UNIT # 03

VECTOR CALCULUS

Introduction:

In this chapter, we shall discuss the vector functions, limits and continuity, differentiation and integration of a vector function.

Vector Function:

A vector function $\vec{f}$ *from set D to set R* $[\vec{f} : D \longrightarrow R]$ *is a rule or corresponding that assigns to each Element t in set D exactly one element y in set R. It is written as* $y = \vec{f}(t)$.

For your information (i) Set D is called domain of $\vec{f}$. *(ii) Set R is called range of* $\vec{f}$.

Limit of Vector Function:

A constant L is called Limit of vector function $\vec{f}(t)$ *by taken t approaches to a* $(t \neq a)$.

It is written as $\lim_{t\to a} \vec{f}(t) = L$ *[It is studied as* $\vec{f}(t) \longrightarrow L$ *as* $t \longrightarrow a$ *]*

Rules of Limit:

(1) $\lim_{t\to a} [k . \vec{f}(t)] = k . \lim_{t\to a} \vec{f}(t)$ *(k is any scalar number)*

(2) $\lim_{t\to a} [\vec{f}(t) \pm \vec{g}(t)] = [\lim_{t\to a} \vec{f}(t)] \pm [\lim_{t\to a} \vec{g}(t)]$

(3) $\lim_{t\to a} [\vec{f}(t) \times \vec{g}(t)] = [\lim_{t\to a} \vec{f}(t)] \times [\lim_{t\to a} \vec{g}(t)]$

(4) $\lim_{t\to a} \left[\frac{\vec{f}(t)}{\vec{g}(t)}\right] = \frac{[\lim_{t\to a} \vec{f}(t)]}{[\lim_{t\to a} \vec{g}(t)]}$

(5) $\lim_{t\to a} [\vec{f}(t)]^n = [\lim_{t\to a} \vec{f}(t)]^n$

Continuity of a Vector Function:

Let $\vec{f}(t)$ *is a vector function . It is called continuous at t = a . If* $\lim_{t\to a} \vec{f}(t) = \vec{f}(a)$.

Otherwise we saysthat $\vec{f}(t)$ *is discontinuous.*

Differentiation of a Vector Function:

Let $\vec{r} = \vec{f}(t)$ *be a vector function. Then*

$$\vec{f}'(t) = \lim_{\delta t \to 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t} \quad \textit{or} \quad \frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t}$$

Is called Differentiation of a vector function. It also called 1st derivative .

Its 2nd , 3rd and so on nth-order derivative are written as

$\vec{f}''(t)$ *or* $\frac{d^2\vec{r}}{dt^2}$

$\vec{f}'''(t)$ *or* $\frac{d^3\vec{r}}{dt^3}$

: :

: :

$\vec{f}^{\,n}(t)$ *or* $\frac{d^n\vec{r}}{dt^n}$

Example#02: If $\vec{r}(t) = \sin t\hat{\imath} + \cos t\hat{\jmath} + t\hat{k}$. ***Find (i)*** $\frac{d\vec{r}}{dt}$ ***(ii)*** $\frac{d^2\vec{r}}{dt^2}$ ***(iii)*** $\left|\frac{d\vec{r}}{dt}\right|$ ***(iv)*** $\left|\frac{d^2\vec{r}}{dt^2}\right|$

Solution: Given $\vec{r}(t) = \sin t\hat{\imath} + \cos t\hat{\jmath} + t\hat{k}$

(i) $\frac{d\vec{r}}{dt} = \frac{d}{dt}[\sin t\hat{\imath} + \cos t\hat{\jmath} + t\hat{k}] = \cos t\hat{\imath} - \sin t\hat{\jmath} + 1\hat{k}$

(ii) $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}[\cos t\hat{\imath} - \sin t\hat{\jmath} + 1\hat{k}] = -\sin t\hat{\imath} - \cos t\hat{\jmath} + 0\hat{k}$

(iii) $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + 1^2} = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{1+1} = \sqrt{2}$

(iv) $\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + 0^2} = \sqrt{\sin^2 t + \cos^2 t + 0} = \sqrt{1} = \mathbf{1}$

Exercise # 3.1

Q#01: Evaluate $\lim_{t\to t_0}\left(\sin^2 t\ \hat{i} + 25\, t^3\hat{j} + \tan t\,\hat{k}\right)$

Solution: Let $L = \lim_{t\to t_0}\left[\sin^2 t\ \hat{i} + 25\, t^3\hat{j} + \tan t\,\hat{k}\right]$

$$L = \left[\lim_{t\to t_0}\sin^2 t\right]\hat{i} + 25\left[\lim_{t\to t_0} t^3\right]\hat{j} + \left[\lim_{t\to t_0}\tan t\right]\hat{k}$$

$$L = \sin^2 t_0\ \hat{i} + 25\, {t_0}^3\hat{j} + \tan t_0\,\hat{k}$$

Q#02: Evaluate $\lim_{t\to\pi}\left[\sec t\ \hat{i} + \cos t\,\hat{j} + \cot t\,\hat{k}\right]$

Solution: Let $L = \lim_{t\to\pi}\left[\sec t\ \hat{i} + \cos t\,\hat{j} + \cot t\,\hat{k}\right]$

$$L = \left[\lim_{t\to\pi}\sec t\right]\hat{i} + 25\left[\lim_{t\to\pi}\cos t\right]\hat{j} + \left[\lim_{t\to\pi}\cot t\right]\hat{k}$$

$$L = \sin\pi\ \hat{i} + \cos\pi\,\hat{j} + \cot\pi\,\hat{k}$$

$$L = 0\ \hat{i} - 1\,\hat{j} + \infty\hat{k} = \infty$$

Q#03: Example #01: ***If the vector function*** $\vec{f}(t) = \begin{cases} \frac{(a+b)\sin t}{t}\ \hat{i} + 3\cos t\,\hat{j} + 16\,b\,\frac{\tan t}{t}\,\hat{k} & \text{if } t \neq 0 \\ 6\hat{i} + 3\hat{j} + 4\,\hat{k} & \text{if } t = 0 \end{cases}$

Is continuous at t = 0 , then find the value of a and b.

Solution: ***Since the vector function is continuous at t= 0 . then by definition***

$$\lim_{t\to 0}\vec{f}(t) = \vec{f}(0)$$

$$\Rightarrow \lim_{t\to 0}\left(\frac{(a+b)\sin t}{t}\ \hat{i} + 3\cos t\,\hat{j} + 16\,b\,\frac{\tan t}{t}\,\hat{k}\right) = 6\hat{i} + 3\hat{j} + 4\,\hat{k}$$

$$(a+b)\left[\lim_{t\to 0}\frac{\sin t}{t}\right]\hat{i} + 3\left[\lim_{t\to 0}\cos t\right]\hat{j} + 16\,b\left[\lim_{t\to 0}\frac{\tan t}{t}\right]\hat{k} = 6\hat{i} + 3\hat{j} + 4\,\hat{k}$$

$$(a+b)\,[1]\,\hat{i} + 3\,[1]\,\hat{j} + 16\,b\,[1]\,\hat{k} = 6\hat{i} + 3\hat{j} + 4\,\hat{k}$$

$$(a+b)\,\hat{i} + 3\,\hat{j} + \boldsymbol{16\,b}\,\hat{k} = 6\hat{i} + 3\hat{j} + 4\,\hat{k}$$

Comparing coefficients of $\hat{i}, \hat{j}\ \&\ \hat{k}$

$\hat{k}$ **:** $\quad 16\,b = 4 \quad\Rightarrow\quad b = \frac{4}{16} \quad\Rightarrow\quad b = \frac{1}{4}$

$\hat{i}$**:** $\quad a + b = 6 \quad\Rightarrow\quad a + \frac{1}{4} = 6 \quad\Rightarrow\quad a = 6 - \frac{1}{4} = \frac{24-1}{4} \quad\Rightarrow\quad a = \frac{23}{4}$

Q#04: If the vector function $\vec{f}(t)=\begin{cases}(a+3b+2c)t^2\,\hat{i}+(2a-b)t^3\hat{j}+c\,\hat{k} & \text{if } t\neq 2\\ \hat{i}+2\hat{j}+3\,\hat{k} & \text{if } t=2\end{cases}$

Is continuous at t = 2, then find the value of a, b & c.

