Lectures Handout (Volume 1)
Course Title: Convex Analysis
Course Code: MTH424

Definition 1: Continuity

A function $f : I \to \mathbb{R}$, where I is interval in $\mathbb{R}$, is said to be continuous at point $x_0 \in I$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \text{ whenever } |x - x_0| < \delta.$$

A function f is said to be continuous on I if it is continuous on each point of I.

Definition 2: Uniform Continuity

A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is said to be uniformly continuous on I if for all $\varepsilon > 0$ and $x, y \in I$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta.$$

From the definition of uniform continuity, one can derive the following remark:

Remark 3: Uniform Continuity Implies Continuity

If a function f is uniformly continuous on I, then it is continuous on I.

Theorem 4

If $f : I \to \mathbb{R}$ is convex on I, then f is continuous on I°, where I° represents interior of I.

Proof

Let $[a, b] \subseteq I^\circ$. We choose $\varepsilon > 0$ so that $a - \varepsilon$ and $b + \varepsilon$ belong to I. As f is convex, therefore it is bounded on closed interval $[a - \varepsilon, b + \varepsilon]$. So assume m and

M are the lower and upper bounds of f on $[a-\varepsilon, b+\varepsilon]$ respectively.

If x, y are different points of $[a,\ b]$, set

$$z = y + \frac{\varepsilon}{|y-x|}\,(y-x) \text{ and } \lambda = \frac{|y-x|}{\varepsilon + |y-x|}.$$

As $\dfrac{y-x}{|y-x|} = \pm 1$, therefore $z \in [a-\varepsilon, b+\varepsilon]$.

Now take

$$\begin{aligned}
\lambda z + (1-\lambda)x &= \frac{|y-x|}{\varepsilon+|y-x|}\left(y + \frac{\varepsilon(y-x)}{|y-x|}\right) + \left(1 - \frac{|y-x|}{\varepsilon+|y-x|}\right)x \\
&= \frac{|y-x|}{\varepsilon+|y-x|}\left(\frac{y|y-x| + \varepsilon(y-x)}{|y-x|}\right) + \left(\frac{\varepsilon+|y-x|-|y-x|}{\varepsilon+|y-x|}\right)x \\
&= \frac{1}{\varepsilon+|y-x|}\left(y|y-x| + \varepsilon(y-x) + \varepsilon x\right) \\
&= \frac{1}{\varepsilon+|y-x|}\left(y|y-x| + \varepsilon y\right) \\
&= \frac{1}{\varepsilon+|y-x|}\left(y\left(|y-x|+\varepsilon\right)\right) = y,
\end{aligned}$$

that is we have that $y = \lambda z + (1-\lambda)x$, so we have

$$\begin{aligned}
f(y) &= f(\lambda z + (1-\lambda)x) \\
&\leq \lambda f(z) + (1-\lambda) f(x) \quad \text{as } f \text{ is convex} \\
&= \lambda f(z) - \lambda f(x) + f(x).
\end{aligned}$$

This implies

$$\begin{aligned}
f(y) - f(x) &\leq \lambda\left(f(z) - f(x)\right) \\
&\leq \lambda(M-m) \\
&= \frac{|y-x|}{\varepsilon+|y-x|}(M-m) \\
&< \frac{M-m}{\varepsilon}|y-x| \\
&= K|y-x|, \text{ where } K := \frac{(M-m)}{\varepsilon}.
\end{aligned}$$

That is

$$f(y) - f(x) < K|y - x|.$$

Since this is true for any $x,\ y \in [a, b]$, we conclude that

$$|f(y) - f(x)| < K|y - x|$$

Now if $\varepsilon_1 > 0$, the above expression gives us

$$|f(y) - f(x)| < \varepsilon_1, \quad \text{whenever } |y - x| < \delta := \frac{\varepsilon_1}{K}.$$

Thus f is uniformly continuous on $[a, b]$ and hence f is continuous on $[a, b]$.

Since a and b are arbitrary, therefore f is continuous on interior I° of I. □

Definition 5: Increasing Function

A function $f : I \to \mathbb{R}$ is said to be *increasing* if for any $x, y \in I$ such that $x < y$, there holds the inequality

$$f(x) \leq f(y). \tag{1}$$

A function is said to be *strictly increasing* on I if strict inequality holds in (1).

Definition 6: Left & Right Derivatives

Let $f : I \to \mathbb{R}$ be a function. The left and right derivatives of f at $x \in I$ are defined as follows:

$$f'_-(x) = \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x}$$

$$f'_+(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x}.$$

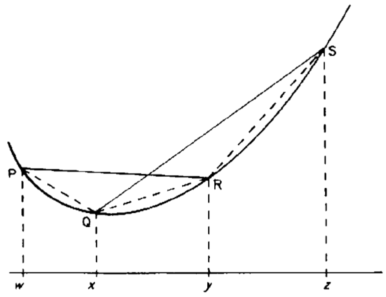
Theorem 7

If $f : I \to \mathbb{R}$ is convex, then $f'_-(x)$ and $f'_+(x)$ exist and are increasing on I°

Proof

Consider four points $w, x, y, z \in I^\circ$ such that

$$w < x < y < z.$$



Also let P, Q, R and S be the corresponding points on the graph of f.

