n\}$ or by (s_n).

But it is not limited to above notations only.

The values s_n are called the *terms* or the *elements* of the sequence $\{s_n\}$.

e.g. i) $\{n\} = \{1, 2, 3, \ldots\}$.

ii) $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

iii) $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \ldots\}$.

iv) $\{2, 3, 5, 7, 11, ...\}$, a sequence of positive prime numbers.

Subsequence

It is a sequence whose terms are contained in given sequence.
A subsequence of $\{s_n\}$ is usually written as $\{s_{n_k}\}$.

Increasing Sequence

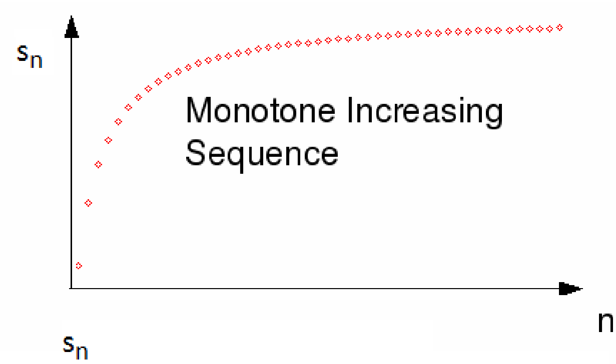
A sequence $\{s_n\}$ is said to be an increasing sequence if $s_{n+1} \geq s_n \quad \forall \; n \geq 1$.

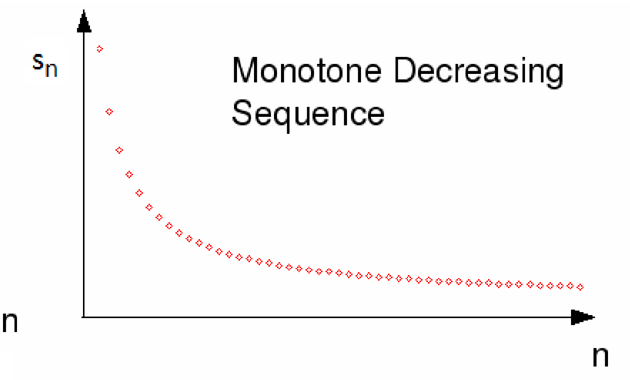
Decreasing Sequence

A sequence $\{s_n\}$ is said to be a decreasing sequence if $s_{n+1} \leq s_n \quad \forall \; n \geq 1$.

Monotonic Sequence

A sequence $\{s_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.





Remarks:

- A sequence $\{s_n\}$ is monotonically increasing if $s_{n+1} - s_n \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically increasing if $\dfrac{s_{n+1}}{s_n} \geq 1, \quad \forall \quad n \geq 1$.
- A sequence $\{s_n\}$ is monotonically decreasing if $s_n - s_{n+1} \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically decreasing if $\dfrac{s_n}{s_{n+1}} \geq 1 \ , \quad \forall \ n \geq 1$.

Strictly Increasing or Decreasing

A sequence $\{s_n\}$ is called strictly increasing or decreasing according as

$$s_{n+1} > s_n \quad \text{or} \quad s_{n+1} < s_n \qquad \forall \ n \geq 1.$$

Examples:

- $\{n\} = \{1, 2, 3, \ldots\}$ is an increasing sequence.
- $\left\{\dfrac{1}{n}\right\}$ is a decreasing sequence.
- $\{\cos n\pi\} = \{-1, 1, -1, 1, \ldots\}$ is neither increasing nor decreasing.

Questions:

1) Prove that $\left\{1 + \dfrac{1}{n}\right\}$ is a decreasing sequence.

2) Is $\left\{\dfrac{n+1}{n+2}\right\}$ is increasing or decreasing sequence?

Bounded Sequence

A sequence $\{s_n\}$ is said to be bounded if there is a number λ such that

$$|s_n| < \lambda \quad \forall \ n \in \mathbb{N}.$$

For such a sequence, every term belongs to the interval $[-\lambda, \lambda]$.
It can be noted that if the sequence is bounded then its supremum and infimum exist.
If S and s are the supremum and infimum of the bounded sequence $\{s_n\}$, then we write

$S = \sup s_n$ and $s = \inf s_n$.

Examples

(i) $\{u_n\} = \left\{\frac{(-1)^n}{n}\right\}$ is a bounded sequence

(ii) $\{v_n\} = \{\sin nx\}$ is also bounded sequence. Its supremum is 1 and infimum is -1.