Solution: ***Since the vector function is continuous at t = 2 . then by definition***

$$\lim_{t\to 2}\vec{f}(t)=\vec{f}(2)$$

$$\lim_{t\to 2}\left[(a+3b+2c)t^2\,\hat{i}+(2a-b)t^3\hat{j}+c\,\hat{k}\right]=\hat{i}+2\hat{j}+3\,\hat{k}$$

$$(a+3b+2c)\ [\ \lim_{t\to 2}t^2]\,\hat{i}+(2a-b)\ [\ \lim_{t\to 0}t^3\ ]\,\hat{j}+c[\ \lim_{t\to 0}1\ ]\,\hat{k}=\hat{i}+2\hat{j}+3\,\hat{k}$$

$$(a+3b+2c)(2)^2\,\hat{i}+(2a-b)\,(2)^3\,\hat{j}+c\ [1]\,\hat{k}=\hat{i}+2\hat{j}+3\,\hat{k}$$

$$4(a+3b+2c)\hat{i}+8(2a-b)\,\hat{j}+c\hat{k}=\hat{i}+2\hat{j}+3\,\hat{k}$$

Comparing coefficients of $\hat{i}$, $\hat{j}$ & $\hat{k}$

$\hat{k}$ **:** $c = 3$

$\hat{i}$**:** $4(a+3b+2c)=1 \implies 4a+12b+8c=1$

$\implies 4a+12b+8(3)=1 \implies 4a+12b+24=1$

$\implies 4a+12b=1-24$

$4a+12b=-23$ ------------**(i)**

$\hat{j}$**:** $8(2a-b)=2 \implies 4(2a-b)=1 \implies 8a-4b=1$

Multiplying by 3: $24a-12b=3$---------------**(ii)**

Adding (i) & (ii)

$$4a+12b=-23$$
$$24a-12b=3$$
$$28a=-20 \implies a=-20/28 \implies a=(-5)/7$$

using in (ii) $8a-4b=1$

$$\implies 8\left(\frac{-5}{7}\right)-4b=1$$

$$\implies \frac{-40}{7}-1=4b$$

$$\implies 4b=\frac{-40-7}{7}$$

$$\implies b=(-47)/28$$

Q#05:If the vector function $\vec{f}(t)=\begin{cases}(a+3b+2c)t^2\,\hat{i}+(2a-b)t^3\hat{j}+(a+b+c)t\,\hat{k} & \text{if } t\neq 1\\ 5\,\hat{i}+6\hat{j}+3\,\hat{k} & \text{if } t=1\end{cases}$

Is continuous at t =1 , then find the value of a , b & c.

Solution: ***Since the vector function is continuous at t = 1 . then by definition***

$$\lim_{t\to1}\vec{f}(t)=\vec{f}(1)$$

$$\lim_{t\to1}\left[(a+3b+2c)t^2\,\hat{i}+(2a-b)t^3\hat{j}+(a+b+c)t\,\hat{k}\right]=5\,\hat{i}+6\hat{j}+3\,\hat{k}$$

$$(a+3b+2c)[\lim_{t\to1}t^2]\hat{i}+(2a-b)[\lim_{t\to1}t^3]\hat{j}+(a+b+c)[\lim_{t\to1}t]\,\hat{k}=5\,\hat{i}+6\hat{j}+3\,\hat{k}$$

$$(a+3b+2c)(1)^2\,\hat{i}+(2a-b)\,(1)^3\,\hat{j}+(a+b+c)\;[1]\,\hat{k}=5\,\hat{i}+6\hat{j}+3\,\hat{k}$$

$$(a+3b+2c)\hat{i}+(2a-b)\,\hat{j}+(a+b+c)\hat{k}=5\,\hat{i}+6\hat{j}+3\,\hat{k}$$

Comparing coefficients of $\hat{i}$, $\hat{j}$ & $\hat{k}$

$$a+3b+2c=5 \quad \text{-------------}(i)$$

$$2a-b=6 \quad \text{------------}(ii)$$

$$a+b+c=3$$

Multiplying by 2: $2a+2b+2c=6$ -------------***(iii)***

Subtracting (i) & (iii

$$\begin{array}{r} a+3b+2c=5\\ \pm2a\pm2b\pm2c=\pm6\\ \hline -a+b=-1 \end{array}$$

$$b=a-1 \text{-------}(iv)$$

Using (iv) in(ii) $2a-(a-1)=6$

$$2a-a+1=6$$

$$a=6-1 \quad \Rightarrow \quad a=5$$

Using a= 5 in (iv) $b=5-1 \quad \Rightarrow \quad b=4$

Using a =5 & b=4 in (iii) $a+b+c=3 \quad \Rightarrow \quad 5+4+c=3$

$$c=3-9 \quad \Rightarrow \quad c=-6$$

Q#06: If $\vec{f}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + 9\,\hat{k}$. **Find** $|\vec{f}(t)|$

Solution: Given $\vec{f}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + 9\,\hat{k}$.

Now $|\vec{f}(t)| = \sqrt{(\cos t)^2 + (-\sin t)^2 + 9^2} = \sqrt{\cos^2 t + \sin^2 t + 81} = \sqrt{1+81}$

$|\vec{f}(t)| = \sqrt{82}$

Q#07: If $\vec{f}(t) = \hat{i} + 2\tan t\,\hat{j} + 2\tan^2 t\,\hat{k}$. **Find** $|\vec{f}(t)|$

Solution: Given $\vec{f}(t) = \hat{i} + 2\tan t\,\hat{j} + 2\tan^2 t\,\hat{k}$

Now $|\vec{f}(t)| = \sqrt{(1)^2 + (2\tan t)^2 + (2\tan^2 t)^2} = \sqrt{(1)^2 + 4\tan^2 t + (2\tan^2 t)^2}$

$= \sqrt{(1)^2 + 2(1)(2\tan^2 t) + (2\tan^2 t)^2} = \sqrt{(1 + 2\tan^2 t)^2}$

$|\vec{f}(t)| = 1 + 2\tan^2 t$

Q#08: If $\vec{f}(t) = t^2\hat{i} + (t-1)\hat{j} + (t^2+t+1)\hat{k}$ **&** $\vec{g}(t) = (t^2+1)\,\hat{i} + t\,\hat{j} - \hat{k}$ **Find(i)** $\vec{f}(t)\cdot\vec{g}(t)$ **(ii)** $\vec{f}(t)\times\vec{g}(t)$

Solution: Given that $\vec{f}(t) = t^2\hat{i} + (t-1)\hat{j} + (t^2+t+1)\hat{k}$ **&** $\vec{g}(t) = (t^2+1)\,\hat{i} + t\,\hat{j} - \hat{k}$

(i) $\vec{f}(t)\cdot\vec{g}(t) = [t^2\hat{i} + (t-1)\hat{j} + (t^2+t+1)\hat{k}]\cdot[(t^2+1)\,\hat{i} + t\,\hat{j} - \hat{k}]$

$= t^2(t^2+1) + (t-1)\,t + (t^2+t+1)(-1) = t^4 + t^2 + t^2 - t - t^2 - t - 1$

$= t^4 + t^2 - 2t - 1$

(ii) $$\vec{f}(t)\times\vec{g}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & t-1 & t^2+t+1 \\ t^2+1 & t & -1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} t-1 & t^2+t+1 \\ t & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} t^2 & t^2+t+1 \\ t^2+1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} t^2 & t-1 \\ t^2+1 & t \end{vmatrix}$$

$= \hat{i}\,[(-1)(t-1) - t(t^2+t+1)] - \hat{j}[(-1)t^2 - (t^2+1)(t^2+t+1)] + \hat{k}[t.t^2 - (t^2+1)(t-1)]$

$= \hat{i}\,[-t+1-t^3-t^2-t] - \hat{j}[-t^2-t^4-t^3-t^2-t^2-t-1] + \hat{k}[t^3-t^3+t^2-t+1]$

$= \hat{i}\,[-2t-t^3-t^2+1] + \hat{j}[3t^2+t^4+t^3+t+1] + \hat{k}[t^2-t+1]$

$= [1-2t-t^2-t^3]\hat{i} - [1+t3t^2+t^3+t^4]\hat{j} + [1-t+t^2]\hat{k}$

Q#09: if $\vec{f}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t^2 \sec t\,\hat{k}$ ***. calculate*** $|\vec{f}'(t)|^2$

Solution: Given $\vec{f}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t^2 \sec t\,\hat{k}$

Differentiate w. r . t t

$$\vec{f}'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + [\,t^2 \sec t . \tan t + 2t \sec t]\hat{k}$$

Now $|\vec{f}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (\,t^2 \sec t . \tan t + 2t \sec t)^2}$

$$= \sqrt{\sin^2 t + \cos^2 t + (\,t^2 \sec t . \tan t + 2t \sec t)^2}$$

$$|\vec{f}\,'(t)| = \sqrt{1 + (\,t^2 \sec t . \tan t + 2t \sec t)^2}$$

Taking square on both sides

$$|\vec{f}'(t)|^2 = 1 + (\,t^2 \sec t . \tan t + 2t \sec t)^2$$

Q#10: If $\vec{f}(t) = (t^2 + 2t - 1)\hat{\imath} + (3t^2 - 2)\hat{\jmath} + (5 - 6t)\hat{k}$ ***Find (i)*** $\vec{f}'(t)$ (ii) $\vec{f}''(t)$

Solution: Given $\vec{f}(t) = (t^2 + 2t - 1)\hat{\imath} + (3t^2 - 2)\hat{\jmath} + (5 - 6t)\hat{k}$

(i) ***Differentiate w. r . t t***

$$\vec{f}'(t) = (\,2t + 2)\hat{\imath} + (6t)\hat{\jmath} + (-6)\hat{k} = (\,2t + 2)\hat{\imath} + 6\,t\,\hat{\jmath} - 6\hat{k}$$