Then we have

$$slope\ \overline{PQ} \leqslant slope\ \overline{PR} \leqslant slope\ \overline{QR} \leqslant slope\ \overline{QS} \leqslant slope\ \overline{RS}$$

Consider

$$slope\ \overline{QR} \leqslant slope\ \overline{RS},$$

this gives

$$\frac{f(y) - f(x)}{y - x} \leqslant \frac{f(z) - f(y)}{z - y} \tag{2}$$

when Q moves towards R, then $x \uparrow y$ and when S moved towards R then $z \downarrow y$. As f is continuous on I°, therefore when $x \uparrow y$, then $f'_{-}(y)$ exists and when $z \downarrow y$ then $f'_{+}(y)$ exists.

Also from (2), one can conclude

$$f'_{-}(y) \leqslant f'_{+}(y) \ \text{ for all } y \in I^\circ. \tag{3}$$

Now we consider

$$slope\ \overline{PQ} \leqslant slope\ \overline{QR},$$

that is

$$\frac{f(x) - f(w)}{x - w} \leqslant \frac{f(y) - f(x)}{y - x}.$$

When x decreased toward w and x increased toward y, we get

$$f'_{+}(w) \leqslant f'_{-}(y) \ \text{ for all } \ w < y. \tag{4}$$

Using (3) and (4), we have for all $w < y$,

$$f'_-(w) \leqslant f'_+(w) \leqslant f'_-(y) \leqslant f'_+(y),$$

So we have proved that for $w < y$,

$$f'_-(w) \leqslant f'_-(y)$$

and

$$f'_+(w) \leqslant f'_+(y).$$

This implies f'_- and f'_+ are increasing. □

Remark 8

If $f : I \to \mathbb{R}$ is strictly convex, then $f'_-(x)$ and $f'_+(x)$ exist and are strictly increasing on I°.

Review

- Assume that the function f is differentiable on interval I. Then f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.
- Assume that the function f is differentiable on interval I. Then f is strictly increasing on I if and only if $f'(x) > 0$ for all $x \in I$.
- Suppose f is differentiable on (a, b). Then f is convex [strictly convex] if, and only if, f' is increasing [strictly increasing] on (a, b).

Theorem 9

Let f is twice differentiable on (a, b). Then f is convex on (a, b) iff $f''(x) \geq 0$ for all $x \in (a, b)$. If $f''(x) > 0$ for all $x \in (a, b)$, then f is strictly convex on (a, b).

Exercises 1. Prove that a function e^x is convex on $(-\infty, \infty)$.
2. Prove that a function $\sin x$ is convex on interval $[\pi, 2\pi]$.
3. Find the value of p for which x^p is convex on $(0, \infty)$.

In the following theorem, we prove that sum of two convex functions is convex.

Theorem 10

If $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are convex then $f + g$ is convex on I.

Proof

Since f and g are convex therefore for $x, y \in I$ and $\lambda \in (0, 1)$

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) \tag{5}$$

and

$$g(\lambda x + (1 - \lambda) y) \leq \lambda g(x) + (1 - \lambda) g(y). \tag{6}$$

Now we consider

$$\begin{aligned}(f + g)(\lambda x + (1 - \lambda) y) &= f(\lambda x + (1 - \lambda) y) + g(\lambda x + (1 - \lambda) y) \\ &\leq \lambda f(x) + (1 - \lambda) f(y) + \lambda g(x) + (1 - \lambda) g(y) \\ &= \lambda (f(x) + g(x)) + (1 - \lambda)(f(y) + g(y)) \\ &= \lambda (f + g)(x) + (1 - \lambda)(f + g)(y).\end{aligned}$$

Hence $(f + g)$ is convex on I. □

In the similar way, one can prove the following:

Theorem 11

If $f : I \to \mathbb{R}$ is convex and $\alpha \geq 0$, then αf is convex on I.

Definition 12: Line of Support

A function f defined on I has support at $x_0 \in I$ if there exists a function

$$A(x) = f(x_0) + m(x - x_0)$$

such that $A(x) \leq f(x)$ for every $x \in I$.

The graph of the support function A is called a line of support for f at x_0.

Theorem 13

A function $f : (a,b) \to \mathbb{R}$ is convex if and only if there is at least one line of support for f at each $x_0 \in (a,b)$.

Proof

Suppose f is convex and $x_0 \in (a,b)$. Then f'_-, f'_+ exist and $f'_-(x_0) \leq f'_+(x_0)$ for all $x_0 \in (a,b)$.