(iii) The geometric sequence $\{ar^{n-1}\}$, $r > 1$ is an unbounded above sequence. It is bounded below by a.

(iv) $\left\{\tan\frac{n\pi}{2}\right\}$ is an unbounded sequence.

Convergence of the sequence

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

is getting closer and closer to the number 0. We say that this sequence converges to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined?

The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like

A sequence $\{s_n\}$ converges to a number L if the terms of the sequence get closer and closer to L.

However, this is too vague and too weak to serve as definition but a rough guide for the intuition, this is misleading in other respects. What about the sequence

0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, 0.00001, 0.00002, ...?

Surely this should converge to 0 but the terms do not get steadily "closer and closer" but back off a bit at each second step.

The definition that captured the idea in the best way was given by Augustin Cauchy in the 1820s. He found a formulation that expressed the idea of "arbitrarily close" using inequalities.

Definition

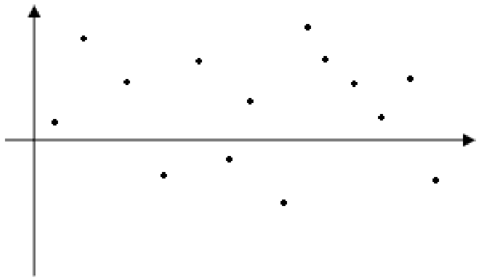
A sequence $\{s_n\}$ of real numbers is said to convergent to limit 's' as $n \to \infty$, if for every real number $\varepsilon > 0$, there exists a positive integer n_0, depending on ε, so that

$$|s_n - s| < \varepsilon \quad \text{whenever } n > n_0.$$

A sequence that converges is said to be *convergent*. A sequence that fails to converge is said to *divergent*.

We will try to understand it by graph of some sequence. Graphs of any four sequences is drawn in the picture below.

1.



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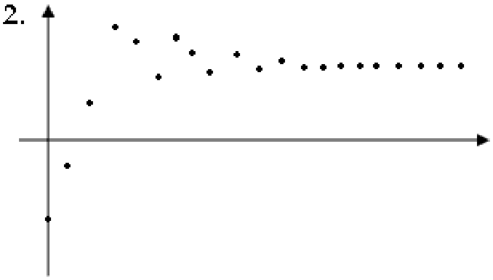

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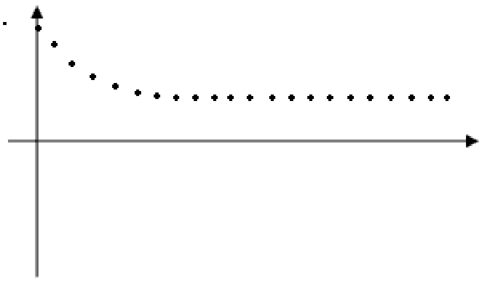

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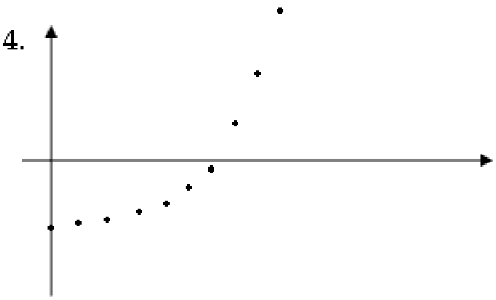

Examples

a) Prove that $\lim_{n\to\infty}\frac{1}{n}=0$ (or $\left\{\frac{1}{n}\right\}$ converges to 0).

Solution: Let $\varepsilon>0$ be given. By the Archimedean Property, there is a positive integer $n_0=n_0(\varepsilon)$ such that $n_0\cdot\varepsilon>1$, that is, $\frac{1}{n_0}<\varepsilon$. Then, if $n\geq n_0$, we have

$$\frac{1}{n}<\frac{1}{n_0}<\varepsilon.$$

Thus we proved that for all $\varepsilon>0$, there exists n_0, depending upon ε, such that

$$\left|\frac{1}{n}-0\right|=\frac{1}{n}<\varepsilon \quad \text{whenever} \quad n\geq n_0.$$

Hence $\left\{\frac{1}{n}\right\}$ converges to point '0'.

b) Prove that $\lim_{n\to\infty}\frac{1}{n^2+1}=0$ (by definition).