(ii) ***Again Differentiate w. r . t t***

$$\vec{f}''(t) = 2\hat{\imath} + 6\,\hat{\jmath} + 0\hat{k}$$

Q#11: if $\vec{f}(t) = \cos t\hat{\imath} + \sin\, t\hat{\jmath} + 8\,\hat{k}$. ***Show that*** $\vec{f}'(t) . \vec{f}''(t) = 0$

Solution: Given $\vec{f}(t) = \cos t\hat{\imath} + \sin\, t\hat{\jmath} + 8\,\hat{k}$

Differentiate w. r. t t $\vec{f}'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 0\hat{k}$

Again Differentiate w. r. t t $\vec{f}''(t) = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath} + 0\hat{k}$

Now

$$\vec{f}\,'(t) . \vec{f}\,''(t) = (-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 0\hat{k}) . (-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath} + 0\hat{k})$$

$$= (-\sin t)(-\cos t) + (\cos t)(-\sin t) + (0)(0)$$

$$= \sin t . \cos t - \sin t . \cos t + 0$$

Hence proved $\vec{f}'(t) . \vec{f}''(t) = 0.$

Q#12: *If* $\vec{f}(t) = (t - \sin t)\,\hat{\imath} + (1 - \cos t)\hat{\jmath} + (t\sin t + \cos t)\,\hat{k}$. *Find* $\vec{f}'(t)$ & $\vec{f}''(t)$ *at* $t=0$ & $t = \frac{\pi}{2}$.

Solution: Given $\vec{f}(t) = (t - \sin t)\,\hat{\imath} + (1 - \cos t)\hat{\jmath} + (t\sin t + \cos t)\,\hat{k}$.

Differentiate w. r. t t

$$\vec{f}'(t) = (1 - \cos t)\hat{\imath} - (-\sin t)\hat{\jmath} + (t\cos t + \sin t - \sin t)\,\hat{k}$$

$$\vec{f}'(t) = (1 - \cos t)\hat{\imath} + \sin t\,\hat{\jmath} + t\cos t\,\hat{k}$$

At t = 0 **:** $\vec{f}'(t) = (1-1)\hat{\imath} + 0\hat{\jmath} + 0\,\hat{k} = \boldsymbol{0}\,\hat{\imath} + 0\hat{\jmath} + 0\,\hat{k}$

At $t = \frac{\pi}{2}$ **:** $\vec{f}'(t) = (1-0)\hat{\imath} + 1\hat{\jmath} + 0\,\hat{k} = \boldsymbol{1}\,\hat{\imath} + 1\hat{\jmath} + 0\,\hat{k}$

Again Differentiate w. r. t t

$$\vec{f}''(t) = (0 + \sin t)\hat{\imath} + \cos t\,\hat{\jmath} + (\cos t - t\sin t)\,\hat{k}$$

$$\vec{f}''(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + (\cos t - t\sin t)\,\hat{k}$$

At t = 0 **:** $\vec{f}''(t) = 0\hat{\imath} + 1\hat{\jmath} + 1\,\hat{k}$

At $t = \frac{\pi}{2}$ **:** $\vec{f}''(t) = 1\hat{\imath} + 0\hat{\jmath} + (0 - \frac{\pi}{2})\hat{k} = 1\,\hat{\imath} + 0\hat{\jmath} - \frac{\pi}{2}\,\hat{k}$

Q#13: *If* $\vec{f}(t) = \left(\frac{t^2+1}{t}\right)\hat{\imath} + \left(\frac{1}{1+t}\right)\hat{\jmath} + t\,\hat{k}$. *Find* $\vec{f}'(t)$ and $\vec{f}(t)\,.\,\vec{f}'(t)$.

Solution: Given $\vec{f}(t) = \left(\frac{t^2+1}{t}\right)\hat{\imath} + \left(\frac{1}{1+t}\right)\hat{\jmath} + t\,\hat{k}$ ***Then***

$$\vec{f}'(t) = \left(\frac{t(2t) - (t^2+1)1}{t^2}\right)\hat{\imath} + \left(\frac{-1}{(1+t)^2}\right)\hat{\jmath} + 1\,\hat{k}$$

$$= \left(\frac{2t^2 - t^2 - 1}{t^2}\right)\hat{\imath} - \frac{1}{(1+t)^2}\hat{\jmath} + 1\,\hat{k}$$

$$\vec{f}'(t) = \left(\frac{t^2-1}{t^2}\right)\hat{\imath} - \frac{1}{(1+t)^2}\hat{\jmath} + 1\,\hat{k}$$

Now

$$\vec{f}(t).\vec{f}'(t) = \left[\left(\frac{t^2+1}{t}\right)\hat{\imath} + \left(\frac{1}{1+t}\right)\hat{\jmath} + t\,\hat{k}\right].\left[\left(\frac{t^2-1}{t^2}\right)\hat{\imath} - \frac{1}{(1+t)^2}\hat{\jmath} + 1\,\hat{k}\right]$$

$$= \left(\frac{t^2+1}{t}\right)\left(\frac{t^2-1}{t^2}\right) + \left(\frac{1}{1+t}\right)\left(\frac{-1}{(1+t)^2}\right) + (t)(1)$$

$$\vec{f}(t).\vec{f}'(t) = \frac{t^4-1}{t^3} - \frac{1}{(1+t)^3} + t$$

Q#14: If $\vec{f}(t)$ & $\vec{g}(t)$ are continuouse at $t = t_0$. Prove that $\vec{f}(t) + \vec{g}(t)$ is also continuouse at $t = t_0$.

Solution: Given $\vec{f}(t)$ & $\vec{g}(t)$ are continuouse at $t = t_0$

Then there exist a number $\varepsilon > 0$. $\left|\vec{f}(t) - \vec{f}(t_0)\right| < \varepsilon$ -------***(i)***

And $|\vec{g}(t) - \vec{g}(t_0)| < \varepsilon$ ------***(ii)***

Adding (i) &(ii) $\left|\vec{f}(t) - \vec{f}(t_0)\right| + |\vec{g}(t) - \vec{g}(t_0)| < \varepsilon + \varepsilon$

$$\left|\vec{f}(t) + \vec{g}(t)\right| - \left|\vec{f}(t_0) + \vec{g}(t_0)\right| < 2\varepsilon$$

Here $2\varepsilon > 0$ ***Then show that*** $\vec{f}(t) + \vec{g}(t)$ is also continuouse at $t = t_0$.

Q#15: Is $\vec{f}(t) = t\,\hat{i} + t^2\hat{j} + \frac{1}{t}\,\hat{k}$ ***is continuous function at t=0?***

Solution: Given $\vec{f}(t) = t\,\hat{i} + t^2\hat{j} + \frac{1}{t}\,\hat{k}$

Now $\lim_{t\to 0^+} \vec{f}(t) = \lim_{t\to 0^+}\left[t\,\hat{i} + t^2\hat{j} + \frac{1}{t}\,\hat{k}\right] = +0\,\hat{i} + (+0)^2\hat{j} + \frac{1}{+0}\,\hat{k} = +0\,\hat{i} + 0\hat{j} + \infty\hat{k} = \infty$ ------***(i)***

$\lim_{t\to 0^-} \vec{f}(t) = \lim_{t\to 0^-}\left[t\,\hat{i} + t^2\hat{j} + \frac{1}{t}\,\hat{k}\right] = -0\,\hat{i} + (-0)^2\hat{j} + \frac{1}{-0}\,\hat{k} = -0\,\hat{i} + 0\hat{j} - \infty\hat{k} = -\infty$ -----***(ii)***

$\vec{f}(0) = 0\,\hat{i} + 0\hat{j} + \frac{1}{0}\,\hat{k} = 0\,\hat{i} + 0\hat{j} + \infty\,\hat{k} = \infty$ ---***(iii)***

From (i) ,(ii) & (iii) this shows that the given vector function is discontinuous at t = 0.

