Chapter 3 – Series

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Infinite Series

Given a sequence $\{a_n\}$, we use the notation $\sum\limits_{i=1}^{\infty} a_n$ or simply $\sum a_n$ to denotes the sum $a_1 + a_2 + a_3 + \ldots$ and called a infinite series or just series.

The numbers $s_n = \sum\limits_{k=1}^{n} a_k$ are called the partial sum of the series.

If the sequence $\{s_n\}$ converges to *s,* we say that the series converges and write $\sum\limits_{n=1}^{\infty} a_n = s$, the number *s* is called the sum of the series but it should be clearly understood that the '*s*' is the limit of the sequence of sums and is not obtained simply by addition.

If the sequence $\{s_n\}$ diverges then the series is said to be diverge.

Note:

The behaviors of the series remain unchanged by addition or deletion of the certain terms

Theorem

If $\sum\limits_{n=1}^{\infty} a_n$ converges then $\lim\limits_{n\to\infty} a_n = 0$.

Proof

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$

Take $\lim\limits_{n\to\infty} s_n = s = \sum a_n$

Since $a_n = s_n - s_{n-1}$

Therefore
$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(s_n - s_{n-1}\right)\\ &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}\\ &= s - s = 0\end{aligned}$$
❑

Note:

The converse of the above theorem is false. For example consider the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$.

We know that the sequence $\{s_n\}$ where $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}$ is divergent therefore $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ is divergent series, although $\lim\limits_{n\to\infty} a_n = 0$.

This implies that if $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent (It is known as basic divergent test).

Theorem (General Principle of Convergence)

A series $\sum a_n$ is convergent if and only if for any real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left| \sum_{i=m+1}^{n} a_i \right| < \varepsilon \qquad \forall \;\; n > m > n_0$$

Proof

Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$

then $\{s_n\}$ is convergent if and only if for $\varepsilon > 0$ $\exists$ a positive integer n_0 such that

$$|\; s_n - s_m \;| < \varepsilon \qquad \forall \;\; n > m > n_0$$

$$\Rightarrow \left| \sum_{i=m+1}^{n} a_i \right| = |\; s_n - s_m \;| < \varepsilon$$

❑

Theorem

Let $\sum a_n$ be an infinite series of non-negative terms and let $\{s_n\}$ be a sequence of its partial sums then $\sum a_n$ is convergent if $\{s_n\}$ is bounded and it diverges if $\{s_n\}$ is unbounded.

Proof

Since $a_n \geq 0 \;\; \forall \;\; n \geq 0$ and $s_n = s_{n-1} + a_n > s_{n-1} \;\; \forall \;\; n \geq 0$,

therefore the sequence $\{s_n\}$ is monotonic increasing and hence it is converges if $\{s_n\}$ is bounded and it will diverge if it is unbounded.

Hence we conclude that $\sum a_n$ is convergent if $\{s_n\}$ is bounded and it divergent if $\{s_n\}$ is unbounded.

❑

Theorem (Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are infinite series such that $a_n > 0$, $b_n > 0 \;\; \forall \;\; n$. Also suppose that for a fixed positive number λ and positive integer k, $a_n < \lambda b_n \;\; \forall \; n \geq k$

Then $\sum a_n$ converges if $\sum b_n$ is converges and $\sum b_n$ is diverges if $\sum a_n$ is diverges.

Proof

Suppose $\sum b_n$ is convergent and

$$a_n < \lambda b_n \qquad \forall \; n \geq k \;\; \ldots\ldots\ldots\ldots \; (i)$$

then for any positive number $\varepsilon > 0$ there exists n_0 such that

$$\sum_{i=m+1}^{n} b_i < \frac{\varepsilon}{\lambda} \qquad n > m > n_0$$

from (i)

$$\Rightarrow \sum_{i=m+1}^{n} a_i < \lambda \sum_{i=m+1}^{n} b_i < \varepsilon \qquad , \qquad n > m > n_0$$

$$\Rightarrow \sum a_n \text{ is convergent.}$$

Now suppose $\sum a_n$ is divergent then $\{S_n\}$ is unbounded.