Solution: Let $\varepsilon>0$ be given. Now consider

$$\left|\frac{1}{n^2+1}-0\right|=\frac{1}{n^2+1}<\frac{1}{n^2}<\frac{1}{n}. \quad (\because\ n^2+1>n^2>0)$$

Now if we choose n_0 such that $\frac{1}{n_0}<\varepsilon$, then the above expression gives us

$$\left|\frac{1}{n^2+1}-0\right|<\varepsilon \quad \text{whenever} \quad n\geq n_0.$$

Hence, we conclude that, $\lim_{n\to\infty}\frac{1}{n^2+1}=0$.

c) Prove that $\lim_{n\to\infty}\frac{3n+2}{n+1}=3$ (by definition).

Solution: Let $\varepsilon>0$ be given. Now consider

$$\left|\frac{3n+2}{n+1}-3\right| = \left|\frac{3n+2-3n-3}{n+1}\right|$$

$$= \left|\frac{-1}{n+1}\right| < \frac{1}{n+1} < \frac{1}{n} \qquad (\because\ n+1>n>0)$$

Now if we choose n_0 such that $\frac{1}{n_0}<\varepsilon$, then the above expression gives us

$$\left|\frac{3n+2}{n+1}-3\right|<\varepsilon \quad \text{whenever} \quad n\geq n_0.$$

Hence, we conclude that $\lim_{n\to\infty}\frac{3n+2}{n+1}=3$.

Questions:

Use definition of the limits to prove the followings:

a) $\lim_{n\to\infty}\frac{2n}{n+1}=2$ b) $\lim_{n\to\infty}\frac{n^2-1}{2n^2+3}=\frac{1}{2}$

Review

- Triangular inequality: If $a,b\in\mathbb{R}$, then $\big||a|-|b|\big|\leq|a\pm b|\leq|a|+|b|$.
- If $0\leq a<\varepsilon$ for all $\varepsilon>0$, then $a=0$.

Theorem

A convergent sequence of real number has one and only one limit (i.e. limit of the sequence is unique.)

Proof:

Suppose $\{s_n\}$ converges to two limits s and t, where $s\neq t$.

Then for all $\varepsilon>0$, there exists two positive integers n_1 and n_2 such that

$$|s_n-s|<\frac{\varepsilon}{2} \qquad \forall\ n>n_1 \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

and

$$|s_n-t|<\frac{\varepsilon}{2} \qquad \forall\ n>n_2. \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

As (1) and (2) hold simultaneously for all $n>\max(n_1,n_2)$.

Thus for all $n>\max(n_1,n_2)$ we have

$$0\leq|s-t|=|s-s_n+s_n-t|$$

$$\leq|s_n-s|+|s_n-t|$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

As ε is arbitrary, we get $s=t$, that is, the limit of the sequence is unique. ❑

Question

Prove that if $\lim\limits_{n\to\infty} s_n = t$, then $\lim\limits_{n\to\infty} |s_n| = |t|$ but converse is not true in general.

Review:

- For all $a,b,c \in \mathbb{R}$, $|a-b| < c \Leftrightarrow c-b < a < c+b$ or $c-a < b < c+a$.

Theorem (Sandwich Theorem or Squeeze Theorem)

Suppose that $\{s_n\}$ and $\{t_n\}$ be two convergent sequences such that $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n$. If $s_n < u_n < t_n \quad \forall\ n \geq n_0$, then the sequence $\{u_n\}$ also converges to s.

Proof:

Since the sequence $\{s_n\}$ and $\{t_n\}$ converge to the same limit s (say), therefore for given $\varepsilon > 0$ there exists two positive integers $n_1, n_2 > n_0$ such that

$$|s_n - s| < \varepsilon \qquad \forall\ n > n_1,$$

$$|t_n - s| < \varepsilon \qquad \forall\ n > n_2.$$

i.e.

$$s - \varepsilon < s_n < s + \varepsilon \qquad \forall\ n > n_1,$$

$$s - \varepsilon < t_n < s + \varepsilon \qquad \forall\ n > n_2.$$

Since we have given

$$s_n < u_n < t_n \qquad \forall\ n > n_0$$

$$\therefore\ s - \varepsilon < s_n < u_n < t_n < s + \varepsilon \qquad \forall\ n > \max(n_0, n_1, n_2)$$

$$\Rightarrow\ s - \varepsilon < u_n < s + \varepsilon \qquad \forall\ n > \max(n_0, n_1, n_2)$$

i.e. $|u_n - s| < \varepsilon \quad \forall\ n > \max(n_0, n_1, n_2)$

i.e. $\lim\limits_{n\to\infty} u_n = s$. ❑

Example

Show that $\lim\limits_{n\to\infty}\left(\dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \dots + \dfrac{1}{(2n)^2}\right) = 0$.

