CHAPTER 02
Sequences and Series of Functions

Course Instructor: Dr. Atiq ur Rehman

Course URL: http://www.mathcity.org/atiq/sp19-mth322

We have seen that a function f that is the sum of two or more functions will share certain desirable properties with those functions. For example, our study of continuity, differentiation, and integration allows us to state if

$$f = f_1 + f_2 + \ldots + f_n$$

on an interval $I = [a, b]$, then

- If $f_1, f_2, \ldots, f_n$ are continuous on I, so is f.
- If $f_1, f_2, \ldots, f_n$ are differentiable on I, so is f, and
$$f' = f_1' + f_2' + \ldots. + f_n'.$$
- If $f_1, f_2, \ldots, f_n$ are integrable on I, so is f, and
$$\int_a^b f(x)dx = \int_a^b f_1 dx + \int_a^b f_2(x)dx + \ldots. + \int_a^b f_n(x)dx.$$

It is natural to ask whether the corresponding results hold when f is the sum of an infinite series of functions,

$$f = f_1 + f_2 + f_3 + \ldots = \sum_{k=1}^{\infty} f_k.$$

Such type of questions lead us to the theory of *sequence of functions* and *series of functions*. If $f_1, f_2, f_3, \ldots$ are real valued function defined on an interval I of the reals numbers. We say that $\{f_n\}$ is an *infinite sequence of functions on I* and $\sum_{k=1}^{\infty} f_k$ or $\sum f_k$ represents the *infinite series of functions on I*.

Definition 1: Pointwise convergence of sequences of functions

Suppose that $\{f_n\}$ is a sequence of functions on an interval I and the sequence of values $\{f_n(x)\}$ converges for each $x \in I$. Then we say that $\{f_n\}$ converges pointwise on I to the limit function f, defined by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad x \in I.$$

Thus if f is the pointwise limit of a sequence of function $\{f_n\}$ define on $[a, b]$, then to each $\varepsilon > 0$ and to each $x \in [a, b]$, there correspond an integer m such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\ n \geq m. \tag{1}$$

Definition 2: Pointwise convergence of series of functions

Suppose that $\sum_{k=1}^{\infty} f_k$ is a series of functions on an interval I. If the series $\sum_{k=1}^{\infty} f_k(x)$ converges for every point $x \in I$, then we say $\sum_{k=1}^{\infty} f_k$ converges pointwise on I. We define

$$f(x) = \sum_{k=1}^{\infty} f_k, \quad x \in I,$$

the function f is called the *sum or the pointwise sum* of the series $\sum f_n$ on I.

Examples:

(1) Let $\{f_n\}$ be a sequence of functions on $\mathbb{R}$ define by

$$f_n(x) = \frac{x}{n}.$$

This sequence converges pointwise to the zero function on $\mathbb{R}$. Indeed, given an $\varepsilon > 0$, choose $N > \left|\frac{x}{\varepsilon}\right|$ then

$$|f_n(x) - 0| = \left|\frac{x}{n}\right| < \left|\frac{x}{N}\right| < \varepsilon, \qquad \text{for} \quad n > N.$$

(2) Consider a sequence $\{f_n(x)\}$ define by $f_n(x) = x^n$ on $[0,1]$. One can note that $\lim_{n\to\infty} f_n(x) = 0$, when $x \in [0,1)$ and $\lim_{n\to\infty} f_n(1) = 1$. Thus we have

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1, \\ 1 & \text{if} \quad \quad x = 1. \end{cases}$$

Remark: Note that the pointwise limit f of the sequence of continuous functions $\{f_n\}$ is discontinuous at $x = 1$.

(3) Consider a sequence $\{f_n\}$, where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ for real x. Since $-1 \le \sin nx \le 1$ and $\sqrt{n} > 0$, therefore we have

$$-\frac{1}{\sqrt{n}} \le \frac{\sin nx}{\sqrt{n}} \le \frac{1}{\sqrt{n}}.$$

This give $f(x) := \lim_{n\to\infty} f_n(x) = 0$.