Q#16: If ω, a, b ***are constant and if*** $\vec{f}(t) = a\cos\omega t + b\sin\omega t$. ***Show that*** $\vec{f}''(t) + \omega^2\vec{f}(t) = 0$

Solution: Given $\vec{f}(t) = a\cos\omega t + b\sin\omega t$----------------***(i)***

Differentiate w. r. t t $\vec{f}'(t) = -a\,\omega\sin\omega t + b\omega\cos\omega t$

Again differentiate w. r. t t $\vec{f}''(t) = -a\,\omega^2\cos\omega t - b\omega^2\sin\omega t$

$$\vec{f}''(t) = -\omega^2[a\cos\omega t + b\sin\omega t]$$

$$\vec{f}''(t) = -\omega^2\vec{f}(t) \qquad \because \textit{From (i)}$$

$$\vec{f}''(t) + \omega^2\vec{f}(t) = 0$$ ***Hence proved.***

Rules of Differentiation:

If $\vec{a}, \vec{b}$ & $\vec{c}$ ***are differentiable function of scalar variable t.***

(i) $\frac{d}{dt}\left[\vec{a} + \vec{b}\right] = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}$

(ii) $\frac{d}{dt}\left[\vec{a}.\vec{b}\right] = \frac{d\vec{a}}{dt}.\vec{b} + \vec{a}.\frac{d\vec{b}}{dt}$

(iii) $\frac{d}{dt}\left[\vec{a} \times \vec{b}\right] = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$

(iv) $\frac{d}{dt}\left[\varphi\, \vec{a}\right] = \frac{d\varphi}{dt}\vec{a} + \varphi\frac{d\vec{a}}{dt}$

(v) $\frac{d}{dt}\left[\vec{a}.(\vec{b} \times \vec{c})\right] = \frac{d\vec{a}}{dt}.(\vec{b} \times \vec{c}) + \vec{a}.\left(\frac{d\vec{b}}{dt} \times \vec{c}\right) + \vec{a}.\left(\vec{b} \times \frac{d\vec{c}}{dt}\right)$

(vi) $\frac{d}{dt}\left[\vec{a} \times (\vec{b} \times \vec{c})\right] = \frac{d\vec{a}}{dt} \times (\vec{b} \times \vec{c}) + \vec{a} \times \left(\frac{d\vec{b}}{dt} \times \vec{c}\right) + \vec{a} \times \left(\vec{b} \times \frac{d\vec{c}}{dt}\right)$

(vii) ***Derivative of a constant vector:***

Let $\vec{r}$ *be constant vector. Then* $\frac{d\vec{r}}{dt} = 0$

(viii) ***Derivative of a vector function in terms of its component.***

Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ *Then* $\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

Theorem #I :

Show that Necessary and sufficient condition for a vector $\vec{a}$ ***of scalar variable t to be a constant is*** $\frac{d\vec{a}}{dt} = 0$.

Proof: ***By given condition. That*** $\vec{a}$ ***be constant vector. i.e.***

$$\vec{a} = \textit{constant}$$

Differentiate w.r.t t $\frac{d\vec{a}}{dt} = \frac{d}{dt}(\textit{constant}) \Rightarrow \frac{d\vec{a}}{dt} = 0$

Conversely, suppose that $\frac{d\vec{a}}{dt} = 0 \Rightarrow d\vec{a} = 0\, dt$

on integrating both sides $\int d\vec{a} = \int 0\, dt$

$$\vec{a} = 0.t + \text{constant} \Rightarrow \vec{a} = \text{constant}$$

Hence prove that

The Necessary and sufficient condition for a vector $\vec{a}$ ***of scalar variable t to be a constant is*** $\frac{d\vec{a}}{dt} = 0$.

Theorem #II: Show that Necessary and sufficient condition for a vector $\vec{a}$ of scalar variable t to have a constant magnitude is $\vec{a}.\frac{d\vec{a}}{dt}= 0$.

***Proof:** By given condition. That vector $\vec{a}$ have a constant magnitude.*

$$|\vec{a}| = constant$$

Taking square on both sides $|\vec{a}|^2 = (\text{constant})^2 = constant$

We know that $\vec{a}.\vec{a} = |\vec{a}|^2$

$$\vec{a}.\vec{a} = constant$$

Differentiate w.r.t t $\frac{d}{dt}(\vec{a}.\vec{a}) = \frac{d}{dt}(constant)$

$$\frac{d\vec{a}}{dt}.\vec{a} + \vec{a}.\frac{d\vec{a}}{dt} = 0$$

$$2\,\vec{a}.\frac{d\vec{a}}{dt} = 0 \qquad \therefore \vec{a}.\frac{d\vec{a}}{dt} = \frac{d\vec{a}}{dt}.\vec{a}$$

$$\vec{a}.\frac{d\vec{a}}{dt} = 0$$

Conversely, suppose that $\vec{a}.\frac{d\vec{a}}{dt} = 0 \qquad \therefore \vec{a}.\frac{d\vec{a}}{dt} = a\frac{d\,a}{dt}$

$$a\frac{d\,a}{dt} = 0$$

$$a\,da = 0\,dt$$

on integrating both sides $\int a\,d\vec{a} = \int 0\,dt$

$$\frac{|\vec{a}|^2}{2} = 0\,.\,t + \text{constant}$$

$$|\vec{a}|^2 = 2(\text{constant})$$

Taking square-root on both sides $|\vec{a}| = \sqrt{2(\text{constant})}$

$$|\vec{a}| = constant$$

Hence prove that

The Necessary and sufficient condition for a vector $\vec{a}$ of scalar variable t to have a constant magnitude is $\vec{a}.\frac{d\vec{a}}{dt}= 0$

Theorem #III: Show that Necessary and sufficient condition for a vector $\vec{a}$ of scalar variable t to have a constant direction is $\vec{a} \times \frac{d\vec{a}}{dt} = 0$.

Proof: *Let $\hat{r}$ be unit vector in the direction of vector $\vec{a}$.*

By given condition, that direction is constant $\therefore$ $\hat{r}$ *= constant*

Then $\frac{d\hat{r}}{dt} = 0$ --*(i)*

As we know that $\hat{r} = \frac{\vec{a}}{a} \Rightarrow \vec{a} = a\,\hat{r}$ ---------- ----------*(ii)* $\therefore |\vec{a}| = a$

Differentiate w.r.t t $\frac{d\vec{a}}{dt} = \frac{d}{dt}(a\,\hat{r}) = \frac{da}{dt}\,\hat{r} + a\frac{d\hat{r}}{dt}$-------------*(iii)*

Taking cross product of equation (ii) & (iii)

$$\vec{a} \times \frac{d\vec{a}}{dt} = a\,\hat{r} \times \left(\frac{da}{dt}\,\hat{r} + a\,\frac{d\hat{r}}{dt}\right) = a\frac{da}{dt}(\hat{r}\times\hat{r}) + a^2\left(\hat{r}\times\frac{d\hat{r}}{dt}\right)$$

$$\vec{a} \times \frac{d\vec{a}}{dt} = a\frac{da}{dt}(0) + a^2\left(\hat{r}\times\frac{d\hat{r}}{dt}\right) \qquad \therefore \hat{r}\times\hat{r} = 0$$

$$\vec{a} \times \frac{d\vec{a}}{dt} = 0 + a^2\left(\hat{r}\times\frac{d\hat{r}}{dt}\right)$$

$$\vec{a} \times \frac{d\vec{a}}{dt} = a^2\left(\hat{r}\times\frac{d\hat{r}}{dt}\right) \text{----------}(iv)$$

$$\vec{a} \times \frac{d\vec{a}}{dt} = a^2(\hat{r}\times 0) \qquad \therefore \text{From (i)} \quad \frac{d\hat{r}}{dt} = 0$$

$$\vec{a} \times \frac{d\vec{a}}{dt} = 0$$

Conversely , suppose that

$$\vec{a} \times \frac{d\vec{a}}{dt} = 0$$

Then equation (iv) will become $a^2\left(\hat{r}\times\frac{d\hat{r}}{dt}\right) = 0 \Rightarrow \hat{r}\times\frac{d\hat{r}}{dt} = 0$

Here $\hat{r} \neq 0$ *but* $\frac{d\hat{r}}{dt} = 0$ *Therefore* $\hat{r}$ *= constant*

Hence prove that

The Necessary and sufficient condition for a vector $\vec{a}$ of scalar variable t to have a constant direction is $\vec{a} \times \frac{d\vec{a}}{dt} = 0$.

Example#01: $\vec{r} = (t+1)\,\hat{\imath} + (t^2+t+1)\hat{\jmath} + (t^3+t^2+t+1)\hat{k}$ ***. Find*** $\frac{d\vec{r}}{dt}$ ***&*** $\frac{d^2\vec{r}}{dt^2}$.

Solution: Given vector function is $\vec{r} = (t+1)\,\hat{\imath} + (t^2+t+1)\hat{\jmath} + (t^3+t^2+t+1)\hat{k}$

Then $\frac{d\vec{r}}{dt} = (1)\,\hat{\imath} + (2t+1)\hat{\jmath} + (3t^2+2t+1)\hat{k}$ ***&*** $\frac{d^2\vec{r}}{dt^2} = 0\hat{\imath} + 2\hat{\jmath} + (6t+2)\hat{k}$

Example#02: $\vec{f}$ ***(t)*** $= \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + t\,\hat{k}$ ***.Find (i)*** $\vec{f}'(t)$ (ii)$\vec{f}''(t)$ ***(iii)*** $|\vec{f}'(t)|$ ***(iv)*** $|\vec{f}''(t)|$.

Solution: Given vector function is

$$\vec{f}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + t\,\hat{k}$$

(i) $\vec{f}'(t) = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath} + 1\,\hat{k}$

(ii) $\vec{f}''(t) = -\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + 0\,\hat{k}$

(iii) $|\vec{f}'(t)| = \sqrt{(\cos t)^2 + (-\sin t)^2 + 1^2} = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{1+1} = \sqrt{2}$

(iv) $|\vec{f}''(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + 0^2} = \sqrt{\sin^2 t + \cos^2 t + 0} = \sqrt{1}$ ***=1***

Example#03: If $\vec{r} = \cos nt\,\hat{\imath} + \sin nt\,\hat{\jmath}$ ***. Where n is a constant . show that*** $\vec{r} \times \frac{d\vec{r}}{dt} = \boldsymbol{n}\,\hat{k}$.

