# Summary: Riemann-Stieltjes Integral

Course Title: Real Analysis II — Course Code: MTH322

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## ➢ Partition

Let $[a,b]$ be a given interval. A finite set $P=\{a=x_0,x_1,x_2,\ldots,x_k,\ldots,x_n=b\}$ is said to be a partition of $[a,b]$ which divides it into $n$ such intervals

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n].$$

Each sub-interval is called a *component* of the partition.

Obviously, corresponding to different choices of the points $x_i$ we shall have different partition.

The maximum of the length of the components is defined as the *norm* of the partition and it is denoted by $\|P\|$.

## ➢ Refinement of a Partition

Let $P$ and $P^*$ be two partitions of an interval $[a,b]$ such that $P\subset P^*$ i.e. $P^*$ contains all the points of $P$ and possibly some other points as well. Then $P^*$ is said to be a *refinement* of $P$.

## ➢ Common Refinement

Let $P_1$ and $P_2$ be two partitions of $[a,b]$. Then a partition $P^*$ is said to be their *common refinement* if $P^*=P_1\cup P_2$.

## ➢ Examples

Consider an interval $[1,10]$ and following partitions of this interval.

$P_1=\{1,2,3,10\}$,

$P_2=\{1,2,3,6,9,10\}$,

$$P_3=\left\{1,1+\frac{9}{100},1+2\left(\frac{9}{100}\right),1+3\left(\frac{9}{100}\right),\ldots,1+99\left(\frac{9}{100}\right),10\right\}$$

and more generally for any positive integer $n$, we can write

$$P_4=\left\{1,1+\frac{9}{n},1+2\left(\frac{9}{n}\right),1+3\left(\frac{9}{n}\right),\ldots,1+(n-1)\left(\frac{9}{n}\right),1+n\left(\frac{9}{n}\right)=10\right\}.$$

One can note that $P_2$ is refinement of $P_1$.

Also note that $\|P_1\|=7$, $\|P_2\|=3$, $\|P_3\|=\dfrac{9}{100}$, $\|P_4\|=\dfrac{9}{n}$.

## ➢ Remark

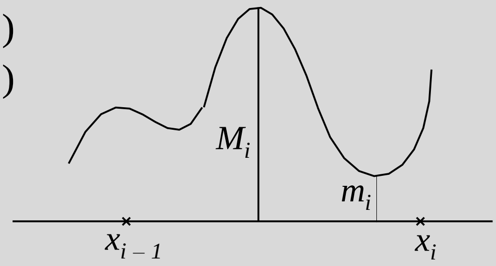
Note that if $P\subseteq P'$ implies $\|P'\|\le\|P\|$. That is, refinement of a partition decreases its norm but the convers does not necessarily hold.

## ➢ *Riemann Integral*

Let $f$ be a real-valued function defined and bounded on $[a,b]$. Corresponding to each partition $P$ of $[a,b]$, we put

$$M_i = \sup f(x) \qquad (x_{i-1} \le x \le x_i)$$

$$m_i = \inf f(x) \qquad (x_{i-1} \le x \le x_i)$$




We define upper and lower sums as

$$U(P,f) = \sum_{i=1}^{n} M_i \,\Delta x_i$$

and $$L(P,f) = \sum_{i=1}^{n} m_i \,\Delta x_i\,.$$

where $\Delta x_i = x_i - x_{i-1} \quad (i = 1,2,\ldots,n)$

and finally $$\overline{\int_a^b} f\,dx = \inf U(P,f) \quad \ldots\ldots\ldots\ldots\ldots\ (i)$$

$$\underline{\int_a^b} f\,dx = \sup L(P,f) \quad \ldots\ldots\ldots\ldots\ldots..(ii)$$

Where the infimum and the supremum are taken over all partitions $P$ of $[a,b]$. Then $\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ are called the upper and lower Riemann Integrals of $f$ over $[a,b]$ respectively.

In case the upper and lower integrals are equal, we say that $f$ is Riemann-Integrable on $[a,b]$ and we write $f \in \mathcal{R}$, where $\mathcal{R}$ denotes the set of Riemann integrable functions.

The common value of $(i)$ and $(ii)$ is denoted by $\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$.

Which is known as the Riemann integral of $f$ over $[a,b]$.

## ➢ *Theorem*

The upper and lower integrals are defined for every bounded function $f$.