Remark: One can note that $f_n'(x) = \sqrt{n}\cos nx$, so that $f_n'(0) = \sqrt{n}$. It is clear that $f_n'(0) \to \infty$ as $n \to \infty$ but $f'(0) = 0$.
Thus at $x = 0$, the sequence $\{f_n'(x)\}$ diverges whereas the limit function $f'(x) = 0$, i.e., the limit of differentials is not equal to the differential of the limit.

(4) The geometric series

$$1 + x + x^2 + x^3 + \ldots$$

converges to $(1-x)^{-1}$ in the interval $-1 < x < 1$.

Remark: Note that all the terms are bounded without the sum being so.

(5) Consider the sequence $\{f_n\}$, where

$$f_n(x) = nx(1-x^2)^n, \quad x \in [0,1].$$

For $x = 0$ or $x = 1$, we have $\lim_{n\to\infty} f_n(x) = 0$.

For $x \in (0,1)$, we have

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n = \lim_{n\to\infty} \frac{nx}{(1-x^2)^{-n}} \\
= \lim_{n\to\infty} \frac{x}{(1-x^2)^{-n}\ln(1-x^2)} \qquad \text{(by L'Hospital rule)} \\
= \lim_{n\to\infty} \frac{x(1-x^2)^n}{\ln(1-x^2)} = 0.
\end{aligned}$$

Thus the limit $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

Remark: Note that $\int_0^1 f(x)dx = 0$ and

$$\begin{aligned}
\int_0^1 f_n(x)dx = \int_0^1 nx(1-x^2)^n dx = \frac{-n}{2}\int_0^1 (1-x^2)^n(-2x)dx \\
= \frac{-n}{2}\left|\frac{(1-x^2)^{n+1}}{n+1}\right|_0^1 = \frac{n}{2(n+1)}.
\end{aligned}$$

So that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2}$.

Thus $\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$.

These few examples should convince us that a quite new category of problems arises with the consideration of sequences (series) of variable terms. We have to to investigate under what supplementary conditions some properties (like boundedness, continuity, differential etc.) of the terms f_n are transferred to the limit function f. A concept of great importance in this respect is that known as *uniform convergence* of sequences (series) in its domain of definition $[a,b]$.

Definition 3: Uniform convergence of sequence of functions

A sequence of functions $\{f_n\}$ is said to *converge uniformly* on an interval $[a,b]$ to a function f if for any $\varepsilon > 0$ and for all $x \in [a,b]$ there exist an integer N (independent of x but dependent of ε) such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\ n \geq N \text{ and } x \in [a,b]. \tag{2}$$

It is clear that every uniformly convergent sequence is pointwise convergent, and the uniform limit function is same as the pointwise limit function.

The difference between the two concepts is this: In case of pointwise convergence, for $\varepsilon > 0$ and for each $x \in [a, b]$ there exist an integer N (depending on ε and x both) such that (1) holds for $n \geq N$; whereas in uniform convergence for each $\varepsilon > 0$, it is possible to find one integer N (depend on ε alone) which will do for all $x \in [a, b]$.

Note: Uniform convergence ⇒ pointwise convergence but not vice-versa.
Also a sequence which is not pointwise convergent cannot be uniformly convergent.

Example:

Consider a sequence of functions $\{f_n(x)\}$ on $[0, b]$, $b > 0$, where $f_n(x) = \dfrac{1}{x+n}$.

Here

$$f(x) := \lim_{n\to\infty} f_n(x) = 0 \qquad \forall\, x \in [0, b],$$

so that the sequence converges pointwise to 0.

