UNIT # 02

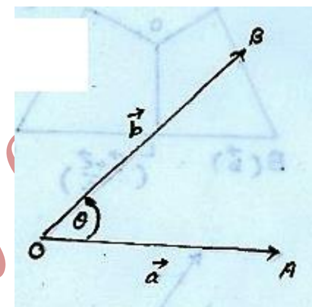
SCALAR AND VECTOR PRODUCT

Scalar product or dot product :

If $\vec{a}$ and $\vec{b}$ be the two vectors. Then the scalar or dot product of two vector is define as

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

Where θ is angle between $\vec{a}$ and $\vec{b}$.



Characteristics:

(i) If $\vec{a} = a_1\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ & $\vec{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\,\mathbf{k}$

Then $\vec{a}.\vec{b} = (a_1\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})\ .\ (b_1\mathbf{i} + b_2\mathbf{j} + b_3\,\mathbf{k}) = a_1 b_1 + a_2 b_2 + a_3 b_3$

(ii) If $\vec{a}$ and $\vec{b}$ are perpendicular ($\theta = 90^0$) vectors then $\vec{a}.\vec{b} = 0$

(iii) If $\vec{a}$ and $\vec{b}$ are parallel ($\theta = 0^0$) vectors then $\vec{a}.\vec{b} = a\,b$

(iv) If $\vec{a}$ and $\vec{b}$ are anti parallel ($\theta = 180^0$) vectors then $\vec{a}.\vec{b} = -\,a\,b$

(v) Dot product is commutative

$$\vec{a}.\vec{b} = \vec{b}.\vec{a}$$

(vi) Dot product of two same vector is

$$\vec{a}.\vec{a} = |\vec{a}|^2$$

(vii) Distributive property of dot product over addition or subtraction .

$\vec{a}.(\vec{b} \pm \vec{c}) = \vec{a}.\vec{b} \pm \vec{a}.\vec{c}$ **Left distributive law**

$(\vec{a} \pm \vec{b}).\vec{c} = \vec{a}.\vec{c} \pm \vec{b}.\vec{c}$ **Right distributive law**

(viii) Scalar multiplication in dot product:

$(\lambda\vec{a}).\vec{b} = \lambda(\vec{a}.\vec{b})$ **or** $\vec{a}.(\lambda\vec{b}) = \lambda(\vec{a}.\vec{b})$

(ix) Relation between $\hat{\imath}, \hat{\jmath}, \hat{k}$ *unit vectors* in dot product

$\hat{\imath}.\hat{\imath} = 1$: $\hat{\imath}.\hat{\jmath} = 0$

$\hat{\jmath}.\hat{\jmath} = 1$: $\hat{\jmath}.\hat{k} = 0$

$\hat{k}.\hat{k} = 1$: $\hat{k}.\hat{\imath} = 0$

(x) *Work done by a force :*

Let $\vec{F}$ be a force , which applied on a particle and displaces it through a displacement $\vec{r}$. then work done is define as $W = \vec{F}.\vec{r}$

(xi) ***Projection of one vector along another vector:***

If $\vec{a}$ and $\vec{b}$ be the two vector. Then

Projection of $\vec{a}$ along $\vec{b}$ = $\vec{a}.\hat{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Projection of $\vec{b}$ along $\vec{a}$ = $\vec{b}.\hat{a} = \frac{\vec{b}.\vec{a}}{|\vec{a}|}$

Theorem:03:If $\vec{a}$ and $\vec{b}$ be the two non-zero vectors are perpendicular if and only if $\vec{a}.\vec{b} = 0$.

Proof: If $\vec{a}$ and $\vec{b}$ are perpendicular vectors . then we have to prove $\vec{a}.\vec{b} = 0$

We know that $\vec{a}.\vec{b} = a\,b\cos\theta$

$\vec{a}.\vec{b} = a\,b\cos(90^0)$ $\therefore$where $\theta = 90^0$

$\vec{a}.\vec{b} = a\,b\,(0)$

$\boxed{\vec{a}.\vec{b} = 0}$

Conversely suppose that $\vec{a}.\vec{b} = 0$

Then we have to prove $\vec{a}$ and $\vec{b}$ are perpendicular vectors. It means ($\theta = 90^0$)

Now takes $\vec{a}.\vec{b} = 0$

$a\,b\cos\theta = 0$

Here $ab \neq 0$ then $\cos\theta = 0$

$\theta = \cos^{-1}(0)$

$\boxed{\theta = 90^0}$

Hence proved .

Example #01 :Determine the magnitude of the vector $\vec{a}$ = 4 i + 3j +12k and also find the a unit vector in the direction of $\vec{a}$.

Solution: *Given vector $\vec{a}$ = 4 i + 3j +12k*

Magnitude: $|\vec{a}| = \sqrt{(4)^2 + (3)^2 + (12)^2} = \sqrt{16+9+144} = \sqrt{169}$ $\Rightarrow$

$|\vec{a}| = 13$

Unit vector: $\hat{a} = \frac{\vec{a}}{a}$

$\hat{a} = \frac{4i+3j+12k}{13} \Rightarrow \hat{a} = \frac{4}{13}i + \frac{3}{13}j + \frac{12}{13}k$

Example#02: If the angle between two vectors whose magnitudes are 14 and 7 is 60^0 . Find their scalar product.

Solution: Let $\vec{a}$ and $\vec{b}$ be the two vectors

Given $|\vec{a}|=14$; $|\vec{b}|=7$ and $\theta= 60^0$ Scalar product =?

As $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = 14.\,7\cos 60^0 = 98\,.\,\frac{1}{2}$ $\Longrightarrow$ $\boxed{\vec{a}.\vec{b} = 49}$

Example#03: Find a unit vector which makes an angle of 45^0 with $\vec{a}$ $[\,2,2,-1]$ and an angle of 60^0 with $\vec{b}$ $[\,0,1,-1]$.

Solution: Let $\hat{u}$ be the required unit vector.

$\hat{u} = xi + yj + z\,k$ ---(A)

$|\hat{u}|^2 = x^2 + y^2 + z^2$ $\Longrightarrow$ $x^2 + y^2 + z^2 = 1$-------------------(i) $\because |\hat{u}|^2 = 1$

Given $\vec{a} = [\,2,2,-1] = 2i + 2j - k$ and $\vec{b} = [\,0,1,-1] = 0i + j - k$

1^st^ condition: The unit vector $\hat{u}$ makes an angle 45^0 with $\vec{a}$.

Then $\vec{a}.\hat{u} = |\vec{a}|\,|\hat{u}|\cos\theta$ $\theta=45^0$

$$(2i + 2j - k).(\,xi + yj + z\,k) = \sqrt{(2)^2 + (2)^2 + (-1)^2}\ .\,1\,.\cos 45^0 \quad \because |\hat{u}| = 1$$

$$2x + 2y - z = \sqrt{4 + 4 + 1}\ .\ \frac{1}{\sqrt{2}} = \sqrt{9}\ .\ \frac{1}{\sqrt{2}}$$

$$2x + 2y - z = \frac{3}{\sqrt{2}} \text{ ---------------------(ii)}$$

2^nd^ condition: The unit vector $\hat{u}$ makes an angle 60^0 with $\vec{b}$.

Then $\vec{b}.\hat{u} = |\vec{b}|\,|\hat{u}|\cos\theta$ $\theta=60^0$

$$(0i + j - k).(\,xi + yj + z\,k) = \sqrt{(0)^2 + (1)^2 + (-1)^2}\ .\,1\,.\cos 60^0 \quad \because |\hat{u}| = 1$$

$$0x + y - z = \sqrt{0 + 1 + 1}\ .\ \frac{1}{2} = \sqrt{2}\ .\ \frac{1}{2} \Longrightarrow y - z = \frac{1}{\sqrt{2}} \text{ -------------(iii)}$$

Subtracting equation (i) and (ii): $\rightarrow$ $(2x + 2y - z) - (\,y - z) = \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$$2x + 2y - z - y + z = \frac{3-1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$2x + y = \sqrt{2} \Longrightarrow y = \sqrt{2} - 2x \text{ -----(iv)}$$

Using Equation (iv) in (ii) $\rightarrow$ $\sqrt{2} - 2x - Z = \frac{1}{\sqrt{2}}$ $\Longrightarrow$ $\sqrt{2} - 2x - \frac{1}{\sqrt{2}} = z$

$\Longrightarrow \sqrt{2} - \frac{1}{\sqrt{2}} - 2x = z \Longrightarrow$ $\frac{2-1}{\sqrt{2}} - 2x = z$ $\Longrightarrow$ $z = \frac{1}{\sqrt{2}} - 2x$ ----------------------(v)

Using equation (iv) and (v) in (i)

$$x^2 + \left[\sqrt{2} - 2x\right]^2 + \left[\frac{1}{\sqrt{2}} - 2x\right]^2 = 1$$

$$x^2 + (\sqrt{2})^2 + (2x)^2 - 2(\sqrt{2})(2x) + \left(\frac{1}{\sqrt{2}}\right)^2 + (2x)^2 - 2\left(\frac{1}{\sqrt{2}}\right)(2x) = 1$$

$$x^2 + 2 + 4x^2 - 4\sqrt{2}\ x + \frac{1}{2} + 4x^2 - 2\sqrt{2}\ x - 1 = 0$$

$$9x^2 - 6\sqrt{2}\ x + 1 + \frac{1}{2} = 0$$

$$9x^2 - 6\sqrt{2}\ x + \frac{3}{2} = 0$$

$$6x^2 - 4\sqrt{2}\ x + 1 = 0 \qquad \{\text{Multiplying equation by } \tfrac{2}{3}\}$$

By using quadratic formula

$$\boldsymbol{x} = \frac{-(-4\sqrt{2}) \pm \sqrt{(4\sqrt{2})^2 - 4(6)(1)}}{2\times 6} = \frac{4\sqrt{2} \pm \sqrt{32-24}}{12} = \frac{4\sqrt{2} \pm \sqrt{8}}{12} \implies x = \frac{4\sqrt{2} \pm 2\sqrt{2}}{12}$$

$$x = \frac{4\sqrt{2}+2\sqrt{2}}{12} = \frac{6\sqrt{2}}{12} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\boxed{x = \frac{1}{\sqrt{2}}}$$

Put in (iv) and (v)

$$y = \sqrt{2} - 2\left(\frac{1}{\sqrt{2}}\right)$$

$$y = \sqrt{2} - \sqrt{2} = 0$$

$$\boxed{y = 0}$$

and

$$z = \frac{1}{\sqrt{2}} - 2\left(\frac{1}{\sqrt{2}}\right)$$

$$z = \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}}$$

$$z = \frac{1-2}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$\boxed{z = \frac{-1}{\sqrt{2}}}$$

$$x = \frac{4\sqrt{2}-2\sqrt{2}}{12} = \frac{2\sqrt{2}}{12} = \frac{\sqrt{2}}{2\times 3} = \frac{1}{3\sqrt{2}}$$

$$\boxed{x = \frac{1}{3\sqrt{2}}}$$

$$y = \sqrt{2} - 2\left(\frac{1}{3\sqrt{2}}\right) = \sqrt{2} - \frac{\sqrt{2}}{3}$$

$$y = \frac{3\sqrt{2}-\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}$$

$$\boxed{y = \frac{2\sqrt{2}}{3}}$$

$$z = \sqrt{2} - 2\left(\frac{1}{3\sqrt{2}}\right)$$

$$z = \frac{1}{\sqrt{2}} - \frac{2}{3\sqrt{2}}$$

$$z = \frac{3-2}{3\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

$$\boxed{z = \frac{1}{3\sqrt{2}}}$$

Using values of x, y ,z in required unit vector represented by equ.(A)

$$\hat{u} = \frac{1}{\sqrt{2}}\ i + 0j - \frac{1}{\sqrt{2}}\ k \quad \text{OR} \quad \hat{u} = \frac{1}{3\sqrt{2}} i + \frac{2\sqrt{2}}{3} j + \frac{1}{3\sqrt{2}}\ k$$

Example#05:For what value of λ , the vector $2i - j + 2k$ and $3i + 2\lambda j$ are perpendicular?

Solution: Let

$$\vec{a} = 2i - j + 2k \quad \text{and} \quad \vec{b} = 3i + 2\lambda j$$

According to given condition $\vec{a} \perp \vec{b}$ then

$$\vec{a} \cdot \vec{b} = 0$$

$$(2i - j + 2k).(3i + 2\lambda j + 0k) = 0$$

$$(2)(3) + (-1)(2\lambda) + (2)(0) = 0$$

$$6 - 2\lambda + 0 = 0$$

$$2\lambda = 6$$

$$\lambda = 6/2$$

$$\boxed{\lambda = 3}$$

Example#06: Find the cosine of the angle between the vectors a and b where $\vec{a} = i + 2j - k$ and $\vec{b} = -i + j - 2k$.

Solution : Given $\vec{a} = i + 2j - k$ and $\vec{b} = -i + j - 2k$.

As $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

Therefore ,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\cos\theta = \frac{(i+2j-k).(-i+j-2k)}{\left(\sqrt{(1)^2+(2)^2+(-1)^2}\right)\left(\sqrt{(1)^2+(1)^2+(-2)^2}\right)}$$

$$\cos\theta = \frac{(1)(-1)+(2)(1)+(-1)(-2)}{(\sqrt{1+4+1})(\sqrt{1+1+4})}$$

$$\cos\theta = \frac{-1+2+2}{(\sqrt{6})(\sqrt{6})} = \frac{3}{6}$$

$$\Longrightarrow \cos\theta = \frac{1}{2}$$

Example#07: If $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$ **. show that** $\vec{a}$ and $\vec{b}$ **are perpendicular.**

Solution: Given

$$\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right| \text{------------------------(i)}$$

We have to prove $\vec{a} \perp \vec{b}$ it means $\vec{a}.\vec{b} = 0$

Squaring equation (i)

$$\left|\vec{a}+\vec{b}\right|^2 = \left|\vec{a}-\vec{b}\right|^2$$

$$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = (\vec{a}-\vec{b}).(\vec{a}-\vec{b})$$

$$\vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = \vec{a}.\vec{a} - \vec{a}.\vec{b} - \vec{b}.\vec{a} + \vec{b}.\vec{b}$$

$$|\vec{a}|^2 + 2\,\vec{a}.\vec{b} + \left|\vec{b}\right|^2 = |\vec{a}|^2 - 2\,\vec{a}.\vec{b} + \left|\vec{b}\right|^2$$

$$|\vec{a}|^2 + 2\,\vec{a}.\vec{b} + \left|\vec{b}\right|^2 - |\vec{a}|^2 + 2\,\vec{a}.\vec{b} - \left|\vec{b}\right|^2 = 0$$

$$4\,\vec{a}.\vec{b} = 0$$

$$\Longrightarrow \quad \vec{a}.\vec{b} = 0$$

Hence proved $\vec{a} \perp \vec{b}$

Example#08: If $\vec{a} = 3i - j - 4k$ **;** $\vec{b} = -2i + 4j - 3k$ **and** $\vec{c} = i + 2j - k$ **.**

Find the projection of($\vec{a}+2\vec{b}$ **) along** $\vec{c}$**.**

Solution: Given $\vec{a} = 3i - j - 4k$; $\vec{b} = -2i + 4j - 3k$ and $\vec{c} = i + 2j - k$

Let $\vec{u} = \vec{a}+2\vec{b}$ Projection of $\vec{u}$ along $\vec{c} = ?$

Now $\vec{u} = \vec{a}+2\vec{b} = (3i - j - 4k) + 2(-2i + 4j - 3k)$

$= 3i - j - 4k - 4i + 8j - 6k$

$\vec{u} = -i + 7j - 10k$

Projection of $\vec{u}$ along $\vec{c} = \vec{u}.\hat{c}$

Projection of $\vec{u}$ along $\vec{c}$ $= \dfrac{\vec{u}.\vec{c}}{|\vec{c}|} = \dfrac{(-i+7j-10k).(i+2j-k)}{\sqrt{(-1)^2+(2)^2+(-1)^2}} = \dfrac{(-1)(1)+(7)(2)+(-10)(-1)}{\sqrt{1+4+1}}$

$$= \frac{-1+14+10}{\sqrt{6}} = \frac{23}{\sqrt{6}}$$

Exercise#2.1

Q#01: If $\vec{a} = 3i + j - k$; $\vec{b} = 2i - j + 2k$ and $\vec{c} = 5i + 3k$. Find

(i) $(2\vec{a}+\vec{b}).\vec{c}$

Solution

$\therefore 2\vec{a}+\vec{b} = 2(3i + j - k) + (2i - j + 2k) = 6i + 2j - 2k + 2i - j + 2k = 8i + j + 0k$

Now

$(2\vec{a}+\vec{b}).\vec{c} = (8i + j + 0k).(5i + 0j + 3k) = (8)(5) + (1)(0) + (0)(3) = 40 + 0 + 0 = 40$

(ii) $(\vec{a} - 2\vec{c}).(\vec{b}+\vec{c})$

Solution:

$\therefore \vec{a} - 2\vec{c} = (3i + j - k) - 2(5i + 3k) = 3i + j - k - 10i - 6k = -7i + j - 7k$

$\therefore \vec{b}+\vec{c} = 2i - j + 2k + 5i + 3k = 7i - j + 5k$

Now

$(\vec{a} - 2\vec{c}).(\vec{b}+\vec{c}) = (-7i + j - 7k).(7i - j + 5k)$

$= (-7)(7) + (1)(-1) + (-7)(5)$

$= -49 - 1 - 35$

$(\vec{a} - 2\vec{c}).(\vec{b}+\vec{c}) = -85$

Q#02: Find x, so that $\vec{a} = 2i + 4j - 7k$ and $\vec{b} = 2i + 6j + xk$ are perpendicular?

Solution: Given

$\vec{a} = 2i + 4j - 7k$ and $\vec{b} = 2i + 6j + xk$

According to given condition $\vec{a} \perp \vec{b}$ then $\vec{a}.\vec{b} = 0$

$(2i + 4j - 7k).(2i + 6j + xk) = 0$

$(2)(2) + (4)(6) + (-7)(x) = 0$

$4 + 24 - 7x = 0$

$28 = 7x$

$x = 28/7$

$\boxed{x = 3}$

Q#03: Find m, for which the angle between $\vec{a} = mi + j - k$ & $\vec{b} = i + mj - k$ is $\frac{\pi}{3}$?

Solution: Given $\vec{a} = mi + j - k$ and $\vec{b} = i + mj - k$

According to given condition that $\theta = \frac{\pi}{3}$ between $\vec{a}$ & $\vec{b}$ then

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$$(mi + j - k).(i + mj - k) = \sqrt{(m)^2 + (1)^2 + (-1)^2}\ \sqrt{(1)^2 + (m)^2 + (-1)^2}\ \cos\frac{\pi}{3}$$

$$(m)(1) + (1)(m) + (-1)(-1) = \sqrt{1 + m^2 + 1}\ \sqrt{1 + m^2 + 1}\ (\tfrac{1}{2})$$

$$m + m + 1 = (\sqrt{m^2 + 2}\,)^2 . (\tfrac{1}{2})$$

$$2\,(2m + 1) = (m^2 + 2)$$

$$4m + 2 = m^2 + 2$$

$$m^2 - 4m = 0$$

$$m\,(m - 4) = 0$$

$$\boxed{m = 0} \quad or \quad m - 4 = 0 \Rightarrow \boxed{m = 4}$$

Q#04: If $\vec{a} = 2i + j - 3k$ & $\vec{b} = i - 2j + k$, find a vector whose magnitude is 5 and perpendicular to both $\vec{a}$ & $\vec{b}$.

Solution: Given $\vec{a} = 2i + j - 3k$ & $\vec{b} = i - 2j + k$

Let $\vec{u} = xi + yj + zk$ ---------------------(A)

$|\vec{u}| = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad x^2 + y^2 + z^2 = |\vec{u}|^2$

$x^2 + y^2 + z^2 = 5^2$ Given $|\vec{u}| = 5$

$x^2 + y^2 + z^2 = 25$---------------------------------------(i)

1st condition: $\vec{u} \perp \vec{a}$ then $\vec{u}.\vec{a} = 0$

$$(xi + yj + zk).(2i + j - 3k) = 0$$

$$(x)(2) + (y)(1) + (z)(-3) = 0$$

$2x + y - 3z = 0$ -----------------(ii)

2st condition: $\vec{u} \perp \vec{b}$ then $\vec{u}.\vec{b} = 0$

$$(xi + yj + zk).(i - 2j + k) = 0$$

$$(x)(1) + (y)(-2) + (z)(1) = 0$$

$x - 2y + z = 0$ ----- --------(iii)

Multiplying equation (iii) by 3 and add in equation (ii)

$$3x - 6y + 3z = 0$$
$$\underline{2x + y - 3z = 0}$$
$$5x - 5y = 0$$
$$5(x - y) = 0$$
$$x - y = 0 \quad \Rightarrow \quad y = x \text{ --------- } (iv)$$

Multiplying equation (ii)by 2 and add in equation (iii)

$$4x + 2y - 6z = 0$$
$$\underline{x - 2y + z = 0}$$
$$5x - 5z = 0$$
$$5(x - z) = 0$$
$$x - z = 0 \Rightarrow z = x \text{ ------------ } (v)$$

using equ. (iv) and (v) in equ. (i)

$$x^2 + x^2 + x^2 = 25$$
$$3x^2 = 25$$
$$x^2 = \frac{25}{3}$$
$$x = \pm\sqrt{\frac{25}{3}} \quad \text{Or} \quad x = \pm\frac{5}{\sqrt{3}}$$

using value of x in equ. (iv) and (v)

$$y = \pm\frac{5}{\sqrt{3}} \quad \text{and} \quad z = \pm\frac{5}{\sqrt{3}}$$

Putting values of x, y and z in (A)

$$\vec{u} = \pm\frac{5}{\sqrt{3}}(i + j + k)$$

Q#05: If the angle between two vectors whose magnitudes are 12 and 4 is 60^0 . Find their scalar product.

Solution: Let $\vec{a}$ and $\vec{b}$ be the two vectors

Given $|\vec{a}| = 12$; $|\vec{b}| = 4$ and $\theta = 60^0$

As
$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$
$$= (12)(4)\cos 60^0$$
$$= 48 \cdot \frac{1}{2}$$
$$\boxed{\vec{a}.\vec{b} = 24}$$

Q#06: Show that $\hat{a} = \frac{2i-2j+k}{3}$, $\hat{b} = \frac{i+2j+2k}{3}$ and $\hat{c} = \frac{2i+j-2k}{3}$ are mutually orthogonal unit vectors.

Solution: Given $\hat{a} = \frac{2i-2j+k}{3}$, $\hat{b} = \frac{i+2j+2k}{3}$ and $\hat{c} = \frac{2i+j-2k}{3}$

For mutually orthogonal condition, we have to prove

$$\hat{a}\,.\,\hat{b} = \hat{b}.\,\hat{c} = \hat{c}\,.\,\hat{a} = 0$$

$$\hat{a}\,.\,\hat{b} = \left(\frac{2i-2j+k}{3}\right).\left(\frac{i+2j+2k}{3}\right) = \frac{(2i-2j+k).(\,i+2j+2k)}{9} = \frac{(2)(1)+(-2)(2)+(1)(2)}{9} = \frac{2-4+2}{9} = \frac{0}{9} \Longrightarrow \hat{a}\,.\,\hat{b}=0$$

$$\hat{b}\,.\,\hat{c} = \left(\frac{i+2j+2k}{3}\right).\left(\frac{2i+j-2k}{3}\right) = \frac{(i+2j+2k).(\,2\,i+j-2k)}{9} = \frac{(1)(2)+(2)(1)+(2)(-2)}{9} = \frac{2+2-4}{9} = \frac{0}{9} \Longrightarrow \hat{b}\,.\,\hat{c}= 0$$

$$\hat{c}\,.\,\hat{a} = \left(\frac{2i+j-2k}{3}\right).\left(\frac{2i-2j+k}{3}\right) = \frac{(2i+j-2k).(\,2\,i-2j+k)}{9} = \frac{(2)(2)+(1)(-2)+(-2)(1)}{9} = \frac{4-2-2}{9} = \frac{0}{9} \Longrightarrow \hat{a}\,.\,\hat{b}=0$$

Hence proved that $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually orthogonal unit vectors.

Q#07: Find the cosine of the angle between $\vec{a} = 2i - 8j + 3k$ and $\vec{b} = 4j + 3k$.

Solution : Given $\vec{a} = 2i - 8j + 3k$ and $\vec{b} = 2j + 4k$.

As $\vec{a}.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$ therefore

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\cos\theta = \frac{(2i-8j+3k\,).(0i+4j+3k)}{\left(\sqrt{(2)^2+(-8)^2+(3)^2}\right)\left(\sqrt{(0)^2+(4)^2+(3)^2}\right)}$$

$$\cos\theta = \frac{(2)(0)+(-8)(4)+(3)(3)}{\left(\sqrt{4+64+9}\right)\left(\sqrt{0+16+9}\right)}$$

$$\cos\theta = \frac{0-32+9}{\left(\sqrt{77}\right)\left(\sqrt{25}\right)}$$

$$\cos\theta = \frac{-23}{5\sqrt{77}}$$

Q#08: (i) If $\vec{a} = 2i - 3j + 4k$ and $\vec{b} = 2j + 4k$, find the component of $\vec{a}$ along $\vec{b}$ and $\vec{b}$ along $\vec{a}$.

Solution: Given $\vec{a} = 2i - 3j + 4k$ and $\vec{b} = 2j + 4k$

Now

Component of $\vec{a}$ along $\vec{b} = \boldsymbol{\vec{a}}.\,\hat{\boldsymbol{b}} = \frac{\vec{a}.\,\vec{b}}{|\vec{b}|} = \frac{(2i-3j+4k\,).(0i+2j+4k)}{\left(\sqrt{(0)^2+(2)^2+(4)^2}\right)} = \frac{(2)(0)+(-3)(2)+\,(4)(4)}{\left(\sqrt{0+4+16}\right)} = \frac{0-6+16}{\left(\sqrt{20}\right)} = \frac{10}{2\sqrt{5}}$

Component of $\vec{b}$ along $\vec{a} = \boldsymbol{\vec{b}}.\,\hat{\boldsymbol{a}} = \frac{\vec{b}.\vec{a}}{|\vec{a}|} = \frac{(0i+2j+4k\,).(2i-3j+4k)}{\left(\sqrt{(2)^2+(-3)^2+(4)^2}\right)} = \frac{(0)(2)+(2)(-3)+\,(4)(4)}{\left(\sqrt{4+9+16}\right)} = \frac{0-6+16}{\left(\sqrt{29}\right)} = \frac{10}{\sqrt{29}}$

Q#08: (ii) If $\vec{a} = 3i - j - 4k$;$\vec{b} = -2i + 4j - 3k$ and $\vec{c} = i + 2j - k$ find the projection of $2\vec{a} + 3\vec{b} - \vec{c}$ along $\vec{a} + \vec{b}$.

Solution: Given $\vec{a} = 3i - j - 4k$; $\vec{b} = -2i + 4j - 3k$ and $\vec{c} = i + 2j - k$

Let

$$\vec{u} = 2\vec{a} + 3\vec{b} - \vec{c} = 2(3i - j - 4k) + 3(-2i + 4j - 3k) - (i + 2j - k)$$

$$= 6i - 2j - 8k - 6i + 12j - 9k - i - 2j + k$$

$$\vec{u} = -i + 8j - 16k$$

And $\vec{v} = \vec{a} + \vec{b} = (3i - j - 4k) + (-2i + 4j - 3k)$

$$= 3i - j - 4k - 2i + 4j - 3k$$

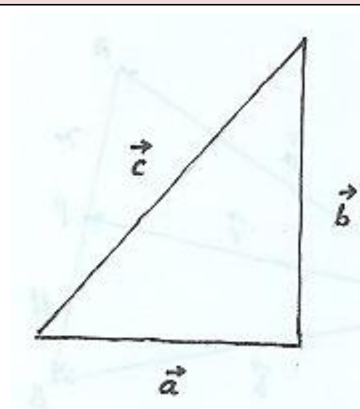
$$\vec{v} = i + 3j - 7k$$

Projection of $\vec{u}$ along $\vec{v} = \vec{u}.\hat{v} = \frac{\vec{u}.\vec{v}}{|\vec{v}|} = \frac{(-i+8j-16k).(i+3j-7k)}{\left(\sqrt{(1)^2+(3)^2+(-7)^2}\right)} = \frac{(-1)(1)+(8)(3)+(-16)(-7)}{(\sqrt{1+9+49})}$

$= \frac{-1+24+112}{(\sqrt{59})} = \frac{135}{\sqrt{59}}$

Q#09: Show that the vectors $\vec{a} = 3i - 2j + k$;$\vec{b} = i - 3j + 5k$ and $\vec{c} = 2i + j - 4k$ form a right angle triangle.

Solution: Given



$$\vec{a} = 3i - 2j + k \;;\vec{b} = i - 3j + 5k \text{ and } \vec{c} = 2i + j - 4k$$

For right angle triangle, we have to prove

$$\vec{a}.\vec{b} = 0 \text{ or } \vec{b}.\vec{c} = 0 \text{ or } \vec{c}.\vec{a} = 0$$

$\vec{a}.\vec{b} = (3i - 2j + k).(i - 3j + 5k)$

$= (3)(1) + (-2)(-3) + (1)(5) = 3 + 6 + 5 = 14$

$\boxed{\vec{a}.\vec{b} \neq 0}$

$\vec{b}.\vec{c} = (i - 3j + 5k).(2i + j - 4k) = (1)(2) + (-3)(1) + (5)(-4) = 2 - 3 - 20 = -21$

$\boxed{\vec{b}.\vec{c} \neq 0}$

$\vec{c}.\vec{a} = (2i + j - 4k).(3i - 2j + k) = (2)(3) + (1)(-2) + (-4)(1) = 6 - 2 - 4$

$\boxed{\vec{c}.\vec{a} = 0}$

So $\vec{c} \perp \vec{a}$

Hence proved that the given vectors form right angle triangle .

Q#10: The vectors $\vec{a} = 2i - j + k$; $\vec{b} = -i + 3j + 5k$ represent two sides of ΔABC. Find its 3rd sides and also the angles of this triangle.

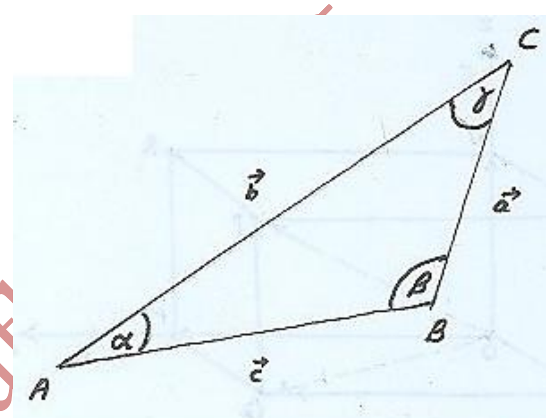
Solution:

Given $\vec{a} = 2i - j + k$ & $\vec{b} = -i + 3j + 5k$

Let $\vec{c}$ be resultant of $\vec{a}$ and $\vec{b}$ in ΔABC. Then

$\vec{c} = \vec{a} + \vec{b} = (2i - j + k) + (-i + 3j + 5k) = 2i - j + k - i + 3j + 5k$

$\vec{c} = i + 2j + 6k$



Now $|\vec{a}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4+1+1} = \sqrt{6}$

$|\vec{b}| = \sqrt{(-1)^2 + (3)^2 + (5)^2} = \sqrt{1+9+25} = \sqrt{35}$

$|\vec{c}| = \sqrt{(1)^2 + (2)^2 + (6)^2} = \sqrt{1+4+36} = \sqrt{41}$

Let α , β and γ be the angle of ΔABC as shown in figure.

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\gamma$$

$$\cos\gamma = \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\cos\gamma = \frac{(2i-j+k).(-i+3j+5k)}{(\sqrt{6})(\sqrt{35})} = \frac{(2)(-1)+(-1)(3)+(1)(5)}{\sqrt{6\times35}} = \frac{-2-3+5}{\sqrt{210}} = \frac{0}{\sqrt{210}}$$

$\cos\gamma = 0$

$\gamma = \cos^{-1}(0)$ $\Rightarrow$ $\boxed{\gamma = 90^0}$

$$\vec{b}.\vec{c} = |\vec{b}|\,|\vec{c}| \cos\alpha$$

$$\cos\alpha = \frac{\vec{b}.\vec{c}}{|\vec{b}|\,|\vec{c}|}$$

$$\cos\alpha = \frac{(-i+3j+5k).(i+2j+6k)}{(\sqrt{35})(\sqrt{41})} = \frac{(-1)(1)+(3)(2)+(5)(6)}{\sqrt{35\times41}} = \frac{-1+6+30}{\sqrt{1435}} = \frac{35}{\sqrt{1435}}$$

$\cos\alpha = 0.923$

$\alpha = \cos^{-1}(0.923)$ $\Rightarrow$ $\boxed{\alpha = 22.49^0}$

We know that

$\alpha + \beta + \gamma = 180^0$

$\beta = 180 - \alpha - \gamma$

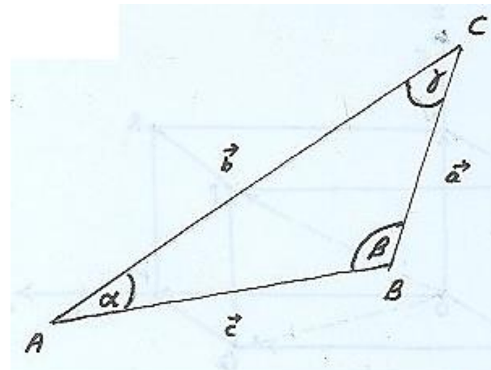
$\beta = 180^0 - 90^0 - 22.49^0$

$\boxed{\beta = 67.51^0}$

Q#11: The vectors $\vec{a} = 3i + 6j - 2k$ & $\vec{b} = 4i - j + 3k$ represent two sides of ΔABC. Find its 3rd sides and also the angles of this triangle.

Solution: Given $\vec{a} = 3i + 6j - 2k$ & $\vec{b} = 4i - j + 3k$

Let $\vec{c}$ be resultant of $\vec{a}$ and $\vec{b}$ $in\ \Delta ABC$. Then



$$\vec{c} = \vec{a} + \vec{b}$$

$$\vec{c} = (3i + 6j - 2k) + (4i - j + 3k)$$

$$\vec{c} = 3i + 6j - 2k + 4i - j + 3k$$

$$\vec{c} = 7i + 5j + k$$

Now

$$|\vec{a}| = \sqrt{(3)^2 + (6)^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

$$|\vec{b}| = \sqrt{(4)^2 + (-1)^2 + (3)^2} = \sqrt{16 + 1 + 9} = \sqrt{26}$$

$$|\vec{c}| = \sqrt{(7)^2 + (5)^2 + (1)^2} = \sqrt{49 + 25 + 1} = \sqrt{75}$$

Let α , β and γ be the angle of ΔABC as shown in figure.

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\gamma$$

$$\cos\gamma = \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\cos\gamma = \frac{(3i+6j-2k).(4i-j+3k)}{(\sqrt{49})(\sqrt{26})} = \frac{(3)(4)+(6)(-1)+(-2)(3)}{\sqrt{49\times 26}} = \frac{12-6-6}{\sqrt{1274}} = \frac{0}{\sqrt{1274}}$$

$$\cos\gamma = 0$$

$$\gamma = \cos^{-1}(0) \implies \boxed{\gamma = 90^0}$$

$$\vec{b}.\vec{c} = |\vec{b}|\,|\vec{c}|\cos\alpha$$

$$\cos\alpha = \frac{\vec{b}.\vec{c}}{|\vec{b}|\,|\vec{c}|}$$

$$\cos\alpha = \frac{(4i-j+3k).(7i+5j+k)}{(\sqrt{26})(\sqrt{75})} = \frac{(4)(7)+(-1)(5)+(3)(1)}{\sqrt{35\times 75}} = \frac{28-5+3}{\sqrt{2625}} = \frac{26}{\sqrt{2625}}$$

$$\cos\alpha = 0.588$$

$$\alpha = \cos^{-1}(0.588) \implies \boxed{\alpha = 54^0}$$

We know that

$$\alpha + \beta + \gamma = 180^0$$

$$\beta = 180 - \alpha - \gamma$$

$$\beta = 180^0 - 90^0 - 54^0$$

$$\boxed{\beta = 36^0}$$

Q#12: Find two unit vectors which makes an angle of 60^0 with vectors $i-j$ and $i-k$.