$\Rightarrow$ $\exists$ a real number $\beta > 0$ such that

$$\sum_{i=m+1}^{n} b_i > \lambda\beta \qquad , \quad n > m$$

from (i)

$$\Rightarrow \sum_{i=m+1}^{n} b_i > \frac{1}{\lambda}\sum_{i=m+1}^{n} a_i > \beta \qquad , \quad n > m$$

$\Rightarrow$ $\sum b_n$ is convergent. ❑

Example

Prove that $\sum \frac{1}{\sqrt{n}}$ is divergent.

We know that $\sum \frac{1}{n}$ is divergent and

$$n \geq \sqrt{n} \qquad \forall \ n \geq 1$$

$$\Rightarrow \frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

$\Rightarrow$ $\sum \frac{1}{\sqrt{n}}$ is divergent as $\sum \frac{1}{n}$ is divergent. ❑

Example

The series $\sum \frac{1}{n^{\alpha}}$ is convergent if $\alpha > 1$ and diverges if $\alpha \leq 1$.

Let $s_n = 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \dots\dots\dots\dots\dots + \frac{1}{n^{\alpha}}$.

If $\alpha > 1$ then

$$s_n < s_{2n} \quad \text{and} \quad \frac{1}{n^{\alpha}} < \frac{1}{(n-1)^{\alpha}}.$$

Now
$$S_{2n} = \left[1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \frac{1}{4^{\alpha}} + \dots + \frac{1}{(2n)^{\alpha}}\right]$$

$$= \left[1 + \frac{1}{3^{\alpha}} + \frac{1}{5^{\alpha}} + \dots + \frac{1}{(2n-1)^{\alpha}}\right] + \left[\frac{1}{2^{\alpha}} + \frac{1}{4^{\alpha}} + \frac{1}{6^{\alpha}} + \dots + \frac{1}{(2n)^{\alpha}}\right]$$

$$= \left[1 + \frac{1}{3^{\alpha}} + \frac{1}{5^{\alpha}} + \dots + \frac{1}{(2n-1)^{\alpha}}\right] + \frac{1}{2^{\alpha}}\left[1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \dots + \frac{1}{(n)^{\alpha}}\right]$$

$$< \left[1 + \frac{1}{2^{\alpha}} + \frac{1}{4^{\alpha}} + \dots + \frac{1}{(2n-2)^{\alpha}}\right] + \frac{1}{2^{\alpha}} s_n \qquad \text{(replacing 3 by 2, 5 by 4 and so on.)}$$

$$= 1 + \frac{1}{2^{\alpha}}\left[1 + \frac{1}{2^{\alpha}} + \dots + \frac{1}{(n-1)^{\alpha}}\right] + \frac{1}{2^{\alpha}} s_n$$

$$=1+\frac{1}{2^\alpha}s_{n-1}+\frac{1}{2^\alpha}s_n \quad =1+\frac{1}{2^\alpha}s_{2n}+\frac{1}{2^\alpha}s_{2n} \qquad \because\ s_{n-1}<s_n<s_{2n}$$

$$=1+\frac{2}{2^\alpha}s_{2n}$$

$$\Rightarrow\ s_{2n}<1+\frac{1}{2^{\alpha-1}}s_{2n}.$$

$$\Rightarrow\ \left(1-\frac{1}{2^{\alpha-1}}\right)s_{2n}<1 \quad \Rightarrow\ \left(\frac{2^{\alpha-1}-1}{2^{\alpha-1}}\right)s_{2n}<1 \quad \Rightarrow\ s_{2n}<\frac{2^{\alpha-1}}{2^{\alpha-1}-1},$$

i.e. $s_n<s_{2n}<\dfrac{2^{\alpha-1}}{2^{\alpha-1}-1}$

$\Rightarrow \{s_n\}$ is bounded and also monotonic. Hence we conclude that $\sum\frac{1}{n^\alpha}$ is convergent when $\alpha>1$.