Choose $m \in [f'_-(x_0), f'_+(x_0)]$. Then we have

$$\frac{f(x) - f(x_0)}{x - x_0} \geq m \quad \text{for} \quad x > x_0$$

and

$$\frac{f(x) - f(x_0)}{x - x_0} \leq m \quad \text{for} \quad x < x_0.$$

That is, we have

$$\begin{aligned} & f(x) - f(x_0) \geq m(x - x_0) \quad \text{for all} \quad x \in (a,b), \\ \Rightarrow \quad & f(x) \geq f(x_0) + m(x - x_0) \quad \text{for all} \quad x \in (a,b) \end{aligned} \tag{7}$$

If we consider $A(x) = f(x_0) + m(x - x_0)$ be support function at $x_0 \in (a,b)$, then from (7), we have

$$f(x) \geq A(x) \quad \text{for all} \quad x \in (a,b).$$

This proves that f has a line of support at each $x_0 \in (a,b)$.

Conversely, suppose that f has a line of support at each point of (a,b) and $A(x)$

define above be support function, then

$$A(x) \le f(x) \quad \text{for all} \quad x \in (a,b).$$

Let $x, y \in (a,b)$ and $x_0 = \lambda x + (1-\lambda)y$, $\lambda \in [0,1]$, then

$$A(x_0) = f(x_0) - m(x_0 - x_0) = f(x_0).$$

Now

$$\begin{aligned} f(x_0) &= A(x_0) \\ &= A(\lambda x + (1-\lambda)y) \\ &= f(x_0) + m(\lambda x + (1-\lambda)y - x_0) \\ &= [\lambda + (1-\lambda)]f(x_0) + m[\lambda x + (1-\lambda)y - \{\lambda + (1-\lambda)\}x_0] \\ &= \lambda[f(x_0) + m(x - x_0)] + (1-\lambda)[f(x_0) + m(y - x_0)] \\ &= \lambda A(x) + (1-\lambda)A(y) \\ &\le \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

That is, we have proved that

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \quad \text{for all} \quad x \in (a,b), \lambda \in [0,1].$$

Hence f is convex on (a,b). □

Remark 14

In previous theorem, we take $f'_-(x_0) \le m \le f'_+(x_0)$. If the function f is differentiable on (a,b), then we have

$$f'_-(x_0) = f'_+(x_0) = f'(x_0).$$

Hence $A(x) = f(x_0) + f'(x_0)(x - x_0)$ will be line of support of f at x_0.

For example: If $f(x) = e^x$ for $x \in \mathbb{R}$, then

$$A(x) = e + e(x-1) = ex$$

is support function for e^x at point $x = 1$.

It can also be written as $y = ex$ or $ex - y = 0$.

In the similar way, what about support function of e^x at $x = 0$?

Review

- A function f defined on I has support at $x_0 \in I$ if there exists a function

$$A(x) = f(x_0) + m(x - x_0)$$

  such that $A(x) \leq f(x)$ for every $x \in I$.

- A function $f : (a, b) \to \mathbb{R}$ is convex if and only if there is at least one line of support for f at each $x_0 \in (a, b)$.

- From the proof of theorem stated in above clause, we have $f'_-(x_0) \leq m \leq f'_+(x_0)$ for line of support at point x_0. If the function f is differentiable on (a, b), then $m = f'(x_0)$.

Exercise: Find the line of supports for the function defined below at $x = 1$.

$$f(x) = \begin{cases} x^2, & x \geq 1; \\ x, & x < 1. \end{cases}$$

Solution. A function

$$A(x) = f(x_0) + m(x - x_0)$$

is line of support at $x = x_0$, where $m \in [f'_-(x_0), f'_+(x_0)]$.

So we have

$$\begin{aligned} & f'_-(x) = 1 \quad \text{and} \quad f'_+(x) = 2x. \\ \Rightarrow \quad & f'_-(1) = 1 \quad \text{and} \quad f'_+(1) = 2. \end{aligned}$$

Thus

$$\begin{aligned} & A(x) = f(1) + m(x - 1), \quad \text{where} \quad m \in [1, 2], \\ \Rightarrow \quad & A(x) = 1 + m(x - 1), \quad \text{where} \quad m \in [1, 2]. \end{aligned}$$

Remark 15

If we are asked to find the line of support at $x = 2$ for the function f defined above, that is, for function

$$f(x) = \begin{cases} x^2, & x \geq 1; \\ x, & x < 1. \end{cases}$$

We see, the function is differentiable at $x = 2$, so $m = f'(2) = 4$. Thus, we have

$$\begin{aligned} & A(x) = f(2) + m(x-2), \\ \Rightarrow \quad & A(x) = 4 + 4(x-2), \\ \Rightarrow \quad & A(x) = 4x - 4. \end{aligned}$$

is required line of support.

References:

- A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.
- C. P. Niculescu and L. E. Persson, *Convex Functions and Their Applications, A Contemporary Approach*, Springer, New York, 2006.

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