***Solution* :** Let $\hat{u}$ be the required unit vector .

$$\hat{u} = xi + yj + z\,k \text{ ------------------(A)}$$

$$|\hat{u}|^2 = x^2 + y^2 + z^2 \implies x^2 + y^2 + z^2 = 1 \text{---------------------(i)} \qquad \therefore\ |\hat{u}|^2 = 1$$

Given $\vec{a} = i - j$ and $\vec{b} = i - k$

***1st condition**:* The unit vector $\hat{u}$ makes an angle 60^0 with $\vec{a}$.

Then $\vec{a}.\hat{u} = |\vec{a}|\,|\hat{u}|\cos\theta \qquad \theta = 60^0$

$$(i - j + 0k).(xi + yj + z\,k) = \sqrt{(1)^2 + (-1)^2 + (0)^2}\ (1)\cos 60^0 \qquad \therefore\ |\hat{u}| = 1$$

$$1.x - 1.y - 0.z = \sqrt{1+1+0}\ .\ \tfrac{1}{2} = \sqrt{2}\ .\ \tfrac{1}{2}$$

$$x - y = \frac{1}{\sqrt{2}} \implies y = x - \frac{1}{\sqrt{2}} \text{ ---------------------(ii)}$$

2nd condition: The unit vector $\hat{u}$ makes an angle 60^0 with $\vec{b}$.

Then $\vec{b}.\hat{u} = |\vec{b}|\,|\hat{u}|\cos\theta \qquad \theta = 60^0$

$$(i + 0j - k).(xi + yj + z\,k) = \sqrt{(0)^2 + (1)^2 + (-1)^2}\ (1)\,.\cos 60^0 \qquad \therefore\ |\hat{u}| = 1$$

$$1.x + 0.y - 1.z = \sqrt{0+1+1}\ .\ \tfrac{1}{2} = \sqrt{2}\ .\ \tfrac{1}{2}$$

$$x - z = \frac{1}{\sqrt{2}} \implies z = x - \frac{1}{\sqrt{2}} \text{ ---------------------(iii)}$$

Using equation (ii) and (iii) in (i)

$$x^2 + \left[x - \frac{1}{\sqrt{2}}\right]^2 + \left[x - \frac{1}{\sqrt{2}}\right]^2 = 1$$

$$x^2 + x^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 2\,x\,\frac{1}{\sqrt{2}} + x^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 2\,x\,\frac{1}{\sqrt{2}} = 1$$

$$3x^2 - \sqrt{2}\,x + \tfrac{1}{2} - \sqrt{2}\,x + \tfrac{1}{2} = 1 \implies 3x^2 - 2\sqrt{2}\,x + 1 = 1 \implies 3x^2 - 2\sqrt{2}\,x = 0 \implies x\,(3x - 2\sqrt{2}) = 0$$

$\boxed{x = 0}$

Put in (ii) and (iii)

$y = 0 - \frac{1}{\sqrt{2}} \implies \boxed{y = -\frac{1}{\sqrt{2}}}$

$z = 0 - \frac{1}{\sqrt{2}} \implies \boxed{z = \frac{-1}{\sqrt{2}}}$

$3x - 2\sqrt{2} = 0 \implies 3x = 2\sqrt{2} \implies \boxed{x = \frac{2\sqrt{2}}{3}}$

$y = \frac{2\sqrt{2}}{3} - \frac{1}{\sqrt{2}} = \frac{4-3}{3\sqrt{2}} \implies \boxed{y = \frac{1}{3\sqrt{2}}}$

$z = \frac{2\sqrt{2}}{3} - \frac{1}{\sqrt{2}} = \frac{4-3}{3\sqrt{2}} \implies \boxed{z = \frac{1}{3\sqrt{2}}}$

Using values of x, y ,z in required unit vector represented by equ.(A)

$$\hat{u} = 0\,i - \frac{1}{\sqrt{2}}j - \frac{1}{\sqrt{2}}\,k \qquad \text{OR} \qquad \hat{u} = \frac{2\sqrt{2}}{3}i + \frac{1}{3\sqrt{2}}j + \frac{1}{3\sqrt{2}}\,k$$

Q#13: Find the projection of vector $2i - 2j + 6k$ On the vector $i + 2j + 2k$.

Solution: Let $\vec{a} = 2i - 2j + 6k$ and $\vec{b} = i + 2j + 2k$

Then

Projection of $\vec{a}$ along $\vec{b} = \vec{a} \cdot \hat{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{(2i-2j+6k).(i+2j+2k)}{\left(\sqrt{(1)^2+(2)^2+(2)^2}\right)} = \frac{(2)(1)+(-2)(2)+(6)(2)}{(\sqrt{1+4+4})} = \frac{2-4+12}{(\sqrt{9})} = \frac{10}{3}$

Q#14: Find the projection of vector $4i - 3j + k$ On the line passing through the points $(2, 3, -1)$ *and* $(-2, -4, 1)$.

Solution: Let $\vec{a} = 4i - 3j + k$ and

Given points $A(2, 3, -1)$; $B(-2, -4, 1)$

Let $\vec{b} = \overrightarrow{AB} = B(-2, -4, 1) - A(2, 3, -1) = (-2-2)i + (-4-3)j + (1+1)k$

$\vec{b} = -4i - 7j + 2k$

Then

Projection of $\vec{a}$ along $\vec{b} = \vec{a} \cdot \hat{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{(4i-3j+k).(-4i-7j+2k)}{\left(\sqrt{(-4)^2+(-7)^2+(2)^2}\right)} = \frac{(4)(-4)+(-3)(-7)+(1)(2)}{(\sqrt{16+49+4})}$

$= \frac{-16+21+2}{(\sqrt{69})} = \frac{7}{\sqrt{69}}$

Q#15: (i) Verify that the scalar product is distributive with respect to the addition of vectors when $\vec{a} = 2i - 3j + 4k$; $\vec{b} = i - j + 2k$ and $\vec{c} = 3i + 2j + k$.

Solution: Given vectors $\vec{a} = 2i - 3j + 4k$; $\vec{b} = i - j + 2k$ and $\vec{c} = 3i + 2j + k$

We have to prove , scalar product is distributive with respect to the addition.

$$\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$$

L.H.S $= \vec{a}.(\vec{b} + \vec{c}) = (2i - 3j + 4k).(i - j + 2k + 3i + 2j + k)$

$= (2i - 3j + 4k).(4i + j + 3k)$

$= (2)(4) + (-3)(1) + (4)(3) = 8 - 3 + 12$

$= 17$ --------------------(i)

R.H.S $= \vec{a}.\vec{b} + \vec{a}.\vec{c} = (2i - 3j + 4k).(i - j + 2k) + (2i - 3j + 4k).(3i + 2j + k)$

$= [(2)(1) + (-3)(-1) + (4)(2)] + [(2)(3) + (-3)(2) + (4)(1)]$

$= [2 + 3 + 8] + [6 - 6 + 4] = 2 + 3 + 8 + 0 + 4$

$= 17$ --------------------(ii)

Hence Verified from (i) and (ii),

That the scalar product is distributive with respect to the addition for vector $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Q#15:(ii) If $\vec{u}$ is a vector such that $\vec{u}\,.\,i = \vec{u}\,.\,j = \vec{u}\,.\,k = 0$, then find $\vec{u}$.

Solution: let $\vec{u} = xi + yj + zk$ ---------------(i)

Given condition: $\vec{u}\,.\,i = \vec{u}\,.\,j = \vec{u}\,.\,k = 0$

$\vec{u}\,.\,i = 0 \implies (xi + yj + zk\,).\,i = 0 \implies \quad x = 0$

$\vec{u}\,.\,j = 0 \implies (xi + yj + zk\,).\,j = 0 \implies \quad y = 0$

$\vec{u}\,.\,k = 0 \implies (xi + yj + zk\,).\,k = 0 \implies \quad z = 0$

Using value of x, y and z in (i)

$\vec{u} = 0i + 0j + 0k$

Q#16: (i) Under what condition does the relation $(\vec{a}\,.\,\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2$ Hold for vector $\vec{a}$ *and* $\vec{b}$.

Solution: By using definition of scalar product

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

Squaring equation on both sides

$$(\vec{a}.\vec{b}) = (\,|\vec{a}|\,|\vec{b}|\cos\theta\,)^2$$

$$(\vec{a}\,.\,\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2\cos^2\theta$$

This condition hold when

$$\cos^2\theta = 1 \implies \cos\theta = \pm 1$$

$\cos\theta = 1 \implies \theta = \cos^{-1}(1) \implies \boxed{\theta = 0^0}$ **and** $\cos\theta = -1 \implies \theta = \cos^{-1}(-1) \implies \boxed{\theta = 180^0}$

Q#16: (ii) If $\vec{a} = i + 2j - 3k$ and $\vec{b} = 3i + j + 2k$ then show that $\vec{a} + \vec{b}$ is perpendicular to $\vec{a} - \vec{b}$.

Solution: Given $\vec{a} = i + 2j - 3k$ and $\vec{b} = 3i + j + 2k$

We have to prove $(\vec{a} + \vec{b}\,) \perp (\vec{a} - \vec{b})$ For this $(\vec{a} + \vec{b}\,).(\vec{a} - \vec{b}) = 0$ ---------(i)

$\vec{a} + \vec{b} = (i + 2j - 3k\,) + (3i + j + 2k\,) = i + 2j - 3k + 3i + j + 2k = 4i + 3j - k$

$\vec{a} - \vec{b} = (i + 2j - 3k\,) - (3i + j + 2k\,) = i + 2j - 3k - 3i - j - 2k = -2i + j - 5k$

Taking L.H.S of (i) $(\vec{a} + \vec{b}\,).(\vec{a} - \vec{b}) = (4i + 3j - k\,).(-2i + j - 5k)$

$= (4)(-2) + (3)(1) + (-1)(-5)$

$= -8 + 3 + 5$

$$(\vec{a} + \vec{b}\,).(\vec{a} - \vec{b}) = 0$$

Hence proved $(\vec{a} + \vec{b}\,) \perp (\vec{a} - \vec{b})$.

Q#17: Example#04: The resultant of two vectors $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{a}$. If $|\vec{b}| = \sqrt{2}\ |a|$ Show that the resultant of 2 $\vec{a}$ and $\vec{b}$ is perpendicular vector $\vec{b}$.

Solution: Given sum of $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{a}$. $(\vec{a}+\vec{b}) \perp \vec{a}$

Then $(\vec{a}+\vec{b}).\vec{a} = 0$

$$\vec{a}.\vec{a}+\vec{b}.\vec{a} = 0$$

$$|\vec{a}|^2+\vec{a}.\vec{b} = 0 \qquad \therefore\ \vec{a}.\vec{a} = |\vec{a}|^2 \ \ \&\ \ \vec{a}.\vec{b} = \vec{b}.\vec{a}$$

$$\vec{a}.\vec{b} = -|\vec{a}|^2 \text{ --------------------(i)}$$

And $|\vec{b}| = \sqrt{2}\ |a|$

Squaring both sides $|\vec{b}|^2 = 2|\vec{a}|^2$ --------------------(ii)

Now we have to prove , resultant of 2 $\vec{a}$ and $\vec{b}$ is perpendicular vector $\vec{b}$. $(2\vec{a}+\vec{b}) \perp \vec{b}$

Then $(2\vec{a}+\vec{b}).\vec{b} = 0$

Now $(2\vec{a}+\vec{b}).\vec{b} = 2\vec{a}.\vec{b}+\vec{b}.\vec{b}$

$$= 2(\vec{a}.\vec{b}) + |\vec{b}|^2$$

$$= 2(-|\vec{a}|^2) + 2|\vec{a}|^2 \qquad \text{From (i) \&(ii)}$$

$$= -2|\vec{a}|^2 + 2|\vec{a}|^2$$

$$(2\vec{a}+\vec{b}).\vec{b} = 0$$

Hence proved that $(2\vec{a}+\vec{b}) \perp \vec{b}$.

Q#18: Prove that $\vec{a} = (\vec{a}.i)i + (\vec{a}.j)j + (\vec{a}.k)k$.

Solution: Let $\vec{a} = a_1 i + a_2 j + a_3 k$ -----------------(i)

Taking dot product $\vec{a}$ with $, j$ *and* k *unit vectors* .

$$\vec{a}.i = (a_1 i + a_2 j + a_3 k).i \quad \Rightarrow \quad \boxed{\vec{a}.i = a_1}$$

$$\vec{a}.j = (a_1 i + a_2 j + a_3 k).j \quad \Rightarrow \quad \boxed{\vec{a}.j = a_2}$$

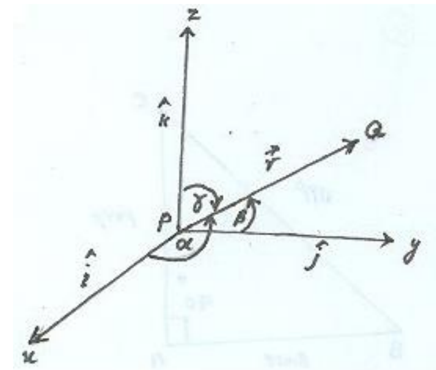
$$\vec{a}.k = (a_1 i + a_2 j + a_3 k).k \quad \Rightarrow \quad \boxed{\vec{a}.k = a_3}$$

Using value of a_1 ,a_2 and a_3 in equation (i)

$$\vec{a} = (\vec{a}.i)i + (\vec{a}.j)j + (\vec{a}.k)k$$

Hence proved.

Q#19: Find the acute angles which the line joining the points $(1,-3,2)$ and $(3,-5,1)$ makes with coordinates axis.



Solution:

Let line joining the points are $P(1,-3,2)$ and $Q(3,-5,1)$.

$\vec{r} = \overrightarrow{PQ} = Q(3,-5,1) - P(1,-3,2)$

$= (3-1)i + (-5+3)j + (1-2)k$

$\vec{r} = 2i - 2j - k$

$|\vec{r}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} \quad \Rightarrow |\vec{r}| = 3$

Let $\vec{r}$ vector makes the acute angles α, β and γ with x,y,z-axis respectivly.

Taking dot product of $\vec{r}$ with i, j and k unit vetors. $\quad \vec{r}.\hat{\imath} = |\vec{r}|\,|\hat{\imath}|\cos\alpha$

$\cos\alpha = \frac{\vec{r}.\hat{\imath}}{|\vec{r}|\,|\hat{\imath}|} = \frac{(2i-2j-k).i}{(3).1} = \frac{2+0+0}{3} = \frac{2}{3} \Rightarrow \alpha = \cos^{-1}\left(\frac{2}{3}\right) \Rightarrow \boxed{\alpha = 48.18^0}$

Similarly

$\cos\beta = \frac{\vec{r}.\hat{\jmath}}{|\vec{r}|\,|\hat{\jmath}|} = \frac{(2i-2j-k).j}{(3).1} = \frac{0-2+0}{3} = \frac{-2}{3} \Rightarrow \beta = \cos^{-1}\left(\frac{2}{3}\right) \Rightarrow \boxed{\beta = 131.81^0}$

$\cos\gamma = \frac{\vec{r}.\hat{k}}{|\vec{r}|\,|\hat{k}|} = \frac{(2i-2j-k).k}{(3).1} = \frac{0+0-1}{3} = \frac{-1}{3} \Rightarrow \gamma = \cos^{-1}\left(\frac{2}{3}\right) \Rightarrow \boxed{\gamma = 109.47^0}$

Q#20: Find the angles which the vector $\vec{a} = 3i - 6j + 2k$ makes with the coordinate axes.

Solution:

Let vector $\vec{a}$ makes makes an angle α, β and γ with x, y and z-axes.

Given vector $\quad \vec{a} = 3i - 6j + 2k$

$|\vec{a}| = \sqrt{(3)^2 + (-6)^2 + (2)^2} = \sqrt{9+36+4} = \sqrt{49}$

$\Rightarrow |\vec{a}| = 7$

Taking dot product of $\vec{a}$ with i, j and k unit vetors.

$\vec{a}.\hat{\imath} = |\vec{a}|\,|\hat{\imath}|\cos\alpha$

$\cos\alpha = \frac{\vec{a}.\hat{\imath}}{|\vec{a}|\,|\hat{\imath}|} = \frac{(3i-6j+2k).i}{(7)(1)} = \frac{3+0+0}{7} = \frac{3}{7} \Rightarrow \alpha = \cos^{-1}\left(\frac{3}{7}\right) \Rightarrow \boxed{\alpha = 64.62^0}$

Similarly

$\cos\beta = \frac{\vec{a}.\hat{\jmath}}{|\vec{a}|\,|\hat{\jmath}|} = \frac{(3i-6j+2k).j}{(7)(1)} = \frac{0-6+0}{7} = \frac{-6}{7} \Rightarrow \beta = \cos^{-1}\left(\frac{-6}{7}\right) \Rightarrow \boxed{\beta = 149^0}$

$\cos\gamma = \frac{\vec{a}.\hat{k}}{|\vec{a}|\,|\hat{k}|} = \frac{(3i-6j+2k).k}{(7)(1)} = \frac{0+0+2}{7} = \frac{2}{7} \Rightarrow \gamma = \cos^{-1}\left(\frac{2}{7}\right) \Rightarrow \boxed{\gamma = 73.39^0}$

Q#21:Prove that $|\vec{r_1}.\vec{r_2}| \le |\vec{r_1}|\,|\vec{r_2}|$ and State the condition for
(i) $\vec{r_1}.\vec{r_2} = |\vec{r_1}|\,|\vec{r_2}|$ (ii) $\vec{r_1}.\vec{r_2} = -\,|\vec{r_1}|\,|\vec{r_2}|$

Solution: By using definition of dot product

$$\vec{r_1}.\vec{r_2} = |\vec{r_1}|\,|\vec{r_2}|\cos\theta$$

If $\cos\theta = 1$ then

$$\vec{r_1}.\vec{r_2} = |\vec{r_1}|\,|\vec{r_2}| \text{ --------------------(i)}$$

If $\cos\theta < 1$ then

$$\vec{r_1}.\vec{r_2} < |\vec{r_1}|\,|\vec{r_2}| \text{ --------------------(ii)}$$

Combining (i) and (ii)

$$\vec{r_1}.\vec{r_2} \le |\vec{r_1}|\,|\vec{r_2}|$$

Taking modulus sign on both sides

$$|\vec{r_1}.\vec{r_2}| \le |\vec{r_1}|\,|\vec{r_2}|$$

Hence proved

(i) $\vec{r_1}.\vec{r_2} = |\vec{r_1}|\,|\vec{r_2}|$

This condition hold, if $\cos\theta = 1$ or $\theta = 0^0$

(ii) $\vec{r_1}.\vec{r_2} = -\,|\vec{r_1}|\,|\vec{r_2}|$

This condition hold, if $\cos\theta = -1$ or $\theta = 180^0$

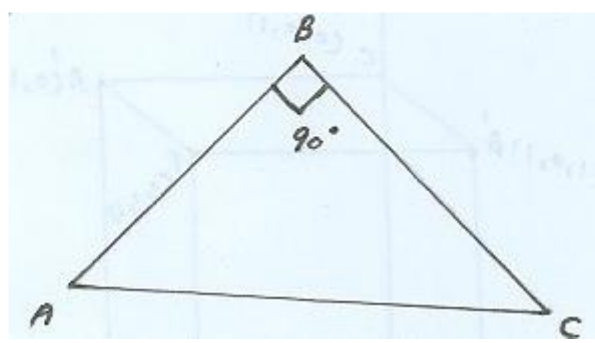
Q#22:Use scalar product to prove that the triangle with vertices A(1,0,1) ,B(1,1,1) and C(1,1,0) is a right isosceles triangle.

Solution: Given vertices of ΔABC are A(1,0,1) ,B(1,1,1) and C(1,1,0)

$\overrightarrow{AB} = p.v\ of\ B - p.v\ of\ A = B(1,1,1) - A(1,0,1)$

$= (1-1)\hat{\imath} + (1-0)\hat{\jmath} + (1-1)\hat{k}$

$= 0\,\hat{\imath} + \hat{\jmath} + 0\,\hat{k} = \hat{\jmath}$ Then $|\overrightarrow{AB}| = 1$ ---------(i)



$\overrightarrow{BC} = p.v\ of\ C - p.v\ of\ B = C(1,1,0) - B(1,1,1)$

$= (1-1)\hat{\imath} + (1-1)\hat{\jmath} + (0-1)\hat{k}$

$= 0\,\hat{\imath} + 0\,\hat{\jmath} - \hat{k} = -\hat{k}$ Then $|\overrightarrow{BC}| = 1$ ---------(ii)

$\overrightarrow{CA} = p.v\ of\ A - p.v\ of\ C = B(1,0,1) - A(1,1,0)$

$= (1-1)\hat{\imath} + (0-1)\hat{\jmath} + (1-0)\hat{k}$

$= 0\,\hat{\imath} - \hat{\jmath} + \hat{k} = -\hat{\jmath} + \hat{k}$, Then $|\overrightarrow{CA}| = \sqrt{2}$ --------(iii)

From (i) ,(ii) & (iii) $|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 = |\overrightarrow{CA}|^2$

Hence proved that the triangle is a right isosceles triangle.

Q#23:The $\vec{a}$ vector of length 5 makes an angle of 30^0 with the z-axis , its vector projection on xy-plane makes an angle 45^0 with x-axis.the vector projection of a 2nd vector $\vec{b}$ on the z-axis has length 4. The vector projection of $\vec{b}$ on xy-plane has length 6 and makes an angle of 120^0 with x-axis.

(a) Write the component of $\vec{a}+\vec{b}$

(b) Determine the angles that the vector $\vec{a}+\vec{b}$ makes with the coordinate axis.

Solution: $\vec{a}$ & $\vec{b}$ be the two vectors.

Given that $|\vec{a}|= 5$ makes angle $\varphi = 30^0$ with z-axis. Then

$$\boldsymbol{a_z} = |\vec{a}| \cos\varphi = 5\cos 30^0 = \frac{5\sqrt{3}}{2} \quad \text{and} \quad \theta = 45^0$$

Projection of $\vec{a}$ on xy-plane $= |\vec{a}| \sin\varphi = 5\cos 30^0 = \frac{5}{2}$

$$\boldsymbol{a_x} = (|\vec{a}|\sin\varphi)\cos\theta = \frac{5}{2}\cos 45^0 = \frac{5}{2\sqrt{2}}$$

$$\boldsymbol{a_y} = (|\vec{a}|\sin\varphi)\sin\theta = \frac{5}{2}\sin 45^0 = \frac{5}{2\sqrt{2}}$$

Projection of $\vec{b}$ $on\ z-axis = |\vec{b}|\cos\varphi = 4$

Projection of $\vec{b}$ $on\ xy-plane = |\vec{a}|\sin\varphi = 6$

$$b_z = |\vec{b}|\cos\varphi = 4$$

$$b_x = (|\vec{b}|\sin\varphi)\cos\theta = 6\cos 120^0 = 6\left(\frac{-1}{2}\right) = -3 \quad \& \quad \theta = 120^0$$

$$b_y = (|\vec{b}|\sin\varphi)\sin\theta = 6\sin 120^0 = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

(i) Let $\vec{R} = \vec{a}+\vec{b}$

Components of $\vec{R}$ are

$$R_x = a_x + b_x = \frac{5}{2\sqrt{2}} - 3 = \frac{5-6\sqrt{2}}{2\sqrt{2}} = -1.23$$

$$R_y = a_y + b_y = \frac{5}{2\sqrt{2}} - 3\sqrt{3} = \frac{5-6\sqrt{6}}{2\sqrt{2}} = -3.43$$

$$R_z = a_z + b_z = \frac{5\sqrt{3}}{2} - 4 = \frac{5\sqrt{3}-8}{2} = 0.33$$

Now $|\vec{R}| = \sqrt{(Rx)^2 + (Ry)^2 + (Rz)^2} = \sqrt{(-1.23)^2 + (-3.43)^2 + (0.33)^2}$

$$|\vec{R}| = \sqrt{1.5129 + 11.7649 + 0.1089} = \sqrt{13.3867}$$

$$|\vec{R}| = 3.66$$

(ii) Let $\vec{R} = \vec{a} + \vec{b}$ *makes* angle α, β *and* γ with coordinate axis.

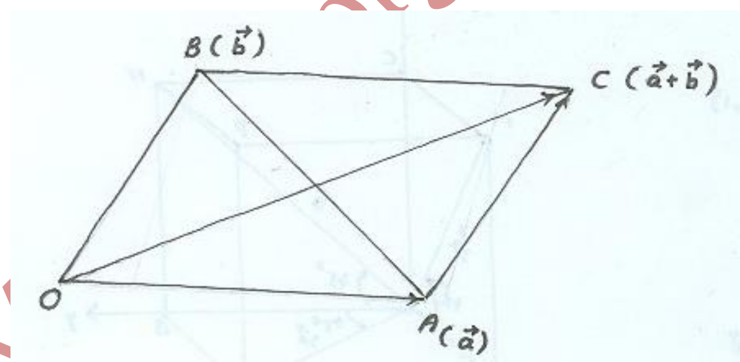
By using direction cosines

$$\cos\alpha = \frac{R_x}{|\vec{R}|} = \frac{-1.23}{3.66} \Rightarrow \alpha = \cos^{-1}\left(\frac{-1.23}{3.66}\right) \Rightarrow \alpha = 90.20^0$$

$$\cos\beta = \frac{R_y}{|\vec{R}|} = \frac{-3.43}{3.66} \Rightarrow \beta = \cos-1\left(\frac{-3.43}{3.66}\right) \Rightarrow \beta = 159.60^0$$

$$\cos\gamma = \frac{R_y}{|\vec{R}|} = \frac{0.33}{3.66} \Rightarrow \gamma = \cos-1\left(\frac{0.33}{3.66}\right) \Rightarrow \gamma = 850^0$$

Q#24: Prove that the sum of the squares of the diagonals of any parallelogram is equal to the sum of squares of it sides.



***Solution*:** Consider a parallelogram as shown in figure

Let O be the origin. hen $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Here $\overrightarrow{AB}$ and $\overrightarrow{OC}$ are the diagonal of parallelogram.

$$\overrightarrow{AB} = \vec{b} - \vec{a}$$

$$\overrightarrow{OC} = \vec{a} + \vec{b}$$

We have to prove

$$|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{OB}|^2 + |\overrightarrow{AC}|^2$$

In this case $|\overrightarrow{OA}| = |BC|$ and $|\overrightarrow{OB}| = |\overrightarrow{AC}|$

$$|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 + |\overrightarrow{OB}|^2$$

$$|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = 2|\overrightarrow{OA}|^2 + 2|\overrightarrow{OB}|^2$$

$$|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = 2\left(|\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2\right) \text{ --------(i)}$$

Now taking L.H.S of (i)

$$|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = |\vec{b} - \vec{a}|^2 + |\vec{a} + \vec{b}|^2 = (\vec{b} - \vec{a}).(\vec{b} - \vec{a}) + (\vec{a} + \vec{b}).(\vec{a} + \vec{b})$$

$$= \vec{b}.\vec{b} - \vec{b}.\vec{a} - \vec{a}.\vec{b} + \vec{a}.\vec{a} + \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}$$

$$= |\vec{b}|^2 - 2\vec{a}.\vec{b} + |\vec{a}|^2 + |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2$$

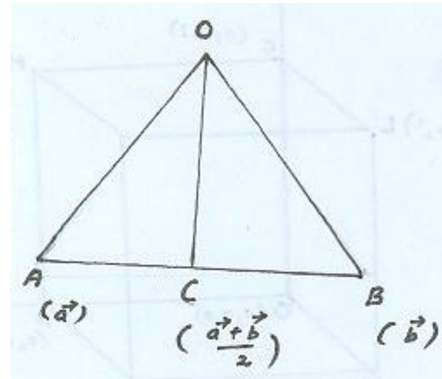
$$= 2|\vec{a}|^2 + 2|\vec{b}|^2$$

$$= 2\left(|\vec{a}|^2 + |\vec{b}|^2\right)$$

$$|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = 2\left(|\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2\right)$$

Hence proved.

Q#25: Show that the median through the vertex of an isosceles triangle is perpendicular to the base.



Solution: Consider an isosceles triangle OACB. Let O be the origin.

$\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$; $\overrightarrow{OC} = \vec{a} + \vec{b}$ and $\overrightarrow{AB} = \vec{b} - \vec{a}$

We have to prove $\overrightarrow{OC} \perp \overrightarrow{AB} \Longrightarrow \overrightarrow{OC}.\overrightarrow{AB} = 0$

Now $\overrightarrow{OC}.\overrightarrow{AB} = (\vec{a} + \vec{b}).(\vec{b} - \vec{a})$

$= \vec{a}.\vec{b} - \vec{a}.\vec{a} + \vec{b}.\vec{b} - \vec{b}.\vec{a}$

$= |\vec{b}|^2 - |\vec{a}|^2$ In isosceles triangle $|\vec{a}| = |\vec{b}|$

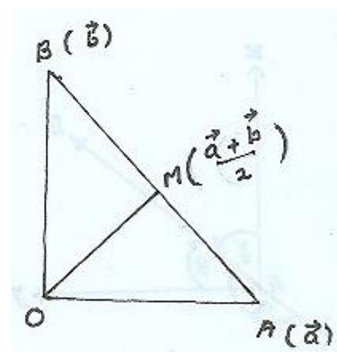
$= |\vec{a}|^2 - |\vec{a}|^2 = 0$

Hence $\overrightarrow{OC}.\overrightarrow{AB} = 0$

Hence proved $\overrightarrow{OC} \perp \overrightarrow{AB}$

Q#26: Prove that in any triangle the median to the hypotenuse is equal to one-half the hypotenuse.

Solution : Let ΔABC and O be the origin. Then $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{AB} = \vec{b} - \vec{a}$



Let M be the midpoint of hypotenuse AB . Then $\overrightarrow{OM} = \frac{\vec{a} + \vec{b}}{2}$

In this case : $\overrightarrow{OA} \perp \overrightarrow{OB} \Longrightarrow \vec{a}.\vec{b} = 0$ --------(i)

We have to prove $|\overrightarrow{OM}| = \frac{1}{2}|\overrightarrow{AB}|$

Now $|\overrightarrow{AB}|^2 = |\vec{b} - \vec{a}|^2 = (\vec{b} - \vec{a}).(\vec{b} - \vec{a}) = \vec{b}.\vec{b} - \vec{b}.\vec{a} - \vec{a}.\vec{b} + \vec{a}.\vec{a}$

$= |\vec{b}|^2 + |\vec{a}|^2 - 2\vec{a}.\vec{b}$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2(0)$ ∴From (i)

$|\overrightarrow{AB}|^2 = |\vec{a}|^2 + |\vec{b}|^2$-----------------------(ii)

Now $|\overrightarrow{OM}|^2 = \left|\frac{\vec{a} + \vec{b}}{2}\right|^2 = \frac{(\vec{a}+\vec{b}).(\vec{a}+\vec{b})}{4} = \frac{\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}}{4} = \frac{|\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}}{4} = \frac{|\vec{a}|^2+|\vec{b}|^2+2(0)}{4}$

$|\overrightarrow{OM}|^2 = \frac{|\vec{a}|^2+|\vec{b}|^2}{4} = \frac{|\overrightarrow{AB}|^2}{4}$ ∴From (ii)

$|\overrightarrow{OM}|^2 = \left(\frac{|\overrightarrow{AB}|}{2}\right)^2$

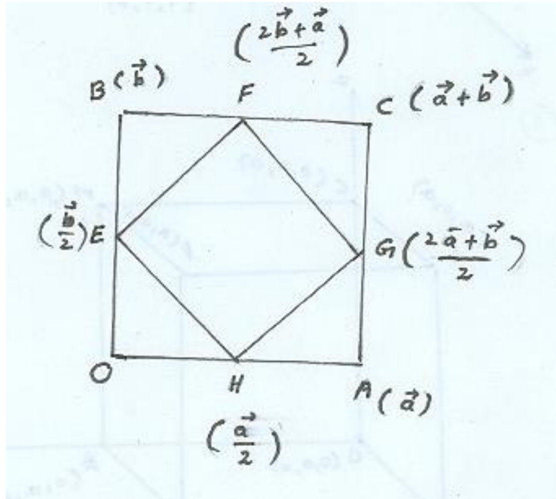
Taking square- root on both sides

$|\overrightarrow{OM}| = \frac{1}{2}|\overrightarrow{AB}|$ Hence proved.

Q#27: Show that the line joining consecutive mid-point of the sides of any square form a square.

Solution :Let OACB be a square whose position vectors are

$\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$; $\overrightarrow{OC} = \vec{a} + \vec{b}$



Let E,F,G and H be the mid points of sides of its square as shown in figure whose position vector are

$\overrightarrow{OE} = \frac{\vec{b}}{2}$; $\overrightarrow{OF} = \frac{2\vec{b}+\vec{a}}{2}$; $\overrightarrow{OG} = \frac{2\vec{a}+\vec{b}}{2}$; $\overrightarrow{OH} = \frac{\vec{a}}{2}$

From figure $\overrightarrow{OA} \perp \overrightarrow{OB} \implies \vec{a}.\vec{b} = 0$ ---------(i)

And $|\vec{a}| = |\vec{b}|$ ---------(ii)

We have to prove $|\overrightarrow{HG}| = |\overrightarrow{GF}| = |\overrightarrow{FE}| = |\overrightarrow{EH}|$ and $\overrightarrow{HG} \perp \overrightarrow{HE}$

$\therefore\ \overrightarrow{HG} = \overrightarrow{OG} - \overrightarrow{OH} = \frac{2\vec{a}+\vec{b}}{2} - \frac{\vec{a}}{2} = \frac{2\vec{a}+\vec{b}-\vec{a}}{2} = \frac{\vec{a}+\vec{b}}{2}$

$|\overrightarrow{HG}|^2 = \left|\frac{\vec{a}+\vec{b}}{2}\right|^2 = \frac{(\vec{a}+\vec{b}).(\vec{a}+\vec{b})}{4} = \frac{\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}}{4} = \frac{|\vec{b}|^2+|\vec{a}|^2+2\vec{a}.\vec{b}}{4} = \frac{|\vec{a}|^2+|\vec{a}|^2+2(0)}{4} = \frac{2|\vec{a}|^2}{4} = \frac{|\vec{a}|^2}{2}$ $\therefore$ From(i)& (ii)

$|\overrightarrow{HG}| = \frac{|\vec{a}|}{\sqrt{2}}$ ---------------(iii)

$\therefore\ \overrightarrow{GF} = \overrightarrow{OF} - \overrightarrow{OG} = \frac{2\vec{b}+\vec{a}}{2} - \frac{2\vec{a}+\vec{b}}{2} = \frac{2\vec{b}+\vec{a}-2\vec{a}-\vec{b}}{2} = \frac{\vec{b}-\vec{a}}{2}$

$|\overrightarrow{GF}|^2 = \left|\frac{\vec{b}-\vec{a}}{2}\right|^2 = \frac{(\vec{b}-\vec{a}).(\vec{b}-\vec{a})}{4} = \frac{\vec{b}.\vec{b}+\vec{b}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{a}}{4} = \frac{|\vec{b}|^2+|\vec{a}|^2-2\vec{a}.\vec{b}}{4} = \frac{|\vec{a}|^2+|\vec{a}|^2-2(0)}{4} = \frac{2|\vec{a}|^2}{4} = \frac{|\vec{a}|^2}{2}$ $\therefore$ From(i)& (ii)

$|\overrightarrow{GF}| = \frac{|\vec{a}|}{\sqrt{2}}$ ---------------(iv)

$\therefore\ \overrightarrow{FE} = \overrightarrow{OE} - \overrightarrow{OF} = \frac{\vec{b}}{2} - \frac{2\vec{b}+\vec{a}}{2} = \frac{\vec{b}-2\vec{b}-\vec{a}}{2} = \frac{-\vec{b}-\vec{a}}{2} = \frac{-(\vec{a}+\vec{b})}{2}$

$|\overrightarrow{FE}|^2 = \left|\frac{-(\vec{a}+\vec{b})}{2}\right|^2 = \frac{(\vec{a}+\vec{b}).(\vec{a}+\vec{b})}{4} = \frac{\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}}{4} = \frac{|\vec{b}|^2+|\vec{a}|^2+2\vec{a}.\vec{b}}{4} = \frac{|\vec{a}|^2+|\vec{a}|^2+2(0)}{4} = \frac{2|\vec{a}|^2}{4} = \frac{|\vec{a}|^2}{2}$ $\therefore$ From(i)& (ii)

$|\overrightarrow{FE}| = \frac{|\vec{a}|}{\sqrt{2}}$ ---------------(v)

$\therefore\ \overrightarrow{EH} = \overrightarrow{OH} - \overrightarrow{OE} = \frac{\vec{a}}{2} - \frac{\vec{b}}{2} = \frac{\vec{a}-\vec{b}}{2}$

$|\overrightarrow{EH}|^2 = \left|\frac{\vec{a}-\vec{b}}{2}\right|^2 = \frac{(\vec{a}-\vec{b}).(\vec{a}-\vec{b})}{4} = \frac{\vec{a}.\vec{a}-\vec{a}.\vec{b}-\vec{b}.\vec{a}+\vec{b}.\vec{b}}{4} = \frac{|\vec{a}|^2+|\vec{b}|^2-2\vec{a}.\vec{b}}{4} = \frac{|\vec{a}|^2+|\vec{a}|^2+2(0)}{4} = \frac{2|\vec{a}|^2}{4} = \frac{|\vec{a}|^2}{2}$ $\therefore$ From(i)& (ii)

$|\overrightarrow{EH}| = \frac{|\vec{a}|}{\sqrt{2}}$ ---------------(vi)

Now $\overrightarrow{HG}\,.\,\overrightarrow{EH} = \left(\frac{\vec{a}+\vec{b}}{2}\right).\left(\frac{\vec{a}-\vec{b}}{2}\right) = \frac{|\vec{a}|^2-|\vec{b}|^2}{4} = \frac{|\vec{a}|^2-|\vec{a}|^2}{4} = \frac{0}{4}$ $\therefore$ From (ii)

$\overrightarrow{HG}\,.\,\overrightarrow{EH} = 0$

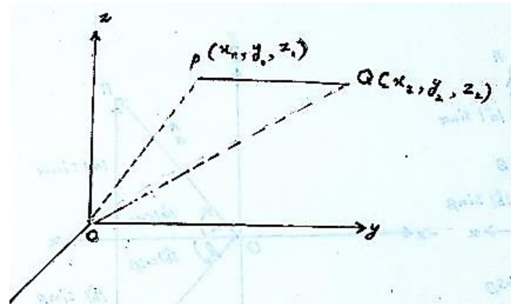
This shows that $\overrightarrow{HG} \perp \overrightarrow{EH}$.