If $\alpha\le 1$ then

$$n^\alpha \le n \quad \forall\ n\ge 1$$

$$\Rightarrow\ \frac{1}{n^\alpha}\ge\frac{1}{n} \quad \forall\ n\ge 1$$

$\because\ \sum\frac{1}{n}$ is divergent therefore $\sum\frac{1}{n^\alpha}$ is divergent when $\alpha\le 1$. ❑

Theorem

Let $a_n>0$, $b_n>0$ and $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lambda\ne 0$ then the series $\sum a_n$ and $\sum b_n$ behave alike.

Proof

Since $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lambda$, therefore for $\varepsilon>0$, there exists integer n_0 such that

$$\left|\frac{a_n}{b_n}-\lambda\right|<\varepsilon \quad \forall\ n\ge n_0.$$

Use $\varepsilon=\dfrac{\lambda}{2}$

$$\Rightarrow\ \left|\frac{a_n}{b_n}-\lambda\right|<\frac{\lambda}{2} \quad \forall\ n\ge n_0. \quad \Rightarrow\ \lambda-\frac{\lambda}{2}<\frac{a_n}{b_n}<\lambda+\frac{\lambda}{2}$$

$$\Rightarrow\ \frac{\lambda}{2}<\frac{a_n}{b_n}<\frac{3\lambda}{2}.$$

Then we got

$$a_n<\frac{3\lambda}{2}b_n \quad \text{and} \quad b_n<\frac{2}{\lambda}a_n.$$

Hence by comparison test we conclude that $\sum a_n$ and $\sum b_n$ converge or diverge together. ❑

Example

Is the series $\sum \frac{1}{n}\sin^2\frac{x}{n}$ is convergent or divergent.

Consider $a_n = \frac{1}{n}\sin^2\frac{x}{n}$ and take $b_n = \frac{1}{n^3}$.

Then $\frac{a_n}{b_n} = n^2\sin^2\frac{x}{n} = \dfrac{\sin^2\frac{x}{n}}{\frac{1}{n^2}} = x^2\left(\dfrac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2$

Applying limit as $n \to \infty$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} x^2\left(\dfrac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2 = x^2\left(\lim_{n\to\infty}\dfrac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2 = x^2(1) = x^2$$

$\Rightarrow$ $\sum a_n$ and $\sum b_n$ have the similar behavior $\forall$ finite values of x except $x = 0$.

Since $\sum\frac{1}{n^3}$ is convergent series therefore the given series is also convergent for finite values of x except $x = 0$. ❑

Theorem (Cauchy Condensation Test)

Let $a_n \ge 0$, $a_n > a_{n+1}$ $\forall$ $n \ge 1$, then the series $\sum a_n$ and $\sum 2^{n-1}a_{2^{n-1}}$ converges or diverges together.

Proof

The condensation test follows from noting that if we collect the terms of the series into groups of lengths 2^n, each of these groups will be less than $2^n a_{2^n}$ by monotonicity. Observe,

$$\sum_{n=1}^{\infty} a_n = a_1 + \underbrace{a_2 + a_3}_{\le a_2+a_2} + \underbrace{a_4 + a_5 + a_6 + a_7}_{\le a_4+a_4+a_4+a_4} + \cdots + \underbrace{a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1}}_{\le a_{2^n}+a_{2^n}+\cdots+a_{2^n}} + \cdots$$

$$\le a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} + \cdots = \sum_{n=0}^{\infty} 2^n a_{2^n}.$$

We have use the fact that a_n is decreasing sequence. The convergence of the original series now follows from direct comparison to this "condensed" series. To see that convergence of the original series implies the convergence of this last series, we similarly put,