For any $\varepsilon > 0$,

$$|f_n(x) - f(x)| = \frac{1}{x+n} < \varepsilon.$$

If $n > \dfrac{1}{\varepsilon} - x$, which decreases with x, the maximum value being $\dfrac{1}{\varepsilon}$. Let N be an integer greater than or equal to $\dfrac{1}{\varepsilon}$, so that for $\varepsilon > 0$, there exists N such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N.$$

Hence the sequence is uniformly convergent in any interval $[0, b]$, $b > 0$.

Definition 4: Uniform convergence of series of functions

A series of functions $\sum f_n$ is said to *converges uniformly* on $[a, b]$ if the sequence $\{s_n\}$ of partial sums, defined by

$$s_n(x) = \sum_{i=1}^{n} f_i(x)$$

converges uniformly on $[a, b]$.

Thus, a series of functions $\sum f_n$ converge uniformly to f on $[a, b]$ if for $\varepsilon > 0$ and all $x \in [a, b]$ there exists an integer N (independent of x and dependent of ε) such that for all x in $[a, b]$

$$|f_1(x) + f_2(x) + \ldots + f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N.$$

Review: (Cauchy's general principle of convergence)
A necessary and sufficient condition for the convergence of a sequence of numbers $\{s_n\}$ is that, for each $\varepsilon > 0$ there exists a positive integer m such that

$$\left|s_{n+p} - s_n\right| < \varepsilon, \quad \forall\, n \geq m \;\wedge\; p \geq 1.$$

Note: The proof of above result can be seen in [1, p.73]. It is equivalent to the statement; "A sequence of real numbers is convergent if and only if it is Cauchy sequence".

Theorem 5: Cauchy's criterion for uniform convergence of sequence

A sequence of functions $\{f_n\}$ defined on $[a,b]$ converges uniformly on $[a,b]$ if and only if for every $\varepsilon > 0$ and for all $x \in [a,b]$, there exist an integer N such that

$$\left|f_{n+p}(x) - f_n(x)\right| < \varepsilon, \quad n \geq N, p \geq 1 \text{ and } x \in [a,b]. \tag{3}$$

Proof. Let the sequence $\{f_n\}$ uniformly converge on $[a,b]$ to the limit function f, so that for a given $\varepsilon > 0$ and for all $x \in [a,b]$, there exist integers m_1, m_2 such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall\, n \geq m_1$$

and

$$\left|f_{n+p}(x) - f(x)\right| < \frac{\varepsilon}{2} \quad \forall\, n \geq m_2, p \geq 1.$$

Let $N = \max(m_1, m_2)$. Then

$$\begin{aligned}
\left|f_{n+p}(x) - f_n(x)\right| &= \left|f_{n+p}(x) - f(x) + f(x) - f_n(x)\right| \\
&\leq \left|f_{n+p}(x) - f(x)\right| + |f_n(x) - f(x)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \qquad \forall\, n \geq N, p \geq 1.
\end{aligned}$$

Conversely, suppose that the given condition (3) holds.

By Cauchy's general principle of convergence, $\{f_n\}$ converges for each $x \in [a,b]$ to a limit, say f. Thus the sequence converges pointwise to f. Let us now prove that the convergence is uniform.

For a given $\varepsilon > 0$, let us choose an integer N such that (3) holds. Fix n, and consider $p \to \infty$ in (3). This gives us $f_{n+p} \to f$ as $p \to \infty$, so we get

$$|f(x) - f_n(x)| < \varepsilon \quad n \geq N, \text{ all } x \in [a,b],$$

which proves that $f_n(x) \to f(x)$ uniformly on $[a,b]$. □

Theorem 6: Cauchy's criterion for uniform convergence of series

A series of functions $\sum f_n$ defined on $[a,b]$ converges uniformly on $[a,b]$ if and only if for every $\varepsilon > 0$ and for all $x \in [a,b]$, there exist an integer N such that

$$\left|f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x)\right| < \varepsilon, \quad n \geq N, p \geq 1. \tag{4}$$

The proof of the above theorem is left for the readers.