Hence proved.

Q#28: Derive a formula for distance between two points in space.

Solution:

Let P(x_1 ,y_1, z_1) and Q(x_2 ,y_2, z_2) be the two points in the space and O be the origin. Let Position vectors.



$\overrightarrow{OP} = x_1\,\hat{\imath} + y_1\,\hat{\jmath} + z_1\,\hat{k}$ and $\overrightarrow{OQ} = x_2\,\hat{\imath} + y_2\,\hat{\jmath} + z_2\,\hat{k}$ then

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

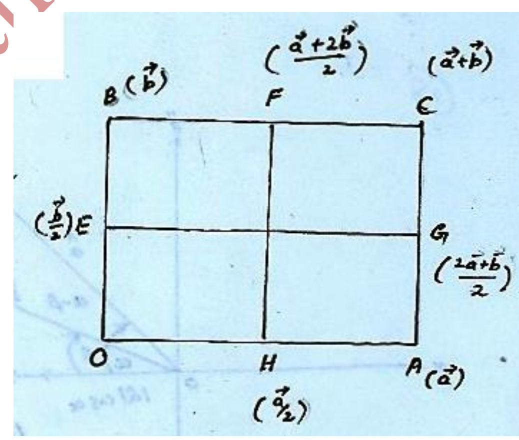
$$= (x_2 - x_1)\,\hat{\imath} + (y_2 - y_1)\,\hat{\jmath} + (z_2 - z_1)\,\hat{k}$$

Now

Distance from P to Q = $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Q#29: (i) Show that the sum of the squares of the diagonals of any quadrilateral is equal two twice the sum of the squares of the line segments joining the mid points of the opposite sides.

Solution: Let OACB be a quadrilateral whose position vectors are $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$



$\overrightarrow{OC}$ and $\overrightarrow{AB}$ be diagonals of quadrilateral. As shown in the figure.

$$\overrightarrow{OC} = \vec{a} + \vec{b} \quad \text{and } \overrightarrow{AB} = \vec{b} - \vec{a}$$

Let E,F,G and H be the mid points of sides of its quadrilateral as shown in figure whose position vector are

$$\overrightarrow{OE} = \frac{\vec{b}}{2};\ \overrightarrow{OF} = \frac{2\vec{b}+\vec{a}}{2};\ \overrightarrow{OG} = \frac{2\vec{a}+\vec{b}}{2};\ \overrightarrow{OH} = \frac{\vec{a}}{2}$$

We have to prove $|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = 2\left(|\overrightarrow{GE}|^2 + |\overrightarrow{FH}|^2\right)$

$\therefore \overrightarrow{FH} = \overrightarrow{OH} - \overrightarrow{OF} = \frac{\vec{a}}{2} - \frac{2\vec{b}+\vec{a}}{2} = \frac{\vec{a}-2\vec{b}-\vec{a}}{2} = -\frac{2\vec{b}}{2} = -\vec{b} \implies |\overrightarrow{FH}|^2 = |\vec{b}|^2$ -----(i)

$\therefore \overrightarrow{GE} = \overrightarrow{OE} - \overrightarrow{OG} = \frac{\vec{b}}{2} - \frac{2\vec{a}+\vec{b}}{2} = \frac{\vec{b}-2\vec{a}-\vec{b}}{2} = -\frac{2\vec{a}}{2} = -\vec{a} \implies |\overrightarrow{GE}|^2 = |\vec{a}|^2$ ----(ii)

Now $|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = |\vec{b} - \vec{a}|^2 + |\vec{a} + \vec{b}|^2 = (\vec{b} - \vec{a}).(\vec{b} - \vec{a}) + (\vec{a} + \vec{b}).(\vec{a} + \vec{b})$

$$= \vec{b}.\vec{b} - \vec{b}.\vec{a} - \vec{a}.\vec{b} + \vec{a}.\vec{a} + \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}$$

$$= |\vec{b}|^2 - 2\vec{a}.\vec{b} + |\vec{a}|^2 + |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2$$

$$= 2|\vec{a}|^2 + 2|\vec{b}|^2 = 2\left(|\vec{a}|^2 + |\vec{b}|^2\right)$$

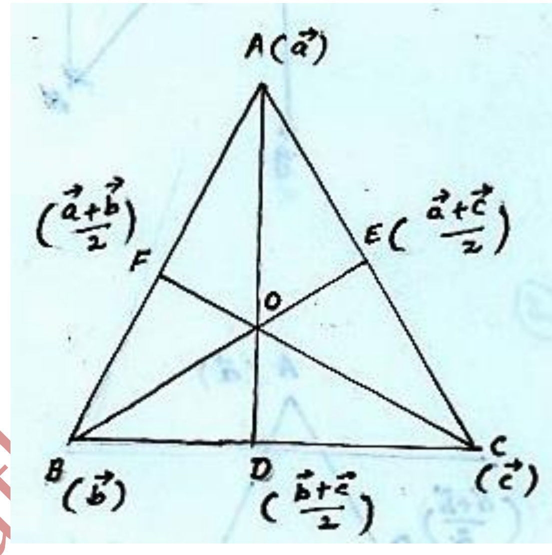
$|\overrightarrow{AB}|^2 + |\overrightarrow{OC}|^2 = 2\left(|\overrightarrow{GE}|^2 + |\overrightarrow{FH}|^2\right)$ Hence proved.

Q#29: (ii)Prove that the altitudes of a triangle are concurrent .

Solution: Let $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ be the position vectors of ΔABC .

Let O be concurrent point. $\overrightarrow{AD}$, $\overrightarrow{BE}$ and $\overrightarrow{CF}$ be the altitude of triangle .

From figure $\overrightarrow{OA} \parallel \overrightarrow{AD}$ then $\overrightarrow{AD} = \lambda\, \overrightarrow{OA} = \lambda\vec{a}$



$\overrightarrow{AD} \perp \overrightarrow{BC}$ then

$\overrightarrow{AD} . \overrightarrow{BC} = 0$

$\lambda\vec{a} . (\vec{c} - \vec{b}) = 0$

$\vec{a} . (\vec{c} - \vec{b}) = 0$

$\vec{a} . \vec{c} - \vec{a}.\vec{b} = 0$

$\vec{a} . \vec{b} = \vec{a}.\vec{c}$ ----------(i)

Again From figure $\overrightarrow{BE} \parallel \overrightarrow{OB}$ then $\overrightarrow{BE} = \lambda\, \overrightarrow{OB} = \lambda\vec{b}$

$\overrightarrow{BE} \perp \overrightarrow{CA}$ then

$\overrightarrow{BE} . \overrightarrow{CA} = 0$

$\lambda\vec{b} . (\vec{a} - \vec{c}) = 0$

$\vec{b} . (\vec{a} - \vec{c}) = 0$

$\vec{b} . \vec{a} - \vec{b}.\vec{c} = 0$

$\vec{a} . \vec{b} = \vec{b}.\vec{c}$ ----------(ii)

Comparing (i) and (ii)

$\vec{a} . \vec{c} = \vec{b}.\vec{c}$

$\vec{a} . \vec{c} - \vec{b}.\vec{c} = 0$

$(\vec{a} - \vec{b}).\vec{c} = 0$

$\lambda\vec{c} . (\vec{a} - \vec{b}) = 0$

$\overrightarrow{CF} . \overrightarrow{AB} = 0$

This shows that $\overrightarrow{CF} \perp \overrightarrow{AB}$

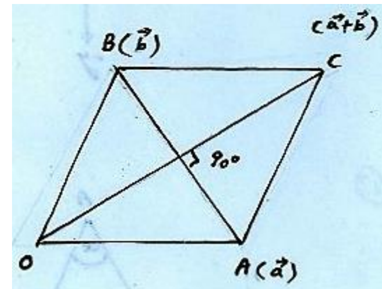
here $\overrightarrow{CF} = \lambda\vec{c} = \lambda\overrightarrow{OC}$ then $\overrightarrow{CF} \parallel \overrightarrow{OC}$

Hence proved.

Q#29: (iii) Example#03: Prove that the diagonal of a rhombus intersect each other at right angle.

Solution: Consider a rhombus OACB. Suppose $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$

Since sides of rhombus are equal, therefore $|\vec{a}| = |\vec{b}|$ ------(i)



Let $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \vec{b}$ are the diagonal of a rhombus.

We have to prove. $\overrightarrow{OC} \perp \overrightarrow{AB}$ for this $\overrightarrow{OC} . \overrightarrow{AB} = 0$

Now $\overrightarrow{OC} . \overrightarrow{AB} = (\vec{a} + \vec{b}).(\vec{b} - \vec{a})$

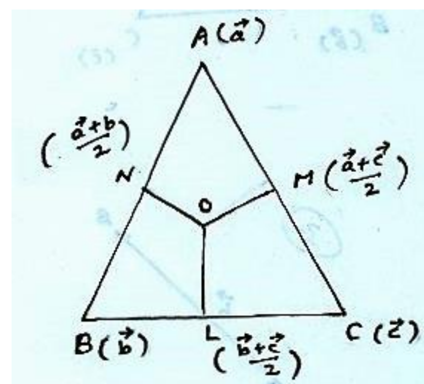
$= (\vec{b} + \vec{a}).(\vec{b} - \vec{a}) = \vec{b}.\vec{b} - \vec{b}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{a} = |\vec{b}|^2 - |\vec{a}|^2 = |\vec{b}|^2 - |\vec{b}|^2$ ∴From (i)

$\overrightarrow{OC} . \overrightarrow{AB} = 0$ hence proved $\overrightarrow{OC} \perp \overrightarrow{AB}$.

Q#29: (iv) Example#02:Prove that the right bisectors of the sides of a triangle are concurrent.

Solution: Consider a $\triangle ABC$ and O be the origin. L, M and N be the mid points of sides of triangle ABC after drawing the perpendicular bisectors of each side. If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$

Let $\overrightarrow{OM} \perp \overrightarrow{AC}$ and $\overrightarrow{ON} \perp \overrightarrow{AB}$ then we have to prove that $\overrightarrow{OL} \perp \overrightarrow{BC}$



$\overrightarrow{OM} = \frac{\vec{a}+\vec{c}}{2}$, $\overrightarrow{ON} = \frac{\vec{a}+\vec{b}}{2}$ and $\overrightarrow{OL} = \frac{\vec{b}+\vec{c}}{2}$

$\overrightarrow{AB} = \vec{b} - \vec{a}$; $\overrightarrow{BC} = \vec{c} - \vec{b}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$

Now $\overrightarrow{OM} \perp \overrightarrow{AC}$

Then $\overrightarrow{OM} . \overrightarrow{AC} = 0 \Rightarrow \left(\frac{\vec{a} + \vec{c}}{2}\right).(\vec{c} - \vec{a}) = 0 \Rightarrow (\vec{c} + \vec{a}).(\vec{c} - \vec{a}) = 0$

$c^2 - a^2 = 0$ ----------------(i)

Now $\overrightarrow{ON} \perp \overrightarrow{AB}$

Then $\overrightarrow{ON} . \overrightarrow{AB} = 0 \Rightarrow \left(\frac{\vec{a} + \vec{b}}{2}\right).(\vec{b} - \vec{a}) = 0 \Rightarrow (\vec{b} + \vec{a}).(\vec{b} - \vec{a}) = 0$

$b^2 - a^2 = 0$ ----------------(ii)

Subtracting (i) & (ii)

$c^2 - b^2 = 0$

$(\vec{c} + \vec{b}).(\vec{c} - \vec{b}) = 0$

$\left(\frac{\vec{b} + \vec{c}}{2}\right).(c - \vec{b}) = 0$

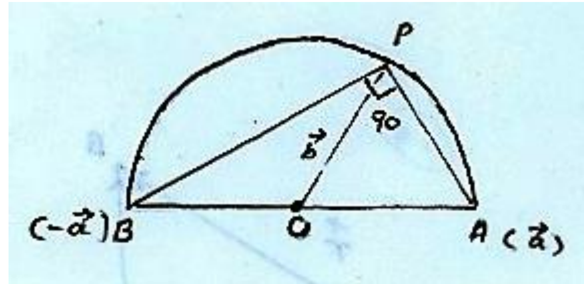
$\overrightarrow{OL} . \overrightarrow{BC} = 0$

This shows that $\overrightarrow{OL} \perp \overrightarrow{BC}$

Hence proved that the right bisectors of the sides of a triangle are concurrent.

Q#29:(v) Example#06:Prove that an angle inscribed in a semi-circle is a right angle.

***Solution*:** Consider a semi-circle as shown in the figure.



Suppose $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = -\vec{a}$ and $\overrightarrow{OP} = \vec{b}$

Since $|\vec{a}| = |\vec{b}|$ = radius of a circle ------(i)

Let $\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = \vec{a} - \vec{b}$

$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = \vec{b} - (-\vec{a}) = \vec{b} + \vec{a}$

Be the diagonal of a rhombus.

We have to prove. $\overrightarrow{BP} \perp \overrightarrow{PA}$ for this $\overrightarrow{BP} . \overrightarrow{PA} = 0$

Now $\overrightarrow{BP} . \overrightarrow{PA} = (\vec{b} + \vec{a}).(\vec{b} - \vec{a}) = \vec{b}.\vec{b} - \vec{b}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{a} = |\vec{b}|^2 - |\vec{a}|^2$

$= |\vec{b}|^2 - |\vec{b}|^2$ from (i)

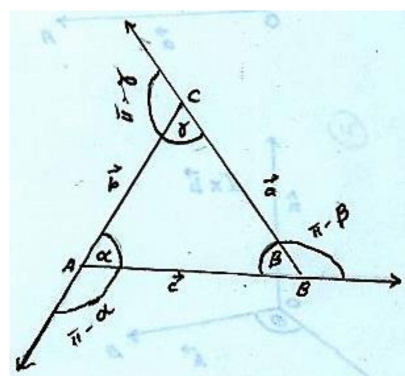
$\overrightarrow{BP} . \overrightarrow{PA} = 0$ Hence $\overrightarrow{BP} \perp \overrightarrow{PA}$

Hence proved that an angle inscribed in a semi-circle is a right angle.

Q#30: Prove that by using vectors

(i) $\boldsymbol{a = b\cos\gamma + c\cos\beta}$

***Solution*:** Let $\triangle ABC$ and $\vec{a}, \vec{b}$ *and* $\vec{c}$ be the three vectors along sides of triangle AB,BC and CA respectively , taken one way round.



Then $\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{a} = -\vec{b} - \vec{c}$

$\vec{a} = -(\vec{b} + \vec{c})$

Taking dot product with $\vec{a}$ vector

$\vec{a}.\vec{a} = -(\vec{b} + \vec{c}).\vec{a}$

$|\vec{a}|^2 = -\vec{b}.\vec{a} - \vec{c}.\vec{a}$

$|\vec{a}|^2 = -|\vec{b}||\vec{a}|\cos(\pi - \gamma) - |\vec{c}||\vec{a}|\cos(\pi - \beta)$

Dividing both sides by $|\vec{a}|$

$|\vec{a}| = |\vec{b}|\cos\gamma + |\vec{c}|\cos\beta$

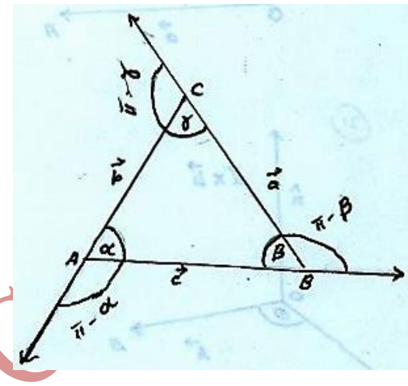
$(ii) b = c\ \cos\alpha + a\cos\gamma$

Solution: Let $\triangle ABC$ and $\vec{a}, \vec{b}$ *and* $\vec{c}$ be the three vectors along sides of triangle AB,BC and CA respectively , taken one way round.

Then $\vec{a} + \vec{b} + \vec{c} = 0$

$$\vec{b} = -\vec{c} - \vec{a}$$

$$\vec{b} = -(\vec{c} + \vec{a})$$



Taking dot product with $\vec{b}$ vector

$$\vec{b}\,.\,\vec{b} = -(\vec{c} + \vec{a})\,.\,\vec{b}$$

$$|\vec{b}|^2 = -\vec{c}.\vec{b} - \vec{a}\,.\,\vec{b}$$

$$|\vec{b}|^2 = -|\vec{c}||\vec{b}|\cos(\pi - \alpha) - |\vec{a}||\vec{b}|\cos(\pi - \gamma)$$

Dividing both sides by $|\vec{b}|$

$$|\vec{b}| = |\vec{c}|\cos\alpha + |\vec{a}|\cos\gamma$$

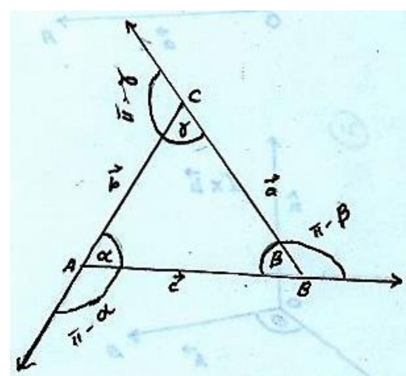
$(iii) c = a\ \cos\beta + b\cos\alpha$

Solution: Let $\triangle ABC$ and $\vec{a}, \vec{b}$ *and* $\vec{c}$ be the three vectors along sides of triangle AB,BC and CA respectively , taken one way round.

Then $\vec{a} + \vec{b} + \vec{c} = 0$

$$\vec{c} = -\vec{a} - \vec{b}$$

$$\vec{c} = -(\vec{a} + \vec{b})$$



Taking dot product with $\vec{c}$ vector

$$\vec{c}\,.\,\vec{c} = -(\vec{a} + \vec{b})\,.\,\vec{c}$$

$$|\vec{c}|^2 = -\vec{a}.\vec{c} - \vec{b}\,.\,\vec{c}$$

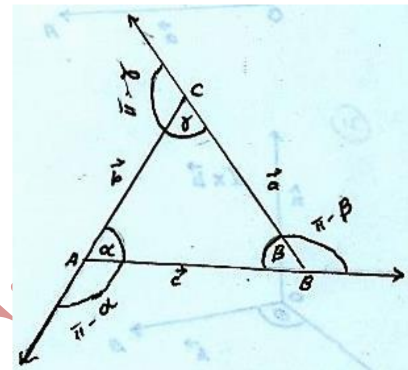
$$|\vec{c}|^2 = -|\vec{a}||\vec{c}|\cos(\pi - \beta) - |\vec{b}||\vec{c}|\cos(\pi - \alpha)$$

Dividing both sides by $|\vec{a}|$

$$|\vec{c}| = |\vec{a}|\cos\beta + |\vec{b}|\cos\alpha$$

$(iv)\, a^2 = b^2 + c^2 - 2bc\cos\alpha$

Solution: Let $\triangle ABC$ and $\vec{a}, \vec{b}$ and $\vec{c}$ be the three vectors along sides of triangle AB,BC and CA respectively , taken one way round.



Then $\vec{a} + \vec{b} + \vec{c} = 0$

$$\vec{a} = -\vec{b} - \vec{c}$$

$$\vec{a} = -(\vec{b} + \vec{c})$$

Taking dot product with $\vec{c}$ vector

$$\vec{a}\,.\,\vec{a} = [-(\vec{b} + \vec{c})]\,.[-(\vec{b} + \vec{c})] = (\vec{b} + \vec{c})\,.\,(\vec{b} + \vec{c})$$

$$|\vec{a}|^2 = \vec{b}.\vec{b} + \vec{b}.\vec{c} + \vec{c}\,.\,\vec{b} + \vec{c}\,.\,\vec{c}$$

$$|\vec{a}|^2 = \left|\vec{b}\right|^2 + |\vec{c}|^2 + 2\,\vec{b}.\vec{c}$$

$$|\vec{a}|^2 = \left|\vec{b}\right|^2 + |\vec{c}|^2 + 2\left|\vec{b}\right||\vec{c}|\cos(\pi - \alpha)$$

$$|\vec{a}|^2 = \left|\vec{b}\right|^2 + |\vec{c}|^2 - 2\left|\vec{b}\right||\vec{c}|\cos\,\alpha$$

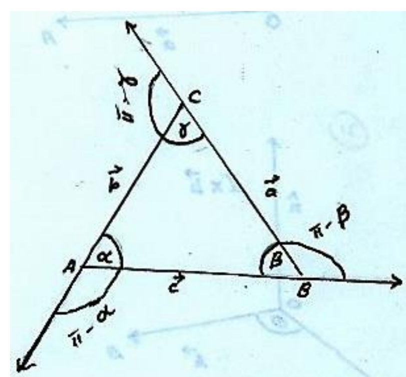
(v) $b^2 = a^2 + c^2 - 2ac\cos\beta$

Solution**:** Let $\triangle ABC$ and $\vec{a}, \vec{b}$ and $\vec{c}$ be the three vectors along sides of triangle AB,BC and CA respectively , taken one way round.

Then $\vec{a} + \vec{b} + \vec{c} = 0$

$$\vec{b} = -\vec{a} - \vec{c}$$

$$\vec{b} = -(\vec{a} + \vec{c})$$



Taking dot product with $\vec{c}$ vector

$$\vec{b}\,.\,\vec{b} = [-(\vec{a} + \vec{c})]\,.[-(\vec{a} + \vec{c}) = (\vec{a} + \vec{c})\,.\,(\vec{a} + \vec{c})$$

$$\left|\vec{b}\right|^2 = \vec{a}.\vec{a} + \vec{a}.\vec{c} + \vec{c}\,.\,\vec{a} + \vec{c}\,.\,\vec{c}$$

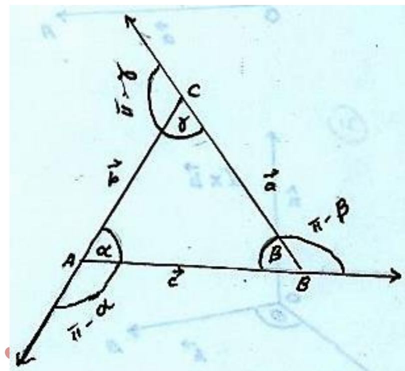
$$\left|\vec{b}\right|^2 = |\vec{a}|^2 + |\vec{c}|^2 + 2\,\vec{a}.\vec{c}$$

$$\left|\vec{b}\right|^2 = |\vec{a}|^2 + |\vec{c}|^2 + 2|\vec{a}||\vec{c}|\cos(\pi - \beta\,)$$

$$\left|\vec{b}\right|^2 = |\vec{a}|^2 + |\vec{c}|^2 - 2\left|\vec{b}\right||\vec{c}|\cos\,\beta$$

$(vi)\, c^2 = a^2 + b^2 - 2ab\cos\gamma$

***Solution*:** Let $\triangle ABC$ and $\vec{a}, \vec{b}$ *and* $\vec{c}$ be the three vectors along sides of triangle AB, BC and CA respectively , taken one way round.



Then $\vec{a} + \vec{b} + \vec{c} = 0$

$$\vec{c} = -\vec{a} - \vec{b}$$

$$\vec{c} = -(\vec{a} + \vec{b})$$

Taking dot product with $\vec{c}$ vector

$$\vec{c}\,.\vec{c} = [-(\vec{a} + \vec{b})]\,.[-(\vec{a} + \vec{b})] = (\vec{a} + \vec{b})\,.\,(\vec{a} + \vec{b})$$

$$|\vec{c}|^2 = \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}\,.\vec{a} + \vec{b}\,.\vec{b}$$

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\ \vec{a}.\vec{b}$$

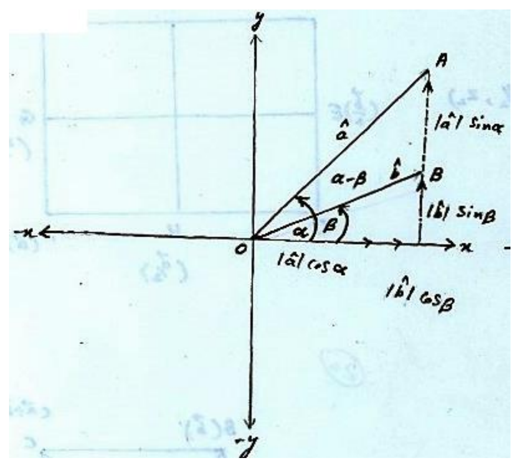
$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos(\pi - \gamma)$$

$$|\vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\gamma$$

(vii) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\gamma$

Solution: Let $\hat{a}$ = OA and $\hat{b} = OB$ be the two unit vectors makes angles α and β makes with x-axis.

From figure:



$$\hat{a} = \widehat{OA} = |\hat{a}|\cos\alpha\ \ \hat{\imath} + |\hat{a}|\sin\alpha\ \ \hat{\jmath}$$

$$= \cos\alpha\ \ \hat{\imath} + \sin\alpha\ \ \hat{\jmath}$$

$$\hat{b} = \widehat{OB} = |\hat{b}|\cos\beta\ \ \hat{\imath} + |\hat{b}|\sin\beta\ \ \hat{\jmath}$$

$$= \cos\beta\ \ \hat{\imath} + \sin\beta\ \ \hat{\jmath}$$

Taking dot product of $\hat{a}$ with $\hat{b}$ unit vectors.

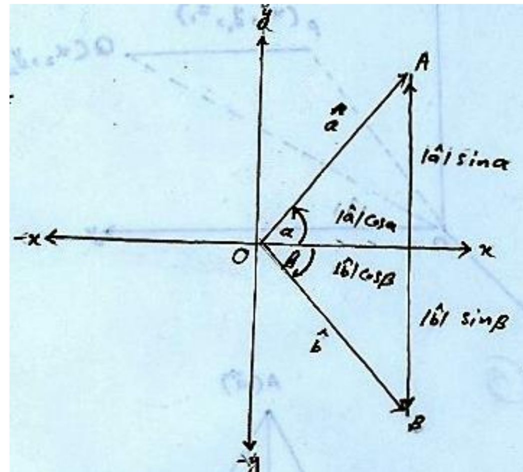
$$\hat{a}\,.\,\hat{b} = (\cos\alpha\ \ \hat{\imath} + \sin\alpha\ \ \hat{\jmath}\ )\,.\,(\cos\beta\ \ \hat{\imath} + \sin\beta\ \ \hat{\jmath}\ )$$

$$|\hat{a}||\hat{b}|\ \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\gamma \qquad \therefore\ |\hat{a}| = |\hat{b}| = 1$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\gamma$$

(viii) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\gamma$

Solution: Let $\hat{a}$ = OA and $\hat{b} = OB$ be the two unit vector makes angles α and β makes with x-axis.



From figure:

$\hat{a}$ = OA= $|\hat{a}|\cos\alpha\ \hat{\imath} + |\hat{a}|\sin\alpha\ \hat{\jmath}$

$= \cos\alpha\ \hat{\imath} + \sin\alpha\ \hat{\jmath}$

$\hat{b}$ = OB= $|\hat{b}|\cos\beta\ \hat{\imath} - |\hat{b}|\sin\beta\ \hat{\jmath}$

$= \cos\beta\ \hat{\imath} - \sin\beta\ \hat{\jmath}$

Taking dot product of $\hat{a}$ with $\hat{b}$ unit vectors.

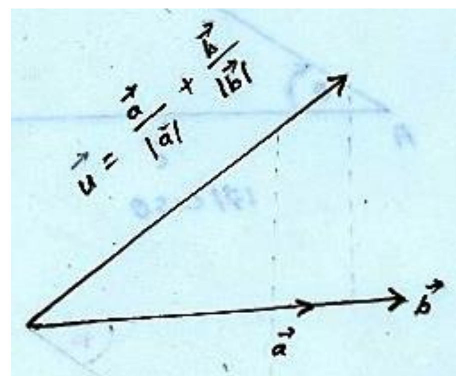
$\hat{a}\,.\,\hat{b} = (\cos\alpha\ \hat{\imath} + \sin\alpha\ \hat{\jmath}\ ).(\cos\beta\ \hat{\imath} - \sin\beta\ \hat{\jmath}\ )$

$|\hat{a}||\hat{b}|\ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\gamma$ $\qquad \therefore\ |\hat{a}| = |\hat{b}| = 1$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\gamma$

Hence proved.

Q#31:Proved that $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{a}|}$ is equally inclined with $\vec{a}$ *and* $\vec{b}$.



Solution: Let $\vec{u} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$

And α be the angle between $\vec{a}$ *and* $\vec{b}$.

1st. $\vec{u}$ is inclined at $\vec{a}$ vector.

Projection $\vec{u}$ along $\vec{a} = \vec{u}\,.\,\hat{a} = \left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right).\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}}{|\vec{a}|}.\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}.\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}.\vec{a}}{|\vec{a}|^2} + \frac{\vec{b}.\vec{a}}{|\vec{b}||\vec{a}|}$

$= \frac{|\vec{a}|^2}{|\vec{a}|^2} + \frac{|\vec{b}||\vec{a}|\cos\alpha}{|\vec{b}||\vec{a}|}$

$= 1 + \cos\alpha$ -------------(i)

2nd. $\vec{u}$ is inclined at $\vec{b}$ vector.

Projection $\vec{u}$ along $\vec{b} = \vec{u}\,.\,\hat{b} = \left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right).\frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a}}{|\vec{a}|}.\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{b}}{|\vec{b}|}.\frac{\vec{b}}{|\vec{b}|} = \frac{\vec{b}.\vec{b}}{|\vec{b}|^2} + \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

$= \frac{|\vec{b}|^2}{|\vec{b}|^2} + \frac{|\vec{a}||\vec{b}|\cos\alpha}{|\vec{b}||\vec{a}|}$

$= 1 + \cos\alpha$ -------------(ii)

From (i) &(ii) hence proved that $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{a}|}$ is equally inclined with $\vec{a}$ *and* $\vec{b}$.

Q# 32: The resultant of two vectors $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{a}$. Show that the resultant of $25\,\vec{a}$ and $\vec{b}$ is perpendicular vector $\vec{b}$ if $|\vec{b}|=5|\vec{a}|$.

Solution: Given Resultant of $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{a}$. $(\vec{a}+\vec{b}) \perp \vec{a}$

Then $(\vec{a}+\vec{b})\,.\,\vec{a} = 0$

$$\vec{a}.\vec{a} + \vec{b}\,.\vec{a} = 0$$

$$|\vec{a}|^2 + \vec{a}.\vec{b} = 0 \qquad \therefore \vec{a}.\vec{a} = |\vec{a}|^2 \;\;\&\;\; \vec{a}.\vec{b} = \vec{b}.\vec{a}$$

$$\vec{a}.\vec{b} = -|\vec{a}|^2 \text{ ---------------------(i)}$$

And $|\vec{b}|=5|\vec{a}|$ or $|\vec{b}|^2 = 25|\vec{a}|^2$ --------------------(ii)

Now we have to prove $25\,\vec{a}+\vec{b}$ is perpendicular vector $\vec{b}$. $(25\,\vec{a}+\vec{b}) \perp \vec{b}$

Then $(25\,\vec{a}+\vec{b})\,.\,\vec{b} = 0$

Taking L.H.S $(25\,\vec{a}+\vec{b})\,.\,\vec{b} = 25\,\vec{a}\,.\,\vec{b} + \vec{b}\,.\,\vec{b} = 25(\vec{a}\,.\,\vec{b}) + |\vec{b}|^2$

$$= 25\,(-|\vec{a}|^2) + 25|\vec{a}|^2 \qquad \therefore From\ (i)\ \&(ii)$$

$$= -25|\vec{a}|^2 + 25|\vec{a}|^2$$

$$(2\,\vec{a}+\vec{b})\,.\,\vec{b} = 0$$

Hence proved $(25\,\vec{a}+\vec{b}) \perp \vec{b}$

Q#33: Find a unit vector parallel to the xy-plane and perpendicular to a vector $4\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}$.

Solution: Let $\hat{u}$ be a required parallel to the xy-plane.

$$\hat{u} = x\hat{\imath} + y\,\hat{\jmath} \text{ ------------------(i)}$$

$$|\hat{u}| = \sqrt{x^2+y^2} \text{ or } |\hat{u}|^2 = x^2 + y^2$$

$$x^2 + y^2 = 1 \text{ ---------------(ii)}$$

Let $\vec{v} = 4\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}$

According to given condition. $\hat{u} \perp \vec{v}$ $\quad \hat{u}\,.\vec{v} = 0$

$$(x\hat{\imath} + y\,\hat{\jmath})\,.\,(4\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}) = 0$$

$$4\,x - 3y = 0$$

$$4x = 3y$$

$$x = \frac{3}{4}\,y \text{ --------------(iii)}$$

Using equation (iii) in (ii)

$$\left(\frac{3}{4}\,y\right)^2 + y^2 = 1$$

$$\frac{9}{16}y^2 + y^2 = 1$$

Multiplying by 16 $\quad 9y^2 + 16y^2 = 16$

$$25y^2 = 16$$

$$y^2 = \frac{16}{25}$$

Taking square-root on both sides $\quad y = \pm\frac{4}{5}$

Using value of y in equation (iii)

$$x = \frac{3}{4}\left(\pm\frac{4}{5}\right)$$

$$x = \pm\frac{3}{5}$$

Using value of x & y in (i)

$$\hat{u} = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath} \quad \text{or} \quad \hat{u} = -\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$$

Q#34: Example #09: (i) Find a work done by the force $\vec{F} = 4i - 3j + 2k$ on moving particle from $(3,2,-1)$ to $B(2,-1,4)$.

Solution : Given $\vec{F} = 4i - 3j + 2k$ and displacement $\vec{r}$ from A(3,2,-1) to B(2,-1 ,4) is

$$\vec{r} = \overrightarrow{AB} = P.v's\ of\ B - P.v's\ of\ A = B(2,-1,4) - A(3,2,-1)$$

$$= (2-3)i + (-1-2)j + (4+1)k$$

$$\vec{r} = -i - 3j + 5k$$

We know that

$$W = \vec{F}.\vec{r} = (4i - 3j + 2k).(-i - 3j + 5k)$$

$$= (4)(-1) + (-3)(-3) + (2)(5)$$

$$= -4 + 9 + 1$$

W= 15 joule

Q#35:(ii) A particle is displaced from point A (2, -3 , 1) to B(4, 2 ,1) under the action of constant forces $\overrightarrow{F_1} = 12\hat{\imath} - 5\hat{\jmath} + 6\hat{k}$; $\overrightarrow{F_2} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ and $\overrightarrow{F_3} = 2\hat{\imath} + 8\hat{\jmath} + \hat{k}$. Find the work done by the forces on the particle.