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \underbrace{a_1 + a_2}_{\le a_1+a_1} + \underbrace{a_2 + a_4 + a_4 + a_4}_{\le a_2+a_2+a_3+a_3} + \cdots + \underbrace{a_{2^n} + a_{2^{n+1}} + \cdots + a_{2^{n+1}}}_{\le a_{2^n}+a_{2^n}+a_{(2^n+1)}+a_{(2^n+1)}+\cdots+a_{(2^{n+1}-1)}} + \cdots$$

$$\le a_1 + a_1 + a_2 + a_2 + a_3 + a_3 + \cdots + a_n + a_n + \cdots = 2\sum_{n=1}^{\infty} a_n.$$

And we have convergence, again by direct comparison. And we are done. Note that we have obtained the estimate

$$\sum_{n=1}^{\infty} a_n \le \sum_{n=0}^{\infty} 2^n a_{2^n} \le 2\sum_{n=1}^{\infty} a_n . \qquad \square$$

Example

Find value of p for which $\sum \frac{1}{n^p}$ is convergent or divergent.

If $p \le 0$ then $\lim_{n\to\infty} \frac{1}{n^p} \ne 0$, therefore the series diverges when $p \le 0$.

If $p > 0$ then the condensation test is applicable and we are lead to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{2^{kp-k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{(p-1)k}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{(p-1)}} \right)^k = \sum_{k=0}^{\infty} 2^{(1-p)k} .$$

Now $2^{1-p} < 1$ iff $1-p < 0$ i.e. when $p > 1$.

And the result follows by comparing this series with the geometric series having common ratio less than one.

The series diverges when $2^{1-p} = 1$ (i.e. when $p = 1$).

The series is also divergent if $0 < p < 1$. $\square$

Example

Prove that if $p > 1$, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges and if $p \le 1$ the series is divergent.

$\because \{\ln n\}$ is increasing $\therefore \left\{ \frac{1}{n \ln n} \right\}$ decreases

and we can use the condensation test to the above series.

We have $a_n = \dfrac{1}{n(\ln n)^p}$

$$\Rightarrow a_{2^n} = \frac{1}{2^n \left(\ln 2^n\right)^p} \qquad \Rightarrow 2^n a_{2^n} = \frac{1}{(n \ln 2)^p}$$

Now $\sum 2^n a_{2^n} = \sum \frac{1}{(n\ln 2)^p} = \frac{1}{(\ln 2)^p} \sum \frac{1}{n^p}$.

This converges when $p > 1$ and diverges when $p \le 1$. $\square$

Example

Prove that $\sum \frac{1}{\ln n}$ is divergent.

Since $\{\ln n\}$ is increasing there $\left\{ \frac{1}{\ln n} \right\}$ decreases.

And we can apply the condensation test to check the behavior of the series

$$\because \; a_n = \frac{1}{\ln n} \qquad \therefore \; a_{2^n} = \frac{1}{\ln 2^n}$$

so $\quad 2^n a_{2^n} = \dfrac{2^n}{\ln 2^n} \quad \Rightarrow \quad 2^n a_{2^n} = \dfrac{2^n}{n \ln 2}$

since $\quad \dfrac{2^n}{n} > \dfrac{1}{n} \qquad \forall \; n \geq 1$

and $\sum \dfrac{1}{n}$ is diverges therefore the given series is also diverges. ❑

Alternating Series

A series in which successive terms have opposite signs is called an alternating series.

e.g. $\sum \dfrac{(-1)^{n+1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots$ is an alternating series.

Theorem (Alternating Series Test or Leibniz Test)

Let $\{a_n\}$ be a decreasing sequence of positive numbers such that $\lim\limits_{n\to\infty} a_n = 0$ then the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$ converges.