Solution: Given vector function is $\vec{r} = \cos nt\,\hat{\imath} + \sin nt\,\hat{\jmath}$ ***Then*** $\frac{d\vec{r}}{dt} = -n\sin nt\,\hat{\imath} + n\cos nt\,\hat{\jmath}$

Now $\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix} = \boldsymbol{0}\,\hat{\imath} - 0\hat{\jmath} + \hat{k}\begin{vmatrix} \cos nt & \sin nt \\ -n\sin nt & n\cos nt \end{vmatrix}$

$= \hat{k}\,[(n\cos nt)(\cos nt) - (-n\sin nt)(\sin nt)]$

$= n\,\hat{k}\,[\cos^2 nt + \sin^2 nt]$

$\vec{r} \times \frac{d\vec{r}}{dt} = n\,\hat{k}$ ***Hence proved***

Example#04: If $\vec{a}$ ***be differentiable vector function of scalar variable t then show that*** $\frac{d}{dt}\left(\vec{a} \times \frac{d\vec{a}}{dt}\right) = \vec{a} \times \frac{d^2\vec{a}}{dt^2}$

Solution: ***L.H.S*** $= \frac{d}{dt}\left(\vec{a} \times \frac{d\vec{a}}{dt}\right) = \frac{d\vec{a}}{dt} \times \frac{d\vec{a}}{dt} + \vec{a} \times \frac{d^2\vec{a}}{dt^2}$

$= \boldsymbol{0} + \vec{a} \times \frac{d^2\vec{a}}{dt^2}$ $\therefore \frac{d\vec{a}}{dt} \times \frac{d\vec{a}}{dt} = 0$

$= \vec{a} \times \frac{d^2\vec{a}}{dt^2}$ ***= R.H.S***

Hence proved L.H.S = R.H.S

***Example#05:** If $\vec{a}$, $\vec{b}$ are constant vectors, ω is a constant and $\vec{r}$ be a vector function is given by*

$\vec{r} = \cos\omega t\ \vec{a} + \sin\omega t\ \vec{b}$. ***Then Show that (i)*** $\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0$ ***(ii)*** $\vec{r} \times \frac{d\vec{r}}{dt} = \omega\left(\vec{a} \times \vec{b}\right)$

Solution: Given vector function $\vec{r} = \cos\omega t\ \vec{a} + \sin\omega t\ \vec{b}$ ----------------***(i)***

Then $\frac{d\vec{r}}{dt} = -\omega\sin\omega t\ \vec{a} + \omega\cos\omega t\ \vec{b}$ -------***(ii)***

(i)

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\ \cos\omega t\ \vec{a} - \omega^2\sin\omega t\ \vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\left[\cos\omega t\ \vec{a} + \ \sin\omega t\vec{b}\right]$$

$$\frac{d^2\vec{r}}{dt^2} = -\ \omega^2\ \vec{r} \qquad \Longrightarrow \qquad \frac{d^2\vec{r}}{dt^2} + \ \omega^2\ \vec{r} = 0$$ ***Hence proved.***

(ii) ***Now taking cross product of equation (i) &(ii)***

$$\vec{r} \times \frac{d\vec{r}}{dt} = (\cos\omega t\ \vec{a} + \sin\omega t\ \vec{b}) \times \left(-\omega\sin\omega t\ \vec{a} + \omega\cos\omega t\ \vec{b}\right)$$

$$= -\omega\cos\omega t\sin\omega t(\vec{a}\times\vec{a}) + \omega\cos^2\omega t\left(\vec{a}\times\vec{b}\right) + \omega\sin^2\omega t\left(-\vec{b}\times\vec{a}\right) + \omega\cos\omega t\sin\omega t\left(\vec{b}\times\vec{b}\right)$$

$$= -\omega\cos\omega t\sin\omega t(0) + \omega\cos^2\omega t\left(\vec{a}\times\vec{b}\right) + \omega\sin^2\omega t\left(\vec{a}\times\vec{b}\right) + \omega\cos\omega t\sin\omega t\,(0)$$

$$= 0 + \omega\left(\vec{a} \times \vec{b}\right)[\cos^2\omega t + \sin^2\omega t] + \ 0$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = \ \omega\left(\vec{a} \times \vec{b}\right)$$ ***Hence proved.***

Example# 06: if $\frac{d\vec{u}}{dt} = \vec{\omega} \times \vec{u}$ & $\frac{d\vec{r}}{dt} = \ \vec{\omega} \times \vec{r}$***.Then show that*** $\frac{d}{dt}\ (\vec{u} \times \vec{r}) = \vec{\omega} \times (\vec{u} \times \vec{r})$***.***

Solution: Taking L.H.S $\frac{d}{dt}\ (\vec{u} \times \vec{r}) = \frac{d\vec{u}}{dt} \times \vec{r} + \vec{u} \times \frac{d\vec{r}}{dt}$

Using given values $\frac{d\vec{u}}{dt} = \vec{\omega} \times \vec{u}$ & $\frac{d\vec{r}}{dt} = \ \vec{\omega} \times \vec{r}$

$$\frac{d}{dt}\ (\vec{u} \times \vec{r}) \ = (\vec{\omega} \times \vec{u}) \times \vec{r} + \vec{u} \times (\vec{\omega} \times \vec{r})$$

$$= (\vec{\omega}.\vec{r})\vec{u} - (\vec{u}.\vec{r})\vec{\omega} + (\vec{u}.\vec{r})\vec{\omega} - (\vec{u}.\vec{\omega})\vec{r}$$

$$\frac{d}{dt}\ (\vec{u} \times \vec{r}) \ = (\vec{\omega}.\vec{r})\vec{u} - (\vec{u}.\vec{\omega})\vec{r}$$ ---------------***(i)***

Now taking R.H.S $\vec{\omega} \times (\vec{u} \times \vec{r}) = (\vec{\omega}.\vec{r})\vec{u} - (\vec{u}.\vec{\omega})\vec{r}$---------------***(ii)***

From (i) & (ii) Hence proved that

$$\frac{d}{dt}\ (\vec{u} \times \vec{r}) = \vec{\omega} \times (\vec{u} \times \vec{r})$$

Example: 07: Differentiate the following w. r. t t. where $\vec{r}$ vector function of scalar variable t.

$\vec{a}$ be constant vector and m is any scalar. (i) $\vec{r}\cdot\vec{a}$ (ii) $\vec{r}\times\vec{a}$ (iii) $\vec{r}\cdot\frac{d\vec{r}}{dt}$

(iv) $\vec{r}\times\frac{d\vec{r}}{dt}$ (v) $r^2+\frac{1}{r^2}$ (vi) $m\left(\frac{d\vec{r}}{dt}\right)^2$ (vii) $\frac{\vec{r}+\vec{a}}{r^2+a^2}$ (viii) $\frac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}$

(i) ***Let*** $\vec{f}(t)=\vec{r}\cdot\vec{a}$

Differentiate w.r.t t $\vec{f}'(t)=\frac{d}{dt}(\vec{r}\cdot\vec{a})=\frac{d\vec{r}}{dt}\cdot\vec{a}+\vec{r}\cdot\frac{d\vec{a}}{dt} \Rightarrow \vec{f}'(t)=\frac{d\vec{r}}{dt}\cdot\vec{a} \quad \because \frac{d\vec{a}}{dt}=0$

(ii) ***Let*** $\vec{f}(t)=\vec{r}\times\vec{a}$

Differentiate w.r.t t $\vec{f}'(t)=\frac{d}{dt}(\vec{r}\times\vec{a})=\frac{d\vec{r}}{dt}\times\vec{a}+\vec{r}\times\frac{d\vec{a}}{dt} \Rightarrow \vec{f}'(t)=\frac{d\vec{r}}{dt}\times\vec{a} \quad \because \frac{d\vec{a}}{dt}=0$

(iii) ***Let*** $\vec{f}(t)=\vec{r}\cdot\frac{d\vec{r}}{dt}$

Differentiate w.r.t t $\vec{f}'(t)=\frac{d}{dt}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)=\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}+\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} \Rightarrow \vec{f}'(t)=\left(\frac{d\vec{r}}{dt}\right)^2+\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} \quad \because \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}=\left(\frac{d\vec{r}}{dt}\right)^2$

(iv) ***Let*** $\vec{f}(t)=\vec{r}\times\frac{d\vec{r}}{dt}$

Differentiate w.r.t t $\vec{f}'(t)=\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}+\vec{r}\times\frac{d^2\vec{r}}{dt^2}$

$\vec{f}'(t)=\mathbf{0}+\vec{r}\times\frac{d^2\vec{r}}{dt^2}$

$\vec{f}'(t)=\vec{r}\times\frac{d^2\vec{r}}{dt^2} \quad \because \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}=0$

(v) ***Let*** $\vec{f}(t)=r^2+\frac{1}{r^2}$

Differentiate w.r.t t $\vec{f}'(t)=\frac{d}{dt}\left(r^2+\frac{1}{r^2}\right)=\frac{d}{dt}(r^2)+\frac{d}{dt}(r^{-2})=2r\frac{dr}{dt}+\left(-2r^{-3}\frac{dr}{dt}\right)$