Solution : Given $\overrightarrow{F_1} = 12\hat{\imath} - 5\hat{\jmath} + 6\hat{k}$; $\overrightarrow{F_2} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ and $\overrightarrow{F_3} = 2\hat{\imath} + 8\hat{\jmath} + \hat{k}$

Let F be the resultant of these forces then

$$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$$

$$= 12\hat{\imath} - 5\hat{\jmath} + 6\hat{k} + \hat{\imath} + 2\hat{\jmath} - 2\hat{k} + 2\hat{\imath} + 8\hat{\jmath} + \hat{k}$$

$$\vec{F} = 15\hat{\imath} + 5\hat{\jmath} + 5\hat{k}$$

And displacement $\vec{r}$ from A(2,-3 ,1) to B(4,2 ,1) is

$\vec{r} = \overrightarrow{AB}$ = P.v's of B – P.v's of A = B(4,2 ,1) - A(2 ,-3,1)

$= (4-2)\,\hat{\imath} + (2+3)\,\hat{\jmath} + (1-1)\,\hat{k}$

$\vec{r} = 2\,\hat{\imath} + 5\,\hat{\jmath} + 0\,\hat{k}$

We know that

$W = \vec{F}.\vec{r} = (15\,\hat{\imath} + 5\,\hat{\jmath} + 5\,\hat{k}).(2\hat{\imath} + 5\,\hat{\jmath} + 0\,\hat{k})$

$= (15)(2) + (5)(5) + (5)(0)$

$= 30+25+0$

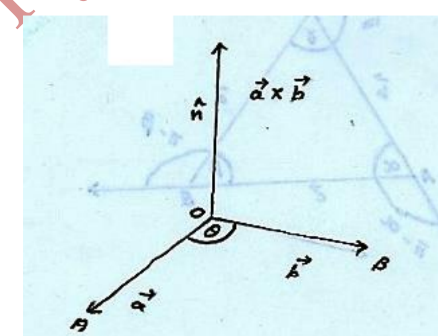
$\boxed{W = 55 \text{ joule}}$

Vector Product Or Cross Product:

If $\vec{a}$ and $\vec{b}$ be the two vectors. Then the vector or cross product of two vector is define as

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$$

Where θ is the angle between $\vec{a}$ and $\vec{b}$ **and** $\hat{n}$ is a unit vector which is perpendicular to both vectors $\vec{a}$ and $\vec{b}$ **.** **{** $\vec{a} \times \vec{b}$ is also perpendicular vector of $\vec{a}$ *and* $\vec{b}$**.}**



Formula:

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$ -------------(i)

Taking magnitude on both sides

$|\vec{a} \times \vec{b}| = \left||\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}\right|$

$|\vec{a} \times \vec{b}| = \left||\vec{a}|\,|\vec{b}| \sin\theta\right|.|\hat{n}|$

$|\vec{a} \times \vec{b}| = \left||\vec{a}|\,|\vec{b}| \sin\theta\right|$:$|\hat{n}|=1$

$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$ -------------(ii)

$$\boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}}$$

From (i)

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a}|\,|\vec{b}| \sin\theta}$$

$$\boxed{\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}}$$: From (ii)

Characteristics:

(i) If $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\vec{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then $\vec{a} \times \vec{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(ii) If $\vec{a}$ and $\vec{b}$ are parallel 0r anti parallel vectors ($\theta = 0^0$ or 180^0) then $\vec{a} \times \vec{b} = 0$

(iii) *Cross product is non-commutative:*

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \quad \text{but} \quad \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

(iv) *Cross product of two same vectors is zero*.

$$\vec{a} \times \vec{a} = 0$$

(v) *Area of parallelogram :*

If $\vec{a}$ *and* $\vec{b}$ be the two sides of parallelogram . Then

Area of parallelogram = $|\vec{a} \times \vec{b}|$ 0r

If $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ be the two diagonals of **a** parallelogram . Then

Area of parallelogram = $\frac{1}{2}|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|$

(vi) ***Area of triangle :***

Area of triangle = $\frac{1}{2}$(Area of parallelogram) = $\frac{1}{2}|\vec{a} \times \vec{b}|$

(vii) ***Three collinear vectors:***

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the three vectors . these are said to be collinear if

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$$

(viii) ***Distributive property of cross product over addition or subtraction .***

$$\vec{a} \times (\vec{b} \pm \vec{c}) = (\vec{a} \times \vec{b}) \pm (\vec{a} \times \vec{c}) \qquad \textbf{Left distributive law}$$

$$(\vec{a} \pm \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) \pm (\vec{b} \times \vec{c}) \qquad \textbf{Right distributive law}$$

(ix) *Scalar multiplication in cross product:*

$$(\lambda\vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b}) \quad \textbf{or} \quad \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$$

(x) Relation between $\hat{\imath}, \hat{\jmath}, \hat{k}$ unit vectors in cross product.

$\hat{\imath}\times\hat{\imath} = 0$: $\hat{\imath}\times\hat{\jmath} = \hat{k}$ and $\hat{\jmath}\times\hat{\imath} = -\hat{k}$

$\hat{\jmath}\times\hat{\jmath} = 0$: $\hat{\jmath}\times\hat{k} = \hat{\imath}$ and $\hat{k}\times\hat{\jmath} = -\hat{\imath}$

$\hat{k}\times\hat{k} = 0$: $\hat{k}\times\hat{\imath} = \hat{\jmath}$ and $\hat{\imath}\times\hat{k} = -\hat{\jmath}$

Note :For this we can use a cyclic process as shown in figure.

(xi) Moment of a force :

If $\vec{r}$ be the position vector of P from O and $\vec{F}$ is the force acting at P. then moment of force $\vec{M}$ is define as

$$\vec{M} = \vec{r}\times\vec{F}$$

Example#01: For vectors $\vec{a}=5\,\hat{\imath}-3\,\hat{\jmath}+4\,\hat{k}$ & $\vec{b}=0\hat{\imath}+\hat{\jmath}-\hat{k}$ determine (i) $\vec{a}\times\vec{b}$ (ii) Sine of the angle between $\vec{a}$ & $\vec{b}$.

Solution: Given $\vec{a}=5\,\hat{\imath}-3\,\hat{\jmath}+4\,\hat{k}$ & $\vec{b}=0\hat{\imath}+\hat{\jmath}-\hat{k}$

(i) $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -3 & 4 \\ 0 & 1 & -1 \end{vmatrix}$

$= \hat{\imath}\begin{vmatrix} -3 & 4 \\ 1 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 5 & 4 \\ 0 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 5 & -3 \\ 0 & 1 \end{vmatrix}$

$= \hat{\imath}(3-4) - \hat{\jmath}(-5-0) + \hat{k}(5-0)$

$= -\hat{\imath} + 5\hat{\jmath} + 5\hat{k}$

(ii) Since $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|} = \frac{\sqrt{(-1)^2+(5)^2+(5)^2}}{\sqrt{(5)^2+(-3)^2+(4)^2}\ \sqrt{(0)^2+(1)^2+(-1)^2}} = \frac{\sqrt{1+25+25}}{\sqrt{25+9+16}\ \sqrt{0+1+1}} = \frac{\sqrt{51}}{\sqrt{50}\,\sqrt{2}} = \frac{\sqrt{51}}{\sqrt{100}}$$

$$\sin\theta = \frac{\sqrt{51}}{10}$$

Example #02:Find a vector perpendicular to both line AB & CD . where $A(0,-1,3)$, $B(2,0,4)$ $C(2,-1,4)$ *and* $D(3,3,2)$ are given points.

Solution: Here $A(0,-1,3)$, $B(2,0,4)$, $C(2,-1,4)$ *and* $D(3,3,2)$ are given points.

Now $\overrightarrow{AB}$ = p.v's of B – p.v's of A= $B(2,0,4) - A(0,-1,3)$

$= (2-0)\,\hat{\imath} + (0+1)\hat{\jmath} + (4-3)\,\hat{k}$

$= 2\hat{\imath} + \hat{\jmath} + \hat{k}$

$\overrightarrow{CD}$ = p.v's of D – p.v's of C= $D(3,3,2) - A(2,-1,4)$

$= (3-2)\,\hat{\imath} + (3+1)\hat{\jmath} + (2-4)\,\hat{k}$

$= \hat{\imath} + 4\hat{\jmath} - 2\hat{k}$

We know that perpendicular vector of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= \hat{\imath}(-2-4) - \hat{\jmath}(-4-1) + \hat{k}(8-1)$$

$$= -6\hat{\imath} + 5\hat{\jmath} + 7\hat{k}$$

Example#03: Find a unit vector perpendicular to $\vec{a} = \hat{\imath} + \hat{\jmath}$ & $\vec{b} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$.

Solution: Let $\hat{n}$ be the unit vector perpendicular to $\vec{a} = \hat{\imath} + \hat{\jmath}$ & $\vec{b} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ vectors.

Then $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ **--------------(i)**

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= \hat{\imath}(1-0) - \hat{\jmath}(1-0) + \hat{k}(2-3)$$

$$= \hat{\imath} - \hat{\jmath} - \hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

From (i) $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \dfrac{\hat{\imath} - \hat{\jmath} - \hat{k}}{\sqrt{3}}$

$$\hat{n} = \frac{1}{\sqrt{3}}\hat{\imath} - \frac{1}{\sqrt{3}}\hat{\jmath} - \frac{1}{\sqrt{3}}\hat{k}$$

Example#04: Find the area of parallelogram with adjacent sides $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$ & $\vec{b} = 2\hat{\jmath} - 3\hat{k}$.

Solution **:**Given sides $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$ & $\vec{b} = 2\hat{\jmath} - 3\hat{k}$.

we know that Area of parallelogram = $|\vec{a} \times \vec{b}|$ -------------(i)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 0 & 2 & -3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= \hat{\imath}(3-2) - \hat{\jmath}(-3-0) + \hat{k}(2-0)$$

$$= \hat{\imath} + 3\hat{\jmath} + 2\hat{k}$$

From (i)

Area of parallelogram = $|\vec{a} \times \vec{b}|$

$$= \sqrt{(1)^2 + (3)^2 + (2)^2} = \sqrt{1+9+4}$$

$$= \sqrt{14} \text{ sq. units}$$

Example#05:Find the area of parallelogram determined by the side $2\hat{\imath}+\hat{\jmath}+5\hat{k}$ & Diagonal $\hat{\imath}-3\hat{\jmath}+\hat{k}$.

Solution: consider a parallelogram ABCDA.

Given side $\overrightarrow{AB}=2\hat{\imath}+\hat{\jmath}+5\hat{k}$ & diagonal $\overrightarrow{AC}=\hat{\imath}-3\hat{\jmath}+\hat{k}$.

By using head to tail rule

$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$$

$$\overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}=(\hat{\imath}-3\hat{\jmath}+\hat{k})-(2\hat{\imath}+\hat{\jmath}+5\hat{k})=\hat{\imath}-3\hat{\jmath}+\hat{k}-2\hat{\imath}-\hat{\jmath}-5\hat{k}$$

$$\overrightarrow{BC}=-\hat{\imath}-4\hat{\jmath}-4\hat{k}$$

For Area of parallelogram $=\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$ -------------(i)

$$\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 1 & 5\\ -1 & -4 & -4\end{vmatrix}=\hat{\imath}\begin{vmatrix}1 & 5\\ -4 & -4\end{vmatrix}-\hat{\jmath}\begin{vmatrix}2 & 5\\ -1 & -4\end{vmatrix}+\hat{k}\begin{vmatrix}2 & 1\\ -1 & -4\end{vmatrix}$$

$$=\hat{\imath}(-4+20)-\hat{\jmath}(-8+5)+\hat{k}(-8+1)$$

$$=16\hat{\imath}+3\hat{\jmath}-7\hat{k}$$

From (i) Area of parallelogram $=\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(16)^2+(3)^2+(-7)^2}=\sqrt{256+9+49}$

$$=\sqrt{314}\text{ sq. units}$$

Example#06 : Find the area of triangle ABC with adjacent sides $\vec{a}=\widehat{3\imath}+2\hat{\jmath}$ & $\vec{b}=2\hat{\jmath}-4\hat{k}$.

Solution: Given sides $\vec{a}=\widehat{3\imath}+2\hat{\jmath}$ & $\vec{b}=2\hat{\jmath}-4\hat{k}$ of Δ ABC.

We know that Area of triangle $=\frac{1}{2}$(Area of parallelogram) $=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$---------(i)

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 3 & 2 & 0\\ 0 & 2 & -4\end{vmatrix}=\hat{\imath}\begin{vmatrix}2 & 0\\ 2 & -4\end{vmatrix}-\hat{\jmath}\begin{vmatrix}3 & 0\\ 0 & -4\end{vmatrix}+\hat{k}\begin{vmatrix}3 & 2\\ 0 & 2\end{vmatrix}$$

$$=\hat{\imath}(-8-0)-\hat{\jmath}(-12-0)+\hat{k}(6-0)$$

$$=-8\hat{\imath}+12\hat{\jmath}+6\hat{k}$$

From (i)

Area of triangle $=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$

$$=\frac{1}{2}\left(\sqrt{(-8)^2+(12)^2+(6)^2}\right)=\frac{1}{2}\left(\sqrt{64+144+36}\right)=\frac{1}{2}\left(\sqrt{244}\right)$$

$$=\frac{1}{2}\left(2\sqrt{61}\right)$$

$$=\sqrt{61}\text{ sq. units}$$

Exercise #2.2

Q#01: Compute the following cross- product.

(i) $\hat{\imath} \times (2\hat{\jmath} + 3\hat{k})$

$= \hat{\imath} \times 2\hat{\jmath} + \hat{\imath} \times 3\hat{k}$

$= 2(\hat{\imath} \times \hat{\jmath}) + 3(\hat{\imath} \times \hat{k})$

$= 2\hat{k} + 3(-\hat{\jmath})$ $\quad \therefore \hat{\imath} \times \hat{\jmath} = \hat{k}$ & $\hat{\imath} \times \hat{k} = -\hat{\jmath}$

$= 2\hat{k} - 3\hat{\jmath}$

(ii) $(2\hat{\imath} - 5\hat{k}) \times \hat{\jmath}$

$= 2\hat{\imath} \times \hat{\jmath} - 5\hat{k} \times \hat{\jmath}$

$= 2(\hat{\imath} \times \hat{\jmath}) - 5(\hat{k} \times \hat{\jmath})$

$= 2\hat{k} - 5(-\hat{\imath})$ $\quad \therefore \hat{\imath} \times \hat{\jmath} = \hat{k}$ & $\hat{k} \times \hat{\jmath} = -\hat{\imath}$

$= 2\hat{k} + 5\hat{\imath}$

(iii) $(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) \times (6\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$

$$(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) \times (6\hat{\imath} + 2\hat{\jmath} - 3\hat{k}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & 5 \\ 6 & 2 & -3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -3 & 5 \\ 2 & -3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 5 \\ 6 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -3 \\ 6 & 2 \end{vmatrix}$$

$$= \hat{\imath}(9 - 10) - \hat{\jmath}(-6 - 30) + \hat{k}(4 + 18)$$

$$= \hat{\imath}(-1) - \hat{\jmath}(-36) + \hat{k}(22)$$

$$= -\hat{\imath} + 36\hat{\jmath} + 22\hat{k}$$

Q#02: Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$

Solution: Taking L.H.S $= (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= \vec{a} \times (\vec{a} + \vec{b}) - \vec{b} \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$ $\quad \therefore \vec{a} \times \vec{a} = 0$

$= 0 + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} + 0$ $\quad \therefore \vec{b} \times \vec{b} = 0$

$= 2(\vec{a} \times \vec{b}) =$ R.H.S $\quad \therefore -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}$

Hence proved

L.H.S = R.H.S

Q#03:If $\vec{a} = 2\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$; $\vec{b} = 3\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ and $\vec{c} = 2\hat{\imath} + 7\hat{\jmath} + 4\hat{k}$.

Find $(\vec{a} - \vec{b}) \times (\vec{c} - \vec{a})$ and $|(\vec{a} - \vec{b}) \times (\vec{c} - \vec{a})|$.

Solution: Given $\vec{a} = 2\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$; $\vec{b} = 3\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ and $\vec{c} = 2\hat{\imath} + 7\hat{\jmath} + 4\hat{k}$

$$\therefore \vec{a} - \vec{b} = 2\hat{\imath} + 5\hat{\jmath} + 3\hat{k} - 3\hat{\imath} - 3\hat{\jmath} - 6\hat{k} = -\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$$

$$\therefore \vec{c} - \vec{a} = 2\hat{\imath} + 7\hat{\jmath} + 4\hat{k} - 2\hat{\imath} - 5\hat{\jmath} - 3\hat{k} = 0\hat{\imath} + 2\hat{\jmath} + \hat{k}$$

Now $$(\vec{a} - \vec{b}) \times (\vec{c} - \vec{a}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & -3 \\ 0 & 2 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -1 & -3 \\ 0 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$= \hat{\imath}(2 + 6) - \hat{\jmath}(-1 - 0) + \hat{k}(2 - 0)$$

$$= \hat{\imath}(8) - \hat{\jmath}(-1) + \hat{k}(2)$$

$$= 8\hat{\imath} + 1\hat{\jmath} + 2\hat{k}$$

And $$|(\vec{a} - \vec{b}) \times (\vec{c} - \vec{a})| = \sqrt{(8)^2 + (1)^2 + (2)^2} = \sqrt{64 + 1 + 4} = \sqrt{69}$$

Q#04: Prove that $(\vec{a} - \vec{b}).(\vec{a} + \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

Solution: Taking L.H.S.= $(\vec{a} - \vec{b}).(\vec{a} + \vec{b})$

$$= \vec{a}.\vec{a} + \vec{a}.\vec{b} - \vec{b}.\vec{a} - \vec{b}.\vec{b}$$

$$= |\vec{a}|^2 + \vec{a}.\vec{b} - \vec{a}.\vec{b} - |\vec{b}|^2 \qquad \therefore \vec{a}.\vec{b} = \vec{b}.\vec{a}$$

$$= |\vec{a}|^2 - |\vec{b}|^2 = \text{R.H.S}$$

Hence proved . L.H.S = R.H.S

Q#05: Find a unit vector perpendicular to $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ & $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$.

Solution: let $\hat{n}$ be the unit vector perpendicular to $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ & $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ vectors.

Then $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ **--------------(i)**

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 3 & -1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= \hat{\imath}(-1 - 3) - \hat{\jmath}(-1 - 2) + \hat{k}(3 - 2) = \hat{\imath}(-4) - \hat{\jmath}(-3) - \hat{k}(1)$$

$$= -4\hat{\imath} + 3\hat{\jmath} - \hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + (3)^2 + (-1)^2} = \sqrt{16 + 9 + 1} = \sqrt{26}$$

From (i) $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{-4\hat{\imath} + 3\hat{\jmath} - \hat{k}}{\sqrt{26}}$ **or** $\hat{n} = \frac{-4}{\sqrt{26}}\hat{\imath} + \frac{3}{\sqrt{26}}\hat{\jmath} - \frac{1}{\sqrt{26}}\hat{k}$

Q#06: (i) When $(\vec{a}+\vec{b})$ is perpendicular to $(\vec{a}-\vec{b})$? When are they parallel?

***Solution:* 1st condition**: $(\vec{a}+\vec{b}) \perp (\vec{a}-\vec{b})$

$$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 0$$

$$\vec{a}.\vec{a} + \vec{a}.\vec{b} - \vec{b}.\vec{a} - \vec{b}.\vec{b} = 0$$

$$|\vec{a}|^2 + \vec{a}.\vec{b} - \vec{a}.\vec{b} - |\vec{b}|^2 = 0 \qquad \because \vec{a}.\vec{b} = \vec{b}.\vec{a}$$

$$|\vec{a}|^2 - |\vec{b}|^2 = 0$$

$$|\vec{a}|^2 = |\vec{b}|^2$$

Taking square-root on both sides $|\vec{a}| = |\vec{b}|$

When $|\vec{a}| = |\vec{b}|$ then $(\vec{a}+\vec{b})$ is perpendicular to $(\vec{a}-\vec{b})$.

2nd condition: $(\vec{a}+\vec{b}) \parallel (\vec{a}-\vec{b})$

$$(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b}) = 0$$

$$\vec{a} \times (\vec{a}+\vec{b}) - \vec{b} \times (\vec{a}+\vec{b}) = 0$$

$$\vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b} = 0 \qquad \because \vec{a}\times\vec{a} = 0 \ \& \ \vec{b}\times\vec{b} = 0$$

$$0 + \vec{a}\times\vec{b} + \vec{a}\times\vec{b} + 0 = 0 \qquad \because \ -\vec{b}\times\vec{a} = \vec{a}\times\vec{b}$$

$$2(\vec{a}\times\vec{b}) = 0$$

$$\vec{a}\times\vec{b} = 0$$

When $\vec{a}\times\vec{b} = 0$ then $(\vec{a}+\vec{b})$ is parallel to $(\vec{a}-\vec{b})$

Q#06: (ii) If $\vec{a} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ & $\vec{b} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$. Then prove that $(\vec{a}+\vec{b})$ and $\vec{a}\times\vec{b}$ are perpendicular.

Solution: Given $\vec{a} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ & $\vec{b} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$

We have to prove $(\vec{a}+\vec{b}) \perp \vec{a}\times\vec{b}$

Now $(\vec{a}+\vec{b}) = \hat{\imath} + 2\hat{\jmath} - 3\hat{k} + 3\hat{\imath} - \hat{\jmath} + 2\hat{k} = 4\hat{\imath} + \hat{\jmath} - \hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 3 & -1 & 2 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= \hat{\imath}(4-3) - \hat{\jmath}(2+9) + \hat{k}(-1-6) = \hat{\imath}(1) - \hat{\jmath}(11) + \hat{k}(-7)$$

$$= \hat{\imath} - 11\hat{\jmath} - 7\hat{k}$$

Now taking dot product

$$(\vec{a}+\vec{b}).(\vec{a}\times\vec{b}) = (4\,\hat{\imath}+\hat{\jmath}-\hat{k}).(\hat{\imath}-11\hat{\jmath}-7\hat{k})$$

$$= (4)\,(1) + (1)(-11) + (-1)\,(-7)$$

$$= 4 - 11 + 7$$

$$(\vec{a}+\vec{b}).(\vec{a}\times\vec{b}) = 0$$

Hence proved $\quad (\vec{a}+\vec{b}) \perp \vec{a}\times\vec{b}$.

Q#07: Show that $|\vec{a}\,.\,\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}\times\vec{b}|^2$

Solution. Given $|\vec{a}\,.\,\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}\times\vec{b}|^2$

$$|\vec{a}\,.\,\vec{b}|^2 + |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 \text{ ----------(i)}$$

Taking L.H.S of (i)

$$|\vec{a}\,.\,\vec{b}|^2 + |\vec{a}\times\vec{b}|^2 = \left||\vec{a}|\,|\vec{b}|\cos\theta\right|^2 + \left||\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}\right|^2$$

$$= |\vec{a}|^2|\vec{b}|^2\cos^2\theta + |\vec{a}|^2|\vec{b}|^2\sin^2\theta\,|\hat{n}|^2$$

$$= |\vec{a}|^2|\vec{b}|^2\cos^2\theta + |\vec{a}|^2|\vec{b}|^2\sin^2\theta \qquad \therefore |\hat{n}|^2=1$$

$$= |\vec{a}|^2|\vec{b}|^2[\cos^2\theta + \sin^2\theta]$$

$$|\vec{a}.\vec{b}|^2 + |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$$

Hence proved

$$|\vec{a}\,.\,\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}\times\vec{b}|^2$$

Q#08: Example#07: prove that (i) $|\vec{a}\times\vec{b}|^2 + |\vec{a}\,.\,\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$

Solution: Taking L.H.S and by using definition of dot and cross product.

$$|\vec{a}\times\vec{b}|^2 + |\vec{a}\,.\,\vec{b}|^2 = \left||\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}\right|^2 + \left||\vec{a}|\,|\vec{b}|\cos\theta\right|^2$$

$$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta\,|\hat{n}|^2 + |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta + |\vec{a}|^2|\vec{b}|^2\cos^2\theta \qquad \therefore |\hat{n}|^2=1$$

$$= |\vec{a}|^2|\vec{b}|^2[\sin^2\theta + \cos^2\theta]$$

$$|\vec{a}\times\vec{b}|^2 + |\vec{a}\,.\,\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 \qquad \therefore \sin^2\theta + \cos^2\theta = 1$$

Hence proved

Q#08: Example#07: (ii) $|\vec{a}.\vec{b}|^2 - |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 \cos 2\theta$

Solution: Taking L.H.S and by using definition of dot and cross product.

$$|\vec{a}.\vec{b}|^2 - |\vec{a} \times \vec{b}|^2 = \left||\vec{a}|\,|\vec{b}|\cos\theta\right|^2 - \left||\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}\right|^2$$

$$= |\vec{a}|^2|\vec{b}|^2\cos^2\theta - |\vec{a}|^2|\vec{b}|^2\sin^2\theta\,|\hat{n}|^2$$

$$= |\vec{a}|^2|\vec{b}|^2\cos^2\theta - |\vec{a}|^2|\vec{b}|^2\sin^2\theta \qquad \therefore |\hat{n}|^2=1$$

$$= |\vec{a}|^2|\vec{b}|^2[\cos^2\theta - \sin^2\theta]$$

$$|\vec{a}.\vec{b}|^2 - |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2\cos 2\theta \qquad \therefore \sin^2\theta + \cos^2\theta = \cos 2\theta$$

Q#09: If $\vec{a}\,'$ and $\vec{b}\,'$ are vector components of vector $\vec{a}$ and $\vec{b}$ respectively on a plane perpendicular to a vector $\vec{c}$. Then show that

(i) $\vec{a} \times \vec{c} = \vec{a}\,' \times \vec{c}$ (ii) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a}\,' + \vec{b}\,') \times \vec{c}$

***Solution:* (i)** Given condition: $\vec{a}\,'$ and $\vec{b}\,'$ $\perp \vec{c}$ $\quad \therefore \theta = 90^0$ Then $\sin 90^0 = 1$

$\vec{a}\,' \times \vec{c} = |\vec{a}\,'||\vec{c}|\sin 90^0\,\hat{n} = |\vec{a}\,'||\vec{c}|\,\hat{n}$ ----------(i)

$\vec{b}\,' \times \vec{c} = |\vec{b}\,'||\vec{c}|\sin 90^0\,\hat{n} = |\vec{b}\,'||\vec{c}|\,\hat{n}$ -----------(ii)

We have to prove

$$\vec{a} \times \vec{c} = \vec{a}\,' \times \vec{c}$$

Taking L.H.S

$\vec{a} \times \vec{c} = |\vec{a}|\,|\vec{c}|\sin\theta\,\hat{n}$

$= (|\vec{a}|\sin\theta)\,|\vec{c}|\,\hat{n}$ $\quad \therefore \vec{a}\,'$ is component of $\vec{a}$: $|\vec{a}\,'| = |\vec{a}|\sin\theta$

$= |\vec{a}\,'||\vec{c}|\,\hat{n}$

$\vec{a} \times \vec{c} = \vec{a}\,' \times \vec{c}$ $\qquad \therefore$From (i)

(ii)We have to prove $\quad (\vec{a} + \vec{b}) \times \vec{c} = (\vec{a}\,' + \vec{b}\,') \times \vec{c}$

Now $\quad (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

$= |\vec{a}|\,|\vec{c}|\sin\theta\,\hat{n} + |\vec{b}|\,|\vec{c}|\sin\theta\,\hat{n}$

$= (|\vec{a}|\sin\theta)\,|\vec{c}|\,\hat{n} + (|\vec{b}|\sin\theta)\,|\vec{c}|\,\hat{n}$ $\therefore \vec{a}\,'$ is component of $\vec{a}$: $|\vec{a}\,'| = |\vec{a}|\sin\theta$

$= |\vec{a}\,'||\vec{c}|\,\hat{n} + |\vec{a}\,'||\vec{c}|\,\hat{n}$ $\qquad \therefore \vec{b}\,'$ is component of $\vec{b}$: $|\vec{b}\,'| = |\vec{b}|\sin\theta$

$= \vec{a}\,' \times \vec{c} + \vec{b}\,' \times \vec{c}$

$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a}\,' + \vec{b}\,') \times \vec{c}$ $\qquad$ Hence proved .

Q#10: Show that (i) $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$

S0lution: Let $\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$; $\vec{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$ & $\vec{c}=c_1\hat{\imath}+c_2\hat{\jmath}+c_3\hat{k}$

$$\therefore \vec{a}+\vec{b}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}+b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}=(a_1+b_1)\hat{\imath}+(a_2+b_2)\hat{\jmath}+(a_3+b_3)\hat{k}$$

$$(\vec{a}+\vec{b})\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1+b_1 & a_2+b_2 & a_3+b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}+\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$ ∴By using determinant property

$$(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$$

Hence proved

Q#10: Show that (ii) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

S0lution:Let $\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$; $\vec{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$ & $\vec{c}=c_1\hat{\imath}+c_2\hat{\jmath}+c_3\hat{k}$

$$\therefore \vec{b}+\vec{c}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}+c_1\hat{\imath}+c_2\hat{\jmath}+c_3\hat{k}=(b_1+c_1)\hat{\imath}+(b_2+c_2)\hat{\jmath}+(b_3+c_3)\hat{k}$$

$$\vec{a}\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$$

$$=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}+\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$ ∴By using determinant property

$$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence proved

Q#11: Show that $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times\vec{c}+\vec{a})+)\ \vec{c}\times(\vec{a}+\vec{b})=0$

Solution: L.H.S$=\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

$$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b} \quad \therefore \begin{cases}\vec{b}\times\vec{a}=-\vec{a}\times\vec{b}\\ \vec{c}\times\vec{a}=-\vec{a}\times\vec{c}\\ \vec{c}\times\vec{b}=-\vec{b}\times\vec{c}\end{cases}$$

$$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}$$

= o = R.H.S

Hence proved .

Q#12(i) If $\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b} \neq 0$ then Show that $\vec{a} + \vec{b} + \vec{c} = 0$

Solution: Given $\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b} \neq 0$

We have to prove $.\vec{a} + \vec{b} + \vec{c} = 0$

Let $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$$\vec{b} \times \vec{c} = \vec{c} \times \vec{a} - \vec{c} \times \vec{c} \qquad \therefore\ \vec{c} \times \vec{c} = 0$$

$$\vec{b} \times \vec{c} = -\vec{a} \times \vec{c} - \vec{c} \times \vec{c} \qquad \therefore\ \vec{c} \times \vec{a} = -\vec{a} \times \vec{c}$$

$$\vec{b} \times \vec{c} = (-\vec{a} - \vec{c}) \times \vec{c}$$

By using right cancellation property

$$\vec{b} = -\vec{a} - \vec{c}$$

$$\vec{a} + \vec{b} + \vec{c} = 0$$ Hence proved.

Q#12(ii) if $\vec{a} + \vec{b} + \vec{c} = 0$ then show that $\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b}$.

Solution: Given $\vec{a} + \vec{b} + \vec{c} = 0$

We have to prove $\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b}$

Let $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{b} = -\vec{a} - \vec{c}$

Taking cross product with $\vec{c}$

$$\vec{b} \times \vec{c} = (-\vec{a} - \vec{c}) \times \vec{c}$$

$$\vec{b} \times \vec{c} = -\vec{a} \times \vec{c} - \vec{c} \times \vec{c} \qquad \therefore \vec{c} \times \vec{c} = 0$$

$$\vec{b} \times \vec{c} = \vec{c} \times \vec{a} - 0 \qquad \therefore -\vec{a} \times \vec{c} = \vec{c} \times \vec{a}$$

$$\vec{b} \times \vec{c} = \vec{c} \times \vec{a} \text{ ----------------(i)}$$

Again Let $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{b} = -\vec{a} - \vec{c}$

Taking cross product with $\vec{a}$

$$\vec{a} \times \vec{b} = \vec{a} \times (-\vec{a} - \vec{c})$$

$$\vec{a} \times \vec{b} = -\vec{a} \times \vec{a} - \vec{a} \times \vec{c} \qquad \therefore \vec{c} \times \vec{c} = 0$$

$$\vec{a} \times \vec{b} = 0 + \vec{c} \times \vec{a} \qquad \therefore -\vec{a} \times \vec{c} = \vec{c} \times \vec{a}$$

$$\vec{a} \times \vec{b} = \vec{c} \times \vec{a} \text{ ----------------(i)}$$

Combining (i) & (ii)

$$\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b}$$ Hence proved.

Q#13: if (i) $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{c}$ or (ii) $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

Where $\vec{a}$ is a non-zero arbitrary vector then show that in either case $\vec{b} = \vec{c}$.

(i) *Solution*: Given $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{c}$ --------------(i)

We have to prove $\vec{b} = \vec{c}$.

From (i) $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{c}$

$$\vec{a}\,.\,\vec{b} - \vec{a}\,.\,\vec{c} = 0$$

$$\vec{a}\,.\,(\vec{b} - \vec{c}) = 0$$ ∴Left distributive law of dot product

Here $\vec{a} \neq o$ but $\vec{b} - \vec{c} = 0$

$$\Rightarrow \vec{b} = \vec{c}$$

(ii) ***Solution***: Given $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ --------(i)

We have to prove $\vec{b} = \vec{c}$.

From (i) $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

$$\vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0$$

$$\vec{a} \times (\vec{b} - \vec{c}) = 0$$ ∴Left distributive law of cross product

Here $\vec{a} \neq o$ but $\vec{b} - \vec{c} = 0$

$$\Rightarrow \vec{b} = \vec{c}$$

Q#14: Show that the vector $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$; $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ & $\vec{c} = -7\hat{\imath} + 0\hat{\jmath} + 10\hat{k}$ are collinear.

Solution: Given

$\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$; $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ & $\vec{c} = -7\hat{\imath} + 0\hat{\jmath} + 10\hat{k}$

For collinear vectors, we have to prove.

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -2 & 3 \\ 3 & -4 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$$

$$= \hat{\imath}(8 - 9) - \hat{\jmath}(-4 - 6) + \hat{k}(3 + 4) = \hat{\imath}(-1) - \hat{\jmath}(-10) + \hat{k}(7)$$

$$= -\hat{\imath} + 10\hat{\jmath} + 7\hat{k}$$ --------(i)

$$\therefore \vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -4 \\ 0 & -7 & 10 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 3 & -4 \\ -7 & 10 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & -4 \\ 0 & 10 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 3 \\ 0 & -7 \end{vmatrix}$$

$$= \hat{\imath}(30-28) - \hat{\jmath}(20-0) + \hat{k}(-14-0) = \hat{\imath}(2) - \hat{\jmath}(20) + \hat{k}(-7)$$

$$= 2\hat{\imath} - 20\hat{\jmath} - 7\hat{k} \text{-------(ii)}$$

$$\therefore \vec{c} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -7 & 10 \\ 1 & -2 & 3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -7 & 10 \\ -2 & 3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 0 & 10 \\ 1 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & -7 \\ 1 & -2 \end{vmatrix}$$

$$= \hat{\imath}(-21+20) - \hat{\jmath}(0-10) + \hat{k}(0+7) = \hat{\imath}(-1) - \hat{\jmath}(-10) + \hat{k}(7)$$

$$= -\hat{\imath} + 10\hat{\jmath} + 7\hat{k} \text{ --------(iii)}$$

Adding (i) , (ii) & (iii)

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$$

Hence proved that the given vectors are collinear.

Q#15: Find a vector perpendicular to both line AB & CD . Where $A(0,2,4)$, $B(3,-1,2)$ $C(2,0,1)$ *and* $D(4,2,0)$ are given points.

***Solution*:** Here $A(0,2,4)$, $B(3,-1,2)$, $C(2,0,1)$ *and* $D(4,2,0)$ are given points.

Now $\overrightarrow{AB}$ = p.v's of B – p.v's of A= $B(3,-1,2) - A(0,2,4)$

$$= (3-0)\,\hat{\imath} + (-1-2)\hat{\jmath} + (2-4)\,\hat{k}$$

$$= 3\hat{\imath} - 3\hat{\jmath} - 2\,\hat{k}$$

$\overrightarrow{CD}$ = p.v's of D – p.v's of C= $D(4,2,0) - C(2,0,1)$

$$= (4-2)\,\hat{\imath} + (2-0)\hat{\jmath} + (0-1)\,\hat{k}$$

$$= 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$$

We know that perpendicular vector of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -3 & -2 \\ 2 & 2 & -1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -3 & -2 \\ 2 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & -3 \\ 2 & 2 \end{vmatrix}$$

$$= \hat{\imath}(3+4) - \hat{\jmath}(-3+4) + \hat{k}(6+6)$$

$$= 7\hat{\imath} - \hat{\jmath} + 12\hat{k}$$

Q#16: (i)Find the area of a triangle whose vertices are A(0,0,0) ,B(1,1,1) & C(0,2,3).