Proof

Looking at the odd numbered partial sums of this series we find that

$$s_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \ldots + (a_{2n-1} - a_{2n}) + a_{2n+1}$$

Since $\{a_n\}$ is decreasing therefore all the terms in the parenthesis are non-negative

$$\Rightarrow \; s_{2n+1} > 0 \quad \forall \; n$$

Moreover

$$\begin{aligned} s_{2n+3} &= s_{2n+1} - a_{2n+2} + a_{2n+3} \\ &= s_{2n+1} - \left(a_{2n+2} - a_{2n+3}\right) \end{aligned}$$

Since $\; a_{2n+2} - a_{2n+3} \geq 0 \;$ therefore $\; s_{2n+3} \leq s_{2n+1}$

Hence the sequence of odd numbered partial sum is decreasing and is bounded below by zero. (as it has +ive terms)

It is therefore convergent.

Thus s_{2n+1} converges to some limit l (say).

Now consider the even numbered partial sum. We find that

$$s_{2n+2} = s_{2n+1} - a_{2n+2}$$

and

$$\begin{aligned} \lim_{n\to\infty} s_{2n+2} &= \lim_{n\to\infty}\left(s_{2n+1} - a_{2n+2}\right) \\ &= \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} a_{2n+2} = l - 0 = l \qquad \because \; \lim_{n\to\infty} a_n = 0. \end{aligned}$$

so that the even partial sum is also convergent to l.

$\Rightarrow$ both sequences of odd and even partial sums converge to the same limit.

Hence we conclude that the corresponding series is convergent. ❑

Absolute Convergence

$\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

Theorem

An absolutely convergent series is convergent.

Proof:

If $\sum |a_n|$ is convergent then for a real number $\varepsilon > 0$, $\exists$ a positive integer n_0 such that

$$\left| \sum_{i=m+1}^{n} a_i \right| < \sum_{i=m+1}^{n} |a_i| < \varepsilon \qquad \forall\ n, m > n_0$$

$\Rightarrow$ the series $\sum a_n$ is convergent. (Cauchy Criterion has been used)

Note

The converse of the above theorem does not hold.

e.g. $\sum \dfrac{(-1)^{n+1}}{n}$ is convergent but $\sum \dfrac{1}{n}$ is divergent. ❑

Theorem (The Root Test)

Let $\limsup\limits_{n\to\infty} |a_n|^{1/n} = p$

Then $\sum a_n$ converges absolutely if $p < 1$ and it diverges if $p > 1$.

Proof

Let $p < 1$ then we can find the positive number $\varepsilon > 0$ such that $p + \varepsilon < 1$

$\Rightarrow |a_n|^{1/n} < p + \varepsilon < 1 \quad \forall\ n > n_0$

$\Rightarrow |a_n| < (p + \varepsilon)^n < 1$

$\because \sum (p+\varepsilon)^n$ is convergent because it is a geometric series with $|r| < 1$.

$\therefore \sum |a_n|$ is convergent

$\Rightarrow \sum a_n$ converges absolutely.

Now let $p > 1$ then we can find a number $\varepsilon_1 > 0$ such that $p - \varepsilon_1 > 1$.

$\Rightarrow |a_n|^{1/n} > p + \varepsilon > 1$

$\Rightarrow |a_n| > 1$ for infinitely many values of *n*.

$\Rightarrow \lim\limits_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum a_n$ is divergent. ❑

Note:

The above test give no information when $p = 1$.

e.g. Consider the series $\sum \dfrac{1}{n}$ and $\sum \dfrac{1}{n^2}$.

For each of these series $p=1$, but $\sum\frac{1}{n}$ is divergent and $\sum\frac{1}{n^2}$ is convergent.

Theorem (Ratio Test)

The series $\sum a_n$

(i) Converges if $\limsup\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

(ii) Diverges if $\left|\frac{a_{n+1}}{a_n}\right|>1$ for $n\geq n_0$, where n_0 is some fixed integer.

