Lecture 11: Discrete Mathematics

Course Title: Discrete Mathematics **Course Code:** MTH211 **Class:** BSM-II

Objectives

The main aim of the lecture is to discuss

- *Types of relations.*
- *Reflexive, symmetric and antisymmetric.*
- *Give some examples related to these.*

References:

- S. Lipschutz and M. Lipson, Schaum's Outlines Discrete Mathematics, Third Edition, McGraw-Hil, 2007.
- K.H. Rosen, Discrete Mathematics and its Application, MeGraw-Hill, 6th edition. 2007.
- K.A. Ross, C.R.B. Wright, Discrete Mathematics, Prentice Hall. New Jersey, 2003.

Types of Relations:

Here we discusses a number of important types of relations defined on a set *A*.

Reflexive Relations

A relation *R* on a set *A* is reflexive if *aRa* for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$.

Thus *R* is not reflexive if there exists $a \in A$ such that $(a, a) \notin R$.

Example 1

Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$

$R_2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R_3 = \{(1, 3), (2, 1)\}$

$R_4 = \emptyset$, the empty relation

$R_5 = A\times A$, the universal relation

Since *A* contains the four elements 1, 2, 3, and 4, a relation *R* on *A* is reflexive if it contains the four pairs (1, 1), (2, 2), (3, 3), and (4, 4). Thus only R_2 and the universal relation $R_5 = A\times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive since, for example, (2, 2) does not belong to any of them.

Example 2

Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set $\mathbb{Z}$ of integers.
(2) Set inclusion $\subseteq$ on a collection C of sets.
(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.
(5) Relation $|$ of divisibility on the set $\mathbb{N}$ of positive integers.
(Recall $x \mid y$ if there exists z such that $xz = y$.)

The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every $x \in \mathbb{Z}$, $A \subseteq A$ for any set $A \in C$, and $n \mid n$ for every positive integer $n \in \mathbb{N}$.

Symmetric Relations

A relation *R* on a set *A* is symmetric if whenever *aRb* then *bRa*, that is, if whenever (*a*, *b*) ∈ *R* then (*b*, *a*) ∈ R.

A relation *R* is not symmetric if there exists *a*, *b* ∈ *A* such that (*a*, *b*) ∈ *R* but (*b*, *a*) ∉ *R*.

Example 3

Consider the following five relations on the set *A* = {1, 2, 3, 4}:

R_1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)}

R_2 = {(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R_3 = {(1, 3), (2, 1)}

R_4 = ∅, the empty relation

R_5 = *A*×*A*, the universal relation

A relation R_1 is not symmetric since (1, 2) ∈ R_1 but (2, 1) ∉ R_1. R_3 is not symmetric since (1, 3) ∈ R_3 but (3, 1) ∉ R_3.

The other relations are symmetric.

Example 4

Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set $\mathbb{Z}$ of integers.

(2) Set inclusion $\subseteq$ on a collection C of sets.

(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.

(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

(5) Relation $|$ of divisibility on the set $\mathbb{N}$ of positive integers.

(Recall $x \mid y$ if there exists z such that $xz = y$.)

The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a.
Also, $\parallel$ is symmetric since if line a is parallel to line b then b is parallel to line a.
The other relations are not symmetric.
For example:

$$3 \leq 4 \text{ but } 4 \not\leq 3; \ \{1, 2\} \subseteq \{1, 2, 3\} \text{ but } \{1, 2, 3\} \not\subseteq \{1, 2\}; \text{ and } 2 \mid 6 \text{ but } 6 \nmid 2.$$

Antisymmetric Relations

A relation *R* on a set *A* is antisymmetric if whenever *aRb* and *bRa* then *a* = *b*, that is, if $a \neq b$ and *aRb* then $b \not R a$.

Thus, *R* is not antisymmetric if there exist distinct elements *a* and *b* in *A* such that *aRb* and *bRa*.

Example 5

Consider the following five relations on the set *A* = {1, 2, 3, 4}:

R_1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)}

R_2 = {(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R_3 = {(1, 3), (2, 1)}

R_4 = ∅, the empty relation

R_5 = *A*×*A*, the universal relation

A relation R_2 is not antisymmetric since (1, 2) and (2, 1) belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_3 is not antisymmetric. All the other relations are antisymmetric.

Example 6

Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set $\mathbb{Z}$ of integers.

(2) Set inclusion $\subseteq$ on a collection C of sets.

(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.

(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

(5) Relation $|$ of divisibility on the set $\mathbb{N}$ of positive integers.

(Recall $x \mid y$ if there exists z such that $xz = y$.)

The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$.
Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$.
Also, divisibility on $\mathbb{N}$ is antisymmetric since whenever $m \mid n$ and $n \mid m$ then $m = n$. (Note that divisibility on $\mathbb{Z}$ is not antisymmetric since $3 \mid -3$ and $-3 \mid 3$ but $3 \neq -3$.)
The relations $\perp$ and $\parallel$ are not antisymmetric.

Remark

The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R' = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

THANKS FOR YOUR ATTENTION