Solution: Consider a triangle ABC. Whose AB and AC are adjacent sides.

$\overrightarrow{AB}$ = p.v's of B – p.v's of A= $B(1,1,1) - A(0,0,0)$

$= (1-0)\,\hat{\imath} + (1-0)\hat{\jmath} + (1-0)\,\hat{k}$

$= \hat{\imath} + \hat{\jmath} + \hat{k}$

$\overrightarrow{AC}$ = p.v's of C – p.v's of A= $C(0,2,3) - A(0,0,0)$

$= (0-0)\,\hat{\imath} + (2-0)\hat{\jmath} + (3-0)\,\hat{k}$

$= 0\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

We know that

Area of triangle =$\frac{1}{2}$(Area of parallelogram) $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$---------(i)

$$\therefore \quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$= \hat{\imath}(3-2) - \hat{\jmath}(3-0) + \hat{k}(2-0)$

$= \hat{\imath} - 3\hat{\jmath} + 2\hat{k}$

From (i)

Area of triangle = $\frac{1}{2}\left|\vec{a} \times \vec{b}\right|$

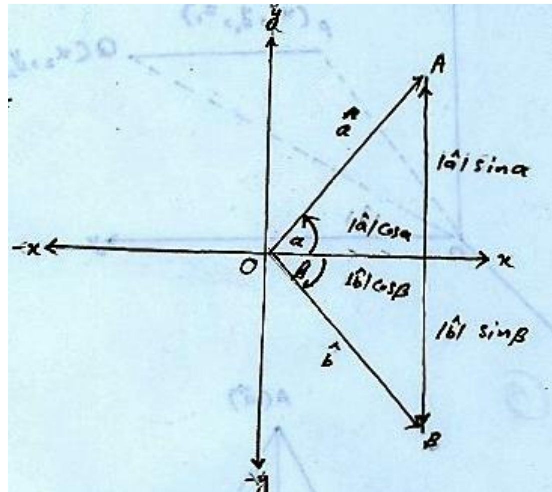
$= \frac{1}{2}\left(\sqrt{(1)^2 + (-3)^2 + (2)^2}\right) = \frac{1}{2}\left(\sqrt{1+9+4}\right)$

$= \frac{1}{2}\left(\sqrt{14}\right)$

$= \frac{\sqrt{14}}{2} \quad sq.units.$

Q#16;(ii)*Prove that* $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Solution: Let $\hat{a} = \widehat{OA}$ and $\hat{b} = \widehat{OB}$ be the two unit vectors makes an angle α and β with x-axis.



From figure:

$$\hat{a} = \widehat{OA} = |\hat{a}|\cos\alpha\ \hat{\imath} + |\hat{a}|\sin\alpha\ \hat{\jmath}$$
$$= \cos\alpha\ \hat{\imath} + \sin\alpha\ \hat{\jmath}$$
$$\hat{b} = \widehat{OB} = |\hat{b}|\cos\beta\ \hat{\imath} + |\hat{b}|\sin\beta\ \hat{\jmath}$$
$$= \cos\beta\ \hat{\imath} + \sin\beta\ \hat{\jmath}$$

Taking cross product of $\hat{b}$ with $\hat{a}$ unit vectors.

$$\hat{b}\times\hat{a} = (\cos\beta\ \hat{\imath} + \sin\beta\ \hat{\jmath}) \times (\cos\alpha\ \hat{\imath} + \sin\alpha\ \hat{\jmath})$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \cos\alpha & 0 \end{vmatrix}$$

$$|\hat{b}||\hat{a}|\ \sin(\alpha-\beta)\ \hat{k} = 0\hat{\imath} - 0\hat{\jmath} + \hat{k}\begin{vmatrix} \cos\beta & \sin\beta \\ \cos\alpha & \sin\alpha \end{vmatrix}$$

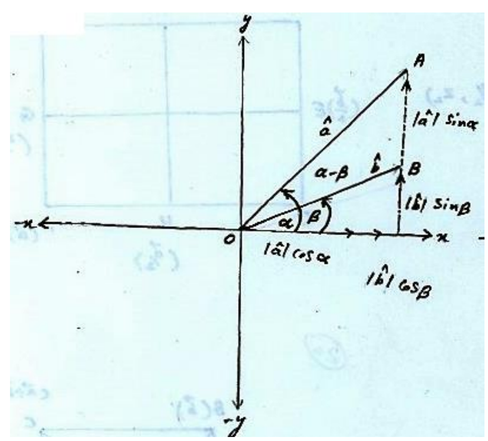
$$\sin(\alpha-\beta)\ \hat{k} = (\cos\beta\ \sin\alpha - \sin\alpha\sin\beta)\ \hat{k} \qquad \therefore |\hat{b}| = |\hat{a}| = 1$$

$\boxed{\sin(\alpha-\beta) = \cos\beta\ \sin\alpha - \sin\alpha\sin\beta}$ Hence proved.

Q#16: (iii)*Prove that* $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Solution: Let $\hat{a}$ = OA and $\hat{b} = OB$ be the two unit vectors makes an angle α and β makes with x-axis.

From figure:



$$\hat{a} = \text{OA} = |\hat{a}|\cos\alpha\ \hat{\imath} + |\hat{a}|\sin\alpha\ \hat{\jmath}$$
$$= \cos\alpha\ \hat{\imath} + \sin\alpha\ \hat{\jmath}$$
$$\hat{b} = \text{OB} = |\hat{b}|\cos\beta\ \hat{\imath} - |\hat{b}|\sin\beta\ \hat{\jmath}$$
$$= \cos\beta\ \hat{\imath} - \sin\beta\ \hat{\jmath}$$

Taking cross product of $\hat{b}$ with $\hat{a}$ unit vectors.

$$\hat{b}\times\hat{a} = (\cos\beta\ \hat{\imath} - \sin\beta\ \hat{\jmath}) \times (\cos\alpha\ \hat{\imath} + \sin\alpha\ \hat{\jmath})$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \cos\alpha & 0 \end{vmatrix}$$

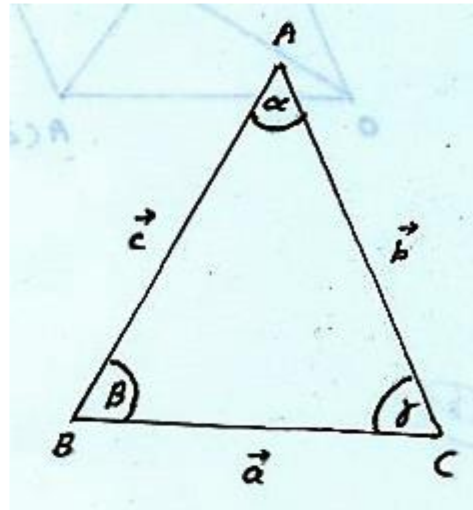
$$|\hat{b}||\hat{a}|\ \sin(\alpha+\beta)\ \hat{k} = 0\hat{\imath} - 0\hat{\jmath} + \hat{k}\begin{vmatrix} \cos\beta & -\sin\beta \\ \cos\alpha & \sin\alpha \end{vmatrix}$$

$$\sin(\alpha+\beta)\ \hat{k} = (\cos\beta\ \sin\alpha + \sin\alpha\sin\beta)\ \hat{k} \qquad \therefore |\hat{b}| = |\hat{a}| = 1$$

$\boxed{\text{Sin}(\alpha+\beta) = \cos\beta\sin\alpha + \sin\alpha\sin\beta}$ Hence proved.

Q#16: (iv) Prove that sin law of trigonometry by using vector .

Solution: Consider a ΔABC as shown in the figure.



If $\overrightarrow{AB} = \vec{c}$; $\overrightarrow{BC} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$

We have to prove $\frac{|\vec{a}|}{\sin\alpha} = \frac{|\vec{b}|}{\sin\beta} = \frac{|\vec{c}|}{\sin\gamma}$

We know that $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \quad \vec{a}=-\vec{b}-\vec{c}$ ----------(i)

Taking cross product of (i) with $\vec{b}$

$$\vec{a}\times\vec{b} = (-\vec{b}-\vec{c})\times\vec{b}$$

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{b}-\vec{c}\times\vec{b}$$

$$\vec{a}\times\vec{b} = 0 \ -\vec{c}\times\vec{b}$$

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}$$

Taking magnitude $|\vec{a}\times\vec{b}|=|\vec{b}\times\vec{c}|$

$$|\vec{a}|\,|\vec{b}|\sin\gamma = |\vec{b}|\,|\vec{c}|\sin\alpha$$

$$|\vec{a}|\sin\gamma = |\vec{c}|\sin\alpha$$

$$\frac{|\vec{a}|}{\sin\alpha} = \frac{|\vec{c}|}{\sin\gamma} \text{ -----------------(ii)}$$

Taking cross product of (i) with $\vec{c}$

$$\vec{a}\times\vec{c} = (-\vec{b}-\vec{c})\times\vec{c}$$

$$\vec{a}\times\vec{c} = -\vec{b}\times\vec{c}-\vec{c}\times\vec{c}$$

$$\vec{a}\times\vec{c} = -\vec{b}\times\vec{c}-0$$

$$\vec{a}\times\vec{c} = \vec{c}\times\vec{b}$$

Taking magnitude $|\vec{a}\times\vec{c}|=|\vec{c}\times\vec{b}|$

$$|\vec{a}|\,|\vec{c}|\sin\beta = |\vec{c}|\,|\vec{b}|\sin\alpha$$

$$|\vec{a}|\sin\beta = |\vec{b}|\sin\alpha$$

$$\frac{|\vec{a}|}{\sin\alpha} = \frac{|\vec{b}|}{\sin\beta} \text{ -----------------(ii)}$$

Combining (i) &(ii)

$$\frac{|\vec{a}|}{\sin\alpha} = \frac{|\vec{b}|}{\sin\beta} = \frac{|\vec{c}|}{\sin\gamma}$$

This is called law of sine of trigonometry.

Q#16:(v)If the diagonals of a given parallelogram are taken as its adjacent sides of a second parallelogram, then prove that the area of the second parallelogram is twice the area of given parallelogram.

Solution: Let $\vec{a}$ & $\vec{b}$ be the adjacent sides of a given parallelogram and $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ be the diagonal expression taken as the adjacent sides of second parallelogram.

We have prove.

(Area of parallelogram with diagonal as sides) = 2(Area of parallelogram with original sides)

$$|(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})| = 2|\vec{a}\times\vec{b}|$$

$$\text{L.H.S} = |(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})|$$

$$= |\vec{a}\times(\vec{a}+\vec{b}) - \vec{b}\times(\vec{a}+\vec{b})|$$

$$= |\vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b}| \qquad \therefore \vec{a}\times\vec{a} = 0$$

$$= |0 + \vec{a}\times\vec{b} + \vec{a}\times\vec{b} + 0| \qquad \therefore \vec{b}\times\vec{b} = 0$$

$$= 2\,|\vec{a}\times\vec{b}| = \text{R.H.S} \qquad \therefore -\vec{b}\times\vec{a} = \vec{a}\times\vec{b}$$

Hence proved.

Q#17:If $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$; $\vec{b} = -\hat{\imath} + \hat{k}$ & $\vec{c} = 2\hat{\jmath} - 10\hat{k}$. Then find the Area of a parallelogram whose diagonals are $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

Solution: Given $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$; $\vec{b} = -\hat{\imath} + 0\hat{\jmath} + \hat{k}$ & $\vec{c} = 0\hat{\imath} + 2\hat{\jmath} - 10\hat{k}$

If $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ be the two diagonals of a parallelogram . Then

Area of parallelogram $= \frac{1}{2}|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|$ -------(i)

$$\therefore \vec{a}+\vec{b} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k} - \hat{\imath} + 0\hat{\jmath} + \hat{k} = \hat{\imath} - 3\hat{\jmath} + 2\hat{k}$$

$$\therefore \vec{a}-\vec{b} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k} + \hat{\imath} - 0\hat{\jmath} - \hat{k} = 3\hat{\imath} - 3\hat{\jmath} + 0\hat{k}$$

Now $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & 2 \\ 3 & -3 & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -3 & -2 \\ -3 & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -3 \\ 3 & -3 \end{vmatrix}$

$$= \hat{\imath}(0-6) - \hat{\jmath}(0-6) + \hat{k}(-3+9)$$

$$= -6\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$$

From (i) Area of parallelogram $= \frac{1}{2}|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|$

$$= \frac{1}{2}\left(\sqrt{(-6)^2 + (6)^2 + (6)^2}\right) = \frac{1}{2}\left(\sqrt{36+36+36}\right)$$

$$= \frac{1}{2}\left(\sqrt{108}\right) = \frac{6\sqrt{3}}{2} = 3\sqrt{3}\ sq.units.$$

Q#18: Find two unit vectors which makes an angle of 60^0 with vectors $[1,-1,0]$ *and* $[1,0,-1]$.

***Solution* :** Let $\hat{u}$ be the required unit vector . Let $\hat{u} = xi + yj + z\,k$ -------------------(A)

Then $|\hat{u}|^2 = x^2 + y^2 + z^2 \Rightarrow x^2 + y^2 + z^2 = 1$---------------------(i) $\therefore |\hat{u}|^2 = 1$

Given $\vec{a} = [1,-1,0] = i - j$ and $\vec{b} = [1,0,-1] = i - k$

1st condition: The unit vector $\hat{u}$ makes an angle 60^0 with $\vec{a}$.

Then $\vec{a}.\hat{u} = |\vec{a}|\,|\hat{u}|\cos\theta$ where $\theta = 60^0$

$$(i - j + 0k).(xi + yj + z\,k) = \sqrt{(1)^2 + (-1)^2 + (0)^2}\ .\ 1\ .\ \cos 60^0 \qquad \therefore |\hat{u}| = 1$$

$$1.x - 1.y - 0.z = \sqrt{1+1+0}\ .\ \tfrac{1}{2} = \sqrt{2}\ .\ \tfrac{1}{2}$$

$$x - y = \tfrac{1}{\sqrt{2}} \Rightarrow y = x - \tfrac{1}{\sqrt{2}} \text{ ----------------------(ii)}$$

***2nd condition*:** The unit vector $\hat{u}$ makes an angle 60^0 with $\vec{b}$.

Then $\vec{b}.\hat{u} = |\vec{b}|\,|\hat{u}|\cos\theta$ where $\theta = 60^0$

$$(i + 0j - k).(xi + yj + z\,k) = \sqrt{(0)^2 + (1)^2 + (-1)^2}\ .\ 1\ .\ \cos 60^0 \qquad \therefore |\hat{u}| = 1$$

$$1.x + 0.y - 1.z = \sqrt{0+1+1}\ .\ \tfrac{1}{2} = \sqrt{2}\ .\ \tfrac{1}{2}$$

$$x - z = \tfrac{1}{\sqrt{2}} \Rightarrow z = x - \tfrac{1}{\sqrt{2}} \text{ ---------------------(iii)}$$

Using equation (ii) and (iii) in (i)

$$x^2 + \left[x - \tfrac{1}{\sqrt{2}}\right]^2 + \left[x - \tfrac{1}{\sqrt{2}}\right]^2 = 1$$

$$x^2 + x^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 - 2\,x\,\tfrac{1}{\sqrt{2}} + x^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 - 2\,x\,\tfrac{1}{\sqrt{2}} = 1$$

$$3x^2 - \sqrt{2}\,x + \tfrac{1}{2} - \sqrt{2}\,x + \tfrac{1}{2} = 1 \Rightarrow 3x^2 - 2\sqrt{2}\,x + 1 = 1 \Rightarrow 3x^2 - 2\sqrt{2}\,x = 0 \quad x(3x - 2\sqrt{2}) = 0$$

$\boxed{x=0}$

Put in (ii) and (iii)

$y = 0 - \frac{1}{\sqrt{2}} \Rightarrow \boxed{y = -\frac{1}{\sqrt{2}}}$

&

$z = 0 - \frac{1}{\sqrt{2}} \Rightarrow \boxed{z = \frac{-1}{\sqrt{2}}}$

$3x - 2\sqrt{2} = 0 \Rightarrow 3x = 2\sqrt{2} \Rightarrow \boxed{x = \frac{2\sqrt{2}}{3}}$

$y = \frac{2\sqrt{2}}{3} - \frac{1}{\sqrt{2}} = \frac{4-3}{3\sqrt{2}} \Rightarrow \boxed{y = \frac{1}{3\sqrt{2}}}$

$z = \frac{2\sqrt{2}}{3} - \frac{1}{\sqrt{2}} = \frac{4-3}{3\sqrt{2}} \Rightarrow \boxed{z = \frac{1}{3\sqrt{2}}}$

Using values of x, y ,z in required unit vector represented by equ.(A)

$$\hat{u} = 0\,i - \tfrac{1}{\sqrt{2}}j - \tfrac{1}{\sqrt{2}}\,k \qquad \text{OR} \qquad \hat{u} = \tfrac{2\sqrt{2}}{3}i + \tfrac{1}{3\sqrt{2}}j + \tfrac{1}{3\sqrt{2}}\,k$$

Q#19:Prove by using cross product that the points $(5, 2, -3), (6, 1, 4), (-2, -3, 6)$ *and* $(-3, -2, 1)$ Are the vertices of a parallelogram then find its area.

Solution: Let $A(5,2,-3)$; $B(6,1,4)$; $C(-2,-3,6)$ *and* $D(-3,-2,1)$ are the vertices of parallelogram ABCDA. AB & AD are its adjacent sides.

Now $\overrightarrow{AB}$ = p.v's of B – p.v's of A= $B(6,1,4) - A(5,2,-3)$

$= (6-5)\,\hat{\imath} + (1-2)\hat{\jmath} + (4+3)\,\hat{k}$

$= \hat{\imath} - \hat{\jmath} + 7\,\hat{k}$

$\overrightarrow{AD}$ = p.v's of D – p.v's of A= $D(-3,-2,1) - A(5,2,-3)$

$= (-3-5)\,\hat{\imath} + (-2-2)\hat{\jmath} + (1+3)\,\hat{k}$

$= -8\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$

We know that perpendicular vector of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 7 \\ -8 & -4 & 4 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & 7 \\ -4 & 4 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 7 \\ -8 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ -8 & -4 \end{vmatrix}$$

$= \hat{\imath}(-4+28) - \hat{\jmath}(2+56) + \hat{k}(-4-8)$

$= 24\hat{\imath} - 58\hat{\jmath} - 12\hat{k}$

Area of parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AD}| = \left(\sqrt{(24)^2 + (-58)^2 + (-12)^2}\right) = \left(\sqrt{576 + 3364 + 144}\right)$

$= (\sqrt{4084})$ *sq.units.*

Q#20: Find the area of parallelogram having diagonals $\vec{a} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ & $\vec{b} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$

Solution: For diagonal expression

Area of parallelogram= $\frac{1}{2}|\vec{a} \times \vec{b}|$--------------(i)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}$$

$= \hat{\imath}(4-6) - \hat{\jmath}(12+2) + \hat{k}(-9-1)$

$= -2\hat{\imath} - 14\hat{\jmath} - 10\hat{k}$

Area of parallelogram=

$\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\left(\sqrt{(-2)^2 + (-14)^2 + (-10)^2}\right) = \frac{1}{2}\left(\sqrt{4 + 196 + 100}\right)$

$= \frac{1}{2}(\sqrt{300}) = \frac{10\sqrt{3}}{2}$

$= 5\sqrt{3}$ *sq.units.*

Q#21: Find area of triangle with vertices at $(3,-1,2);(1,-1,-3)$ *and* $(4,-3,1)$.

Solution: Let $A(3,-1,2); B(1,-1,-3)$ and $C(4,-3,1)$ are the vertices of triangle ABC.

If AB & AC be the adjacent sides of its triangle. Then

$\overrightarrow{AB} = p.v's\ of\ B - p.v's\ of\ A = B(1,-1,-3) - A(3,-1,2) = (1-3)\,\hat{\imath} + (-1+1)\hat{\jmath} + (-3-2)\,\hat{k}$

$= -2\hat{\imath} + 0\hat{\jmath} - 5\,\hat{k}$

$\overrightarrow{AC} = p.v's\ of\ C - p.v's\ of\ A = C(4,-3,1) - A(3,-1,2) = (4-3)\,\hat{\imath} + (-3+1)\hat{\jmath} + (1-2)\,\hat{k}$

$= \hat{\imath} - 2\hat{\jmath} - \hat{k}$

We know that perpendicular vector of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & -5 \\ -2 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -2 & -5 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix}$$

$= \hat{\imath}(0-10) - \hat{\jmath}(2+5) + \hat{k}(4-0)$

$= -10\hat{\imath} - 7\hat{\jmath} + 4\hat{k}$

Area of triangle $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left(\sqrt{(-10)^2 + (-7)^2 + (4)^2}\right) = \frac{1}{2}\left(\sqrt{100+49+16}\right)$

$= \frac{1}{2}(\sqrt{165})$

$= \frac{\sqrt{165}}{2}$ $\quad sq.units$

Q#22: If $\vec{a} = 2\hat{\imath} - \hat{\jmath}$; $\vec{b} = \hat{\jmath} + \hat{k}$ & $|\vec{c}| = 12$ and $\vec{c}$ is perpendicular to both $\vec{a}$ *and* $\vec{b}$, write the component form of $\vec{c}$.

Solution: If $\vec{a} = 2\hat{\imath} - \hat{\jmath} + 0\hat{k}$; $\vec{b} = 0\hat{\imath} + \hat{\jmath} + \hat{k}$ & $|\vec{c}| = 12$

Let $\hat{c}$ be the unit vector perpendicular to both $\vec{a}$ *and* $\vec{b}$. Then $\hat{c} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ --------------------(i)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = \hat{\imath}(-1-0) - \hat{\jmath}(2-0) + \hat{k}(2-0)$$

$= -\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

$\left|\vec{a} \times \vec{b}\right| = \sqrt{(-1)^2 + (-2)^2 + (2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

From (i) $\quad \hat{c} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{-\hat{\imath}-2\hat{\jmath}+2\hat{k}}{3}$ --------------(ii)

Now by using definition of unit vector

$\hat{c} = \frac{\vec{c}}{|\vec{c}|} \implies \vec{c} = |\vec{c}|\ \hat{c}$

$\vec{c} = 12\left(\frac{-\hat{\imath}-2\hat{\jmath}+2\hat{k}}{3}\right) = 4\left(-\hat{\imath} - 2\hat{\jmath} + 2\hat{k}\right) \implies \vec{c} = -4\hat{\imath} - 8\hat{\jmath} + 8\hat{k}$

Q#23: Show that $\vec{a} \times \vec{b} = \hat{\imath} \times a_1 \vec{b} + \hat{\imath} \times a_2\vec{b} + \hat{\imath} \times a_3\vec{b}$ where $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$.

***Solution*:** Given $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

We have to prove $\vec{a} \times \vec{b} = \hat{\imath} \times a_1 \vec{b} + \hat{\imath} \times a_2\vec{b} + \hat{\imath} \times a_3\vec{b}$

Now R.H.S $= \hat{\imath} \times a_1 \vec{b} + \hat{\imath} \times a_2 \vec{b} + \hat{\imath} \times a_3\vec{b}$

$= a_1\, \hat{\imath} \times \vec{b} + a_2\hat{\imath} \times \vec{b} + a_3\, \hat{\imath} \times \vec{b}$

$= (a_1\, \hat{\imath} + a_2\hat{\imath} + a_3\, \hat{\imath}) \times \vec{b}$

$= \vec{a} \times \vec{b} =$ L.H.S

Hence proved . R.H.S= L.H.S

Q#24: If $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 7\hat{k}$;$\vec{b} = \hat{\imath} - \hat{\jmath} + 10\hat{k}$ &$\vec{c} = 3\hat{\imath} - 5\hat{\jmath} + 4\hat{k}$ and these vector have a common initial point, Determine whether the terminal points lies on a straight line.

***Solution*:** Given if $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 7\hat{k}$;$\vec{b} = \hat{\imath} - \hat{\jmath} + 10\hat{k}$ &$\vec{c} = 3\hat{\imath} - 5\hat{\jmath} + 4\hat{k}$

We have to prove $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & 7 \\ 1 & -1 & 10 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -3 & 7 \\ -1 & 10 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 7 \\ 1 & 10 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix}$$

$= \hat{\imath}(-30 + 7) - \hat{\jmath}(20 - 7) + \hat{k}(-2 + 3)$

$= -23\hat{\imath} - 13\hat{\jmath} + \hat{k}$---------(i)

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 10 \\ 3 & -5 & 4 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & 10 \\ -5 & 4 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 10 \\ 3 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix}$$

$= \hat{\imath}(-4 + 50) - \hat{\jmath}(4 - 30) + \hat{k}(-5 + 3)$

$= 46\hat{\imath} + 26\hat{\jmath} - 2\hat{k}$ ---------(ii)

$$\vec{c} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -5 & 4 \\ 2 & -3 & 7 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -5 & 4 \\ -3 & 7 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & -5 \\ 2 & -3 \end{vmatrix}$$

$= \hat{\imath}(-35 + 12) - \hat{\jmath}(21 - 8) + \hat{k}(-9 + 10)$

$= -23\hat{\imath} - 13\hat{\jmath} + \hat{k}$ --------(iii)

Adding (i), (ii) & (iii)

$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

Yes, the terminal point lies on the straight line.

Q#25: Let $\hat{a}$ & $\hat{b}$ be the unit vectors and θ be the angle between $\hat{a}$ & $\hat{b}$.

Show that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{b} - \hat{a}|$.

Solution: Let $\hat{a}$ & $\hat{b}$ be the unit vectors and θ be the angle between $\hat{a}$ & $\hat{b}$.

Then we have to prove $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{b} - \hat{a}|$.

Let $|\hat{b} - \hat{a}|^2 = (\hat{b} - \hat{a}).(\hat{b} - \hat{a})$

$= \hat{b}.\hat{b} - \hat{b}.\hat{a} - \hat{a}.\hat{b} + \hat{a}.\hat{a}$

$= |\hat{b}|^2 + |\hat{a}|^2 - 2|\hat{b}||\hat{a}|\cos\theta$ $\qquad |\hat{b}| = |\hat{a}| = 1$

$= 1 + 1 - 2\cos\theta = 2 - 2\cos\theta = 2(1 - \cos\theta)$

$= 2\left(2\sin^2\frac{\theta}{2}\right)$

$|\hat{b} - \hat{a}|^2 = \left(2\sin\frac{\theta}{2}\right)^2$

Taking square-root on the both sides

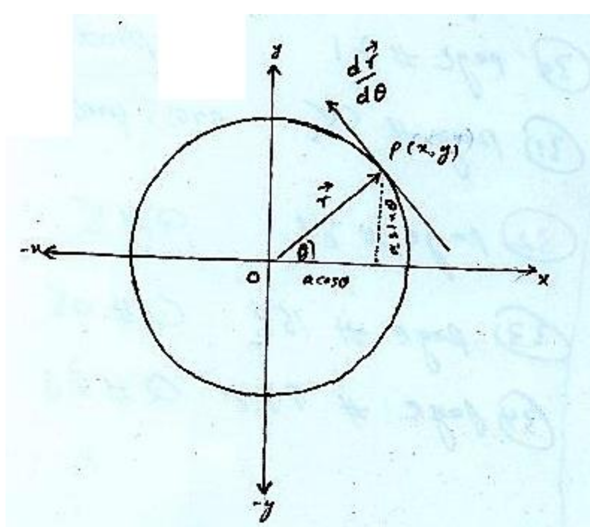
$|\hat{b} - \hat{a}| = 2\sin\frac{\theta}{2}$

$\frac{1}{2}|\hat{b} - \hat{a}| = \sin\frac{\theta}{2}$

Henc proved that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{b} - \hat{a}|$.

Q#26: Show that the component form of a unit tangent vector to a circle $x^2 + y^2 = a^2$ is given by $\pm\frac{1}{a}(-y\hat{\imath} + x\hat{\jmath})$.

Solution: Let $\vec{r}$ be the radius vector of a circle. Let $\vec{r} = x\hat{\imath} + y\hat{\jmath}$



Put $x = a\cos\theta$ & $y = a\sin\theta$

$\vec{r} = a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath}$ --------(i)

For tangent vector, differentiate equation (i) w.r.t θ

$$\frac{d\vec{r}}{d\theta} = -a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath}$$

Required unit vector of tangent vector is

$$\frac{d\hat{r}}{d\theta} = \frac{\frac{d\vec{r}}{d\theta}}{\left|\frac{d\vec{r}}{d\theta}\right|} = \frac{-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath}}{\sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2}} = \frac{-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath}}{\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}}$$

$$= \frac{-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath}}{a\sqrt{\sin^2\theta + \cos^2\theta}} = \frac{-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath}}{a\sqrt{1}}$$

$$= \frac{-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath}}{a\,(\pm 1)} = \frac{-y\hat{\imath} + x\hat{\jmath}}{\pm a}$$

$\frac{d\hat{r}}{d\theta} = \pm\frac{1}{a}(-y\hat{\imath} + x\hat{\jmath})$. *Hence proved.*

Q#27:Prove that: $|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}+\vec{b}|^2 + |\vec{b}+\vec{c}|^2 + |\vec{c}+\vec{a}|^2$

Solution: L.H.S$=|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{a}+\vec{b}+\vec{c}|^2$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\,\vec{a}.\vec{b} + 2\,\vec{b}.\vec{c} + 2\,\vec{c}.\vec{a}$

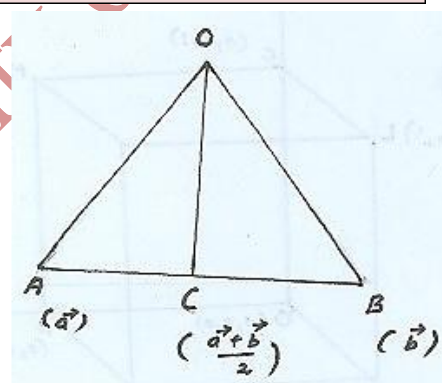
$=\left[|\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}.\vec{b}\right] + \left[|\vec{b}|^2 + |\vec{c}|^2 + 2\,\vec{b}.\vec{c}\right] + [|\vec{c}|^2 + |\vec{a}|^2 + 2\,\vec{c}.\vec{a}]$

$=|\vec{a}+\vec{b}|^2 + |\vec{b}+\vec{c}|^2 + |\vec{c}+\vec{a}|^2$= R.H.S

Hence proved R.H.S= L.H.S

Q#28: Show that the median through the vertex of an isosceles triangle is perpendicular to the base.

Solution: Consider an isosceles triangle OACB. Let O be the origin .



Let $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$; $\overrightarrow{OC} = \vec{a} + \vec{b}$ and $\overrightarrow{AB} = \vec{b} - \vec{a}$

We have to prove $\overrightarrow{OC} \perp \overrightarrow{AB} \Longrightarrow \overrightarrow{OC}.\overrightarrow{AB} = 0$

Now $\overrightarrow{OC}.\overrightarrow{AB} = (\vec{a} + \vec{b}).(\vec{b} - \vec{a})$

$= \vec{a}.\vec{b} - \vec{a}.\vec{a} + \vec{b}.\vec{b} - \vec{b}.\vec{a}$

$= |\vec{b}|^2 - |\vec{a}|^2$

$= |\vec{a}|^2 - |\vec{a}|^2$ In isosceles triangle: $|\overrightarrow{OA}| = |\overrightarrow{OB}| \Longrightarrow |\vec{a}| = |\vec{b}|$

$\overrightarrow{OC}.\overrightarrow{AB} = 0$

Hence proved. $\overrightarrow{OC} \perp \overrightarrow{AB}$

Q#29: In triangle ABC, D & E are mid points of the sides AB & AC respectively. Show that DE is Parallel to BC.

Solution: Consider a ΔABC. Let Position vectors are A($\vec{a}$), B($\vec{b}$) , C ($\vec{c}$) and Mid points D$\left(\frac{\vec{a}+\vec{b}}{2}\right)$ & E$\left(\frac{\vec{a}+\vec{c}}{2}\right)$.

We have to prove. $\overrightarrow{DE} \parallel \overrightarrow{BC}$

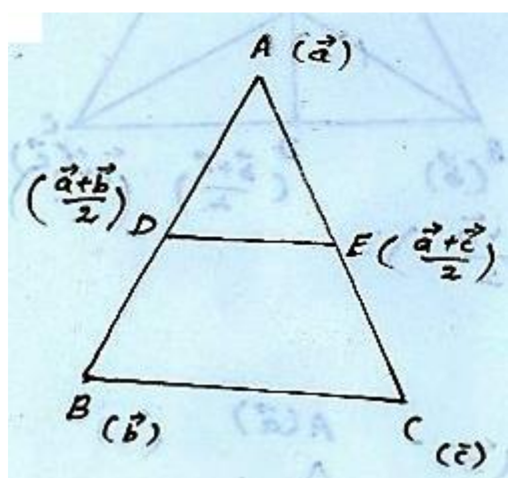
$\overrightarrow{BC} = \vec{c} - \vec{b}$-------------------(i)

& $\overrightarrow{DE} = \frac{\vec{a}+\vec{c}}{2} - \frac{\vec{a}+\vec{b}}{2} = \frac{\vec{a}+\vec{c}-\vec{a}-\vec{b}}{2} = \frac{\vec{c}-\vec{b}}{2}$

$= \frac{1}{2}(\vec{c} - \vec{b})$

$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$ ∴ From (i)

This shows that $\overrightarrow{DE} \parallel \overrightarrow{BC}$



Q#30:{Example}: Find the moment about the point $A(5,-1,3)$ of a force $4\hat{\imath} + 2\hat{\jmath} + \hat{k}$ through point $B(5,2,4)$.

Solution: Let $\vec{F} = 4\hat{\imath} + 2\hat{\jmath} + \hat{k}$ be a force & $\vec{r}$ be a position vector from point $A(5,-1,3)$ to $B(5,2,4)$.

$\vec{r} = P.v\ of\ B - P.v\ of\ A = B(5,2,4) - A(5,-1,3)$

$= (5-5)\hat{\imath} + (2+1)\hat{\jmath} + (4-3)\hat{k}$

$\vec{r} = 0\hat{\imath} + 3\hat{\jmath} + \hat{k}$

We know that

Moment of Force $= \vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & 1 \\ 4 & 2 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & 3 \\ 4 & 2 \end{vmatrix}$

$= \hat{\imath}(3-2) - \hat{\jmath}(0-4) + \hat{k}(0-12)$

$= \hat{\imath} + 4\hat{\jmath} - 12\hat{k}$

Q#31: Find the moment about the point origin of a force $4\hat{\imath} + 2\hat{\jmath} + \hat{k}$ through point $(5,2,4)$.

Solution: Let $\vec{F} = 4\hat{\imath} + 2\hat{\jmath} + \hat{k}$ be a force & $\vec{r}$ be a position vector from origin O(0,0,0) to $A(5,2,4)$.

$\vec{r}$ = P.v of A – P.v of 0= $A(5,2,4) - O(0,0,0)$

$\vec{r} = 5\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

We know that

Moment of Force $= \vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 2 & 4 \\ 4 & 2 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 5 & 4 \\ 4 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 5 & 2 \\ 4 & 2 \end{vmatrix}$

$= \hat{\imath}(2-8) - \hat{\jmath}(5-4) + \hat{k}(10-8)$

$= -6\hat{\imath} - \hat{\jmath} + 2\hat{k}$

Scalar Triple Product:

If $\vec{a}, \vec{b}$ & $\vec{c}$ be any three vectors, then $[\vec{a} \quad \vec{b} \quad \vec{c}]$ or $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called scalar triple product of $\vec{a}, \vec{b}$ & $\vec{c}$.

