Ch 02: Functions Defined by Improper Integrals

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In this chapter, a function constructed by an improper integral is discussed. Such functions have many applications in mathematics and related sciences. For examples Laplace Transform, Gamma function, Bessel's integrals and etc. Sometime these functions appears as solution of differential equations, which help us study and solve many real life problems.

➢ *Definition*

Let f be a real valued function of two variables x & y, $x \in [a,+\infty)$, $y \in S$ where $S \subset \mathbb{R}$. Suppose further that, for each y in S, the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ is convergent. If F denotes the function defined by the equation

$$F(y) = \int_a^\infty f(x,y)\,d\alpha(x) \qquad \text{for} \quad y \in S,$$

Then the integral is said to converge *pointwise* to F on S or we say it converges *pointwise* on S.

➢ *Example:*

Consider the function $f(x,y) = ye^{-|y|x}$ on $[0,\infty)\times(-\infty,\infty)$.

Now $$F(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty ye^{-|y|x}\,dx$$

$$= \lim_{b\to\infty}\int_0^b ye^{-|y|x}\,dx = \lim_{b\to\infty}\left[\frac{ye^{-|y|x}}{-|y|}\right]_0^b, \quad \text{when } y \neq 0,$$

$$= \lim_{b\to\infty}\frac{y}{|y|}\left(1-\frac{1}{e^{b|y|}}\right) \quad = \frac{y}{|y|} = \begin{cases} -1 & y<0, \\ 1 & y>0. \end{cases}$$

When $y=0$, then we have $F(0)=0$, that is, $F(y) = \begin{cases} -1 & y<0, \\ 0 & y=0, \\ 1 & y>0. \end{cases}$

➢ *Example:*

Consider the function $f(x,y) = y^3e^{-y^2x}$ on $[0,\infty)\times(-\infty,\infty)$.

Now $$F(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty y^3e^{-y^2x}\,dx$$

$$= \lim_{b\to\infty}\int_0^b y^3e^{-y^2x}\,dx = \lim_{b\to\infty}\left[\frac{y^3e^{-y^2x}}{-y^2}\right]_0^b = \lim_{b\to\infty} y\left(1-\frac{1}{e^{y^2x}}\right)$$

$$= y \quad \text{for all } y \in (-\infty, \infty).$$

➢ *Definition*

Let f be a real valued function of two variables x and y, $x \in [a, +\infty)$, $y \in S$ where $S \subset \mathbb{R}$. Denote $F(y) = \int_a^\infty f(x,y)\,d\alpha(x)$ for $y \in S$ and assume that the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges pointwise to F on S. The integral is said to converge uniformly on S if, for every $\varepsilon > 0$ there exists a $B > 0$ (depending only on ε) such that $b > B$ implies

$$\left| F(y) - \int_a^b f(x,y)\,d\alpha(x) \right| < \varepsilon \qquad \forall \;\; y \in S.$$

➢ *Remarks:*

- Pointwise convergence means convergence when y is fixed but uniform convergence is for every $y \in S$.

➢ *Theorem (Cauchy condition for uniform convergence.)*

The integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S, iff, for every $\varepsilon > 0$ there exists a $B > 0$ (depending on ε) such that $c > b > B$ implies

$$\left| \int_b^c f(x,y)\,d\alpha(x) \right| < \varepsilon \qquad \forall \;\; y \in S.$$

Proof

Proceed as in the proof for Cauchy condition for infinite integral $\int_a^\infty f\,d\alpha$.

➢ *Review*

If $f, |f| \in R(\alpha, a, b)$, then one has $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$.

➢ *Theorem (Weierstrass M-test)*

Assume that $\alpha \uparrow$ on $[a, +\infty)$ and suppose that the integral $\int_a^b f(x,y)\,d\alpha(x)$ exists for every $b \ge a$ and for every y in S. If there is a positive function M defined on $[a, +\infty)$ such that the integral $\int_a^\infty M(x)\,d\alpha(x)$ converges and $|f(x,y)| \le M(x)$ for each $x \ge a$ and every y in S, then the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

Proof

Since $|f(x,y)| \le M(x)$ for each $x \ge a$ and every y in S,
therefore For every $c \ge b$, we have

$$\left|\int_b^c f(x,y)\,d\alpha(x)\right| \le \int_b^c |f(x,y)|\,d\alpha(x) \le \int_b^c M\,d\alpha \;\;\ldots\ldots\ldots\ldots (i)$$

Since $I := \int_a^\infty M\,d\alpha$ is convergent,

for given $\varepsilon > 0$, $\exists\; B > 0$ such that $b > B$ implies

$$\left|\int_a^b M\,d\alpha - I\right| < \varepsilon/2 \;\;\ldots\ldots\ldots\ldots\ldots (ii)$$

Also if $c > b > B$, then

$$\left|\int_a^c M\,d\alpha - I\right| < \varepsilon/2 \;\;\ldots\ldots\ldots\ldots\ldots (iii)$$