Solution

Consider

$$s_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}$$

As

$$\underbrace{\frac{1}{(2n)^2} + \frac{1}{(2n)^2} + \dots + \frac{1}{(2n)^2}}_{n \text{ times}} \leq s_n < \underbrace{\frac{1}{n^2} + \frac{1}{n^2} + \dots + \frac{1}{n^2}}_{n \text{ times}},$$

that is,

$$n \cdot \frac{1}{(2n)^2} \leq s_n < n \cdot \frac{1}{n^2}$$

$$\Rightarrow\ \frac{1}{4n} \leq s_n < \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{4n} \le \lim_{n\to\infty} s_n < \lim_{n\to\infty}\frac{1}{n}$$

$$\Rightarrow 0 \le \lim_{n\to\infty} s_n < 0$$

$$\Rightarrow \lim_{n\to\infty} s_n = 0.$$ ❑

Cauchy Sequence

A sequence $\{s_n\}$ of real number is said to be a *Cauchy sequence* if for given number $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$|s_n - s_m| < \varepsilon \qquad \forall\ m,n > n_0$$

Example

The sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

Suppose $s_n = \frac{1}{n}$ and $\varepsilon > 0$ be given. We choose a positive integer $n_0 = n_0(\varepsilon)$ such that $n_0 > \frac{2}{\varepsilon}$.

Then if $m,n \ge n_0$, we have $\frac{1}{n} \le \frac{1}{n_0} < \frac{\varepsilon}{2}$ and similarly $\frac{1}{m} \le \frac{1}{n_0} < \frac{\varepsilon}{2}$. Therefore, it follows that if $m,n \ge n_0$, then

$$|s_n - s_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \le \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\left\{\frac{1}{n}\right\}$ is Cauchy sequence.

Theorem

A Cauchy sequence of real numbers is bounded.

Proof:

Let $\{s_n\}$ be a Cauchy sequence. Then for given number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$|s_n - s_m| < \varepsilon \qquad \forall\ m,n > n_0.$$

Take $\varepsilon = 1$, then we have

$$|s_n - s_m| < 1 \qquad \forall\ m,n > n_0.$$

Fix $m = n_0 + 1$ then

$$\begin{aligned} |s_n| &= |s_n - s_{n_0+1} + s_{n_0+1}| \\ &\le |s_n - s_{n_0+1}| + |s_{n_0+1}| \\ &< 1 + |s_{n_0+1}| \qquad \forall\ n > n_0 \end{aligned}$$

$$= \lambda \qquad \forall \ n > 1 \text{ , and } \lambda = 1 + \left| s_{n_0+1} \right| \quad (n_0 \text{ changes as } \varepsilon \text{ changes})$$

Hence we conclude that $\{s_n\}$ is a Cauchy sequence, which is bounded one. ❑

Note:

(i) Convergent sequence is bounded.

(ii) The converse of the above theorem does not hold.

i.e. every bounded sequence is not Cauchy.

Consider the sequence $\{s_n\}$ where $s_n = (-1)^n$, $n \geq 1$. It is bounded sequence because

$$\left| (-1)^n \right| = 1 < 2 \qquad \forall \ n \geq 1.$$

But it is not a Cauchy sequence if it is then for $\varepsilon = 1$ we should be able to find a positive integer n_0 such that $\left| s_n - s_m \right| < 1$ for all $m, n > n_0$.

But with $m = 2k+1$, $n = 2k+2$ when $2k+1 > n_0$, we arrive at

$$\left| s_n - s_m \right| = \left| (-1)^{2n+2} - (-1)^{2k+1} \right|$$

$$= \left| 1 + 1 \right| = 2 < 1 \qquad \text{is absurd.}$$

Hence $\{s_n\}$ is not a Cauchy sequence. Also this sequence is not a convergent sequence. (it is an oscillatory sequence).

Question:

Prove that every Cauchy sequence of real number is bounded but converse is not true.

Theorem

If the sequence $\{s_n\}$ converges to s then $\exists$ a positive integer n_1 such that $\left| s_n \right| > \frac{1}{2} s$ for all $n > n_1$.