Note: Relation (4) in the statement may be replaced by

$$|f_{m+1}(x) + f_{m+2}(x) + \ldots + f_n(x)| < \varepsilon, \quad n, m \geq N.$$

Example: Consider a sequence of function $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad \text{for all } x \in \mathbb{R}.$$

Prove that $\{f_n\}$ is pointwise convergent but not uniformly convergent on an interval containing 0.

Solution.

$$\begin{aligned} f(x) := \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{1/n + nx^2} \\ &= 0 \qquad \forall\, x \in \mathbb{R}. \end{aligned}$$

Hence sequence $\{f_n\}$ converges pointwise to $f(x) = 0$ for all real x.

Let $\{f_n\}$ converges uniformly in any interval $[a, b]$, so that the pointwise limit is also the uniform limit. Therefore for given $\varepsilon > 0$, there exists an integer N such that for all $x \in [a, b]$

$$\left|\frac{nx}{1+n^2x^2} - 0\right| < \varepsilon \quad \forall\, n \geq N.$$

In particular, we take $\varepsilon = \frac{1}{3}$, then we have

$$\left|\frac{nx}{1+n^2x^2}\right| < \frac{1}{3} \quad \forall\, n \geq N.$$

Let m be an integer greater than N such that $\frac{1}{m} \in [a, b]$. Now if we take $n = m$ and $x = \frac{1}{m}$, then we have

$$\left|\frac{nx}{1+n^2x^2}\right| = \left|\frac{m \cdot (1/m)}{1+m^2 \cdot (1/m^2)}\right| = \frac{1}{2} \nless \frac{1}{3} = \varepsilon.$$

We thus arrive at a contradiction and so the sequence is not uniformly convergent in the interval $[a, b]$, which contains the point $1/m$. Since $1/m$ can tends to 0, therefore the interval $[a, b]$ contains 0.

Hence the sequence is not uniformly convergent on any interval $[a, b]$ containing 0.

Theorem 7

Let $\{f_n\}$ be a sequence of functions, such that

$$\lim_{n\to\infty} f_n(x) = f(x), \quad x \in [a,b]$$

and let

$$M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly on $[a,b]$ if and only if $M_n \to 0$ as $n \to \infty$.

Proof. Let $f_n \to f$ uniformly on $[a,b]$, so that for a given $\varepsilon > 0$, there exists an integer N such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N,\ \forall\, x \in [a,b]$$

$$\Rightarrow \quad M_n := \sup_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N.$$

$$\Rightarrow \quad M_n \to 0 \ \text{ as } \ n \to \infty.$$

Conversely, suppose that $M_n \to 0$ as $n \to \infty$, i.e. $\lim_{n\to\infty} M_n = 0$.

This gives for all $\varepsilon > 0$, there exists an integer N such that

$$|M_n - 0| < \varepsilon, \quad \forall\, n \geq N,$$

$$\Rightarrow \quad M_n < \varepsilon \quad \forall\, n \geq N,$$

that is

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N,$$

$$\Rightarrow \quad |f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N,\ \forall\, x \in [a,b],$$

$$\Rightarrow \quad f_n \to f \text{ unifromly on } [a,b].$$

This complete the proof. □

Question: Use the above theorem to prove that a sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

is not uniformly convergent on any interval containing zero.

Solution of the above question left for the reader.

Question: Prove that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{x}{1+nx^2}$$

is uniformly convergent on any interval I.