Since $\int_a^c M\,d\alpha = \int_a^b M\,d\alpha + \int_b^c M\,d\alpha$,

then $\left|\int_b^c M\,d\alpha\right| = \left|\int_a^c M\,d\alpha - \int_a^b M\,d\alpha\right|$

$$= \left|\int_a^c M\,d\alpha - I + I - \int_a^b M\,d\alpha\right|$$

$$\le \left|\int_a^c M\,d\alpha - I\right| + \left|\int_a^b M\,d\alpha - I\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{(By } ii \text{ \& } iii\text{)}$$

$$\Rightarrow \left|\int_b^c f(x,y)\,d\alpha(x)\right| < \varepsilon\,, \quad c > b > B \text{ \& for each } y \in S$$

Cauchy condition for convergence (uniform) being satisfied.
Therefore the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

➢ *Example*

Consider $\int_0^\infty e^{-xy}\sin x\,dx$

$$\left|e^{-xy}\sin x\right| \le \left|e^{-xy}\right| = e^{-xy} \qquad \left(\because\ |\sin x| \le 1\right)$$

and $\quad e^{-xy} \le e^{-xc} \quad$ if $\quad c \le y$

Now take $\quad M(x) = e^{-cx}$

The integral $\int_0^\infty M(x)\,dx = \int_0^\infty e^{-cx}\,dx = \lim_{b\to\infty}\int_0^b e^{-cx}\,dx = \lim_{b\to\infty}\left|\dfrac{e^{-cx}}{-c}\right|_0^b$ if $c \ne 0$.

That is

$$\int_0^\infty M(x)\,dx = \lim_{b\to\infty}\frac{1}{c}\left(1-\frac{1}{e^{cb}}\right) = \begin{cases} 1/c & \text{if } c>0, \\ \infty & \text{if } c<0. \end{cases}$$

If $c=0$, then $\int_0^\infty M(x)\,dx$ is also divergent.

Since the conditions of M-test are satisfied, therefore $\int_0^\infty e^{-xy}\sin x\,dx$ converges uniformly on $[c,+\infty)$ for every $c>0$.

> ***Theorem (Dirichlet's test for uniform convergence)***
>
> Assume that α is bounded on $[a,+\infty)$ and suppose the integral $\int_a^b f(x,y)\,d\alpha(x)$ exists for every $b \ge a$ and for every y in S. For each fixed y in S, assume that $f(x,y) \le f(x',y)$ if $a \le x' < x < +\infty$. Furthermore, suppose there exists a positive function g, defined on $[a,+\infty)$, such that $g(x)\to 0$ as $x\to+\infty$ and such that $x \ge a$ implies
>
> $$|f(x,y)| \le g(x) \quad \text{for every } y \text{ in } S.$$
>
> Then the integral $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

Proof

Let $M>0$ be an upper bound for $|\alpha|$ on $[a,+\infty)$.

Given $\varepsilon>0$, choose $B>a$ such that

$$|g(x)-0| < \frac{\varepsilon}{4M} \quad \text{for } x \ge B$$

$$\Rightarrow\ g(x) < \frac{\varepsilon}{4M} \quad \text{for } x \ge B.$$

We define $f_y(t) = f(t,y)$ for all $t\in[a,+\infty)$.

If $c>b$, integration by parts yields

$$\int_b^c f(x,y)\,d\alpha(x) = \int_b^c f_y(x)\,d\alpha(x) = \left| f_y(x)\cdot\alpha(x)\right|_b^c - \int_b^c \alpha\, df_y$$

$$= f(c,y)\alpha(c) - f(b,y)\alpha(b) + \int_b^c \alpha\, d(-f_y) \quad \ldots\ldots\ldots\ldots (i)$$

But, since $-f_y$ is increasing (for each fixed y), we have

$$\left|\int_b^c \alpha\, d(-f_y)\right| \le M\int_b^c d(-f_y) \qquad (\because \text{ upper bound of } |\alpha| \text{ is } M)$$

$$= M\ f(b,y) - M\ f(c,y) \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Now if $c>b>B$, from (i) we have

$$\left|\int_b^c f\,d\alpha\right| \le \left| f(c,y)\alpha(c) - f(b,y)\alpha(b)\right| + \left|\int_b^c \alpha\, d(-f_y)\right|$$
$$\le |\alpha(c)||f(c,y)| + |f(b,y)||\alpha(b)| + M\,|f(b,y) - f(c,y)| \qquad \text{from } (ii)$$
$$\le |\alpha(c)||f(c,y)| + |\alpha(b)||f(b,y)| + M\,|f(b,y)| + M\,|f(c,y)|$$
$$\le M\,g(c) + M\,g(b) + M\,g(b) + M\,g(c)$$
$$= 2M\left[g(b) + g(c)\right]$$
$$< 2M\left[\frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}\right] = \varepsilon$$
$$\Rightarrow \left|\int_b^c f\,d\alpha\right| < \varepsilon \quad \text{for every } y \text{ in } S.$$

Therefore the Cauchy condition is satisfied and $\int_a^\infty f(x,y)\,d\alpha(x)$ converges uniformly on S.