Proof

If (i) holds we can find $\beta<1$ and integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \quad \text{for } n\geq N$$

In particular

$$\left|\frac{a_{N+1}}{a_N}\right| < \beta$$

$$\Rightarrow \left|a_{N+1}\right| < \beta\left|a_N\right|$$

$$\Rightarrow \left|a_{N+2}\right| < \beta\left|a_{N+1}\right| < \beta^2\left|a_N\right|$$

$$\Rightarrow \left|a_{N+3}\right| < \beta^3\left|a_N\right|$$

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$$\Rightarrow \left|a_{N+p}\right| < \beta^p\left|a_N\right|$$

$$\Rightarrow \left|a_n\right| < \beta^{n-N}\left|a_N\right| \qquad \text{we put } N+p=n.$$

i.e. $\left|a_n\right| < \left|a_N\right|\beta^{-N}\beta^n$ for $n\geq N$.

$\because \sum\beta^n$ is convergent because it is geometric series with common ration <1.

Therefore $\sum a_n$ is convergent (by comparison test)

Now if

$$\left|a_{n+1}\right|\geq\left|a_n\right| \qquad \text{for } n\geq n_0$$

then $\lim\limits_{n\to\infty}a_n\neq 0$

$\Rightarrow \sum a_n$ is divergent. ❑

Note

The knowledge $\left|\frac{a_{n+1}}{a_n}\right|=1$ implies nothing about the convergent or divergent of series.

Example

Prove that series $\sum a_n$ with $a_n = \left[\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^{n+1}\right]^{-n}$, is divergent.

Since $\frac{n}{n+1} < 1$, therefore $a_n > 0 \quad \forall \; n$.

Also $(a_n)^{\frac{1}{n}} = \left[\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^{n+1}\right]^{-1}$

$$= \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n}{n+1}\right)^{n}\right]^{-1} = \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n+1}{n}\right)^{-n}\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \lim_{n\to\infty}\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= 1\cdot\left[1-e^{-1}\right]^{-1} = \left[1-\frac{1}{e}\right]^{-1} = \left[\frac{e-1}{e}\right]^{-1} = \left[\frac{e}{e-1}\right] > 1$$

$\Rightarrow$ the series is divergent. ❑

Theorem (Dirichlet)

Suppose that $\{s_n\}$, $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ is bounded. Let $\{b_n\}$ be positive term decreasing sequence such that $\lim\limits_{n\to\infty} b_n = 0$, then $\sum a_n b_n$ is convergent.

Proof

Since $\{s_n\}$ is bounded,

therefore there exists a positive number λ such that

$$|s_n| < \lambda \qquad \forall \; n \geq 1.$$

Then $\quad a_i b_i = (s_i - s_{i-1}) b_i \qquad$ for $i \geq 2$

$$= s_i b_i - s_{i-1} b_i$$

$$= s_i b_i - s_{i-1} b_i + s_i b_{i+1} - s_i b_{i+1}$$

$$= s_i (b_i - b_{i+1}) - s_{i-1} b_i + s_i b_{i+1}$$

$$\Rightarrow \sum_{i=m+1}^{n} a_i b_i = \sum_{i=m+1}^{n} s_i (b_i - b_{i+1}) - (s_m b_{m+1} - s_n b_{n+1})$$

$\because \; \{b_n\}$ is decreasing

$$\therefore \left|\sum_{i=m+1}^{n} a_i b_i\right| = \left|\sum_{i=m+1}^{n} s_i\left(b_i - b_{i+1}\right) - s_m b_{m+1} + s_n b_{n+1}\right|$$

$$< \sum_{i=m+1}^{n} \{|s_i|(b_i - b_{i+1})\} + |s_m| b_{m+1} + |s_n| b_{n+1}$$

$$< \sum_{i=m+1}^{n} \{\lambda(b_i - b_{i+1})\} + \lambda b_{m+1} + \lambda b_{n+1} \qquad \because |s_i| < \lambda$$

$$= \lambda\left(\sum_{i=m+1}^{n} (b_i - b_{i+1}) + b_{m+1} + b_{n+1}\right)$$

$$= \lambda\left((b_{m+1} - b_{n+1}) + b_{m+1} + b_{n+1}\right) = 2\lambda(b_{m+1})$$