Proof:

We fix $\varepsilon = \frac{1}{2} |s| > 0$

$\Rightarrow \exists$ a positive integer n_1 such that

$$|s_n - s| < \varepsilon \qquad \text{for } n > n_1$$

$$\Rightarrow |s_n - s| < \frac{1}{2}|s|$$

Now

$$\frac{1}{2}|s| = |s| - \frac{1}{2}|s|$$

$$< |s| - |s_n - s| \quad \leq \left| s + (s_n - s) \right|$$

$$\Rightarrow \frac{1}{2}|s| < |s_n|.$$

❑

Theorem

Let a and b be fixed real numbers if $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively, then

(i) $\{as_n + bt_n\}$ converges to $as + bt$.

(ii) $\{s_n t_n\}$ converges to st.

(iii) $\left\{\dfrac{s_n}{t_n}\right\}$ converges to $\dfrac{s}{t}$, provided $t_n \neq 0 \;\; \forall \; n$ and $t \neq 0$.

Proof:

Since $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively,

$$\therefore \;\; |s_n - s| < \varepsilon \qquad \forall \; n > n_1 \in \mathbb{N}$$

$$|t_n - t| < \varepsilon \qquad \forall \; n > n_2 \in \mathbb{N}$$

Also $\exists \;\; \lambda > 0$ such that $|s_n| < \lambda \quad \forall \; n > 1$ $\qquad$ ($\because$ $\{s_n\}$ is bounded)

(i) We have

$$\begin{aligned}
\left|(as_n + bt_n) - (as + bt)\right| &= \left|a(s_n - s) + b(t_n - t)\right| \\
&\leq \left|a(s_n - s)\right| + \left|b(t_n - t)\right| \\
&< |a|\varepsilon + |b|\varepsilon \qquad\qquad \forall \;\; n > \max(n_1, n_2) \\
&= \varepsilon_1,
\end{aligned}$$

where $\varepsilon_1 = |a|\varepsilon + |b|\varepsilon$ a certain number.

This implies $\{as_n + bt_n\}$ converges to $as + bt$.

(ii) $\quad |s_n t_n - st| = |s_n t_n - s_n t + s_n t - st|$

$$\begin{aligned}
&= \left|s_n(t_n - t) + t(s_n - s)\right| \quad \leq |s_n| \cdot |(t_n - t)| + |t| \cdot |(s_n - s)| \\
&< \lambda\varepsilon + |t|\varepsilon \qquad\qquad \forall \;\; n > \max(n_1, n_2) \\
&= \varepsilon_2, \qquad\qquad \text{where } \varepsilon_2 = \lambda\varepsilon + |t|\varepsilon \text{ a certain number.}
\end{aligned}$$

This implies $\{s_n t_n\}$ converges to st.

(iii) $\quad \left|\dfrac{1}{t_n} - \dfrac{1}{t}\right| = \left|\dfrac{t - t_n}{t_n t}\right|$

$$\begin{aligned}
&= \frac{|t_n - t|}{|t_n||t|} < \frac{\varepsilon}{\frac{1}{2}|t||t|} \qquad \forall \;\; n > \max(n_1, n_2) \qquad \because \; |t_n| > \frac{1}{2}t \\
&= \frac{\varepsilon}{\frac{1}{2}|t|^2} = \varepsilon_3, \qquad \text{where } \varepsilon_3 = \frac{\varepsilon}{\frac{1}{2}|t|^2} \text{ a certain number.}
\end{aligned}$$

This implies $\left\{\dfrac{1}{t_n}\right\}$ converges to $\dfrac{1}{t}$.

Hence $\left\{\dfrac{s_n}{t_n}\right\} = \left\{s_n \cdot \dfrac{1}{t_n}\right\}$ converges to $s \cdot \dfrac{1}{t} = \dfrac{s}{t}$. $\;$ (from (ii)) $\qquad$ ❑

Theorem

For each irrational number x, there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim_{n\to\infty} r_n = x$.

Proof:

Since x and $x + 1$ are two different real numbers

$\because$ $\exists$ a rational number r_1 such that

$$x < r_1 < x+1$$

Similarly $\exists$ a rational number $r_2 \neq r_1$ such that

$$x < r_2 < \min\left(r_1, x+\frac{1}{2}\right) < x+1$$

Continuing in this manner we have

$$x < r_3 < \min\left(r_2, x+\frac{1}{3}\right) < x+1$$

$$x < r_4 < \min\left(r_3, x+\frac{1}{4}\right) < x+1$$

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$$x < r_n < \min\left(r_{n-1}, x+\frac{1}{n}\right) < x+1$$

This implies that there is a sequence $\{r_n\}$ of the distinct rational number such that

$$x < r_n < x+\frac{1}{n}.$$

Since

$$\lim_{n\to\infty}(x) = \lim_{n\to\infty}\left(x+\frac{1}{n}\right) = x.$$

Therefore

$$\lim_{n\to\infty} r_n = x.$$ ❑

Theorem

Let a sequence $\{s_n\}$ be a bounded sequence.