$\vec{f}'(t)=2r\frac{dr}{dt}-\frac{2}{r^3}\frac{dr}{dt}=2\frac{dr}{dt}\left(r-\frac{2}{r^3}\right)$

(vi) ***Let*** $\vec{f}(t)=m\left(\frac{d\vec{r}}{dt}\right)^2$

Differentiate w.r.t t $\vec{f}'(t)=m\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2=\boldsymbol{2m}\left(\frac{d\vec{r}}{dt}\right)\left[\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)\right] \Rightarrow \vec{f}'(t)=\boldsymbol{2m}\frac{d\vec{r}}{dt}\frac{d^2\vec{r}}{dt^2}$

(vii) ***Let*** $\vec{f}(t)=\frac{\vec{r}+\vec{a}}{r^2+a^2}$

Differentiate w.r.t t

$\vec{f}'(t)=\frac{d}{dt}\left(\frac{\vec{r}+\vec{a}}{r^2+a^2}\right)=\frac{(r^2+a^2)\frac{d}{dt}(\vec{r}+\vec{a})-(\vec{r}+\vec{a})\frac{d}{dt}(r^2+a^2)}{(r^2+a^2)^2}=\frac{(r^2+a^2)\left[\frac{d\vec{r}}{dt}+\frac{d\vec{a}}{dt}\right]-(\vec{r}+\vec{a})\left[2r\frac{dr}{dt}\right]}{(r^2+a^2)^2} \quad \because \frac{d\vec{a}}{dt}=0$

(viii) ***Let*** $\vec{f}(t)=\frac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}$

Differentiate w.r.t t

$$\vec{f}'(t)=\frac{d}{dt}\left(\frac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}\right)=\frac{(\vec{r}.\vec{a})\frac{d}{dt}(\vec{r}\times\vec{a})-(\vec{r}\times\vec{a})\frac{d}{dt}(\vec{r}.\vec{a})}{(\vec{r}.\vec{a})^2}=\frac{(\vec{r}.\vec{a})\left[\frac{d\vec{r}}{dt}\times\vec{a}+\vec{r}\times\frac{d\vec{a}}{dt}\right]-(\vec{r}\times\vec{a})\left[\frac{d\vec{r}}{dt}.\vec{a}+\vec{r}.\frac{d\vec{a}}{dt}\right]}{(\vec{r}.\vec{a})^2}$$

$$\vec{f}'(t)=\frac{(\vec{r}.\vec{a})\left[\frac{d\vec{r}}{dt}\times\vec{a}+0\right]-(\vec{r}\times\vec{a})\left[\frac{d\vec{r}}{dt}.\vec{a}+0\right]}{(\vec{r}.\vec{a})^2} \qquad \therefore \frac{d\vec{a}}{dt}=0$$

$$\vec{f}'(t)=\frac{(\vec{r}.\vec{a})\left[\frac{d\vec{r}}{dt}\times\vec{a}\right]-(\vec{r}\times\vec{a})\left[\frac{d\vec{r}}{dt}.\vec{a}\right]}{(\vec{r}.\vec{a})^2}$$

$$\vec{f}'(t)=\frac{(\vec{r}.\vec{a})\left[\frac{d\vec{r}}{dt}\times\vec{a}\right]}{(\vec{r}.\vec{a})^2}-\frac{(\vec{r}\times\vec{a})\left[\frac{d\vec{r}}{dt}.\vec{a}\right]}{(\vec{r}.\vec{a})^2}=\frac{\left[\frac{d\vec{r}}{dt}\times\vec{a}\right]}{(\vec{r}.\vec{a})}-\frac{(\vec{r}\times\vec{a})\left[\frac{d\vec{r}}{dt}.\vec{a}\right]}{(\vec{r}.\vec{a})^2}$$

Example#08: A particle that move along a curve . $x=4\cos t, y=4\sin t, z=6t$ ***.Find velocity and acceleration at t = 0 & t =*** $\frac{\pi}{2}$***.***

Solution: Let $\vec{r}(t)$ be a position vector. Then

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

Putting $x=4\cos t, y=4\sin t, z=6t$, ***we get***

$$\vec{r}=4\cos t\,\hat{i}+4\sin t\,\hat{j}+6t\,\hat{k}$$

Velocity: Differentiate $\vec{r}$ ***w. r. t t.***

$$\vec{v}=\frac{d\vec{r}}{dt}=-4\sin t\,\hat{i}+4\cos t\,\hat{j}+6\,\hat{k}$$

At t = 0: $\vec{v}=-4\sin 0\,\hat{i}+4\cos 0\,\hat{j}+6\,\hat{k}=0\hat{i}+4\hat{j}+6\hat{k} \implies \vec{v}=4\hat{j}+6\hat{k}$

At $t=\frac{\pi}{2}$***:*** $\vec{v}=-4\sin\frac{\pi}{2}\hat{i}+4\cos\frac{\pi}{2}\hat{j}+6\,\hat{k}=-4\hat{i}+0\hat{j}+6\hat{k} \implies \vec{v}=-4\hat{i}+6\hat{k}$

Acceleration: Differentiate $\vec{v}$ ***w. r. t t.***

$$\vec{a}=\frac{d\vec{r}}{dt}=-4\cos t\,\hat{i}-4\sin t\,\hat{j}+0\,\hat{k}$$

At t = 0: $\vec{a}=-4\cos 0\,\hat{i}-4\sin 0\,\hat{j}+0\,\hat{k}=-4\hat{i}+0\hat{j}+0\hat{k} \implies \vec{a}=-4\hat{i}$

At $t=\frac{\pi}{2}$***:*** $\vec{a}=-4\cos\frac{\pi}{2}\hat{i}-4\sin\frac{\pi}{2}\hat{j}+0\,\hat{k}=0\hat{i}-4\hat{j}+0\hat{k} \implies \vec{a}=-4\hat{j}$

Exercise#3.2

Q#01 :If $\vec{f}(t) = (2t+1)\hat{\imath} + (3-2t^2)\hat{\jmath} + (t^2-1)\hat{k}$ **&** $\vec{g}(t) = (3+2t^2)\hat{\imath} + (3t+1)\hat{\jmath} + (2t-t^3)\hat{k}$.

Find $\frac{d}{dt}[\vec{f}+\vec{g}]$.

Solution: Given

$$\vec{f}(t) = (2t+1)\hat{\imath} + (3-2t^2)\hat{\jmath} + (t^2-1)\hat{k} \quad \& \quad \vec{g}(t) = (3+2t^2)\hat{\imath} + (3t+1)\hat{\jmath} + (2t-t^3)\hat{k}$$

Then $\vec{f}+\vec{g} = [(2t+1)\hat{\imath} + (3-2t^2)\hat{\jmath} + (t^2-1)\hat{k}] + [(3+2t^2)\hat{\imath} + (3t+1)\hat{\jmath} + (2t-t^3)\hat{k}]$

$$= (2t+1+3+2t^2)\hat{\imath} + (3-2t^2+3t+1)\hat{\jmath} + (t^2-1+2t-t^3)\hat{k}$$

$$\vec{f}+\vec{g} = (4+2t+2t^2)\hat{\imath} + (4+3t-2t^2)\hat{\jmath} + (-1+2t+t^2-t^3)\hat{k}$$

Now taking derivative w. r . t t

$$\frac{d}{dt}[\vec{f}+\vec{g}] = (2+4t)\hat{\imath} + (3-4t)\hat{\jmath} + (2+2t-3t^2)\hat{k}$$

Q#02: Find $\frac{d}{dt}[\vec{f}\cdot\vec{g}]$ **&** $\frac{d}{dt}[\vec{f}\times\vec{g}]$

(i) if $\vec{f}(t) = (3t^2+1)\hat{\imath} + (2t^3-1)\hat{\jmath} + (2t^2+3t^3)\hat{k}$ **&** $\vec{g}(t) = t\,\hat{\imath} + (t^2-2t)\hat{\jmath} + (3t-t^3)\hat{k}$

Solution: Given

$$\vec{f}(t) = (3t^2+1)\hat{\imath} + (2t^3-1)\hat{\jmath} + (2t^2+3t^3)\hat{k} \quad \& \quad \vec{g}(t) = t\,\hat{\imath} + (t^2-2t)\hat{\jmath} + (3t-t^3)\hat{k}$$