Characteristics:

(i) **If** $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Then the scalar triple product can be finding by the following method.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(ii) ***Volume of the parallelepiped*** :

Let $\vec{a}, \vec{b}$ & $\vec{c}$ be the three vectors along the edges of parallelepiped. Then

Volume of the parallelepiped = V = $\vec{a} \cdot (\vec{b} \times \vec{c})$

(iii) ***Volume of the tetrahedron:***

Let $\vec{a}, \vec{b}$ & $\vec{c}$ be the three vectors along the edges of tetrahedron . Then

Volume of the tetrahedron = V = $\frac{1}{6}[\vec{a} \cdot (\vec{b} \times \vec{c})]$

(iv) ***Coplanar vectors***:

If $\vec{a}, \vec{b}$ & $\vec{c}$ be the three non-zero vectors . These vectors are said to be coplanar if

$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the four non-zero vectors . These vectors are said to be coplanar if

$(\vec{b} - \vec{a}) \cdot (\vec{c} - \vec{a}) \times (\vec{d} - \vec{c}) = 0$

(v) $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

(vi) If two vectors are same in scalar triple product, then the scalar triple product is equal to zero.

As $\vec{a} \cdot (\vec{b} \times \vec{a}) = 0$

Example#01:Find the volume of parallelepiped whose edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$. where $\vec{a}= 3\hat{\imath} + 2\hat{k}$; $\vec{b}= \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{c}= -\hat{\jmath} + 4\hat{k}$.

Solution: Given $\vec{a}= 3\hat{\imath} + 2\hat{k}$; $\vec{b}= \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{c}= -\hat{\jmath} + 4\hat{k}$.

We know that

Volume of the parallelepiped = V = $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix}$

$$= 3\begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} - 0\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$$

$$= 3(8-1) - 0(4-0) + 2(-1-0)$$

$$= 3(7) - 0 + 2(-1)$$

$$= 21 - 0 - 2$$

=19 cubic units

Example#02: Find p such that the vectors $\vec{a}= 2\hat{\imath} - \hat{\jmath} + \hat{k}$; $\vec{b}= \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ & $\vec{c}= 3\hat{\imath} + p\,\hat{\jmath} + 5\hat{k}$ are coplanar .

Solution: Given $\vec{a}= 2\hat{\imath} - \hat{\jmath} + \hat{k}$; $\vec{b}= \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ and $\vec{c}=3\hat{\imath} + p\,\hat{\jmath} + 5\hat{k}$

According to given condition , the vectors are coplanar. Therefore

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix} = 0$$

$$2\begin{vmatrix} 2 & -3 \\ p & 5 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & p \end{vmatrix} = 0$$

$$2(10 + 3p) + 1(5 + 9) + 1(p - 6) = 0$$

$$20 + 6p + 14 + p - 6 = 0$$

$$7p + 28 = 0$$

$$7p = 28$$

$$p = 28/7$$

$$\boxed{p = 4}$$

Example#03:Prove that the four ponts $(4\hat{\imath}+5\hat{\jmath}+\hat{k});(-\hat{\jmath}-\hat{k});(3\hat{\imath}+9\hat{\jmath}+4\hat{k})$ & $4(-\hat{\imath}+\hat{\jmath}+\hat{k})$ are coplanar.

Solution: Let $A(4\hat{\imath}+5\hat{\jmath}+\hat{k});B(-\hat{\jmath}-\hat{k});C(3\hat{\imath}+9\hat{\jmath}+4\hat{k})$& $D(-4\hat{\imath}+4\hat{\jmath}+4\hat{k})$ are given four points.

If these four points are coplanar then we have to prove coplanar condition

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})=0$$

$\therefore \overrightarrow{AB}=B(-\hat{\jmath}-\hat{k})-A(4\hat{\imath}+5\hat{\jmath}+\hat{k}) = -\hat{\jmath}-\hat{k}-4\hat{\imath}-5\hat{\jmath}-\hat{k} = -4\hat{\imath}-6\hat{\jmath}-2\hat{k}$

$\therefore \overrightarrow{AC}=C(3\hat{\imath}+9\hat{\jmath}+4\hat{k})-A(4\hat{\imath}+5\hat{\jmath}+\hat{k}) = 3\hat{\imath}+9\hat{\jmath}+4\hat{k}-4\hat{\imath}-5\hat{\jmath}-\hat{k} = -\hat{\imath}+4\hat{\jmath}+3\hat{k}$

$\therefore \overrightarrow{AD}=D(-4\hat{\imath}+4\hat{\jmath}+4\hat{k})-A(4\hat{\imath}+5\hat{\jmath}+\hat{k}) = -4\hat{\imath}+4\hat{\jmath}+4\hat{k}-4\hat{\imath}-5\hat{\jmath}-\hat{k} = -8\hat{\imath}-\hat{\jmath}+3\hat{k}$

Now $\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})=\begin{vmatrix}-4 & -6 & -2\\ -1 & 4 & 3\\ -8 & -1 & 3\end{vmatrix} = -4\begin{vmatrix}4 & 3\\ -1 & 3\end{vmatrix}-(-6)\begin{vmatrix}-1 & 3\\ -8 & 3\end{vmatrix}+(-2)\begin{vmatrix}-1 & 4\\ -8 & -1\end{vmatrix}$

$$= -4(12+3)+6(-3+24)-2(1+32)$$

$$= -4(15)+6(21)-2(33)$$

$$= -60+126-66$$

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) = 0$$

This shows that the given four points are coplanar.

Example#04: Prove that $[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]=2[\vec{a}\quad \vec{b}\quad \vec{c}]$

Solution: L.H.S= $[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]$

$=(\vec{a}+\vec{b}).[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$=(\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$

$=(\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+0+\vec{c}\times\vec{a}]$ $\quad\therefore \vec{c}\times\vec{c}=0$

$=(\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$

$=\vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$

$=\vec{a}.(\vec{b}\times\vec{c})+0+0+0+0+\vec{b}.(\vec{c}\times\vec{a})$

$=\vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{c})$ $\quad\therefore \vec{b}.(\vec{c}\times\vec{a})=\vec{a}.(\vec{b}\times\vec{c})$

$=2\ \vec{a}.(\vec{b}\times\vec{c})$

$=2\,[\vec{a}\quad\vec{b}\quad\vec{c}]$ =R.H.S

Hence proved L.H.S= R.H.S

Example#05 : if $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vector of A,B,C. Prove that $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is a vector perpendicular to the plan of ABC.

Solution: Let $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vector of A,B & C. then

$\overrightarrow{AB} = \vec{b} - \vec{a}$: $\overrightarrow{BC} = \vec{c} - \vec{b}$ & $\overrightarrow{CA} = \vec{a} - \vec{c}$

Let $\vec{d} = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$

We have to prove $\vec{d} \perp \overrightarrow{AB}$ $\quad \vec{d} . \overrightarrow{AB} = 0$

L.H.S $= \vec{d} . \overrightarrow{AB} = [\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}].(\vec{b} - \vec{a})$

$= (\vec{a} \times \vec{b}).\vec{b} + (\vec{b} \times \vec{c}).\vec{b} + (\vec{c} \times \vec{a}).\vec{b} - (\vec{a} \times \vec{b}).\vec{a} - (\vec{b} \times \vec{c}).\vec{a} - (\vec{c} \times \vec{a}).\vec{a}$

$= 0 + 0 + (\vec{c} \times \vec{a}).\vec{b} + 0 - (\vec{b} \times \vec{c}).\vec{a} - 0$

$= (\vec{b} \times \vec{c}).\vec{a} - (\vec{b} \times \vec{c}).\vec{a}$ $\qquad \therefore (\vec{b} \times \vec{c}).\vec{a} = (\vec{c} \times \vec{a}).\vec{b}$

$= 0 =$ R.H.S

Hence proved . L.H.S= R.H.S

Example#06: Prove that $[\vec{l} \;\; \vec{m} \;\; \vec{n}]\,[\vec{a} \;\; \vec{b} \;\; \vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$

Solution: Let $\vec{l} = l_1\hat{i} + l_2\hat{j} + l_3\hat{k}$ & $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{m} = m_1\hat{i} + m_2\hat{j} + m_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$ & $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\text{L.H.S} = [\vec{l} \;\; \vec{m} \;\; \vec{n}]\,[\vec{a} \;\; \vec{b} \;\; \vec{c}] = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} . \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} . \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \therefore \text{ Taking transpose of } 2^{nd} \text{ determinant .}$$

$$= \begin{vmatrix} l_1a_1 + l_2a_2 + l_3a_3 & l_1b_1 + l_2b_2 + l_3b_3 & l_1c_1 + l_2c_2 + l_3c_3 \\ m_1a_1 + m_2a_2 + m_3a_3 & m_1b_1 + m_2b_2 + m_3b_3 & m_1c_1 + m_2c_2 + m_3c_3 \\ n_1a_1 + n_2a_2 + n_3a_3 & n_1a_1 + n_2b_2 + n_3b_3 & n_1c_1 + n_2c_2 + n_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix} = \text{R.H.S}$$

Hence proved that L.H.S= R.H.S

Exercise#2.3

Q#01: If $\vec{a}= 3\hat{\imath} - \hat{\jmath} + 5\hat{k}$; $\vec{b}= 4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ & $\vec{c}= 2\hat{\imath} + 5\hat{\jmath} + \hat{k}$ .Find $\vec{a} \cdot (\vec{b} \times \vec{c})$ and also verify that $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$.

Solution:

$$\therefore\ \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix} = 3\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= 3(3+10) + 1(4+4) + 5(20-6)$$

$$= 3(13) + 8 + 5(14)$$

$$= 39 + 8 + 70$$

$$= 117 \text{--------------(i)}$$

$$\therefore\ \vec{b} \cdot (\vec{c} \times \vec{a}) = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix} = 4\begin{vmatrix} 5 & 1 \\ -1 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix}$$

$$= 4(25+1) - 3(10-3) - 2(-2-15)$$

$$= 4(26) - 3(7) - 2(-17)$$

$$= 104 - 21 + 34$$

$$= 117 \text{--------------(ii)}$$

$$\therefore\ \vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix} = 2\begin{vmatrix} -1 & 5 \\ 3 & -2 \end{vmatrix} - 5\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 2(2-15) - 5(-6-20) + 1(9+4)$$

$$= 3(-13) - 5(-26) + 1(13)$$

$$= 39 + 130 + 13$$

$$= 117 \text{--------------(iii)}$$

From (i),(ii) & (iii) hence verify that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}).$$

Q#02:Find the value of $\hat{\imath} \cdot \hat{\jmath} \times \hat{k}$.

Solution: $\hat{\imath} \cdot (\hat{\jmath} \times \hat{k})$

$= \hat{\imath} \cdot \hat{\imath}$ $\quad\quad \because \hat{\jmath} \times \hat{k} = \hat{\imath}$

$= 1$ $\quad\quad \therefore \quad \hat{\imath} \cdot \hat{\imath} = 1$

Q#03:Prove that $[\hat{i}-\hat{j} \quad \hat{j}-\hat{k} \quad \hat{k}-\hat{i}]=0$

Solution: L.H.S=$[\hat{i}-\hat{j} \quad \hat{j}-\hat{k} \quad \hat{k}-\hat{i}]$

$$=\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

$= 1(1-0)+1(0-1)+0$

$= 1(1)+1(-1) = 1-1$

= R.H.S

Hence proved L.H.S= R.H.S

Q#04:Find the volume of parallelepiped whose edges are represented by

(i) $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$; $\vec{b}=\hat{i}-\hat{j}+2\hat{k}$ *and* $\vec{c}=2\hat{i}+\hat{j}-\hat{k}$

Solution: Given $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$; $\vec{b}=\hat{i}-\hat{j}+2\hat{k}$ *and* $\vec{c}=2\hat{i}+\hat{j}-\hat{k}$

We know that

Volume of the parallelepiped = V = $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$

$$= 2\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= 2(1-2)+3(-1-4)+1(1+2)$

$= 2(-1)+3(-5)+1(3) = -2-15+3$

$= -14$

$V = 14$ *unit Cube* (V is always positive)

(ii) $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$; $\vec{b}=2\hat{i}+\hat{j}-\hat{k}$ *and* $\vec{c}=\hat{j}+\hat{k}$

Solution: Given $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$; $\vec{b}=2\hat{i}+\hat{j}-\hat{k}$ *and* $\vec{c}=\hat{j}+\hat{k}$

We know that

Volume of the parallelepiped = V = $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$

$$= 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$= 1(1+1)+2(2-0)+3(2+0)$

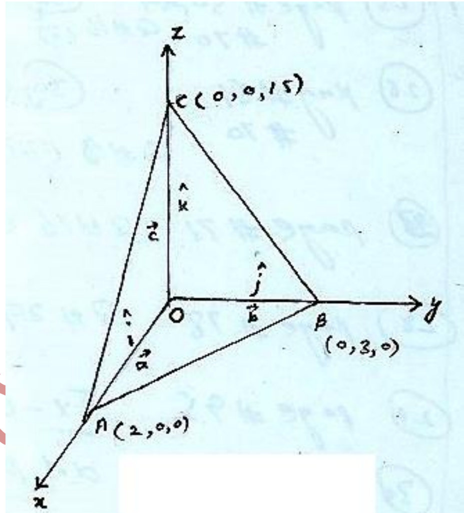
$= 1(2)+2(2)+3(2) = 2+4+6$

$V = 12$ *unit cube*

Q#05: Find the volume of tetrahedron bounded by the coordinate planes and the plane is $15x + 10y + 2z - 30 = 0$

Solution: Given equation of the plane is

$$15x + 10y + 2z - 30 = 0 \text{ ----------(i)}$$



When x=0 & y=0 then

$2z - 30 = 0 \implies 2z=30 \implies \boxed{z=15}$

When y=0 & z=0 then

$15x - 30 = 0 \implies 15x=30 \implies \boxed{x=2}$

When z=0 & x=0 then

$10y - 30 = 0 \implies 10y=30 \implies \boxed{y=3}$

Let $\overrightarrow{OA} = \vec{a} = 2\hat{\imath}$; $\overrightarrow{OB} = \vec{b} = 3\hat{\jmath}$ & $\overrightarrow{OC} = \vec{c} = 15\hat{k}$

along x,y,z-axis as shown in the figure.

We know that

Volume of the tetrahedron $= V = \frac{1}{6}[\vec{a}\cdot(\vec{b}\times\vec{c})] = \frac{1}{6}[2\hat{\imath}\cdot(3\hat{\jmath}\times 15\hat{k})]$

$= \frac{90}{6}[\hat{\imath}\cdot(\hat{\jmath}\times\hat{k})]$

$=15\ [\hat{\imath}\cdot\hat{\imath}]$ $\qquad \therefore \hat{\jmath}\times\hat{k} = \hat{\imath}$

$V = 15$ *unit cube* $\qquad \therefore \hat{\imath}\cdot\hat{\imath}=1$

Q#06: Show that the vectors $\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$; $-2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ & $\hat{\imath} - 3\hat{\jmath} + 5\hat{k}$ are coplanar.

Solution: Let $\vec{a}=\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$; $\vec{b} = -2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ *and* $\vec{c} = \hat{\imath} - 3\hat{\jmath} + 5\hat{k}$

For coplanar vectors, we have to prove $\vec{a}\cdot(\vec{b}\times\vec{c})=0$

$$\therefore \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & -4 \\ -3 & 5 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -4 \\ 1 & 5 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix}$$

$$= 1(15-12) + 2(-10+4) + 3(6-3)$$

$$= 1(3) + 2(-6) + 3(3)$$

$$= 3 - 12 + 9$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$$

Hence proved that the given vectors are coplanar.

Q#07: Show that the vectors $5\vec{a}+6\vec{b}+7\vec{c}$, $7\vec{a}-8\vec{b}+9\vec{c}$ & $3\vec{a}+20\vec{b}+5\vec{c}$ are coplanar, where $\vec{a}$, $\vec{b}$ & $\vec{c}$ are any vectors.

Solution: Let $\vec{u}=5\vec{a}+6\vec{b}+7\vec{c}$; $\vec{v}=7\vec{a}-8\vec{b}+9\vec{c}$ *and* $\vec{w}=3\vec{a}+20\vec{b}+5\vec{c}$

For coplanar vectors, we have to prove $\vec{a}\cdot(\vec{b}\times\vec{c})=0$

$$\therefore\ \vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix}5 & 6 & 7\\ 7 & -8 & 9\\ 3 & 20 & 5\end{vmatrix}=5\begin{vmatrix}-8 & 9\\ 20 & 5\end{vmatrix}-6\begin{vmatrix}7 & 9\\ 3 & 5\end{vmatrix}+7\begin{vmatrix}7 & -8\\ 3 & 20\end{vmatrix}$$

$$=5(-40-180)-6(35-27)+7(140+24)$$

$$=5(-220)-6(8)+7(164)$$

$$=-1100-48+1148$$

$$\vec{u}\cdot(\vec{v}\times\vec{w})=0$$

Hence proved that the given vectors are coplanar.

Q#08: Show that the four points $2\vec{a}+3\vec{b}-\vec{c}$; $\vec{a}-2\vec{b}+3\vec{c}$; $3\vec{a}+4\vec{b}-2\vec{c}$ & $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar.

Solution: Let $A(2\vec{a}+3\vec{b}-\vec{c})$;$B(\vec{a}-2\vec{b}+3\vec{c})$;$C(3\vec{a}+4\vec{b}-2\vec{c})$& $D(\vec{a}-6\vec{b}+6\vec{c})$ are given four points. If these four points are coplanar then we have to prove coplanar condition

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})=0$$

$\therefore\ \overrightarrow{AB}=B(\vec{a}-2\vec{b}+3\vec{c})-A(2\vec{a}+3\vec{b}-\vec{c})=\vec{a}-2\vec{b}+3\vec{c}-2\vec{a}-3\vec{b}+\vec{c}=-\vec{a}-5\vec{b}+4\vec{c}$

$\therefore\ \overrightarrow{AC}=C(3\vec{a}+4\vec{b}-2\vec{c})-A(2\vec{a}+3\vec{b}-\vec{c})=3\vec{a}+4\vec{b}-2\vec{c}-2\vec{a}-3\vec{b}+\vec{c}=\vec{a}+\vec{b}-\vec{c}$

$\therefore\ \overrightarrow{AD}=D(\vec{a}-6\vec{b}+6\vec{c})-A(2\vec{a}+3\vec{b}-\vec{c})=\vec{a}-6\vec{b}+6\vec{c}-2\vec{a}-3\vec{b}+\vec{c}=-\vec{a}-9\vec{b}+7\vec{c}$

Now $$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})=\begin{vmatrix}-1 & -5 & 4\\ 1 & 1 & -1\\ -1 & -9 & 7\end{vmatrix}=-1\begin{vmatrix}1 & -1\\ -9 & 7\end{vmatrix}-(-5)\begin{vmatrix}1 & -1\\ -1 & 7\end{vmatrix}+4\begin{vmatrix}1 & 1\\ -1 & -9\end{vmatrix}$$

$$=-1(7-9)+5(7-1)+4(-9+1)$$

$$=-1(-2)+5(6)+4(-8)$$

$$=2+30-32$$

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})=0$$

This shows that the given four points are coplanar.

Q#09: **(i) If** $\vec{a}\cdot\vec{r}=0;\ \vec{b}\cdot\vec{r}=0\ \&\ \vec{c}\cdot\vec{r}=0$ **then prove that** $\vec{a}\cdot(\vec{b}\times\vec{c})=0$.

(ii) If $\vec{a}\cdot\vec{n}=0;\ \vec{b}\cdot\vec{n}=0\ \&\ \vec{c}\cdot\vec{n}=0$ **then prove that** $\vec{a}\cdot(\vec{b}\times\vec{c})=0$.

Solution: Let $\vec{r}=x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k}$ &

$\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$

$\vec{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$

$\vec{c}=c_1\hat{\imath}+c_2\hat{\jmath}+c_3\hat{k}$

According to given conditions.

$\therefore\ \vec{a}\cdot\vec{r}=0$

$(a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k})\cdot(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k})=0$

$a_1x+a_2y+a_3z=0$ -------------(i)

$\therefore\ \vec{b}\cdot\vec{r}=0$

$(b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k})\cdot(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k})=0$

$b_1x+b_2y+b_3z=0$-------------(ii)

$\therefore\ \vec{c}\cdot\vec{r}=0$

$(c_1\hat{\imath}+c_2\hat{\jmath}+c_3\hat{k})\cdot(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k})=0$

$c_1x+c_2y+c_3z=0$-------------(iii)

Eliminating x, y & z from equation (i) ,(ii) & (iii)

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\vec{a}\cdot(\vec{b}\times\vec{c})=0$$

Note: Part (ii) is similar to part (ii) only $\vec{r}$ replace by $\vec{n}$.

Q#10; (i) is similar to example #04:

(ii)prove that $\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{c}\times\vec{a})+\vec{c}\cdot(\vec{a}\times\vec{b})=3\,[\vec{a}\cdot(\vec{b}\times\vec{c})]$

Solution: L.H.S= $\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{c}\times\vec{a})+\vec{c}\cdot(\vec{a}\times\vec{b})$

Because $\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})$

Therefore $=\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{c})$

$=3[\vec{a}\cdot(\vec{b}\times\vec{c})]$= R.H.S

Hence proved L.H.S= R.H.S

Q#11: Find λ such that the vectors $\hat{i}+\hat{j}-\hat{k}$; $\hat{i}-2\hat{j}+\hat{k}$ and $\vec{c}=\lambda\hat{i}+\hat{j}-\lambda\hat{k}$ are coplanar .

Solution: Let $\vec{a}=\hat{\imath}+\hat{\jmath}-\hat{k}$; $\vec{b}=\hat{\imath}-2\hat{\jmath}+\hat{k}$ *and* $\vec{c}=\lambda\hat{\imath}+\hat{\jmath}-\lambda\hat{k}$

According to given condition , the vectors are coplanar. Therefore

$$\vec{a}\cdot(\vec{b}\times\vec{c})=0$$

$$\begin{vmatrix}1 & 1 & -1\\ 1 & -2 & 1\\ \lambda & 1 & -\lambda\end{vmatrix}=0$$

$$1\begin{vmatrix}-2 & 1\\ 1 & -\lambda\end{vmatrix}-1\begin{vmatrix}1 & 1\\ \lambda & -\lambda\end{vmatrix}+(-1)\begin{vmatrix}1 & -2\\ \lambda & 1\end{vmatrix}=0$$

$$1(2\lambda-1)-1(-\lambda-\lambda)-1(1+2\lambda)=0$$

$$2\lambda-1+2\lambda-1-2\lambda=0$$

$$2\lambda-2=0$$

$$2\lambda=2$$

$$\boxed{\lambda=1}$$

Q#12: If $\vec{a},\vec{b}$ & $\vec{c}$ are three non coplanar vectors , show that

$$\vec{r}=\frac{[\vec{b}\ \vec{c}\ \vec{r}]\,\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}+\frac{[\vec{c}\ \vec{a}\ \vec{r}]\,\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}+\frac{[\vec{a}\ \vec{b}\ \vec{r}]\,\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$ ***for any vector $\vec{r}$.***

Solution: Let $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$ -------- (A)

Taking dot product of equation (A) with $(\vec{b}\times\vec{c})$

$$(\vec{b}\times\vec{c})\cdot\vec{r}=(\vec{b}\times\vec{c})\cdot(x\vec{a}+y\vec{b}+z\vec{c})$$

$$[\vec{b}\ \ \vec{c}\ \ \vec{r}]=x\vec{a}\cdot(\vec{b}\times\vec{c})+y\vec{b}\cdot(\vec{b}\times\vec{c})+z\vec{c}\cdot(\vec{b}\times\vec{c})$$

$$[\vec{b}\ \ \vec{c}\ \ \vec{r}]=x\vec{a}\cdot(\vec{b}\times\vec{c})+0+0$$

$$[\vec{b}\ \ \vec{c}\ \ \vec{r}]=x[\vec{a}\ \ \vec{b}\ \ \vec{c}]$$

$$x=\frac{[\vec{b}\ \ \vec{c}\ \ \vec{r}]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}\ \text{--------------(i)}$$

Taking dot product of equation (i) with $(\vec{c}\times\vec{a})$

$$(\vec{c}\times\vec{a})\cdot\vec{r}=(\vec{c}\times\vec{a})\cdot(x\vec{a}+y\vec{b}+z\vec{c})$$

$$[\vec{c}\ \ \vec{a}\ \ \vec{r}]=x\vec{a}\cdot(\vec{c}\times\vec{a})+y\vec{b}\cdot(\vec{c}\times\vec{a})+z\vec{c}\cdot(\vec{c}\times\vec{a})$$

$$[\vec{c}\ \ \vec{a}\ \ \vec{r}]=0+y\vec{b}\cdot(\vec{c}\times\vec{a})+0$$

$$[\vec{c}\ \ \vec{a}\ \ \vec{r}]=y[\vec{a}\ \ \vec{b}\ \ \vec{c}]$$

$$y=\frac{[\vec{c}\ \ \vec{a}\ \ \vec{r}]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}\ \text{------------------(ii)}$$

Taking dot product of equation (i) with $(\vec{a} \times \vec{b})$

$$(\vec{a} \times \vec{b}) \cdot \vec{r} = (\vec{a} \times \vec{b}) \cdot (x\vec{a} + y\vec{b} + z\vec{c})$$

$$[\vec{a} \quad \vec{b} \quad \vec{r}] = x\vec{a} \cdot (\vec{a} \times \vec{b}) + y\vec{b} \cdot (\vec{a} \times \vec{b}) + z\vec{c} \cdot (\vec{a} \times \vec{b})$$

$$[\vec{a} \quad \vec{b} \quad \vec{r}] = 0 + 0 + z\vec{c} \cdot (\vec{a} \times \vec{b})$$

$$[\vec{a} \quad \vec{b} \quad \vec{r}] = z[\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$z = \frac{[\vec{a} \quad \vec{b} \quad \vec{r}]}{[\vec{a} \quad \vec{b} \quad \vec{c}]} \text{ --------------------(iii)}$$

Using value x, y & z in equation (A)

$$\vec{r} = \frac{[\vec{b} \quad \vec{c} \quad \vec{r}]\,\vec{a}}{[\vec{a} \quad \vec{b} \quad \vec{c}]} + \frac{[\vec{c} \quad \vec{a} \quad \vec{r}]\,\vec{b}}{[\vec{a} \quad \vec{b} \quad \vec{c}]} + \frac{[\vec{a} \quad \vec{b} \quad \vec{r}]\,\vec{c}}{[\vec{a} \quad \vec{b} \quad \vec{c}]}$$

Hence Proved.

Q#13:Solve the following system of equation. $a_r\, x + b_r y + c_r\, z = d_r$ **where r= 1,2,**3.

Solution: Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

$\vec{d} = d_1\hat{\imath} + d_2\hat{\jmath} + d_3\hat{k}$

Given equation $a_r\, x + b_r y + c_r\, z = d_r$

$Put\ r = 1$ $a_1\, x + b_1 y + c_1\, z = d_1$

Multiplying equation with $\hat{\imath}$ *unit vector* .

$$a_1\, x\,\hat{\imath} + b_1 y\,\hat{\imath} + c_1\, z\,\hat{\imath} = d_1\hat{\imath} \text{ -----------------(i)}$$

$Put\ r = 2$ $a_2\, x + b_2 y + c_2\, z = d_2$

Multiplying equation with $\hat{\jmath}$ *unit vector*

$$a_2\, x\hat{\jmath} + b_2 y\,\hat{\jmath} + c_2\, z\,\hat{\jmath} = d_2\,\hat{\jmath} \text{ -----------------(ii)}$$

$Put\ r = 3$ $a_3\, x + b_3 y + c_3\, z = d_3$

Multiplying equation with $\hat{k}$ *unit vector*

$$a_3\, x\hat{k} + b_3 y\,\hat{k} + c_3\, z\,\hat{k} = d_3\hat{k} \text{ -----------------(iii)}$$

Adding equation (i) , (ii) & (iii)

$$(a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k})\,x + (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})\,y + (c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k})\,z = d_1\hat{\imath} + d_2\hat{\jmath} + d_3\hat{k}$$

$$\vec{a}\,x + \vec{b}\,y + \vec{c}\,z = \vec{d} \text{ ---------------------------(vi)}$$

Taking dot product of equation (iv) with $(\vec{b} \times \vec{c})$

$$(\vec{a}\, x + \vec{b}\,y + \vec{c}z).(\vec{b} \times \vec{c}) = \vec{d}.(\vec{b} \times \vec{c})$$

$$\vec{a}.(\vec{b} \times \vec{c})\; x + \vec{b}.(\vec{b} \times \vec{c})\; y + \vec{c}.(\vec{b} \times \vec{c})\; z = \vec{d}.(\vec{b} \times \vec{c})$$

$$\vec{a}.(\vec{b} \times \vec{c})\quad x + 0 + 0 = \vec{d}.(\vec{b} \times \vec{c})$$

$$[\vec{a}\;\; \vec{b}\;\; \vec{c}]\; x = [\vec{d}\;\; \vec{b}\;\; \vec{c}]$$

$$\boxed{x = \frac{[\vec{d}\;\; \vec{b}\;\; \vec{c}]}{[\vec{a}\;\; \vec{b}\;\; \vec{c}]}}$$

Taking dot product of equation (iv) with $(\vec{c} \times \vec{a})$

$$(\vec{a}\, x + \vec{b}\,y + \vec{c}z).(\vec{b} \times \vec{c}) = \vec{d}.(\vec{b} \times \vec{c})$$

$$\vec{a}.(\vec{c} \times \vec{a})\; x + \vec{b}.(\vec{c} \times \vec{a})\; y + \vec{c}.(\vec{c} \times \vec{a})\; z = \vec{d}.(\vec{c} \times \vec{a})$$

$$0 + \vec{a}.(\vec{b} \times \vec{c})\; y + 0 = \vec{d}.(\vec{c} \times \vec{a})$$

$$[\vec{a}\;\; \vec{b}\;\; \vec{c}]\; y = [\vec{d}\;\; \vec{c}\;\; \vec{a}]$$

$$\boxed{y = \frac{[\vec{d}\;\; \vec{c}\;\; \vec{a}]}{[\vec{a}\;\; \vec{b}\;\; \vec{c}]}}$$

Similarly

Taking dot product of equation (iv) with $(\vec{a} \times \vec{b})$

$$\boxed{z = \frac{[\vec{d}\;\; \vec{a}\;\; \vec{b}]}{[\vec{a}\;\; \vec{b}\;\; \vec{c}]}}$$

$$\text{Solution set} = \left\{\left(\frac{[\vec{d}\;\; \vec{b}\;\; \vec{c}]}{[\vec{a}\;\; \vec{b}\;\; \vec{c}]}, \frac{[\vec{d}\;\; \vec{c}\;\; \vec{a}]}{[\vec{a}\;\; \vec{b}\;\; \vec{c}]}, \frac{[\vec{d}\;\; \vec{a}\;\; \vec{b}]}{[\vec{a}\;\; \vec{b}\;\; \vec{c}]}\right)\right\}$$

Q#14: Solve the simultaneous equation $\vec{r} \times \vec{a} = \vec{a} \times \vec{b}$; where $\vec{r} \cdot \vec{c} = 0$ & $\vec{a} \cdot \vec{c} \neq 0$.

Solution: Given conditions

$$\vec{r} \cdot \vec{c} = 0 \quad \& \quad \vec{a} \cdot \vec{c} \neq 0$$

Given simultaneous equation

$$\vec{r} \times \vec{a} = \vec{a} \times \vec{b}$$

$$\vec{r} \times \vec{a} - \vec{a} \times \vec{b} = 0$$

$$\vec{r} \times \vec{a} + \vec{b} \times \vec{a} = 0 \qquad\qquad -\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$$

$$(\vec{r} + \vec{b}) \times \vec{a} = 0$$

This condition hold when

$$\vec{r} + \vec{b} = t\,\vec{a} \qquad\qquad t\,\vec{a} \times \vec{a} = 0$$

$$\vec{r} = t\,\vec{a} - \vec{b} \text{ ----------------------------(i)} \qquad t \text{ is a scalar number}$$

Taking dot product of equation (i) with $\vec{c}$ vector

$$\vec{r} \cdot \vec{c} = (t\,\vec{a} - \vec{b}) \cdot \vec{c}$$

$$\vec{r} \cdot \vec{c} = t\,\vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c}$$

$$0 = t\,\vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c} \qquad \text{Because} \quad \vec{r} \cdot \vec{c} = 0 \quad \& \quad \vec{a} \cdot \vec{c} \neq 0$$

$$t\,(\vec{a} \cdot \vec{c}) = \vec{b} \cdot \vec{c}$$

$$t = \frac{\vec{b} \cdot \vec{c}}{\vec{a} \cdot \vec{c}}$$

using value of t in equation (i)

$$\vec{r} = \frac{\vec{b} \cdot \vec{c}}{\vec{a} \cdot \vec{c}}\,\vec{a} - \vec{b}$$

This is the required solution.

VECTOR TRIPLE PRODUCT:

If $\vec{a}, \vec{b}$ & $\vec{c}$ be any three vectors, then $\vec{a} \times (\vec{b} \times \vec{c})$ is called scalar triple product of $\vec{a}, \vec{b}$ & $\vec{c}$.

Theorem:04:Prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Proof: Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$; $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ & $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \hat{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = (b_2c_3 - b_3c_2)\hat{\imath} - (b_1c_3 - b_3c_1)\hat{\jmath} + (b_1c_2 - b_2c_1)\hat{k}$$

$$= (b_2c_3 - b_3c_2)\hat{\imath} + (b_3c_1 - b_1c_3)\hat{\jmath} + (b_1c_2 - b_2c_1)\hat{k}$$

$$\text{L.H.S} = \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= \hat{\imath}\begin{vmatrix} a_2 & a_3 \\ b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_2c_3 - b_3c_2 & b_1c_2 - b_2c_1 \end{vmatrix} + \hat{\imath}\begin{vmatrix} a_1 & a_2 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 \end{vmatrix}$$

$$= \{(a_2b_1c_2 - a_2b_2c_1) - (a_3b_3c_1 - a_3b_1c_3)\}\hat{\imath} - \{(a_1b_1c_2 - a_1b_2c_1) - (a_3b_2c_3 - a_3b_3c_2)\}\hat{\jmath}$$
$$+\{(a_1b_3c_1 - a_1b_1c_3) - (a_2b_2c_3 - a_2b_3c_2)\}\hat{k}$$

$$= \{a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3\}\hat{\imath} - \{a_1b_1c_2 - a_1b_2c_1 - a_3b_2c_3 + a_3b_3c_2\}\hat{\jmath}$$
$$+\{a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2\}\hat{k}$$

$$= \{a_2b_1c_2 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1\}\hat{\imath} - \{a_1b_1c_2 + a_3b_3c_2 - a_1b_2c_1 - a_3b_2c_3\}\hat{\jmath}$$
$$+\{a_1b_3c_1 + a_2b_3c_2 - a_1b_1c_3 - a_2b_2c_3\}\hat{k}$$

$$= \{a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1\}\hat{\imath}$$
$$- \{a_1b_1c_2 + a_2b_2c_2 + a_3b_3c_2 - a_1b_2c_1 - a_2b_2c_2 - a_3b_2c_3\}\hat{\jmath}$$
$$+\{a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3\}\hat{k}$$

$$= \{b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3\}\hat{\imath}$$
$$- \{c_2(a_1b_1 + a_2b_2 + a_3b_3) - b_2(a_1c_1 + a_2c_2 + a_3c_3\}\hat{\jmath}$$
$$+\{b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 - a_2b_2 - a_3b_3)\}\hat{k}$$

$$= \{b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3)\}\hat{\imath}$$
$$+ \{b_2(a_1c_1 + a_2c_2 + a_3c_3 - c_2(a_1b_1 + a_2b_2 + a_3b_3)\}\hat{\jmath}$$
$$+\{b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 - a_2b_2 - a_3b_3)\}\hat{k}$$

$$= b_1(a_1c_1 + a_2c_2 + a_3c_3)\hat{\imath} - c_1(a_1b_1 + a_2b_2 + a_3b_3)\,\hat{\imath}$$
$$+b_2(a_1c_1 + a_2c_2 + a_3c_3\hat{\jmath} - c_2(a_1b_1 + a_2b_2 + a_3b_3)\,\hat{\jmath}$$
$$+b_3(a_1c_1 + a_2c_2 + a_3c_3)\hat{k} - c_3(a_1b_1 - a_2b_2 - a_3b_3)\,\hat{k}$$

$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k})$

$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ = R.H.S Hence proved.