(i) If $\{s_n\}$ is monotonically increasing then it converges to its supremum.

(ii) If $\{s_n\}$ is monotonically decreasing then it converges to its infimum.

Proof

(i) Let $S = \sup s_n$ and take $\varepsilon > 0$.

Since there exists s_{n_0} such that $S - \varepsilon < s_{n_0}$

Since $\{s_n\}$ is monotonically increasing,

therefore

$$S-\varepsilon < s_{n_0} < s_n < S < S+\varepsilon \qquad \text{for } n > n_0$$
$$\Rightarrow S-\varepsilon < s_n < S+\varepsilon \qquad \text{for } n > n_0$$
$$\Rightarrow |s_n - S| < \varepsilon \qquad \text{for } n > n_0$$
$$\Rightarrow \lim_{n\to\infty} s_n = S$$

(ii) Let $s = \inf s_n$ and take $\varepsilon > 0$.

Since there exists s_{n_1} such that $s_{n_1} < s+\varepsilon$

Since $\{s_n\}$ is monotonically decreasing,

therefore

$$s-\varepsilon < s < s_n < s_{n_1} < s+\varepsilon \qquad \text{for } n > n_1$$
$$\Rightarrow s-\varepsilon < s_n < s+\varepsilon \qquad \text{for } n > n_1$$
$$\Rightarrow |s_n - s| < \varepsilon \qquad \text{for } n > n_1$$

Thus $\lim_{n\to\infty} s_n = s$ ❑

Question:

1. Let $\{s_n\}$ be a sequence and $\lim_{n\to\infty} s_n = s$. Then prove that $\lim_{n\to\infty} s_{n+1} = s$.
2. Prove that a bounded increasing sequence converges to its supremum.
3. Prove that a bounded decreasing sequence converges to its infimum.

Recurrence Relation

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

Example:

Let $t_1 > 1$ and let $\{t_n\}$ be defined by $t_{n+1} = 2 - \dfrac{1}{t_n}$ for $n \geq 1$.

(i) Show that $\{t_n\}$ is decreasing sequence.

(ii) It is bounded below.

(iii) Find the limit of the sequence.

Since $t_1 > 1$ and $\{t_n\}$ is defined by $t_{n+1} = 2 - \dfrac{1}{t_n}$; $n \geq 1$

$$\Rightarrow t_n > 0 \qquad \forall\ n \geq 1$$

Also $t_n - t_{n+1} = t_n - 2 + \dfrac{1}{t_n}$

$$= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n - 1)^2}{t_n} > 0.$$

$$\Rightarrow t_n > t_{n+1} \qquad \forall\ n \geq 1.$$

This implies that t_n is monotonically decreasing.

Since $t_n > 1 \quad \forall\ n \geq 1$,

$\Rightarrow\ t_n$ is bounded below.

Since t_n is decreasing and bounded below therefore t_n is convergent.

Let us suppose $\lim\limits_{n\to\infty} t_n = t$.

$$\text{Then} \quad \lim_{n\to\infty} t_{n+1} = \lim_{n\to\infty} t_n \quad \Rightarrow \lim_{n\to\infty}\left(2 - \frac{1}{t_n}\right) = \lim_{n\to\infty} t_n$$

$$\Rightarrow\ 2 - \frac{1}{t} = t \quad \Rightarrow\ \frac{2t-1}{t} = t \quad \Rightarrow\ 2t - 1 = t^2 \quad \Rightarrow\ t^2 - 2t + 1 = 0$$

$$\Rightarrow\ (t-1)^2 = 0 \quad \Rightarrow\ t = 1.$$

❑

Question:

- Let $\{t_n\}$ be a positive term sequence. Find the limit of the sequence if $4t_{n+1} = \frac{2}{5} - 3t_n$ for all $n \geq 1$.
- Let $\{u_n\}$ be a sequence of positive numbers. Then find the limit of the sequence if $u_{n+1} = \frac{1}{u_n} + \frac{1}{4}u_{n-1}$ for $n \geq 1$.
- The Fibonacci numbers are: $F_1 = F_2 = 1$, and for every $n \geq 3$, F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Find the $\lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}}$ (this limit is known as golden number)

Theorem

Every Cauchy sequence of real numbers has a convergent subsequence.