### *Proof*

Take $M$ and $m$ to be the upper and lower bounds of $f(x)$ in $[a,b]$.

$$\Rightarrow \quad m \le f(x) \le M \qquad (a \le x \le b)$$

Then $M_i \le M$ and $m_i \ge m \qquad (i = 1,2,.....,n)$

Where $M_i$ and $m_i$ denote the supremum and infimum of $f(x)$ in $(x_{i-1}, x_i)$ for certain partition $P$ of $[a,b]$.

$$\Rightarrow \; L(P,f) = \sum_{i=1}^{n} m_i \,\Delta x_i \ge \sum_{i=1}^{n} m\,\Delta x_i \qquad (\Delta x_i = x_{i-1} - x_i)$$

$$\Rightarrow \; L(P,f) \ge m \sum_{i=1}^{n} \Delta x_i$$

But $\sum_{i=1}^{n}\Delta x_i=(x_1-x_0)+(x_2-x_1)+(x_3-x_2)+....+(x_n-x_{n-1})$

$=x_n-x_0=b-a$

$\Rightarrow\ L(P,f)\geq m(b-a)$

Similarity $U(P,f)\leq M(b-a)$

$\Rightarrow\ m(b-a)\leq L(P,f)\leq U(P,f)\leq M(b-a)$

Which shows that the numbers $L(P,f)$ and $U(P,f)$ form a bounded set.

$\Rightarrow$ The upper and lower integrals are defined for every bounded function $f$. ⊙

## ➢ *Riemann-Stieltjes Integral*

It is a generalization of the Riemann Integral. Let $\alpha(x)$ be a monotonically increasing function on $[a,b]$. $\alpha(a)$ and $\alpha(b)$ being finite, it follows that $\alpha(x)$ is bounded on $[a,b]$. Corresponding to each partition $P$ of $[a,b]$, we write

$$\Delta\alpha_i=\alpha(x_i)-\alpha(x_{i-1})$$

( Difference of values of $\alpha$ at $x_i$ & $x_{i-1}$ )

$\because\ \alpha(x)$ is monotonically increasing.

$\therefore\ \Delta\alpha_i\geq 0$

Let $f$ be a real function which is bounded on $[a,b]$.

Put
$$U(P,f,\alpha)=\sum_{i=1}^{n}M_i\,\Delta\alpha_i$$
$$L(P,f,\alpha)=\sum_{i=1}^{n}m_i\,\Delta\alpha_i\ ,$$

where $M_i$ and $m_i$ have their usual meanings.

Define

$$\overline{\int_a^b} f\,d\alpha=\inf U(P,f,\alpha)\ \ldots\ldots\ldots\ldots\ldots\ (i)$$

$$\underline{\int_a^b} f\,d\alpha=\sup L(P,f,\alpha)\ \ldots\ldots\ldots\ldots\ldots\ (ii)$$

Where the infimum and supremum are taken over all partitions of $[a,b]$.

If $\overline{\int_a^b} f\,d\alpha=\underline{\int_a^b} f\,d\alpha$, we denote their common value by $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$.

This is the Riemann-Stieltjes integral or simply the Stieltjes Integral of $f$ w.r.t. $\alpha$ over $[a,b]$.

If $\int_a^b f\,d\alpha$ exists, we say that $f$ is integrable w.r.t. $\alpha$, in the Riemann sense, and write $f\in\mathcal{R}(\alpha)$.

## ➢ *Note*

The Riemann-integral is a special case of the Riemann-Stieltjes integral when we take $\alpha(x) = x$.

$\because$ The integral depends upon $f, \alpha, a$ and $b$ but not on the variable of integration.

$\therefore$ We can omit the variable and prefer to write $\int_a^b f\, d\alpha$ instead of $\int_a^b f(x)\, d\alpha(x)$.

In the following discussion $f$ will be assume to be real and bounded, and $\alpha$ monotonically increasing on $[a,b]$.

### ➢ *Theorem*

If $P^*$ is a refinement of $P$, then following holds:

(i) $L(P, f, \alpha) \leq L(P^*, f, \alpha)$,

(ii) $U(P, f, \alpha) \geq U(P^*, f, \alpha)$.

### ➢ *Theorem*

Let $f$ be a real valued function defined on $[a,b]$ and $\alpha$ be a monotonically increasing function on $[a,b]$. Then

$$\sup L(P, f, \alpha) \leq \inf U(P, f, \alpha)$$

i.e. $$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha .$$

### ➢ *Theorem* *(Condition of Integrability or Cauchy's Criterion for Integrability.)*

A function $f \in R(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition $P$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ .

### ➢ *Theorem*

If $f \in R(\alpha)$ on $[a,b]$, then $|f| \in R(\alpha)$ on $[a,b]$ and $\left| \int_a^b f\, d\alpha \right| \leq \int_a^b |f|\, d\alpha$ .

### ➢ *Theorem (Fundamental Theorem of Calculus)*

If $f \in R$ on $[a,b]$ and if there is a differentiable function $F$ on $[a,b]$ such that $F' = f$ , then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

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