Solution. Here the pointwise limit

$$f(x) := \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0 \quad \forall\, x \in \mathbb{R}.$$

Now let

$$M_n = \sup_{x\in I} |f_n(x) - f(x)| = \sup_{x\in I} \left| \frac{x}{1+nx^2} \right|.$$

If we take $g(x) = \dfrac{x}{1+nx^2}$, then

$$g'(x) = \frac{(1+nx^2)\cdot 1 - x\cdot 2nx}{(1+nx^2)^2} = \frac{1+nx^2-2nx^2}{(1+nx^2)^2}$$
$$= \frac{1-nx^2}{(1+nx^2)^2}.$$

Put $g'(x) = 0$, we get

$$1-nx^2 = 0 \quad \Rightarrow\; nx^2 = 1 \quad \Rightarrow\; x^2 = \frac{1}{n} \quad \Rightarrow\; x = \pm\frac{1}{\sqrt{n}}.$$

This gives $g(x)$ has extreme values at $x = \pm\frac{1}{\sqrt{n}}$.

Now

$$g''(x) = \frac{(1+nx^2)^2\cdot(-2nx) - (1-nx^2)\cdot 2(1+nx^2)(2nx)}{(1+nx^2)^4}$$
$$= \frac{-2nx(1+nx^2)(1+nx^2+2-2nx^2)}{(1+nx^2)^4} = \frac{-2nx(3-nx^2)}{(1+nx^2)^3}.$$

Since

$$g''\left(\frac{1}{\sqrt{n}}\right) = -\frac{\sqrt{n}}{2} < 0 \quad \text{and} \quad g''\left(-\frac{1}{\sqrt{n}}\right) = \frac{\sqrt{n}}{2} > 0,$$

this gives g has extreme value at $x = \pm\frac{1}{\sqrt{n}}$ and $g\left(\pm\frac{1}{\sqrt{n}}\right) = \pm\frac{1}{2\sqrt{n}}$.

Hence

$$M_n = \sup\left|g\left(\pm\frac{1}{\sqrt{n}}\right)\right| = \frac{1}{2\sqrt{n}} \quad \text{and} \quad M_n \to 0 \text{ as } n\to\infty.$$

This implies $\{f_n\}$ converges uniformly on I.

Exercises

1. Show that the sequence $\{f_n\}$, where

$$f_n(x) = nxe^{-nx^2}, \quad x \ge 0,$$

   is not uniformly convergent on $[0,k]$, $k > 0$.

2. Show that the sequence $\{x^n\}$ is not uniformly convergent on $[0, 1]$.

3. Show that the sequence $\{\exp(-nx)\}$ is not uniformly convergent on $[0, k]$, $k > 0$.

4. Test the following sequences for uniform convergence.

   a. $\left\{\dfrac{\sin nx}{\sqrt{n}}\right\}$, $0 \le x \le 2\pi$.
   b. $\left\{\dfrac{x}{n+x}\right\}$, $0 \le x \le k$, where $k > 0$.
   c. $\left\{\dfrac{x}{n+x}\right\}$, $0 \le x < \infty$.

Review: (Cauchy's criterion for convergence of series)
A necessary and sufficient condition for the convergence of a series of numbers $\sum x_n$ is that, for each $\varepsilon > 0$ there exists a positive integer m such that

$$\left|x_{n+1} + x_{n+2} + \ldots + x_{n+p}\right| < \varepsilon \quad \text{for } n > m \text{ and } p \ge 1.$$

Theorem 8: Weierstrass's M-test

A series of functions $\sum f_n$ will converge uniformly (and absolutely) on $[a, b]$ if there exists a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a, b]$

$$|f_n(x)| \le M_n \quad \text{for all } n.$$

Proof. Since $\sum M_n$ is convergent, therefore by Cauchy criterion for convergence of series, for all $\varepsilon > 0$, there exists and integer N such that

$$\left|M_{n+1} + M_{n+2} + \ldots + M_{n+p}\right| < \varepsilon \quad \forall\, n > N \text{ and } p \ge 1,$$

$$\text{i.e.} \quad M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \varepsilon \quad \forall\, n > N \text{ and } p \ge 1 \quad \text{as } M_n > 0\ \forall\, n.$$