Proof:

Suppose $\{s_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ then $\exists$ a positive integer $n_0 \geq 1$ such that

$$\left|s_{n_k} - s_{n_{k-1}}\right| < \frac{\varepsilon}{2^k} \qquad \forall\ n_k, n_{k-1},\ k = 1,2,3,.........$$

$$\text{Put} \quad b_k = \left(s_{n_1} - s_{n_0}\right) + \left(s_{n_2} - s_{n_1}\right) + \ldots + \left(s_{n_k} - s_{n_{k-1}}\right)$$

$$\Rightarrow\ |b_k| = \left|\left(s_{n_1} - s_{n_0}\right) + \left(s_{n_2} - s_{n_1}\right) + \ldots + \left(s_{n_k} - s_{n_{k-1}}\right)\right|$$

$$\leq \left|\left(s_{n_1} - s_{n_0}\right)\right| + \left|\left(s_{n_2} - s_{n_1}\right)\right| + \ldots + \left|\left(s_{n_k} - s_{n_{k-1}}\right)\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^k}$$

$$= \varepsilon\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^k}\right) \quad = \varepsilon\left(\frac{\frac{1}{2}\left(1 - \frac{1}{2^k}\right)}{1 - \frac{1}{2}}\right) = \varepsilon\left(1 - \frac{1}{2^k}\right)$$

$\Rightarrow\ |b_k| < \varepsilon \qquad \forall\ k \geq 1$

This gives $\{b_k\}$ is bounded.

Since $b_{k+1} = b_k + \left(s_{n_{k+1}} - s_{n_k}\right)$

$\Rightarrow\ \{b_k\}$ is convergent

$\because\ b_k = s_{n_k} - s_{n_0} \qquad \therefore\ s_{n_k} = b_k + s_{n_0}$,

where s_{n_0} is a certain fix number therefore $\{s_{n_k}\}$ which is a subsequence of $\{s_n\}$ is convergent. ❑

Theorem (Cauchy's General Principle for Convergence)

A sequence of real number is convergent if and only if it is a Cauchy sequence.

Proof:

Let $\{s_n\}$ be a convergent sequence, which converges to s.

Then for given $\varepsilon > 0$ $\exists$ a positive integer n_0, such that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \forall\ n > n_0$$

Now for $n > m > n_0$

$$\begin{aligned} |s_n - s_m| &= |s_n - s + s - s_m| \\ &\leq |s_n - s| + |s - s_m| = |s_n - s| + |s_m - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\{s_n\}$ is a Cauchy sequence.

Conversely, suppose that $\{s_n\}$ is a Cauchy sequence then for $\varepsilon > 0$, there exists a positive integer m_1 such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \quad \forall\ n, m > m_1 \ \ldots\ldots\ldots\ (i)$$

Since $\{s_n\}$ is a Cauchy sequence,

therefore it has a subsequence $\{s_{n_k}\}$ converging to s (say).

This implies there exists a positive integer m_2 such that

$$|s_{n_k} - s| < \frac{\varepsilon}{2} \qquad \forall\ n > m_2 \ \ldots\ldots\ldots\ (ii)$$

Now

$$\begin{aligned} |s_n - s| &= |s_n - s_{n_k} + s_{n_k} - s| \\ &\leq |s_n - s_{n_k}| + |s_{n_k} - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall\ n > \max(m_1, m_2), \end{aligned}$$

this shows that $\{s_n\}$ is a convergent sequence. ❑

Example

Prove that $\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right\}$ is divergent sequence.

Let $\{t_n\}$ be defined by

$$t_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

For $m,n\in\mathbb{N}$, $n>m$ we have

$$\begin{aligned}|t_n - t_m| &= \frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n}\\ &> \frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n} \quad (n-m \text{ times})\\ &= (n-m)\frac{1}{n} = 1-\frac{m}{n}.\end{aligned}$$

In particular if $n=2m$ then

$$|t_n - t_m| > \frac{1}{2}.$$

This implies that $\{t_n\}$ is not a Cauchy sequence therefore it is divergent. ❑

Theorem (nested intervals)

Suppose that $\{I_n\}$ is a sequence of the closed interval such that $I_n=[a_n,b_n]$, $I_{n+1}\subset I_n \ \forall\ n\geq 1$, and $(b_n-a_n)\to 0$ as $n\to\infty$ then $\bigcap I_n$ contains one and only one point.