Example#05: Prove that $[\vec{a} \quad \vec{b} \quad \vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Solution: Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

& $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

L.H.S $= [\vec{a} \quad \vec{b} \quad \vec{c}]^2$

$= [\vec{a} \quad \vec{b} \quad \vec{c}]\,[\vec{a} \quad \vec{b} \quad \vec{c}]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ ∴ Taking transpose of 2nd determinant .

$$= \begin{vmatrix} a_1a_1 + a_2a_2 + a_3a_3 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1b_1 + b_2b_2 + b_3b_3 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1a_1 + c_2b_2 + c_3b_3 & c_1c_1 + c_2c_2 + c_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} = \text{R.H.S}$$

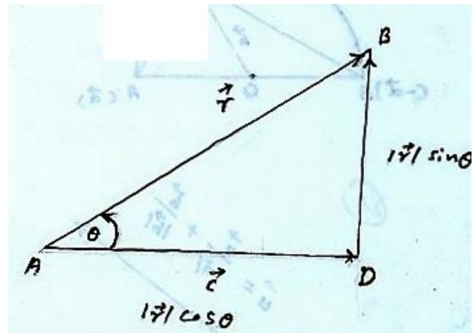
Hence proved that L.H.S= R.H.S

Example#06: Prove that the component of a vector $\vec{r}$ is parallel and perpendicular to $\vec{c}$ in the plane of $\vec{c}$ & $\vec{r}$ are . (i) $\frac{\vec{c}.\vec{r}}{\vec{c}.\vec{c}}\ \vec{c}$ (ii) $\frac{\vec{c}\times(\vec{r}\times\vec{c})}{\vec{c}.\vec{c}}$

Solution: Consider two vectors $\vec{c}$ & $\vec{r}$.

Vector $\vec{c}$ taken along x-axis and vector $\vec{r}$ makes an angle θ with $\vec{c}$ vector.

Draw a perpendicular BD on $\vec{c}$ vector as shown in the figure.



From figure :

$$\overrightarrow{AB} = \vec{r} \quad \& \quad \overrightarrow{AD} = \vec{c}$$

(i) Component of $\vec{r}$ parallel to $\vec{c}$.

Let $\overrightarrow{AD}$ be vector whose magnitude is parallel to the direction of $\vec{c}$ vector.

As

$\overrightarrow{AD} = |\overrightarrow{AD}|\ \hat{c}$ ∴ x-coponent of $\vec{r} = |\overrightarrow{AD}| = |\vec{r}|\cos\theta$ & $\hat{c} = \frac{\vec{c}}{|\vec{c}|}$

$$\overrightarrow{AD} = |\vec{r}|\cos\theta\ \frac{\vec{c}}{|\vec{c}|}$$

$= \frac{|\vec{c}|\ |\vec{r}|\cos\theta\ \vec{c}}{|\vec{c}|\ |\vec{c}|}$ ∴ Multiplying & dividing by $|\vec{c}|$

$= \frac{(\vec{c}.\vec{r})\ \vec{c}}{|\vec{c}|^2}$ ∴ $\vec{c}.\vec{r} = |\vec{c}|\ |\vec{r}|\cos\theta$

$\overrightarrow{AD} = \frac{\vec{c}.\vec{r}}{\vec{c}.\vec{c}}\ \vec{c}$ ∴ $\vec{c}.\vec{c} = |\vec{c}|^2$

(ii) Component of $\vec{r}$ perpedicular to $\vec{c}$.

By using Head To Tail Rule.

$$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$$

$$\overrightarrow{DB} = \overrightarrow{AB} - \overrightarrow{AD}$$

$$\overrightarrow{DB} = \vec{r} - \frac{\vec{c}.\vec{r}}{\vec{c}.\vec{c}}\ \vec{c}$$

$$= \frac{(\vec{c}.\vec{c})\vec{r} - (\vec{c}.\vec{r})\vec{c}}{\vec{c}.\vec{c}}$$

$$\overrightarrow{DB} = \frac{\vec{c}\times(\vec{r}\times\vec{c})}{\vec{c}.\vec{c}}$$

Hence proved

Exercise#2.4

Q#01: Find

(i) $\hat{\imath} \times (\hat{\jmath} \times \hat{k})$

$\hat{\imath} \times (\hat{\jmath} \times \hat{k})$

$= \hat{\imath} \times \hat{\imath}$ $\qquad \therefore \ \hat{\jmath} \times \hat{k} = \hat{\imath}$

$= 0$ $\qquad \therefore \ \hat{\imath} \times \hat{\imath} = 0$

(ii) $\hat{\jmath} \times (\hat{k} \times \hat{\jmath})$

$\hat{\jmath} \times (\hat{k} \times \hat{\jmath})$

$= \hat{\jmath} \times (-\hat{\imath})$ $\qquad \therefore \ \hat{k} \times \hat{\jmath} = -\hat{\imath}$

$= -\hat{\jmath} \times \hat{\imath}$ $\qquad \therefore \ -\hat{\jmath} \times \hat{\imath} = \hat{\imath} \times \hat{\jmath}$

$= \hat{\imath} \times \hat{\jmath}$

$= \hat{k}$ $\qquad \therefore \ \hat{\imath} \times \hat{\jmath} = \hat{k}$

(iii) $(\hat{\imath} \times \hat{k}) \times \hat{\imath}$

$= -\hat{\jmath} \times \hat{\imath}$ $\qquad \therefore \ \hat{\imath} \times \hat{k} = -\hat{\jmath}$

$= \hat{\imath} \times \hat{\jmath}$ $\qquad \therefore \ -\hat{\jmath} \times \hat{\imath} = \hat{\imath} \times \hat{\jmath}$

$= \hat{k}$ $\qquad \therefore \ \hat{\imath} \times \hat{\jmath} = \hat{k}$

Q#02: Evaluate $\vec{a} \times (\vec{b} \times \vec{c})$. If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$; $\vec{b} = -\hat{\imath} + \hat{\jmath} + \hat{k}$ *and* $\vec{c} = 4\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$.

Solution: Given $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$; $\vec{b} = -\hat{\imath} + \hat{\jmath} + \hat{k}$ *and* $\vec{c} = 4\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

We know that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$= \{(2\hat{\imath} + 3\hat{\jmath} - 5\hat{k}).(4\hat{\imath} + 2\hat{\jmath} + 6\hat{k})\}\vec{b} - \{(2\hat{\imath} + 3\hat{\jmath} - 5\hat{k}).(-\hat{\imath} + \hat{\jmath} + \hat{k})\}\vec{c}$$

$$= \{8 + 6 - 30\}\vec{b} - \{-2 + 3 - 5\}\vec{c}$$

$$= (-16)(-\hat{\imath} + \hat{\jmath} + \hat{k}) - (-4)(4\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$$

$$= (-16)(-\hat{\imath} + \hat{\jmath} + \hat{k}) + 4(4\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$$

$$= 16\hat{\imath} - 16\hat{\jmath} - 16\hat{k} + 16\hat{\imath} + 8\hat{\jmath} + 24\hat{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = 32\hat{\imath} - 8\hat{\jmath} + 8\hat{k}$$

Q#03: Verify the formula $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

If $\vec{a} = \hat{\imath} + \hat{\jmath}$; $\vec{b} = -\hat{\imath} + 2\hat{k}$ *and* $\vec{c} = \hat{\jmath} + \hat{k}$

Solution: $\vec{a} = \hat{\imath} + \hat{\jmath} + 0\hat{k}$; $\vec{b} = -\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$ *and* $\vec{c} = 0\hat{\imath} + \hat{\jmath} + \hat{k}$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = \hat{\imath}(0-2) - \hat{\jmath}(-1-0) + \hat{k}(-1-0)$$

$$= -2\hat{\imath} + \hat{\jmath} - \hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ -2 & 1 & -1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}$$

$$= \hat{\imath}(-1-0) - \hat{\jmath}(-1-0) + \hat{k}(1+2)$$

$$= -\hat{\imath} + \hat{\jmath} + 3\hat{k} \text{ ---------------(i)}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = \{(\hat{\imath} + \hat{\jmath} + 0\hat{k}).(0\hat{\imath} + \hat{\jmath} + \hat{k})\}\vec{b} - \{(\hat{\imath} + \hat{\jmath} + 0\hat{k}).(-\hat{\imath} + 0\hat{\jmath} + 2\hat{k})\}\vec{c}$$

$$= \{0 + 1 + 0\}\vec{b} - \{-1 + 0 + 0\}\vec{c}$$

$$= (1)(-\hat{\imath} + 0\hat{\jmath} + 2\hat{k}) - (-1)(0\hat{\imath} + \hat{\jmath} + \hat{k})$$

$$= (1)(-\hat{\imath} + 0\hat{\jmath} + 2\hat{k}) + 1\,(0\hat{\imath} + \hat{\jmath} + \hat{k})$$

$$= -\hat{\imath} + 0\hat{\jmath} + \hat{k} + 0\,\hat{\imath} + \hat{\jmath} + \hat{k}$$

$$= -2\hat{\imath} + \hat{\jmath} + 3\hat{k} \text{ ---------------(ii)}$$

From (i) & (ii) hence verified that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Q#04: Prove that $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{c}$

Solution: L.H.S $= (\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})$

Let $\vec{b}\times\vec{c} = \vec{r}$

$$= \vec{r}\times(\vec{c}\times\vec{a})$$

$$= (\vec{r}.\vec{a})\vec{c} - (\vec{r}.\vec{c})\vec{a}$$

$$= \{(\vec{b}\times\vec{c}).\vec{a}\}\vec{c} - \{(\vec{b}\times\vec{c}).\vec{c}\}\vec{a}$$

$$= \{(\vec{a}\times\vec{b}).\vec{c}\}\vec{c} - \{0\}\vec{a}$$

$$= \{(\vec{a}\times\vec{b}).\vec{c}\}\vec{c} - 0$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{c} = \text{R.H.S}$$

Hence proved

L.H.S = R.H.S

Q#05: (i) Example#04: show that $[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = [\vec{a} \quad \vec{b} \quad \vec{c}]^2$

Solution: we have $[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = (\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$ ------(i) Let $(\vec{b}\times\vec{c}) = \vec{d}$

$$[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = (\vec{a}\times\vec{b})\cdot\{\vec{d}\times(\vec{c}\times\vec{a})\}$$

$$= (\vec{a}\times\vec{b})\cdot\{(\vec{d}\cdot\vec{a})\vec{c} - (\vec{d}\cdot\vec{c})\vec{a}\}$$

$$= (\vec{a}\times\vec{b})\cdot[\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c} - \{(\vec{b}\times\vec{c})\cdot\vec{c}\}\vec{a}]$$

$$= (\vec{a}\times\vec{b})\cdot[\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c} - \{0\}\vec{a}]$$

$$= (\vec{a}\times\vec{b})\cdot[\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c} - 0]$$

$$= (\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c}$$

$$= [(\vec{a}\times\vec{b})\cdot\vec{c}][(\vec{a}\times\vec{b})\cdot\vec{c}]$$

$$[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = [\vec{a} \quad \vec{b} \quad \vec{c}]^2$$ Hence proved

Q#05: (ii)Example #02: Show that $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$.

Solution: We know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}$$

$$\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$$

Now $\quad L.H.S = \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$

$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$$

$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b} = 0 = R.H.S$$

Hence Proved $\quad L.H.S = R.H.S$

Q#05(iii) E#03: If $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$; $\vec{b} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$ *and* $\vec{c} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ then find $\vec{a}\times(\vec{b}\times\vec{c})$.

Solution: Given $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$; $\vec{b} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$ *and* $\vec{c} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

We know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$= \{(\hat{\imath} - 2\hat{\jmath} + \hat{k})\cdot(\hat{\imath} + 2\hat{\jmath} - \hat{k})\}\vec{b} - \{(\hat{\imath} - 2\hat{\jmath} + \hat{k})\cdot(2\hat{\imath} + \hat{\jmath} + \hat{k})\}\vec{c}$$

$$= \{1 - 4 - 1\}\vec{b} - \{2 - 2 + 1\}\vec{c}$$

$$= (-4)(2\hat{\imath} + \hat{\jmath} + \hat{k}) - (1)(\hat{\imath} + 2\hat{\jmath} - \hat{k})$$

$$= -8\hat{\imath} - 4\hat{\jmath} - 4\hat{k} - \hat{\imath} - 2\hat{\jmath} + \hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = -9\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$$

Q#06: Determine the components of $\vec{a} \times (\vec{b} \times \vec{c})$ along the directions of $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$.

Solution: Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$; $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ & $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

We know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$$= (\vec{a}.\vec{c})(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) - (\vec{a}.\vec{b})(c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k})$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = [(\vec{a}.\vec{c})b_1 - (\vec{a}.\vec{b})c_1]\,\hat{\imath} + [(\vec{a}.\vec{c})b_2 - (\vec{a}.\vec{b})c_2]\,\hat{\jmath} + [(\vec{a}.\vec{c})b_3 - (\vec{a}.\vec{b})c_3]\,\hat{k}$$

Hence $[(\vec{a}.\vec{c})b_1 - (\vec{a}.\vec{b})c_1]$, $[(\vec{a}.\vec{c})b_2 - (\vec{a}.\vec{b})c_2]$ & $[(\vec{a}.\vec{c})b_3 - (\vec{a}.\vec{b})c_3]$ are the components of $\vec{a} \times (\vec{b} \times \vec{c})$ along the directions of $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$.

Q#07: Establish the identity $\vec{a} = \frac{1}{2}[\hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k})]$

Solution: Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

R.H.S $= \frac{1}{2}[\hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k})]$

$= \frac{1}{2}[\hat{\imath} \times \{(a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times \hat{\imath}\} + \hat{\jmath} \times \{(a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times \hat{\jmath}\}$
$+ \hat{k} \times \{(a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times \hat{k}\}]$

$= \frac{1}{2}[\hat{\imath} \times \{a_1(\hat{\imath} \times \hat{\imath}) + a_2(\hat{\jmath} \times \hat{\imath}) + a_3(\hat{k} \times \hat{\imath})\} + \hat{\jmath} \times \{a_1(\hat{\imath} \times \hat{\jmath}) + a_2(\hat{\jmath} \times \hat{\jmath})$
$+ a_3(\hat{k} \times \hat{\jmath})\} + \hat{k} \times \{a_1(\hat{\imath} \times \hat{k}) + a_2(\hat{\jmath} \times \hat{k}) + a_3(\hat{k} \times \hat{k})\}]$

$= \frac{1}{2}[\hat{\imath} \times \{a_1(0) + a_2(-\hat{k}) + a_3(\hat{\jmath})\} + \hat{\jmath} \times \{a_1(\hat{k}) + a_2(0) + a_3(-\hat{\imath})\}$
$+ \hat{k} \times \{a_1(-\hat{\jmath}) + a_2(\hat{\imath}) + a_3(0)\}]$

$= \frac{1}{2}[\hat{\imath} \times \{0 - a_2\hat{k} + a_3\hat{\jmath}\} + \hat{\jmath} \times \{a_1\hat{k} + 0 - a_3\hat{\imath}\} + \hat{k} \times \{-a_1\hat{\jmath} + a_2\hat{\imath} + 0\}]$

$= \frac{1}{2}[\hat{\imath} \times \{-a_2\hat{k} + a_3\hat{\jmath}\} + \hat{\jmath} \times \{a_1\hat{k} - a_3\hat{\imath}\} + \hat{k} \times \{-a_1\hat{\jmath} + a_2\hat{\imath}\}]$

$= \frac{1}{2}[-a_2(\hat{\imath} \times \hat{k}) + a_3(\hat{\imath} \times \hat{\jmath}) + a_1(\hat{\jmath} \times \hat{k}) - a_3(\hat{\jmath} \times \hat{\imath}) - a_1(\hat{k} \times \hat{\jmath}) + a_2(\hat{k} \times \hat{\imath})]$

$= \frac{1}{2}[-a_2(-\hat{\jmath}) + a_3(\hat{k}) + a_1(\hat{\imath}) - a_3(-\hat{k}) - a_1(-\hat{\imath}) + a_2(\hat{\jmath})]$

$= \frac{1}{2}[a_2\hat{\jmath} + a_3\hat{k} + a_1\hat{\imath} + a_3\hat{k} + a_1\hat{\imath} + a_2\hat{\jmath}]$

$= \frac{1}{2}[2a_1\hat{\imath} + 2a_2\hat{\jmath} + 2a_3\hat{k}]$

$= \frac{1}{2}.2[a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}]$

$= a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$= \vec{a}$ = L.H.S

Hence proved. L.H.S= R.H.S

Q#08: Show that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ if & only if , the vector $\vec{a}$ & $\vec{c}$ are collinear.

Solution:

Given

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

We have to prove vector $\vec{a}$ & $\vec{c}$ are collinear.

Let

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

By using cancellation property

$$(\vec{b}.\vec{c})\vec{a} = (\vec{a}.\vec{b})\vec{c}$$

Let $\quad \vec{b}.\vec{c} = \lambda \quad \& \quad \vec{a}.\vec{b} = \mu$

$$\lambda\,\vec{a} = \mu\,\vec{c}$$

$$\vec{a} = \frac{\mu}{\lambda}\vec{c}$$

This shows that vector $\vec{a}$ & $\vec{c}$ are collinear.

Conversely, suppose that vector $\vec{a}$ & $\vec{c}$ are collinear.

We have to prove $\quad (\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$

As

$$\vec{a} = \frac{\mu}{\lambda}\vec{c}$$

$$\lambda\,\vec{a} = \mu\,\vec{c}$$

Put $\quad \vec{b}.\vec{c} = \lambda \quad \& \quad \vec{a}.\vec{b} = \mu$

$$(\vec{b}.\vec{c})\vec{a} = (\vec{a}.\vec{b})\vec{c}$$

$$-(\vec{b}.\vec{c})\vec{a} = -(\vec{a}.\vec{b})\vec{c}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

Hence proved.

Q#09: (i) If $\hat{a}$, $\hat{b}$ & $\hat{c}$ be three unit vectors such that $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$

Find the angle which $\hat{a}$ makes with $\hat{b}$ & $\hat{c}$, $\hat{b}$ & $\hat{c}$ being non- parallel.

Solution: Given condition

$$\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$$

$$(\hat{a}.\hat{c})\hat{b} - (\hat{a}.\hat{b})\hat{c} = \frac{1}{2}\hat{b} - 0\,\hat{c} \qquad \text{here } |\hat{a}| = |\hat{b}| = |\hat{c}| = 1$$

Comparing coefficients of $\hat{b}$ & $\hat{c}$.

$$\hat{a}.\hat{c} = \frac{1}{2}$$

$$|\hat{a}|\ |\hat{c}|\ cos\ \alpha = \frac{1}{2}$$

$$cos\ \alpha = \frac{1}{2}$$

$$\alpha = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\alpha = 60^0}$$

$\hat{a}$ makes Angle $\alpha = 60^0$ with $\hat{c}$.

$$\hat{a}.\hat{b} = 0$$

$$|\hat{a}|\ |\hat{b}|\ cos\ \beta = 0$$

$$cos\ \beta = 0$$

$$\beta = \cos^{-1}(0)$$

$$\boxed{\beta = 90^0}$$

$\hat{a}$ makes Angle $\beta = 90^0$ with $\hat{b}$.

Q#09: (ii) **If $\hat{a}$, $\hat{b}$ & $\hat{c}$ be three unit vectors such that $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b} - \frac{\sqrt{3}}{2}\hat{c}$**

Find the angle which $\hat{a}$ makes with $\hat{b}$ & $\hat{c}$, $\hat{b}$ & $\hat{c}$ being non- parallel.

Solution: Given condition

$$\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b} - \frac{\sqrt{3}}{2}\hat{c}$$

$$(\hat{a}.\hat{c})\hat{b} - (\hat{a}.\hat{b})\hat{c} = \frac{1}{2}\hat{b} - \frac{\sqrt{3}}{2}\hat{c} \qquad \text{here } |\hat{a}| = |\hat{b}| = |\hat{c}| = 1$$

Comparing coefficients of $\hat{b}$ & $\hat{c}$.

$$\hat{a}.\hat{c} = \frac{1}{2}$$

$$|\hat{a}|\ |\hat{c}|\ cos\ \alpha = \frac{1}{2}$$

$$cos\ \alpha = \frac{1}{2}$$

$$\alpha = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\alpha = 60^0}$$

$\hat{a}$ makes Angle $\alpha = 60^0$ with $\hat{c}$.

$$\hat{a}.\hat{b} = \frac{\sqrt{3}}{2}$$

$$|\hat{a}|\ |\hat{b}|\ cos\ \beta = \frac{\sqrt{3}}{2}$$

$$cos\ \beta = \frac{\sqrt{3}}{2}$$

$$\beta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\boxed{\beta = 30^0}$$

$\hat{a}$ makes Angle $\beta = 30^0$ with $\hat{b}$.

Q#10: Prove that $[(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})].\vec{d} = (\vec{a}.\vec{d})[\vec{a}\quad\vec{b}\quad\vec{c}]$

Solution: L.H.S $= [(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})].\vec{d}$

Let $\vec{a}\times\vec{b} = \vec{r}$

$$= [\vec{r}\times(\vec{a}\times\vec{c})].\vec{d}$$

$$= [(\vec{r}.\vec{c})\vec{a} - (\vec{r}.\vec{a})\vec{c}].\vec{d}$$

$$= [(\vec{a}\times\vec{b}).\vec{c}\}\vec{a} - \{(\vec{a}\times\vec{b}).\vec{a}\}\vec{c}].\vec{d}$$

$$= [(\vec{a}\times\vec{b}).\vec{c}\}\vec{a} - \{0\}\vec{c}].\vec{d}$$

$$=[\{(\vec{a}\times\vec{b}).\vec{c}\}\vec{a} - 0].\vec{d}$$

$$=[\vec{a}\quad\vec{b}\quad\vec{c}](\vec{a}.\vec{d})$$

$$= (\vec{a}.\vec{d})[\vec{a}\quad\vec{b}\quad\vec{c}] = \text{R.H.S}$$

Hence proved

L.H.S = R.H.S

Q#11: Example#02: Show that $\vec{a}\times(\vec{b}\times\vec{c})$ **,** $\vec{b}\times(\vec{c}\times\vec{a})$ **&** $\vec{c}\times(\vec{a}\times\vec{b})$ **are coplanar.**

Solution:

Let

$$\vec{r_1} = \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$\vec{r_2} = \vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a}$$

$$\vec{r_3} = \vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}$$

Adding $\vec{r_1}$, $\vec{r_2}$ & $\vec{r_3}$

$$\vec{r_1} + \vec{r_2} + \vec{r_3} = \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$$

$$=(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}$$

$$=(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{a}.\vec{b})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{b}.\vec{c})\vec{a} - (\vec{a}.\vec{c})\vec{b}$$

$$\vec{r_1} + \vec{r_2} + \vec{r_3} = 0$$

This shows that $\vec{r_1}$, $\vec{r_2}$ & $\vec{r_3}$ are coplanar.

SCALAR & VECTOR PRODUCT OF FOUR VECTORS:

Scalar Product of Four Vectors:

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ be any four vectors, then the scalar product of these four vectors is define as

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$$

Vector Product of Four Vectors:

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ be any four vectors, then the vector product of these four vectors is define as

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}.(\vec{c}\times\vec{d})]\vec{b} - [\vec{b}.(\vec{c}\times\vec{d})]\vec{a}$$

Reciprocal Vectors:

If $\vec{a}, \vec{b}$ & $\vec{c}$ be any three non coplanar vectors so that $[\vec{a}\ \ \vec{b}\ \ \vec{c}]\neq 0$, then the three reciprocal vectors $\vec{a}\,'$, $\vec{b}\,'$ & $\vec{c}\,'$ will be define as

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} \quad ; \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} \quad ; \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$$

Theorem: I Prove that $\vec{\mathbf{a}}.\vec{\mathbf{a}}\,' = \vec{\mathbf{b}}.\vec{\mathbf{b}}\,' = \vec{\mathbf{c}}.\vec{\mathbf{c}}\,' = 1$

Proof:

We know that $\quad \vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} \quad ; \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} \quad ; \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$

$$\vec{\mathbf{a}}.\vec{\mathbf{a}}\,' = \vec{\mathbf{a}}.\frac{\vec{\mathbf{b}}\times\vec{\mathbf{c}}}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \frac{\vec{\mathbf{a}}.(\vec{\mathbf{b}}\times\vec{\mathbf{c}})}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \frac{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \mathbf{1} \text{ ------(i)}$$

$$\vec{\mathbf{b}}.\vec{\mathbf{b}}\,' = \vec{\mathbf{b}}.\frac{\vec{\mathbf{c}}\times\vec{\mathbf{a}}}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \frac{\vec{\mathbf{b}}.(\vec{\mathbf{c}}\times\vec{\mathbf{a}})}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \frac{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \mathbf{1} \text{------(ii)}$$

$$\vec{\mathbf{c}}.\vec{\mathbf{c}}\,' = \vec{\mathbf{c}}.\frac{\vec{\mathbf{a}}\times\vec{\mathbf{b}}}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \frac{\vec{\mathbf{c}}.(\vec{\mathbf{a}}\times\vec{\mathbf{b}})}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \frac{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]}{[\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\ \ \vec{\mathbf{c}}]} = \mathbf{1} \text{ -----(iii)}$$

From (i) ,(ii) & (iii) Hence proved

$$\vec{a}.\vec{a}\,' = \vec{b}.\vec{b}\,' = \vec{c}.\vec{c}\,' = 1$$

Theorem: II

Prove that $\vec{a}\cdot\vec{b}\,' = \vec{a}\cdot\vec{c}\,' = \vec{b}\cdot\vec{a}\,' = \vec{b}\cdot\vec{c}\,' = \vec{c}\cdot\vec{a}\,' = \vec{c}\cdot\vec{b}\,' = 0$

Proof:

We know that

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$\vec{a}\cdot\vec{b}\,' = \vec{a}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{\vec{a}\cdot(\vec{c}\times\vec{a})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{[\vec{a}\ \ \vec{c}\ \ \vec{a}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{0}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = 0 \text{ -------(i)}$$

$$\vec{a}\cdot\vec{c}\,' = \vec{a}\cdot\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{\vec{a}\cdot(\vec{a}\times\vec{b})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{[\vec{a}\ \ \vec{a}\ \ \vec{b}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{0}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = 0 \text{ -------(ii)}$$

$$\vec{b}\cdot\vec{a}\,' = \vec{b}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{\vec{b}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{[\vec{b}\ \ \vec{b}\ \ \vec{c}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{0}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = 0 \text{ ------(iii)}$$

$$\vec{b}\cdot\vec{c}\,' = \vec{b}\cdot\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{\vec{b}\cdot(\vec{a}\times\vec{b})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{[\vec{b}\ \ \vec{a}\ \ \vec{b}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{0}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = 0 \text{ -----(iv)}$$

$$\vec{c}\cdot\vec{a}\,' = \vec{c}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{\vec{c}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{[\vec{c}\ \ \vec{b}\ \ \vec{c}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{0}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = 0 \text{------ -(v)}$$

$$\vec{c}\cdot\vec{b}\,' = \vec{c}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{\vec{c}\cdot(\vec{c}\times\vec{a})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{[\vec{c}\ \ \vec{c}\ \ \vec{a}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{0}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = 0 \text{ -------(vi)}$$

From (i) ,(ii) ,(iii),(iv),(v) & (vi) Hence proved

$$\vec{a}\cdot\vec{b}\,' = \vec{a}\cdot\vec{c}\,' = \vec{b}\cdot\vec{a}\,' = \vec{b}\cdot\vec{c}\,' = \vec{c}\cdot\vec{a}\,' = \vec{c}\cdot\vec{b}\,' = 0$$

Theorem: III

Prove that $[\vec{a}\ \ \vec{b}\ \ \vec{c}\,][\,\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,'\,] = 1$

Proof: We know that

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\quad ;\ \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\quad ;\ \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

Let

$$[\,\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,'\,] = \vec{a}\,'\ .(\vec{b}\,'\times\vec{c}\,')$$

$$= \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\ .\left(\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\times\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\right)$$

$$= \frac{(\vec{b}\times\vec{c})\ .\ \{\,(\,\vec{c}\times\vec{a}\,)\times(\vec{a}\times\vec{b}\,)\}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{(\vec{b}\times\vec{c})\ .\ \{\,[\vec{c}\,.(\vec{a}\times\vec{b}\,)]\,\vec{a} - [\,\vec{a}\,.(\vec{a}\times\vec{b}\,)]\,\vec{c}\}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{(\vec{b}\times\vec{c})\ .\left\{\,[\vec{c}\,.\ (\vec{a}\times\vec{b}\,)]\vec{a} - [\,0]\vec{c}\right\}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{(\vec{b}\times\vec{c})\ .\left\{[\vec{c}\,.\ (\vec{a}\times\vec{b}\,)]\vec{a} - 0\right\}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{(\vec{b}\times\vec{c}\,).\vec{a}\ [\vec{c}\,.\ (\vec{a}\times\vec{b}\,)]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]\ [\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$[\,\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,'\,] = \frac{1}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$[\vec{a}\ \ \vec{b}\ \ \vec{c}\,][\,\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,'\,] = 1$$

Hence proved.

Example #01:Find the area of a triangle by using result

$$(\vec{b}\times\vec{c}).(\vec{b}\times\vec{c}) = (\vec{c}\times\vec{a}).(\vec{c}\times\vec{a})$$

Solution: Let a triangle ABC. If $\vec{a}$, $\vec{b}$ & $\vec{c}$ be the vectors along the sides of triangle.

We know that

$$\text{Area of triangle} = \frac{1}{2}|\vec{a}\times\vec{b}| \text{------------(i)}$$

Given condition

$$(\vec{b}\times\vec{c}).(\vec{b}\times\vec{c}) = (\vec{c}\times\vec{a}).(\vec{c}\times\vec{a})$$

$$|\vec{b}\times\vec{c}|^2 = |\vec{c}\times\vec{a}|^2$$

Taking square-root on both sides

$$|\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$$

Multiplying both sides by $|\vec{a}\times\vec{b}|$

$$|\vec{a}\times\vec{b}||\vec{b}\times\vec{c}| = |\vec{a}\times\vec{b}||\vec{c}\times\vec{a}|$$

$$|\vec{a}\times\vec{b}|\,|\vec{b}||\vec{c}|\sin\alpha = |\vec{a}||\vec{b}|\sin\gamma\ \ |\vec{c}||\vec{a}|\sin\beta$$

$$|\vec{a}\times\vec{b}| = \frac{|\vec{a}||\vec{b}|\sin\gamma\ \ |\vec{c}||\vec{a}|\sin\beta}{|\vec{b}||\vec{c}|\sin\alpha}$$

$$|\vec{a}\times\vec{b}| = \frac{|\vec{a}|^2\sin\gamma\ \ \sin\beta}{\sin\alpha}$$

Using in equation(i) $\quad \text{Area of triangle} = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{|\vec{a}|^2\sin\gamma\ \ \sin\beta}{2\sin\alpha}$

Example#03: **Prove that** $\vec{d}.[\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}] = (\vec{b}.\vec{d})[\vec{a}\ \ \vec{c}\ \ \vec{d}]$

Solution: L.H.S $= \vec{d}.[\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}]$

$= \vec{d}.[\vec{a}\times\{(\vec{b}.\vec{d})\vec{c} - (\vec{b}.\vec{c})\vec{d}\}]$

$= \vec{d}.[\vec{a}\times(\vec{b}.\vec{d})\vec{c} - \vec{a}\times(\vec{b}.\vec{c})\vec{d}]$

$= \vec{d}.[(\vec{a}\times\vec{c})(\vec{b}.\vec{d}) - (\vec{a}\times\vec{d})(\vec{b}.\vec{c})]$

$= [\vec{d}.(\vec{a}\times\vec{c})](\vec{b}.\vec{d}) - [\vec{d}.(\vec{a}\times\vec{d})](\vec{b}.\vec{c})$

$= \vec{d}.(\vec{a}\times\vec{c})](\vec{b}.\vec{d}) - 0 \qquad \because\ \vec{d}.(\vec{a}\times\vec{d}) = 0$

$= (\vec{b}.\vec{d})[\vec{a}\ \ \vec{c}\ \ \vec{d}] \qquad \because\ \vec{d}.(\vec{a}\times\vec{c}) = \vec{a}.(\vec{c}\times\vec{d})$

= R.H.S

Hence proved

L.H.S = R.H.S

Example#04:Prove that

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})=-2[\vec{b}.(\vec{c}\times\vec{d})]\vec{a}$$

Solution: We know that

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}.(\vec{c}\times\vec{d})]\vec{b}-[\vec{b}.(\vec{c}\times\vec{d})]\vec{a}$$

$$(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})=[\vec{a}.(\vec{d}\times\vec{b})]\vec{c}-[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}$$

$$(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})=[\vec{a}.(\vec{b}\times\vec{c})]\vec{d}-[\vec{d}.(\vec{b}\times\vec{c})]\vec{a}$$

L.H.S$=(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$

$=[\vec{a}.(\vec{c}\times\vec{d})]\vec{b}-[\vec{b}.(\vec{c}\times\vec{d})]\vec{a}+[\vec{a}.(\vec{d}\times\vec{b})]\vec{c}-[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}$

$+[\vec{a}.(\vec{b}\times\vec{c})]\vec{d}-[\vec{d}.(\vec{b}\times\vec{c})]\vec{a}$

$=[\vec{a}.\vec{b}(\vec{c}\times\vec{d})]-[\vec{b}.\vec{a}(\vec{c}\times\vec{d})]+[\vec{a}.(\vec{d}\times\vec{b})]\vec{c}-[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}$

$+[\vec{a}.(-\vec{c}\times\vec{b})]\vec{d}-[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}$

$=[(\vec{a}.\vec{b})(\vec{c}\times\vec{d})]-[(\vec{a}.\vec{b})(\vec{c}\times\vec{d})]+[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}-[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}$

$-[\vec{d}.(\vec{c}\times\vec{b})]\vec{a}-[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}$

$=[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}-[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}-[\vec{d}.(\vec{c}\times\vec{b})]\vec{a}-[\vec{c}.(\vec{d}\times\vec{b})]\vec{a}$

$=-2[\vec{b}.(\vec{c}\times\vec{d})]\vec{a}$ =R.H.S

Hence proved.

L.H.S = R.H.S

Example#05 : If the four vector $\vec{a},\vec{b},\vec{c}$ & $\vec{d}$ are coplanar , show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$

Solution: Let $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ & $\vec{b}$ in the plane .

Similarly , $\vec{c}\times\vec{d}$ is is perpendicular to both $\vec{c}$ & $\vec{d}$ in the plane.Then$(\vec{a}\times\vec{b})$ & $(\vec{c}\times\vec{d})$ both the normal of the same plane.

In this situation $(\vec{a}\times\vec{b})$ *is parallel to* $(\vec{c}\times\vec{d})$.

Therefore

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$

Hence proved.

Example#06: Find a set of vectors reciprocal to the set of $2\hat{i}+3\hat{j}-\hat{k}$; $\hat{i}-\hat{j}-2\hat{k}$ and $-\hat{i}+2\hat{j}+2\hat{k}$.