Then $\vec{f}\cdot\vec{g} = [(3t^2+1)\hat{\imath} + (2t^3-1)\hat{\jmath} + (2t^2+3t^3)\hat{k}]\cdot[t\,\hat{\imath} + (t^2-2t)\hat{\jmath} + (3t-t^3)\hat{k}]$

$$= (3t^2+1)t + (2t^3-1)(t^2-2t) + (2t^2+3t^3)(3t-t^3)$$

$$= 3t^3 + t + 2t^5 - 4t^4 - t^2 + 2t + 6t^3 - 2t^5 + 9t^4 - 3t^6$$

$$\vec{f}\cdot\vec{g} = 3t - t^2 + 9t^3 + 5t^4 - 3t^6$$

Now taking derivative w. r . t t

$$\frac{d}{dt}[\vec{f}\cdot\vec{g}] = 3 - 2t - 18t^2 + 20t^3 - 18t^5$$

$$\vec{f}\times\vec{g} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t^2+1 & 2t^3-1 & 2t^2+3t^3 \\ t & t^2-2t & 3t-t^3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 2t^3-1 & 2t^2+3t^3 \\ t^2-2t & 3t-t^3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 3t^2+1 & 2t^2+3t^3 \\ t & 3t-t^3 \end{vmatrix} + \hat{k}\begin{vmatrix} 3t^2+1 & 2t^3-1 \\ t & t^2-2t \end{vmatrix}$$

$$= \hat{\imath}\,[(2t^3-1)(3t-t^3) - (t^2-2t)(2t^2+3t^3)] - \hat{\jmath}[(3t^2+1)(3t-t^3) - (t)(2t^2+3t^3)]$$

$$+ \hat{k}[(3t^2+1)(t^2-2t) - (t)(2t^3-1)]$$

$= \hat{i}\,[6t^4 - 2t^6 - 3t + t^3 - 2t^4 - 3t^5 + 4t^3 + 6t^4] - \hat{j}[9t^3 - 3t^5 + 3t - t^3 - 2t^3 - 3t^4]$

$+ \hat{k}[3t^4 - 6t^3 + t^2 - 2t - 2t^4 + t]$

$\vec{f} \times \vec{g} = \hat{i}\,[-3t + 5t^3 - 10t^4 - 3t^5 - 2t^6] - \hat{j}[3t + 6t^2 - 3t^4 - 3t^5] + \hat{k}[-t + t^2 - 6t^3 + t^4]$

Now

$\frac{d}{dt}[\vec{f} \times \vec{g}] = [-3 + 15t^2 - 40t^3 - 15t^4 - 12t^5]\hat{i} - [3 + 12t - 12t^3 - 15t^4]\hat{j} + [-1 + 2t - 18t^2 + 4t^3]\hat{k}$

(ii)If $\vec{f}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \hat{k}$ ***&*** $\vec{g}(t) = t\hat{i} + (2t-1)\hat{j} + t^2\hat{k}$ **.**

Solution: Given $\vec{f}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \hat{k}$ ***&*** $\vec{g}(t) = t\hat{i} + (2t-1)\hat{j} + t^2\hat{k}$

Now $\vec{f}.\,\vec{g} = [\cos t\,\hat{i} + \sin t\,\hat{j} + \hat{k}].[t\hat{i} + (2t-1)\hat{j} + t^2\hat{k}] = (\cos t)t + (\sin t)(2t-1) + (1)(t^2)$

$\vec{f}.\,\vec{g} = t\cos t + 2t\sin t - \sin t + t^2$

Now taking derivative w. r . t t

$\frac{d}{dt}[\vec{f}.\,\vec{g}] = -t\sin t + \cos t + 2t\cos t + 2\sin t - \cos t + 2t = (2-t)\sin t + 2t\cos t + 2$

& $\vec{f} \times \vec{g} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 1 \\ t & 2t-1 & t^2 \end{vmatrix} = \hat{i}\begin{vmatrix} \sin t & 1 \\ 2t-1 & t^2 \end{vmatrix} - \hat{j}\begin{vmatrix} \cos t & 1 \\ t & t^2 \end{vmatrix} + \hat{k}\begin{vmatrix} \cos t & \sin t \\ t & 2t-1 \end{vmatrix}$

$= \hat{i}\,[(\sin t)(t^2) - (2t-1)(1)] - \hat{j}[(\cos t)(t^2) - (t)(1)] + \hat{k}[(\cos t)(2t-1) - (t)(\sin t)]$

$\vec{f} \times \vec{g} = \hat{i}\,[t^2\sin t - 2t + 1] - \hat{j}[t^2\cos t - t] + \hat{k}[2t\cos t - \cos t - t\sin t]$

Now taking derivative w. r . t t

$\frac{d}{dt}[\vec{f} \times \vec{g}] = [t^2\cos t + 2t\sin t - 2]\hat{i} - [-t^2\sin t + 2t\cos t - 1]\hat{j} + [-2t\sin t + 2\cos t + \sin t - t\cos t - \sin t]\hat{k}$

$\frac{d}{dt}[\vec{f} \times \vec{g}] = [t^2\cos t + 2t\sin t - 2]\hat{i} - [-t^2\sin t + 2t\cos t - 1]\hat{j} + [-2t\sin t + 2\cos t - t\cos t]\hat{k}$

Q#03: (i) If $\vec{r}$ ***is position vector of moving point then show that*** $\vec{r}\,.\,\frac{d\vec{r}}{dt} = r\,\frac{dr}{dt}$ ***here*** $|\vec{r}| = r$.

(ii)Interpret the relation $\vec{r}.\frac{d\vec{r}}{dt} = 0$ $\&\ \vec{r} \times \frac{d\vec{r}}{dt} = 0$ **.**

(i) If $\vec{r}$ ***is position vector of moving point then show that*** $\vec{r}\,.\,\frac{d\vec{r}}{dt} = r\,\frac{dr}{dt}$ ***here*** $|\vec{r}| = r$.

Solution: If $\vec{r}$ ***is position vector of moving point . Then***

$$\vec{r}\,.\,\vec{r} = r^2$$

Differentiate w.r.t t $\frac{d}{dt}[\vec{r}\,.\,\vec{r}] = \frac{d}{dt}\,r^2$

$$\frac{d\vec{r}}{dt}.\vec{r} + \vec{r}.\frac{d\vec{r}}{dt} = 2r\frac{dr}{dt}$$

$$2\ \vec{r}.\frac{d\vec{r}}{dt} = 2r\frac{dr}{dt} \qquad \therefore \vec{r}.\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dt}.\vec{r}$$

$$\vec{r}.\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$$ ***Hence proved.***

(ii)Interpret the relation $\vec{r}.\frac{d\vec{r}}{dt} = 0 \quad \& \ \vec{r} \times \frac{d\vec{r}}{dt} = 0$ **.**

(a) ***If*** $\vec{r}$ & $\frac{d\vec{r}}{dt}$ ***are perpendicular.*** ($\theta = 90^0$) ***Then***

As $\vec{r}.\frac{d\vec{r}}{dt} = |\vec{r}|\left|\frac{dr}{dt}\right|\cos 90^0 = |\vec{r}|\left|\frac{dr}{dt}\right|(0)$ ***Because*** $\cos 90^0 = 0$

$$\vec{r}.\frac{d\vec{r}}{dt} = 0$$

(b) ***If*** $\vec{r}$ & $\frac{d\vec{r}}{dt}$ ***are parallel .*** ($\theta = 0^0$) ***Then***

As $\vec{r} \times \frac{d\vec{r}}{dt} = |\vec{r}|\left|\frac{dr}{dt}\right|\sin 0^0 = |\vec{r}|\left|\frac{dr}{dt}\right|(0)$ ***Because*** $\sin 0^0 = 0$

$$\vec{r} \times \frac{d\vec{r}}{dt} = 0$$

Q#04: If $\vec{r} = \cos 5t\,\hat{\imath} + \sin 5t\,\hat{\jmath}$ ***. Then show that*** $\vec{r} \times \frac{d\vec{r}}{dt} = 5\hat{k}$.

Solution: Given vector function $\vec{r} = \cos 5t\,\hat{\imath} + \sin 5t\,\hat{\jmath}$ ***Then*** $\frac{d\vec{r}}{dt} = -5\sin 5t\,\hat{\imath} + 5\cos 5t\,\hat{\jmath}$

Now $\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos 5t & \sin 5t & 0 \\ -5\sin 5t & 5\cos 5t & 0 \end{vmatrix} = \hat{k}\begin{vmatrix} \cos 5t & \sin 5t \\ -5\sin 5t & 5\cos 5t \end{vmatrix}$ $\therefore$***Expanding by*** C_3

$$= \hat{k}[(\cos 5t)(5\cos 5t) - (-5\sin 5t)(\sin 5t)]$$

$$= 5\hat{k}[\cos^2 5t + \sin^2 5t]$$

Hence proved $\vec{r} \times \frac{d\vec{r}}{dt} = 5\hat{k}$.

Q#05: If $\vec{f}(t) = \vec{a}\cos\omega t + \vec{b}\sin\omega t$ ***then show that*** $\vec{f} \times \vec{f}' = \omega(\vec{a} \times \vec{b})$.