Hence for all $x \in [a, b]$ and for all $n > N$, $p \ge 1$, we have

$$\begin{aligned} &|f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x)| \\ &\quad \le |f_{n+1}(x)| + |f_{n+2}(x)| + \ldots + |f_{n+p}(x)| \\ &\quad \le M_{n+1} + M_{n+2} + \ldots + M_{n+p} \\ &\quad < \varepsilon \end{aligned} \tag{5}$$

This gives that $\sum f_n$ is uniformly convergent on $[a, b]$. Also from (5), one can conclude that $\sum f_n$ is absolutely convergent on $[a, b]$. □

Remark: The converse of above theorem is not true, i.e. non-convergence of $\sum M_n$ does not imply anything as for as $\sum f_n$ is concerned.

Example: Consider the series $\sum \dfrac{\cos n\theta}{n^p}$ for all $\theta \in \mathbb{R}$. Since we have

$$\left|\frac{\cos n\theta}{n^p}\right| \le \frac{1}{n^p}.$$

We know that $\sum \frac{1}{n^p}$ is convergent for $p > 1$. Hence we conclude that the given series is uniformly convergent on any interval in $\mathbb{R}$.

Exercise: Prove that the following series are uniformly convergent for all real x.

$$(i) \quad \sum \frac{\sin(x^2 + n^2 x)}{n(n+1)} \qquad\qquad (ii) \quad \sum \frac{(-1)^n x^{2n}}{n^{p+1}(1+x^{2n})}, \quad p > 0.$$

Theorem 9: Uniform convergence and continuity

Let $\{f_n\}$ be a sequence of functions defined on an interval I, and $x_0 \in I$. If the sequence $\{f_n\}$ converges uniformly to some function f on I and if each of the function f_n is continuous at x_0, then the function f is also continuous at x_0.

Proof. Since $f_n \to f$ uniformly on I, for given $\varepsilon > 0$, there exists an integer N such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \quad \forall\, n \geq N,\ \forall\, x \in I. \tag{6}$$

As we have given, each f_n is continuous at x_0, there is a $\delta > 0$ such that

$$|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}, \quad \text{whenever } |x - x_0| < \delta. \tag{7}$$

Now for all $x \in I$ and all $n \geq N$ such that $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \qquad \text{by using (6) and (7).} \end{aligned}$$

This conclude that f is continuous at x_0. □

Corollary 9

Let $\{f_n\}$ be a sequence of functions defined on an interval I. If the sequence $\{f_n\}$ converges uniformly to some function f on I and if each of the function f_n is continuous on I, then the function f is also continuous on I.

Theorem 10: Uniform convergence and integration

Let $\{f_n\}$ be a sequence of functions defined on $[a,b]$. If $f_n \to f$ uniformly on $[a,b]$ and each function f_n is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx. \tag{8}$$

Proof. Since each f_n is continuous and $f_n \to f$ uniformly on $[a,b]$, therefore f is continuous on $[a,b]$ and hence $\int_a^b f(x)dx$ exists.

Now

$$\begin{aligned}\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &= \left|\int_a^b (f_n(x)-f(x))dx\right| \\ &\le \int_a^b |f_n(x)-f(x)|\,dx \\ &\le \int_a^b \max_{x\in[a,b]} |f_n(x)-f(x)|\,dx \\ &= \max_{x\in[a,b]} |f_n(x)-f(x)| \int_a^b dx,\end{aligned}$$

that is, we have

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \le (b-a) \max_{x\in[a,b]} |f_n(x)-f(x)|. \tag{9}$$

Since $f_n \to f$ uniformly on $[a,b]$, for all $\varepsilon > 0$, there exists an integer N such that

$$|f_n(x)-f(x)| < \frac{\varepsilon}{b-a} \quad \forall\, n \ge N,\ \forall\, x \in [a,b],$$

this gives

$$\max_{x\in[a,b]} |f_n(x)-f(x)| < \frac{\varepsilon}{b-a} \quad \forall\, n \ge N.$$