$\Rightarrow \left|\sum_{i=m+1}^{n} a_i b_i\right| < \varepsilon$ where $\varepsilon = 2\lambda(b_{m+1})$ a certain number

$\Rightarrow$ The $\sum a_n b_n$ is convergent. (We have use Cauchy Criterion here.) ❑

Theorem

Suppose that $\sum a_n$ is convergent and that $\{b_n\}$ is monotonic convergent sequence then $\sum a_n b_n$ is also convergent.

Proof

Suppose $\{b_n\}$ is decreasing and it converges to b.

Put $c_n = b_n - b$

$\Rightarrow$ $c_n \geq 0$ and $\lim_{n\to\infty} c_n = 0$.

Since $\sum a_n$ is convergent,

therefore $\{s_n\}$, $s_n = a_1 + a_2 + \ldots + a_n$ is convergent.

$\Rightarrow$ It is bounded,

$\Rightarrow$ $\sum a_n c_n$ is bounded.

Since $a_n b_n = a_n c_n + a_n b$ and $\sum a_n c_n$ and $\sum a_n b$ are convergent,

therefore $\sum a_n b_n$ is convergent.

Now if $\{b_n\}$ is increasing and converges to b then we shall put $c_n = b - b_n$. ❑

Example

A series $\sum \frac{1}{(n\ln n)^{\alpha}}$ is convergent if $\alpha > 1$ and divergent if $\alpha \leq 1$.

To see this we proceed as follows

$$a_n = \frac{1}{(n\ln n)^{\alpha}}$$

Take $b_n = 2^n a_{2^n} = \dfrac{2^n}{\left(2^n \ln 2^n\right)^{\alpha}} = \dfrac{2^n}{\left(2^n\ n \ln 2\right)^{\alpha}}$

$$= \frac{2^n}{2^{n\alpha} n^\alpha (\ln 2)^\alpha} = \frac{1}{2^{n\alpha - n} n^\alpha (\ln 2)^\alpha}$$

$$= \frac{1}{(\ln 2)^\alpha} \cdot \frac{\left(\frac{1}{2}\right)^{(\alpha-1)n}}{n^\alpha}$$

Since $\sum \frac{1}{n^\alpha}$ is convergent when $\alpha > 1$ and $\left(\frac{1}{2}\right)^{(\alpha-1)n}$ is decreasing for $\alpha > 1$ and it converges to 0. Therefore $\sum b_n$ is convergent

$\Rightarrow \sum a_n$ is also convergent.

Now $\sum b_n$ is divergent for $\alpha \le 1$ therefore $\sum a_n$ diverges for $\alpha \le 1$. ❑

Example

To check $\sum \frac{1}{n^\alpha \ln n}$ is convergent or divergent.

We have $a_n = \frac{1}{n^\alpha \ln n}$

Take $b_n = 2^n a_{2^n} = \frac{2^n}{(2^n)^\alpha (\ln 2^n)} = \frac{2^n}{2^{n\alpha} (n \ln 2)}$

$$= \frac{1}{\ln 2} \cdot \frac{2^{(1-\alpha)n}}{n} = \frac{1}{\ln 2} \cdot \frac{\left(\frac{1}{2}\right)^{(\alpha-1)n}}{n}$$

$\because \sum \frac{1}{n}$ is divergent although $\left\{\left(\frac{1}{2}\right)^{n(\alpha-1)}\right\}$ is decreasing, tending to zero for $\alpha > 1$ therefore $\sum b_n$ is divergent.

$\Rightarrow \sum a_n$ is divergent.

The series also divergent if $\alpha \le 1$.
i.e. it is always divergent. ❑

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