➢ *Example*

Consider $\int_0^\infty \frac{e^{-xy}}{x}\sin x\,dx$, where $y \in [0,+\infty)$.

Take $\alpha(x) = \cos x$ and $f(x,y) = \frac{e^{-xy}}{x}$ if $x > 0$, $y \ge 0$.

If $S = [0,+\infty)$ and $g(x) = \frac{1}{x}$ on $[\varepsilon,+\infty)$ for every $\varepsilon > 0$ then

i) $f(x,y) \le f(x',y)$ if $x' \le x$ and $\alpha(x)$ is bounded on $[\varepsilon,+\infty)$.

ii) $g(x) \to 0$ as $x \to +\infty$

iii) $|f(x,y)| = \left|\frac{e^{-xy}}{x}\right| \le \frac{1}{x} = g(x) \quad \forall\ y \in S$.

So that the conditions of Dirichlet's theorem are satisfied.
Hence

$$\int_\varepsilon^\infty \frac{e^{-xy}}{x}\sin x\,dx = +\int_\varepsilon^\infty \frac{e^{-xy}}{x}\,d(-\cos x) \quad \text{converges uniformly on } [\varepsilon,+\infty) \text{ if } \varepsilon > 0.$$

$\because \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \therefore \int_0^\varepsilon e^{-xy}\frac{\sin x}{x}dx$ converges being a proper integral.

$\Rightarrow \int_0^\infty e^{-xy}\frac{\sin x}{x}dx$ also converges uniformly on $[0,+\infty)$.

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➢ *Dirichlet's test for convergence*

Let $\phi(x)$ be bounded and monotonic in $[a,+\infty)$ and let $\phi(x)\to 0$, when $x\to\infty$.

Also let $\int_a^X f(x)\,dx$ be bounded when $X \ge a$.

Then $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

➢ *Example*

Consider $\int_0^\infty \frac{\sin x}{x}\,dx$

$\because \frac{\sin x}{x}\to 1$ as $x\to 0$ $\therefore$ 0 is not a point of infinite discontinuity.

Now consider the improper integral $\int_1^\infty \frac{\sin x}{x}\,dx$.

The factor $\frac{1}{x}$ of the integrand is monotonic and $\to 0$ as $x\to\infty$.

Also $\left|\int_1^X \sin x\,dx\right| = \left|-\cos X + \cos(1)\right| \le \left|\cos X\right| + \left|\cos(1)\right| < 2$

So that $\int_1^X \sin x\,dx$ is bounded above for every $X \ge 1$.

$\Rightarrow \int_1^\infty \frac{\sin x}{x}\,dx$ is convergent. Now since $\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral, we see that $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

➢ *Example*

Consider $\int_0^\infty \sin x^2\,dx$.

We write $\sin x^2 = \frac{1}{2x}\cdot 2x\cdot \sin x^2$

Now $\int_1^\infty \sin x^2\,dx = \int_1^\infty \frac{1}{2x}\cdot 2x\cdot\sin x^2\,dx$

$\frac{1}{2x}$ is monotonic and $\to 0$ as $x\to\infty$.

Also $\left|\int_1^X 2x\sin x^2\,dx\right| = \left|-\cos X^2 + \cos(1)\right| < 2$

So that $\int_1^X 2x\sin x^2\,dx$ is bounded for $x \geq 1$.

Hence $\int_1^\infty \frac{1}{2x}\cdot 2x\cdot\sin x^2\,dx$ i.e. $\int_1^\infty \sin x^2\,dx$ is convergent.

Since $\int_0^1 \sin x^2\,dx$ is only a proper integral, we see that the given integral is convergent.

➢ *Example*

Consider $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx$, $a > 0$

Here e^{-ax} is monotonic and bounded and $\int_0^\infty \frac{\sin x}{x}dx$ is convergent.

Hence $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx$ is convergent.

➢ *Question*

Show that $\int_0^\infty \frac{\sin x}{(1+x)^\alpha}dx$ converges for $\alpha > 0$.

Solution

$\int_0^\infty \sin x\,dx$ is bounded because $\int_0^X \sin x\,dx \leq 2 \quad \forall\ x > 0$.

Furthermore the function $\frac{1}{(1+x)^\alpha}$, $\alpha > 0$ is monotonic on $[0,+\infty)$.

$\Rightarrow$ the integral $\int_0^\infty \frac{\sin x}{(1+x)^\alpha}dx$ is convergent.

➢ *Question*

Show that $\int_0^\infty e^{-x}\cos x\,dx$ is absolutely convergent.

Solution

$\because$ $\left|e^{-x}\cos x\right| < e^{-x}$ and $\int_0^\infty e^{-x}\,dx = 1$

$\therefore$ the given integral is absolutely convergent. (comparison test).