Thus for $n \ge N$, expression (9) leads us to

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \le (b-a)\cdot\frac{\varepsilon}{b-a} = \varepsilon,$$

which is equivalent to the required result. □

Review: Mean value theorem (see [5, Page 108])
If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $c \in (a,b)$ at which

$$f(b)-f(a) = (b-a)f'(c).$$

Review: Fundamental theorem of calculus (see [5, Page 134])
If ϕ is integrable over $[a,b]$ and there exists a differentiable function f on $[a,b]$ such that $f' = \phi$ then

$$\int_a^b \phi(t)dt = f(b) - f(a).$$

Theorem 11: Uniform convergence and differentiation

Let $\{f_n\}$ be a sequence of functions defined on $[a,b]$ such that $f_n(x_0)$ converges for some point x_0 on $[a,b]$. If each f_n is differentiable and $\{f'_n\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f'_n(x) \quad (a < x < b).$$

Proof. Let $\varepsilon > 0$ be given. Choose N such that $n, m \geq N$ implies

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \tag{10}$$

and

$$\left|f'_n(t) - f'_m(t)\right| < \frac{\varepsilon}{2(b-a)} \quad (a \leq t \leq b). \tag{11}$$

If we apply the mean value theorem to function $f_n - f_m$, we have

$$\frac{f_n(x) - f_m(x) - f_n(t) + f_m(t)}{x - t} = f'_n(c) - f'_m(c) \tag{12}$$

for any x and t in $[a,b]$, $c \in (a,b)$ and $n, m \geq N$.

Using it in (11), we have

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x-t|\varepsilon}{2(b-a)} \leq \frac{\varepsilon}{2}. \tag{13}$$

Now we have

$$\begin{aligned}|f_n(x) - f_m(x)| &= |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0) + f_n(x_0) - f_m(x_0)| \\ &\leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|.\end{aligned}$$

Using (10) and (13) in above inequality, we have

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (a \leq x \leq b, n, m \geq N),$$

this implies $\{f_n\}$ converges uniformly on $[a,b]$. Let

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (a \leq x \leq b). \tag{14}$$

Since we have given that $\{f'_n\}$ is uniformly convergent, therefore consider

$$\phi(x) = \lim_{n\to\infty} f'_n(x) \quad (a \leq x \leq b).$$

Let us know fix a point x on $[a,b]$. Then we have

$$\begin{aligned}\int_a^x \phi(t)dt &= \int_a^x \lim_{n\to\infty} f_n'(t)dt \\ &= \lim_{n\to\infty}\int_a^x f_n'(t)dx \qquad \text{(as } f_n' \text{ is uniformly convergent)} \\ &= \lim_{n\to\infty}[f_n(x)-f_n(a)] \qquad \text{(by fundamental theorem of calculus)}\end{aligned}$$

By using (14), we get that

$$\int_a^x \phi(t)dt = f(x)-f(a)$$

Now again by using fundamental theorem of calculus, we get that

$$\phi(x) = f'(x) \quad (a \le x \le b).$$

This complete the proof. □

Disclaimer: Most of the contents in these notes are taken from [1]. These notes are made for students and they are encourage to read the books. Also see some other useful books in references.

References

[1] S.C. Malik and S. Arora, *Mathematical analysis*, New Age International, 1992.

[2] William F. Trench, *Introduction to real analysis*, 2013.
URL: http://ramanujan.math.trinity.edu/wtrench/misc/index.shtml

[3] Robert Gardner Bartle, and Donald R. Sherbert, *Introduction to real analysis*, Wiley New York, 1992.

[4] Brian S. Thomson, and Judith B. Bruckner, and Andrew M. Bruckner, *Elementary real analysis*, ClassicalRealAnalysis. com, 2008.

[5] Walter Rudin, *Principles of mathematical analysis*, 3rd Edition, McGraw-Hill, Inc., 1976.