Proof:

Since $I_{n+1}\subset I_n$

$\therefore\ a_1<a_2<a_3<\ldots<a_{n-1}<a_n<b_n<b_{n-1}<\ldots<b_3<b_2<b_1$

$\{a_n\}$ is increasing sequence, bounded above by b_1 and bounded below by a_1.

And $\{b_n\}$ is decreasing sequence bounded below by a_1 and bounded above by b_1.

$\Rightarrow$ $\{a_n\}$ and $\{b_n\}$ both are convergent.

Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b.

But $$\begin{aligned}|a-b| &= |a-a_n+a_n-b_n+b_n-b|\\ &\leq |a_n-a|+|a_n-b_n|+|b_n-b| \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

$\Rightarrow a=b$

and $a_n<a<b_n \quad \forall\ n\geq 1$. ❑

....

Limit Inferior of the sequence

Suppose $\{s_n\}$ is bounded below then we define limit inferior of $\{s_n\}$ as follow

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} u_n, \quad \text{where} \quad u_n = \inf\{s_k : k \geq n\}.$$

If s_n is not bounded below then

$$\liminf_{n\to\infty} s_n = -\infty.$$

Limit Superior of the sequence

Suppose $\{s_n\}$ is bounded above then we define limit superior of $\{s_n\}$ as follow

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} v_n, \quad \text{where} \quad v_n = \sup\{s_k : k \geq n\}$$

If s_n is not bounded above then we have

$$\limsup_{n\to\infty} s_n = +\infty.$$

Note:

(i) A bounded sequence has unique limit inferior and superior

(ii) Let $\{s_n\}$ contains all the rational numbers, then every real number is a subsequencial limit then limit superior of s_n is $+\infty$ and limit inferior of s_n is $-\infty$

(iii) Let $\{s_n\} = (-1)^n\left(1+\frac{1}{n}\right)$

then limit superior of s_n is 1 and limit inferior of s_n is -1.

(iv) Let $s_n = \left(1+\frac{1}{n}\right)\cos n\pi$.

Then $u_k = \inf\{s_n : n \geq k\}$

$$= \inf\left\{\left(1+\frac{1}{k}\right)\cos k\pi, \left(1+\frac{1}{k+1}\right)\cos(k+1)\pi, \left(1+\frac{1}{k+2}\right)\cos(k+2)\pi, \ldots\right\}$$

$$= \begin{cases} \left(1+\frac{1}{k}\right)\cos k\pi & \text{if } k \text{ is odd} \\ \left(1+\frac{1}{k+1}\right)\cos(k+1)\pi & \text{if } k \text{ is even} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty}\left(\inf s_n\right) = \lim_{n\to\infty} u_k = -1$$

Also $v_k = \sup\{s_n : n \geq k\}$

$$= \begin{cases} \left(1+\frac{1}{k+1}\right)\cos(k+1)\pi & \text{if } k \text{ is odd} \\ \left(1+\frac{1}{k}\right)\cos k\pi & \text{if } k \text{ is even} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty}\left(\sup s_n\right) = \lim_{n\to\infty} v_k = 1$$

❑

Theorem

If $\{s_n\}$ is a convergent sequence then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(\inf s_n) = \lim_{n\to\infty}(\sup s_n)$$

Proof:

Let $\lim\limits_{n\to\infty} s_n = s$ then for a real number $\varepsilon > 0$, $\exists$ a positive integer n_0 such that

$$|s_n - s| < \varepsilon \qquad \forall\ n \ge n_0 \ \ldots\ldots\ldots. (i)$$

i.e. $\quad s - \varepsilon < s_n < s + \varepsilon \qquad \forall\ n \ge n_0$

If $\quad v_k = \sup\{s_n : n \ge k\}$

Then $\quad s - \varepsilon < v_n < s + \varepsilon \qquad \forall\ k \ge n_0$

$\Rightarrow \quad s - \varepsilon < \lim\limits_{k\to\infty} v_n < s + \varepsilon \qquad \forall\ k \ge n_0 \ \ldots\ldots\ldots\ldots (ii)$

from (i) and (ii) we have

$$s = \lim_{k\to\infty}\ \sup\{s_n\}$$

We can have the same result for limit inferior of $\{s_n\}$ by taking

$$u_k = \inf\{s_n : n \ge k\}\ .$$ ❑

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