Solution: Let $\vec{a}=2\hat{\imath}+3\hat{\jmath}-\hat{k}$; $\vec{b}=\hat{\imath}-\hat{\jmath}-2\hat{k}$ *and* $\vec{c}=-\hat{\imath}+2\hat{\jmath}+2\hat{k}$

We know that reciprocal vector of $\vec{a},\vec{b},\vec{c}$ are

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$\therefore\ \vec{b}\times\vec{c} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & -1 & -2\\ -1 & 2 & 2\end{vmatrix} = \hat{\imath}\begin{vmatrix}-1 & -2\\ 2 & 2\end{vmatrix} - \hat{\jmath}\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix} + \hat{k}\begin{vmatrix}1 & -1\\ -1 & 2\end{vmatrix}$$

$$= \hat{\imath}(2+4) - \hat{\jmath}(2-2) + \hat{k}(2-1)$$

$$= 6\,\hat{\imath} + \hat{k}$$

$$\therefore\ \vec{c}\times\vec{a} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -1 & 2 & 2\\ 2 & 3 & -1\end{vmatrix} = \hat{\imath}\begin{vmatrix}2 & 2\\ 3 & -1\end{vmatrix} - \hat{\jmath}\begin{vmatrix}-1 & 2\\ 2 & -1\end{vmatrix} + \hat{k}\begin{vmatrix}-1 & 2\\ 2 & 3\end{vmatrix}$$

$$= \hat{\imath}(-2-6) - \hat{\jmath}(1-4) + \hat{k}(-3-4)$$

$$= -8\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$$

$$\therefore\ \vec{a}\times\vec{b} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & -1\\ 1 & -1 & -2\end{vmatrix} = \hat{\imath}\begin{vmatrix}3 & -1\\ -1 & -2\end{vmatrix} - \hat{\jmath}\begin{vmatrix}2 & -1\\ 1 & -2\end{vmatrix} + \hat{k}\begin{vmatrix}2 & 3\\ 1 & -1\end{vmatrix}$$

$$= \hat{\imath}(-6-1) - \hat{\jmath}(-4+1) + \hat{k}(-2-3)$$

$$= -7\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$$

$$\therefore [\vec{a}\ \ \vec{b}\ \ \vec{c}\,] = \begin{vmatrix}2 & 3 & -1\\ 1 & -1 & -2\\ -1 & 2 & 2\end{vmatrix} = 2\begin{vmatrix}-1 & -2\\ 2 & 2\end{vmatrix} - 3\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix} + (-1)\begin{vmatrix}1 & -1\\ -1 & 2\end{vmatrix}$$

$$= 2(-2+4) - 3(2-2) - 1(2-1)$$

$$= 2(2) - 3(0) - 1(1) = 4 - 0 - 1$$

$$= 3$$

Then

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{6\,\hat{\imath}+\hat{k}}{3}$$

$$\vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{-8\hat{\imath}+3\hat{\jmath}-7\hat{k}}{3}$$

$$\vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{-7\hat{\imath}+3\hat{\jmath}-5\hat{k}}{3}$$

Exercise#2.5

Q#01: $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c}).(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d}).(\vec{b}\times\vec{c})=0$

And also show that $\sin(\theta+\varphi).\sin(\theta-\varphi)=\sin^2\theta-\sin^2\varphi$

Solution:

L.H.S= $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c}).(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d}).(\vec{b}\times\vec{c})$

$=\begin{vmatrix}\vec{a}.\vec{c} & \vec{b}.\vec{c}\\ \vec{a}.\vec{d} & \vec{b}.\vec{d}\end{vmatrix}+\begin{vmatrix}\vec{a}.\vec{d} & \vec{c}.\vec{d}\\ \vec{a}.\vec{b} & \vec{c}.\vec{b}\end{vmatrix}+\begin{vmatrix}\vec{a}.\vec{b} & \vec{d}.\vec{b}\\ \vec{a}.\vec{c} & \vec{d}.\vec{c}\end{vmatrix}$

$=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{a}.\vec{d})(\vec{b}.\vec{c})+(\vec{a}.\vec{d})(\vec{c}.\vec{b})-(\vec{a}.\vec{b})(\vec{c}.\vec{d})+(\vec{a}.\vec{b})(\vec{d}.\vec{c})-(\vec{a}.\vec{c})(\vec{d}.\vec{b})$

$=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{a}.\vec{d})(\vec{b}.\vec{c})+(\vec{a}.\vec{d})(\vec{b}.\vec{c})-(\vec{a}.\vec{b})(\vec{c}.\vec{d})+(\vec{a}.\vec{b})(\vec{c}.\vec{d})-(\vec{a}.\vec{c})(\vec{b}.\vec{d})$

= 0 =R.H.S

Hence proved L.H.S= R.H.S

Now let

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c}).(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d}).(\vec{b}\times\vec{c})=0$$

$|\vec{a}||\vec{b}|\sin\theta|\vec{c}||\vec{d}|\sin\theta+|\vec{a}||\vec{c}|(-\sin\varphi)\,|\vec{d}||\vec{b}|\sin\varphi+|\vec{a}||\vec{d}|\sin(\theta+\varphi)|\vec{b}||\vec{c}|\{-\sin(\theta-\varphi)\}=0$

$|\vec{a}||\vec{b}||\vec{c}||\vec{d}|\,[\sin^2\theta-\sin^2\varphi-\sin(\theta+\varphi)\sin(\theta-\varphi)]=0$

$|\vec{a}||\vec{b}||\vec{c}||\vec{d}|\neq 0$ Then $\sin^2\theta-\sin^2\varphi-\sin(\theta+\varphi)\sin(\theta-\varphi)=0$

$$\sin^2\theta-\sin^2\varphi=\sin(\theta+\varphi)\sin(\theta-\varphi)$$

Hence proved $\sin(\theta+\varphi).\sin(\theta-\varphi)=\sin^2\theta-\sin^2\varphi$

Q#02: Expand $[\{\vec{a}\times(\vec{b}\times\vec{c})\}\times\vec{d}\,].\vec{e}$

Solution:

$[\{\vec{a}\times(\vec{b}\times\vec{c})\}\times\vec{d}\,].\vec{e}$

$=[\{(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}\}\times\vec{d}\,].\vec{e}$

$=[(\vec{a}.\vec{c})\vec{b}\times\vec{d}-(\vec{a}.\vec{b})\vec{c}\times\vec{d}\,].\vec{e}$

$=(\vec{a}.\vec{c})[(\vec{b}\times\vec{d}).\vec{e}]-(\vec{a}.\vec{b})[(\vec{c}\times\vec{d}).\vec{e}\,]$

Q#03:Prove that

(i) $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}) = [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{b}$

***Solution*:** L.H.S $= (\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})$

$= [\vec{a}\cdot(\vec{b}\times\vec{c})]\,\vec{b} - [\vec{b}\cdot(\vec{b}\times\vec{c})]\,\vec{a}$

$= [\vec{a}\cdot(\vec{b}\times\vec{c})]\,\vec{b} - (0)\,\vec{a}$ $\quad\therefore\ [\vec{b}\cdot(\vec{b}\times\vec{c})]=0$

$= [\vec{a}\cdot(\vec{b}\times\vec{c})]\,\vec{b}$

$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{b}$ = R.H.S

Hence proved. L.H.S = R.H.S

(ii) $[(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})]\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \ \vec{b}\ \ \vec{c}]^2$

***Solution*:** L.H.S $= [(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})]\cdot(\vec{b}\times\vec{c})$

$= [\{\vec{a}\cdot(\vec{a}\times\vec{c})\}\,\vec{b} - \{\vec{b}\cdot(\vec{a}\times\vec{c})\}\,\vec{a}]\cdot(\vec{b}\times\vec{c})$

$= [\{0\}\,\vec{b} - \{\vec{b}\cdot(\vec{a}\times\vec{c})\}\,\vec{a}]\cdot(\vec{b}\times\vec{c})$ $\quad\therefore\ \vec{a}\cdot(\vec{a}\times\vec{c}) = 0$

$= [-\{\vec{b}\cdot(-\vec{c}\times\vec{a})\}\,\vec{a}]\cdot(\vec{b}\times\vec{c})$ $\quad\therefore\ \vec{a}\times\vec{c} = -\vec{c}\times\vec{a}$

$= [\{\vec{b}\cdot(\vec{c}\times\vec{a})\}\,\vec{a}]\cdot(\vec{b}\times\vec{c})$

$= [\vec{b}\cdot(\vec{c}\times\vec{a})]\,[\vec{a}\cdot(\vec{b}\times\vec{c})]$ $\quad\therefore\ [\vec{b}\cdot(\vec{c}\times\vec{a})] = [\vec{a}\cdot(\vec{b}\times\vec{c})]$

$= [\vec{a}\cdot(\vec{b}\times\vec{c})]\,[\vec{a}\cdot(\vec{b}\times\vec{c})]$

$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]^2$ =R.H.S

Hence proved. L.H.S = R.H.S

(iii) $[\{(\vec{b}\times\vec{c})\times\vec{a}\}\times\vec{a}]\cdot\vec{b} = [\vec{a}\ \ \vec{b}\ \ \vec{c}](\vec{a}\cdot\vec{b})$

***Solution*:** L.H.S $= [\{(\vec{b}\times\vec{c})\times\vec{a}\}\times\vec{a}]\cdot\vec{b}$

$= [\{(\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}\}\times\vec{a}]\cdot\vec{b}$

$= [(\vec{a}\cdot\vec{c})\,\vec{b}\times\vec{a} - (\vec{a}\cdot\vec{b})\,\vec{c}\times\vec{a}]\cdot\vec{b}$

$= (\vec{a}\cdot\vec{c})\,[(\vec{b}\times\vec{a})\cdot\vec{b}] - (\vec{a}\cdot\vec{b})[(\vec{c}\times\vec{a})\cdot\vec{b}]$

$= (\vec{a}\cdot\vec{c})\,[0] - (\vec{a}\cdot\vec{b})[(\vec{c}\times\vec{a})\cdot\vec{b}]$

$= -(\vec{a}\cdot\vec{b})[(-\vec{a}\times\vec{c})\cdot\vec{b}]$

$= (\vec{a}\cdot\vec{b})[(\vec{a}\times\vec{c})\cdot\vec{b}]$

$= [\vec{a}\ \ \vec{b}\ \ \vec{c}](\vec{a}\cdot\vec{b})$ = R.H.S

Hence proved L.H.S=R.H.S

Q#04: Expand $[\{\vec{a}\times(\vec{b}\times\vec{c})\}\times\vec{d}\ ]\times\vec{e}$

Solution: $[\{\vec{a}\times(\vec{b}\times\vec{c})\}\times\vec{d}\ ]\times\vec{e}$

$= [\{(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}\}\times\vec{d}\ ]\times\vec{e}$

$= [(\vec{a}.\vec{c})\vec{b}\times\vec{d} - (\vec{a}.\vec{b})\vec{c}\times\vec{d}\ ] . \vec{e}$

$= (\vec{a}.\vec{c})[(\vec{b}\times\vec{d})\times\vec{e}\,] - (\vec{a}.\vec{b})[(\vec{c}\times\vec{d})\times\vec{e}\,]$

$= (\vec{a}.\vec{c})[(\vec{b}.\vec{e})\vec{d} - (\vec{d}.\vec{e})\vec{b}] - (\vec{a}.\vec{b})[(\vec{c}.\vec{e})\vec{d} - (\vec{d}.\vec{e})\vec{c}\,]$

$= (\vec{a}.\vec{c})(\vec{b}.\vec{e})\vec{d} - (\vec{a}.\vec{c})(\vec{d}.\vec{e})\vec{b} - (\vec{a}.\vec{b})(\vec{c}.\vec{e})\vec{d} - (\vec{a}.\vec{b})(\vec{d}.\vec{e})\vec{c}$

Q#05: Prove that $2[(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})] = \begin{vmatrix} -\vec{a} & -\vec{b} & \vec{c} & \vec{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_2 & b_3 & c_3 & d_3 \end{vmatrix}$

Solution: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$; $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ & $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

$$\text{L.H.S} = \begin{vmatrix} -\vec{a} & -\vec{b} & \vec{c} & \vec{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_2 & b_3 & c_3 & d_3 \end{vmatrix} = -\vec{a}\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_2 \end{vmatrix} - (-\vec{b})\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_2 \end{vmatrix} + \vec{c}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_2 \end{vmatrix} - \vec{d}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_2 \end{vmatrix}$$

Taking transpose of each determinant

$$= -\vec{a}\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \vec{b}\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \vec{c}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} - \vec{d}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= -\vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})] + \vec{b}\,[\vec{a}.(\vec{c}\times\vec{d})] + \vec{c}\,[\vec{a}.(\vec{b}\times\vec{d})] - \vec{b}\,[\vec{a}.(\vec{b}\times\vec{c})]$

$= \vec{b}\,[\vec{a}.(\vec{c}\times\vec{d})] - \vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})] + \vec{c}\,[\vec{a}.(\vec{b}\times\vec{d})] - \vec{b}\,[\vec{a}.(\vec{b}\times\vec{c})]$

$= \vec{b}\,[\vec{a}.(\vec{c}\times\vec{d})] - \vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})] + \vec{c}\,[\vec{b}.(\vec{a}\times\vec{d})] - \vec{a}\,[\vec{d}.(\vec{b}\times\vec{c})]$

$= \vec{b}\,[\vec{a}.(\vec{c}\times\vec{d})] - \vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})] + \vec{b}\,[\vec{c}.(\vec{a}\times\vec{d})] - \vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})]$

$= \vec{b}\,[\vec{a}.(\vec{c}\times\vec{d})] - \vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})] + \vec{b}\,[\vec{a}.(\vec{c}\times\vec{d})] - \vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})]$

$= 2\vec{b}\,[\vec{a}.(\vec{c}\times\vec{d})] - 2\vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})]$ ------------(i)

$\text{L.H.S} = 2[(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})]$

$= 2\{[\vec{a}.(\vec{c}\times\vec{d})]\vec{b} - [\vec{b}.(\vec{c}\times\vec{d})]\vec{a}\ \}$

$= 2\vec{b}\,[\vec{a}.(\vec{c}\times\vec{d})] - 2\vec{a}\,[\vec{b}.(\vec{c}\times\vec{d})]$ ------------(ii)

From (i) &(ii) Hence Proved L.H.S = R.H.S

Q#06: Prove that $\left[\vec{a}\cdot(\vec{b}\times\vec{c})\right](\vec{p}\times\vec{q}) = \begin{vmatrix} \vec{p}\cdot\vec{a} & \vec{q}\cdot\vec{a} & \vec{a} \\ \vec{p}\cdot\vec{b} & \vec{q}\cdot\vec{b} & \vec{b} \\ \vec{p}\cdot\vec{c} & \vec{q}\cdot\vec{c} & \vec{c} \end{vmatrix}$

Solution: Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$; $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$; $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

$\vec{p} = p_1\hat{\imath} + p_2\hat{\jmath} + p_3\hat{k}$ & $\vec{q} = q_1\hat{\imath} + q_2\hat{\jmath} + q_3\hat{k}$

$$\text{L.H.S} = \left[\vec{a}\cdot(\vec{b}\times\vec{c})\right](\vec{p}\times\vec{q})$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \hat{\imath} & p_1 & q_1 \\ \hat{\jmath} & p_2 & q_2 \\ \hat{k} & p_3 & q_3 \end{vmatrix}$$

$\therefore$ Taking transpose of 2nd determinant .

$$= \begin{vmatrix} a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} & a_1p_1 + a_2p_2 + a_3p_3 & a_1q_1 + a_2q_2 + a_3q_3 \\ b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k} & b_1p_1 + b_2p_2 + b_3p_3 & b_1q_1 + b_2q_2 + b_3q_3 \\ c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k} & c_1p_1 + c_2p_2 + c_3p_3 & c_1q_1 + c_2q_2 + c_3q_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a} & \vec{a}\cdot\vec{p} & \vec{a}\cdot\vec{q} \\ \vec{b} & \vec{b}\cdot\vec{p} & \vec{b}\cdot\vec{q} \\ \vec{c} & \vec{b}\cdot\vec{p} & \vec{c}\cdot\vec{q} \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a} & \vec{p}\cdot\vec{a} & \vec{q}\cdot\vec{a} \\ \vec{b} & \vec{p}\cdot\vec{b} & \vec{q}\cdot\vec{b} \\ \vec{c} & \vec{p}\cdot\vec{c} & \vec{q}\cdot\vec{c} \end{vmatrix}$$

$$= -\begin{vmatrix} \vec{q}\cdot\vec{a} & \vec{p}\cdot\vec{a} & \vec{a} \\ \vec{q}\cdot\vec{b} & \vec{p}\cdot\vec{b} & \vec{b} \\ \vec{q}\cdot\vec{c} & \vec{p}\cdot\vec{c} & \vec{c} \end{vmatrix}$$

$\therefore$Interchanging C_1 & C_2

$$= \begin{vmatrix} \vec{p}\cdot\vec{a} & \vec{q}\cdot\vec{a} & \vec{a} \\ \vec{p}\cdot\vec{b} & \vec{q}\cdot\vec{b} & \vec{b} \\ \vec{p}\cdot\vec{c} & \vec{q}\cdot\vec{c} & \vec{c} \end{vmatrix}$$

$\therefore$Interchanging C_1 & C_2

= R.H.S

Hence proved that

L.H.S= R.H.S

Q#07:Prove the identity $\vec{a} \times \{ \vec{a} \times (\vec{a} \times \vec{b})\} = (\vec{a}.\vec{a})(\vec{a} \times \vec{b})$

Solution: L.H.S $= \vec{a} \times \{ \vec{a} \times (\vec{a} \times \vec{b})\}$

$$= \vec{a} \times \{ (\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b} \}$$

$$= \vec{a} \times (\vec{a}.\vec{b})\vec{a} - \vec{a} \times (\vec{a}.\vec{a})\vec{b}$$

$$= (\vec{a}.\vec{b})(\vec{a} \times \vec{a}) - (\vec{a}.\vec{a})(\vec{a} \times \vec{b})$$

$$= (\vec{a}.\vec{b})(0) - (\vec{a}.\vec{a})(-\vec{b} \times \vec{a}) \qquad \because \vec{a} \times \vec{a} = 0 \ \ \& \ \ \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$= 0 + (\vec{a}.\vec{a})(\vec{b} \times \vec{a})$$

$$= (\vec{a}.\vec{a})(\vec{b} \times \vec{a}) = \text{R.H.S}$$

Q#08: Prove that

$$[(\vec{a} \times \vec{p}).\{(\vec{b} \times \vec{q})\times(\vec{c} \times \vec{r})\}] + [(\vec{a} \times \vec{q}).\{(\vec{b} \times \vec{r})\times(\vec{c} \times \vec{p})\}] + [(\vec{a} \times \vec{r}).\{(\vec{b} \times \vec{p})\times(\vec{c} \times \vec{q})\}] = 0$$

Solution: Let

$$[(\vec{a} \times \vec{p}).\{(\vec{b} \times \vec{q})\times(\vec{c} \times \vec{r})\}] = (\vec{a} \times \vec{p}).\{ [\vec{b}.(\vec{c} \times \vec{r})]\vec{q} - [\vec{q}.(\vec{c} \times \vec{r})]\vec{b} \}$$

$$= (\vec{a} \times \vec{p}).[\vec{b}.(\vec{c} \times \vec{r})]\vec{q} - (\vec{a} \times \vec{p}).[\vec{q}.(\vec{c} \times \vec{r})]\vec{b}$$

$$= [(\vec{a} \times \vec{p}).\vec{q}][\vec{b}.(\vec{c} \times \vec{r})] - [(\vec{a} \times \vec{p}).\vec{b}][\vec{q}.(\vec{c} \times \vec{r})]$$

$$= [(\vec{a} \times \vec{p}).\vec{b}][\vec{q}.(\vec{c} \times \vec{r})] - [(\vec{a} \times \vec{p}).\vec{b}][\vec{q}.(\vec{c} \times \vec{r})]$$

$$= 0 \text{--------------(i)}$$

$$[(\vec{a} \times \vec{q}).\{(\vec{b} \times \vec{r})\times(\vec{c} \times \vec{p})\}] = (\vec{a} \times \vec{q}).\{ [\vec{b}.(\vec{c} \times \vec{p})]\vec{r} - [\vec{r}.(\vec{c} \times \vec{p})]\vec{b} \}$$

$$= (\vec{a} \times \vec{q}).[\vec{b}.(\vec{c} \times \vec{p})]\vec{r} - (\vec{a} \times \vec{q}).[\vec{r}.(\vec{c} \times \vec{p})]\vec{b}$$

$$= [(\vec{a} \times \vec{q}).\vec{r}][\vec{b}.(\vec{c} \times \vec{p})] - [(\vec{a} \times \vec{q}).\vec{b}][\vec{r}.(\vec{c} \times \vec{p})]$$

$$= [(\vec{a} \times \vec{q}).\vec{r}][\vec{r}.(\vec{c} \times \vec{p})] - [(\vec{a} \times \vec{q}).\vec{b}][\vec{r}.(\vec{c} \times \vec{p})]$$

$$= 0 \text{--------------(ii)}$$

$$[(\vec{a} \times \vec{r}).\{(\vec{b} \times \vec{p})\times(\vec{c} \times \vec{q})\}] = (\vec{a} \times \vec{r}).\{ [\vec{b}.(\vec{c} \times \vec{q})]\vec{p} - [\vec{p}.(\vec{c} \times \vec{q})]\vec{b} \}$$

$$= (\vec{a} \times \vec{r}).[\vec{b}.(\vec{c} \times \vec{q})]\vec{p} - (\vec{a} \times \vec{r}).[\vec{p}.(\vec{c} \times \vec{q})]\vec{b}$$

$$= [(\vec{a} \times \vec{r}).\vec{p}][\vec{b}.(\vec{c} \times \vec{q})] - [(\vec{a} \times \vec{r}).\vec{b}][\vec{p}.(\vec{c} \times \vec{q})]$$

$$= [(\vec{a} \times \vec{r}).\vec{b}][\vec{p}.(\vec{c} \times \vec{q})] - [(\vec{a} \times \vec{r}).\vec{b}][\vec{p}.(\vec{c} \times \vec{q})]$$

$$= 0 \text{-------------(iii)}$$

Adding (i) ,(ii) & (iii)

$$[(\vec{a} \times \vec{p}).\{(\vec{b} \times \vec{q})\times(\vec{c} \times \vec{r})\}] + [(\vec{a} \times \vec{q}).\{(\vec{b} \times \vec{r})\times(\vec{c} \times \vec{p})\}] + [(\vec{a} \times \vec{r}).\{(\vec{b} \times \vec{p})\times(\vec{c} \times \vec{q})\}] = 0$$

Q#09: Establish the identity $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} + [\vec{c}\ \vec{a}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$

Solution:

Let $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{A}\times(\vec{c}\times\vec{d})$ Put $\vec{a}\times\vec{b} = \vec{A}$

$= (\vec{A}.\vec{d})\vec{c} - (\vec{A}.\vec{c})\vec{d}$

$= \{(\vec{a}\times\vec{b}).\vec{d}\}\vec{c} - \{(\vec{a}\times\vec{b}).\vec{c}\}\vec{d}$ ----------(i)

Let $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\times\vec{B}$ Put $\vec{c}\times\vec{d} = \vec{B}$

$= (\vec{a}.\vec{B})\vec{b} - (\vec{b}.\vec{B})\vec{a}$

$= \{\vec{a}.(\vec{c}\times\vec{d})\}\vec{b} - \{\vec{b}.(\vec{c}\times\vec{d})\}\vec{a}$ ------------(ii)

Comparing (i) & (ii)

$$\{(\vec{a}\times\vec{b}).\vec{d}\}\vec{c} - \{(\vec{a}\times\vec{b}).\vec{c}\}\vec{d} = \{\vec{a}.(\vec{c}\times\vec{d})\}\vec{b} - \{\vec{b}.(\vec{c}\times\vec{d})\}\vec{a}$$

$$\{(\vec{a}\times\vec{b}).\vec{d}\}\vec{c} - \{(\vec{a}\times\vec{b}).\vec{c}\}\vec{d} = \{\vec{a}.(-\vec{d}\times\vec{c})\}\vec{b} - \{\vec{b}.(\vec{c}\times\vec{d})\}\vec{a}$$

$$\{(\vec{a}\times\vec{b}).\vec{d}\}\vec{c} - \{(\vec{a}\times\vec{b}).\vec{c}\}\vec{d} = -\{\vec{a}.(\vec{d}\times\vec{c})\}\vec{b} - \{\vec{b}.(\vec{c}\times\vec{d})\}\vec{a}$$

$$-\{(\vec{a}\times\vec{b}).\vec{c}\}\vec{d} = -\{\vec{a}.(\vec{d}\times\vec{c})\}\vec{b} - \{\vec{b}.(\vec{c}\times\vec{d})\}\vec{a} - \{(\vec{a}\times\vec{b}).\vec{d}\}\vec{c}$$

$$\{(\vec{a}\times\vec{b}).\vec{c}\}\vec{d} = \{\vec{a}.(\vec{d}\times\vec{c})\}\vec{b} + \{\vec{b}.(\vec{c}\times\vec{d})\}\vec{a} + \{(\vec{a}\times\vec{b}).\vec{d}\}\vec{c}$$

$$[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} + [\vec{c}\ \vec{a}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$$

Q#10: Prove that $(\vec{a}\times\vec{b}).\{(\vec{a}\times\vec{c})\times\vec{d}\} = (\vec{a}.\vec{d})[\vec{a}\ \vec{b}\ \vec{c}]$

Solution:

L.H.S $= (\vec{a}\times\vec{b}).\{(\vec{a}\times\vec{c})\times\vec{d}\}$

$= (\vec{a}\times\vec{b}).\{(\vec{a}.\vec{d})\vec{c} - (\vec{c}.\vec{d})\vec{a}\}$

$= (\vec{a}\times\vec{b}).(\vec{a}.\vec{d})\vec{c} - (\vec{a}\times\vec{b}).(\vec{c}.\vec{d})\vec{a}$

$= [(\vec{a}\times\vec{b}).\vec{c}](\vec{a}.\vec{d}) - [(\vec{a}\times\vec{b}).\vec{a}](\vec{c}.\vec{d})$

$= [(\vec{a}\times\vec{b}).\vec{c}](\vec{a}.\vec{d}) - (0)(\vec{c}.\vec{d})$ $\therefore [(\vec{a}\times\vec{b}).\vec{a}] = 0$

$= (\vec{a}.\vec{d})[\vec{a}\ \vec{b}\ \vec{c}]$

= R.H.S

Hence proved L.H.S=R.H.S

Q#11:Prove that $\vec{a} \times [\vec{b} \times (\vec{c} \times \vec{d})] = (\vec{b}.\vec{d})(\vec{a} \times \vec{c}) - (\vec{b}.\vec{c})(\vec{a} \times \vec{d})$

Solution: L.H.S$= \vec{a} \times [\vec{b} \times (\vec{c} \times \vec{d})]$

$= \vec{a} \times [(\vec{b}.\vec{d})\vec{c} - (\vec{b}.\vec{c})\vec{d}]$

$= \vec{a} \times (\vec{b}.\vec{d})\vec{c} - \vec{a} \times (\vec{b}.\vec{c})\vec{d}$

$= (\vec{b}.\vec{d})(\vec{a} \times \vec{c}) - (\vec{b}.\vec{c})(\vec{a} \times \vec{d})$

=R.H.S

Q#12: **Find a set of vectors reciprocal to the set of** $-\hat{\imath} + \hat{\jmath} + \hat{k}$; $\hat{\imath} - \hat{\jmath} + \hat{k}$ *and* $\hat{\imath} + \hat{\jmath} + \hat{k}$.

Solution: Let $\vec{a} = -\hat{\imath} + \hat{\jmath} + \hat{k}$; $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$ *and* $\vec{c} = \hat{\imath} + \hat{\jmath} + \hat{k}$

We know that reciprocal vector of $\vec{a}, \vec{b}, \vec{c}$ are

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a} \quad \vec{b} \quad \vec{c}]} \quad ; \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a} \quad \vec{b} \quad \vec{c}]} \quad ; \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a} \quad \vec{b} \quad \vec{c}]}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = \hat{\imath}(-1-1) - \hat{\jmath}(1-1) + \hat{k}(1+1)$$

$= -2\hat{\imath} + 2\hat{k}\backslash$

$$\vec{c} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = \hat{\imath}(1-1) - \hat{\jmath}(1+1) + \hat{k}(1+1)$$

$= -2\hat{\jmath} + 2\hat{k}\backslash$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = \hat{\imath}(1+1) - \hat{\jmath}(-1-1) + \hat{k}(1-1)$$

$= 2\hat{\imath} + 2\hat{\jmath}$

$$[\vec{a} \quad \vec{b} \quad \vec{c}] = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$= -1(-1-1) - 1(1-1) + 1(1+1)$

$= -1(-2) - 1(0) + 1(2) = 2 - 0 + 2 = 4$

Then

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a} \quad \vec{b} \quad \vec{c}]} = \frac{-2\hat{\imath} + 2\hat{k}}{4} = \frac{2(-\hat{\imath} + \hat{k})}{4} = \frac{-\hat{\imath} + \hat{k}}{2}$$

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a} \quad \vec{b} \quad \vec{c}]} 3\ldots = \frac{-2\hat{\jmath} + 2\hat{k}}{4} = \frac{2(-\hat{\jmath} + \hat{k})}{4} = \frac{-\hat{\jmath} + \hat{k}}{2}$$

$$\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a} \quad \vec{b} \quad \vec{c}]} = \frac{2\hat{\imath} + 2\hat{\jmath}}{4} = \frac{2(\hat{\imath} + \hat{\jmath})}{4} = \frac{\hat{\imath} + \hat{\jmath}}{2}$$

Q#13: If $\vec{a}, \vec{b}, \vec{c}$ be the set of non-coplanar vectors and

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

Then prove that

$$\vec{a} = \frac{\vec{b}\,'\times\vec{c}\,'}{[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']} \quad ; \vec{b} = \frac{\vec{c}\,'\times\vec{a}\,'}{[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']} \quad ; \vec{c} = \frac{\vec{a}\,'\times\vec{b}\,'}{[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']}$$

Solution: Let $[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,'] = \vec{a}\,'\,.\,(\vec{b}\,'\times\vec{c}\,')$

$$= \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\,.\left(\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \times \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\right)$$

$$= \frac{(\vec{b}\times\vec{c})\,.\,\{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{(\vec{b}\times\vec{c})\,.\,\{[\vec{c}.(\vec{a}\times\vec{b})]\,\vec{a} - [\vec{a}.(\vec{a}\times\vec{b})]\,\vec{c}\}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{(\vec{b}\times\vec{c})\,.\,\{[\vec{c}\,.\,(\vec{a}\times\vec{b})]\vec{a} - [0]\vec{c}\}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{(\vec{b}\times\vec{c})\,.\,\{[\vec{c}\,.\,(\vec{a}\times\vec{b})]\vec{a} - 0\}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{(\vec{b}\times\vec{c}).\vec{a}\,[\vec{c}\,.\,(\vec{a}\times\vec{b})]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$= \frac{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]\,[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^3}$$

$$[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,'] = \frac{1}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \Rightarrow [\vec{a}\ \ \vec{b}\ \ \vec{c}\,] = \frac{1}{[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']} \quad \text{---------(i)}$$

Now

$$\vec{b}\,'\times\vec{c}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\times\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = \frac{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2} = \frac{[\vec{c}.(\vec{a}\times\vec{b})]\,\vec{a} - [\vec{a}.(\vec{a}\times\vec{b})]\,\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2} = \frac{[\vec{c}.(\vec{a}\times\vec{b})]\,\vec{a} - [0]\,\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}$$

$$= \frac{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]\,\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}$$

$$\vec{b}\,'\times\vec{c}\,' = \frac{\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$[\vec{b}\,'\times\vec{c}\,']\,[\vec{a}\ \ \vec{b}\ \ \vec{c}\,] = \vec{a} \quad \Rightarrow \quad [\vec{b}\,'\times\vec{c}\,']\,\frac{1}{[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']} = \vec{a} \quad \Rightarrow \quad \vec{a} = \frac{\vec{b}\,'\times\vec{c}\,'}{[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']}$$

Similarly

$$\vec{b} = \frac{\vec{c}\,'\times\vec{a}\,'}{[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']} \quad \& \quad \vec{c} = \frac{\vec{a}\,'\times\vec{b}\,'}{[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']}$$

Q#14: if $\vec{a},\vec{b},\vec{c}$ & $\vec{a}\,',\vec{b}\,',\vec{c}\,'$ are reciprocal system of vectors . Prove that

(i) $\vec{a}\times\vec{a}\,'+\vec{b}\times\vec{b}\,'+\vec{c}\times\vec{c}\,'=0$

Solution: We know that

$$\vec{a}\,'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\quad;\vec{b}\,'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\quad;\vec{c}\,'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$\text{L.H.S}=\vec{a}\times\vec{a}\,'+\vec{b}\times\vec{b}\,'+\vec{c}\times\vec{c}\,'$$

$$=\vec{a}\times\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}+\vec{b}\times\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}+\vec{c}\times\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$=\frac{\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$=\frac{(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}+(\vec{b}.\vec{a})\vec{c}-(\vec{b}.\vec{c})\vec{a}+(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$=\frac{(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}+(\vec{a}.\vec{b})\vec{c}-(\vec{b}.\vec{c})\vec{a}+(\vec{b}.\vec{c})\vec{a}-(\vec{a}.\vec{c})\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$=\frac{0}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}=0\ =\text{R.H.S}$$

(ii) $\vec{a}\,'\times\vec{b}\,'+\vec{b}\,'\times\vec{c}\,'+\vec{c}\,'\times\vec{a}\,'=\dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$

Solution: We know that

$$\vec{a}\,'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\quad;\vec{b}\,'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\quad;\vec{c}\,'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$\text{L.H.S}=\vec{a}\,'\times\vec{b}\,'+\vec{b}\,'\times\vec{c}\,'+\vec{c}\,'\times\vec{a}\,'$$

$$=\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\times\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}+\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\times\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}+\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}\times\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$=\frac{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}+\frac{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}+\frac{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}$$

$$=\frac{[\vec{b}.(\vec{c}\times\vec{a})]\vec{c}-[\vec{c}.(\vec{c}\times\vec{a})]\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}+\frac{[\vec{c}.(\vec{a}\times\vec{b})]\vec{a}-[\vec{a}.(\vec{a}\times\vec{b})]\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}+\frac{[\vec{a}.(\vec{b}\times\vec{c})]\vec{b}-[\vec{b}.(\vec{b}\times\vec{c})]\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}$$

$$=\frac{[\vec{a}.(\vec{b}\times\vec{c})]\vec{c}-[0]\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}+\frac{[\vec{a}.(\vec{b}\times\vec{c})]\vec{a}-[0]\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}+\frac{[\vec{a}.(\vec{b}\times\vec{c})]\vec{b}-[0]\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}$$

$$=\frac{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}+\frac{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}+\frac{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]^2}$$

$$=\frac{\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}+\frac{\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}+\frac{\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}=\frac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}=R.H.S$$

(iii) $\vec{a}\cdot\vec{a}\,' + \vec{b}\cdot\vec{b}\,' + \vec{c}\cdot\vec{c}\,' = 3$

Solution: We know that

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$\text{L.H.S} = \vec{a}\cdot\vec{a}\,' + \vec{b}\cdot\vec{b}\,' + \vec{c}\cdot\vec{c}\,'$$

$$= \vec{a}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} + \vec{b}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} + \vec{c}\cdot\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$= \frac{\vec{a}\cdot(\vec{b}\times\vec{c}\,)+\vec{b}\cdot(\vec{c}\times\vec{a})+\vec{c}\cdot(\vec{a}\times\vec{b})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$= \frac{\vec{a}\cdot(\vec{b}\times\vec{c}\,)+\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

$$= \frac{3[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} = 3 = \text{R.H.S}$$

Q#15: If $\vec{a},\vec{b},\vec{c}$ & $\vec{a}\,',\vec{b}\,',\vec{c}\,'$. Such that

$$\vec{a}\,'\cdot\vec{a} = \vec{b}\,'\cdot\vec{b} = \vec{c}\,'\cdot\vec{c} = 1 \ \ \& \ \ \vec{a}\,'\cdot\vec{b} = \vec{a}\,'\cdot\vec{c} = \vec{b}\,'\cdot\vec{a} = \vec{b}\,'\cdot\vec{c} = \vec{c}\,'\cdot\vec{a} = \vec{c}\,'\cdot\vec{b} = 0$$

Then show that

$$\vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad ; \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

S0lution: Given $\vec{a}\,'\cdot\vec{a} = \vec{b}\,'\cdot\vec{b} = \vec{c}\,'\cdot\vec{c} = 1$------(i)

$$\vec{a}\,'\cdot\vec{b} = \vec{a}\,'\cdot\vec{c} = \vec{b}\,'\cdot\vec{a} = \vec{b}\,'\cdot\vec{c} = \vec{c}\,'\cdot\vec{a} = \vec{c}\,'\cdot\vec{b} = 0$$

Let $\vec{a}\,'\cdot\vec{b} = \vec{a}\,'\cdot\vec{c} = 0$ This shows that $\vec{a}\,'$ *is* $\perp$ to both $\vec{b}$ & $\vec{c}$.

Then $\vec{a}\,' = \lambda\,(\vec{b}\times\vec{c})$ ---------------(ii)

$$\vec{a}\,'\cdot\vec{a} = \lambda\,(\vec{b}\times\vec{c})\cdot\vec{a}$$

$$1 = \lambda[\,(\vec{a}\times\vec{b})\cdot\vec{c}\,]$$

$$1 = \lambda\,[\vec{a}\ \ \vec{b}\ \ \vec{c}\,] \quad\Rightarrow\quad \lambda = \frac{1}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

Using value of λ in equation (ii)

$$\vec{a}\,' = \frac{1}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}(\vec{b}\times\vec{c}) \quad\Rightarrow\quad \vec{a}\,' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$

Similarly ,

$$\vec{b}\,' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]} \quad \& \quad\Rightarrow\quad \vec{c}\,' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]}$$