Solution: Given

$$\vec{f}(t) = \vec{a}\cos\omega t + \vec{b}\sin\omega t$$

Then $\vec{f}'(t) = -\vec{a}\omega\sin\omega t + \vec{b}\omega\cos\omega t$

Now

$$\vec{f} \times \vec{f}' = (\vec{a}\cos\omega t + \vec{b}\sin\omega t) \times (-\vec{a}\omega\sin\omega t + \vec{b}\omega\cos\omega t)$$

$$= -(\vec{a} \times \vec{a})\omega\cos\omega t\sin\omega t + (\vec{a} \times \vec{b})\omega\cos^2\omega t + (-\vec{b} \times \vec{a})\omega\sin^2\omega t + (\vec{b} \times \vec{b})\,\omega\cos\omega t\sin\omega t$$

$= 0 + (\vec{a} \times \vec{b})\omega\cos^2\omega t + (\vec{a} \times \vec{b})\omega\sin^2\omega t + 0$ $\qquad \therefore \vec{a} \times \vec{a} = 0 \;\&\; \vec{b} \times \vec{b} = 0$

$= \omega\,(\vec{a} \times \vec{b})[\cos^2\omega t + \sin^2\omega t]$ $\qquad \therefore \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ $\qquad \vec{f} \times$

$\vec{f}\,' = \omega\,(\vec{a} \times \vec{b})$

Hence proved

Q#06: If $\hat{r}$ is a unit vector then prove that $\left|\hat{r} \times \frac{d\hat{r}}{dt}\right| = \left|\frac{d\hat{r}}{dt}\right|$

Solution: If $\hat{r}$ is a unit vector.

Then $\left|\hat{r} \times \frac{d\hat{r}}{dt}\right| = |\hat{r}|\left|\frac{d\hat{r}}{dt}\right|\sin\theta$

We know that $\hat{r}$ & $\frac{d\hat{r}}{dt}$ ***are perpendicular vectors. Then*** $\theta = 90^0$

$$\left|\hat{r} \times \frac{d\hat{r}}{dt}\right| = |\hat{r}|\left|\frac{d\hat{r}}{dt}\right|\sin 90^0$$

$$\left|\hat{r} \times \frac{d\hat{r}}{dt}\right| = (1)\left|\frac{d\hat{r}}{dt}\right| \;(1) \qquad \therefore |\hat{r}| = 1 \;\&\; \sin 90^0 = 1$$

$$\left|\hat{r} \times \frac{d\hat{r}}{dt}\right| = \left|\frac{d\hat{r}}{dt}\right|$$

Hence proved.

Q#07: If $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t + \frac{\vec{c}}{\omega^2} t\sin\omega t$ ***then prove that*** $\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = \frac{2\vec{c}}{\omega}\cos\omega t.$

Where $\vec{a}, \vec{b}, \vec{c}$ are constant vectors and ω is a scalar.

Solution: Given vector function is $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t + \frac{\vec{c}}{\omega^2} t\sin\omega t$ ------*(i)*

Then $\frac{d\vec{r}}{dt} = \vec{a}\,\omega\cos\omega t - \vec{b}\,\omega\sin\omega t + \frac{\vec{c}}{\omega^2}\omega\, t\cos\omega t + \frac{\vec{c}}{\omega^2}\sin\omega t$

$$\frac{d\vec{r}}{dt} = \vec{a}\,\omega\cos\omega t - \vec{b}\,\omega\sin\omega t + \frac{\vec{c}}{\omega} t\cos\omega t + \frac{\vec{c}}{\omega^2}\sin\omega t$$

& $\frac{d^2\vec{r}}{dt^2} = -\vec{a}\,\omega^2\sin\omega t - \vec{b}\,\omega^2\cos\omega t - \frac{\vec{c}}{\omega}\omega t\sin\omega t + \frac{\vec{c}}{\omega}\cos\omega t + \frac{\vec{c}}{\omega^2}\omega\cos\omega t$

$$\frac{d^2\vec{r}}{dt^2} = -\vec{a}\,\omega^2\sin\omega t - \vec{b}\,\omega^2\cos\omega t - \vec{c}\,t\sin\omega t + \frac{\vec{c}}{\omega}\cos\omega t + \frac{\vec{c}}{\omega}\cos\omega t$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2[\vec{a}\sin\omega t + \vec{b}\cos\omega t + \frac{\vec{c}}{\omega^2} t\sin\omega t] + 2\frac{\vec{c}}{\omega}\cos\omega t$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r} + \frac{2\vec{c}}{\omega}\cos\omega t \qquad \therefore \textit{From(i)}$$

$$\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = \frac{2\vec{c}}{\omega}\cos\omega t$$

Hence proved.

Q#08: If $\vec{r} = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + a\,t\tan\alpha\,\hat{k}$ ***then show that***

(i) $\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right] = a^3\tan\alpha$ ***(ii)*** $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = a^2\sec\alpha$

Solution: Given vector function is $\vec{r} = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + a\,t\,\tan\alpha\,\hat{k}$

Then $\frac{d\vec{r}}{dt} = -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} + a\tan\alpha\,\hat{k}$ ***;***

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\hat{\imath} - a\sin t\,\hat{\jmath} + 0\,\hat{k}$$

& $\frac{d^3\vec{r}}{dt^3} = a\sin t\,\hat{\imath} - a\cos t\,\hat{\jmath} + 0\,\hat{k}$

(i) ***Now taking L.H.S***

$$\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt}\cdot\left(\frac{d^2\vec{r}}{dt^2}\times\frac{d^3\vec{r}}{dt^3}\right) = \begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = a\tan\alpha\begin{vmatrix} -a\cos t & -a\sin t \\ a\sin t & -a\cos t \end{vmatrix}$$ $\therefore$ ***Expandingby*** C_3

$$= a\tan\alpha\,[(-a\cos t)(-a\cos t) - (a\sin t)(-\sin t)] = a\tan\alpha[a^2\cos^2 t + a^2\sin^2 t]$$

$\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right] = a^3\tan\alpha$ ***Hence proved.***

(ii)Now Taking cross product

$$\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} a\cos t & a\tan\alpha \\ -a\sin t & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -a\sin t & a\tan\alpha \\ -a\cos t & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} -a\sin t & a\cos t \\ -a\cos t & -a\sin t \end{vmatrix}$$

$$= \hat{\imath}\,[(a\cos t)(0) - (-a\sin t)(a\tan\alpha)] - \hat{\jmath}[(-a\sin t)(0) - (-a\cos t)(a\tan\alpha)]$$

$$+\hat{k}[(-a\sin t)(-a\sin t) - (-a\cos t)(a\cos t)]$$

$$= \hat{\imath}\,[0 + a^2\sin t\,\tan\alpha] - \hat{\jmath}[0 + a^2\cos t\,\tan\alpha] + \hat{k}[a^2\sin^2 t + a^2\cos^2 t]$$

$$= \hat{\imath}\,[a^2\sin t\,\tan\alpha] - \hat{\jmath}[a^2\cos t\,\tan\alpha] + \hat{k}a^2\,[\sin^2 t + \cos^2 t]$$

$$\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2} = [a^2\sin t\,\tan\alpha]\hat{\imath} - \hat{\jmath}[a^2\cos t\,\tan\alpha]\hat{\jmath} + a^2\hat{k}$$

Taking magnitude $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(a^2\sin t\,\tan\alpha)^2 + (a^2\cos t\,\tan\alpha)^2 + (a^2)^2}$

$$= \sqrt{[(a^2)^2]\,[\sin^2 t\tan^2\alpha + \cos^2 t\,\tan^2\alpha + 1]}$$

$$= \sqrt{[(a^2)^2]\,[(\sin^2 t + \cos^2 t)\tan^2\alpha + 1]} = \sqrt{(a^2)^2[\tan^2\alpha + 1]}$$

$$= \sqrt{(a^2)^2[\sec^2\alpha]} = \sqrt{(a^2\sec\alpha)^2}$$

$\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = a^2\sec\alpha$ ***Hence proved***

Q#09: If $\vec{r} = \cos 2t\ \vec{a} + \sin 2t\ \vec{b}$ ***. Where*** $\vec{a}$ & $\vec{b}$ ***are constant vectors. Show that***

(i) $\frac{d^2\vec{r}}{dt^2} + 4\vec{r} = 0$ ***(ii)*** $\vec{r} \times \frac{d\vec{r}}{dt} = 2\,(\vec{a} \times \vec{b})$

Solution: Given vector $\vec{r} = \cos 2t\ \vec{a} + \sin 2t\ \vec{b}$ -------***(i)***

Then $\frac{d\vec{r}}{dt} = -2\sin 2t\ \vec{a} + 2\cos 2t\ \vec{b}$

(i) $\frac{d^2\vec{r}}{dt^2} = -4\cos 2t\ \vec{a} - 4\sin 2t\ \vec{b}$

$\frac{d^2\vec{r}}{dt^2} = -4\,[\cos 2t\ \vec{a} + \sin 2t\ \vec{b}]$

$\frac{d^2\vec{r}}{dt^2} = -4\vec{r}$ $\therefore$ From (i)

$\frac{d^2\vec{r}}{dt^2} + 4\vec{r} = 0$ ***Hence proved .***

(ii) Now $\vec{r} \times \frac{d\vec{r}}{dt} = (\cos 2t\ \vec{a} + \sin 2t\ \vec{b}) \times (-2\sin 2t\ \vec{a} + 2\cos 2t\ \vec{b})$

$= -2\cos 2t \sin 2t\,(\vec{a} \times \vec{a}) + 2\cos^2 2t(\vec{a} \times \vec{b}) + 2\sin^2 2t(-\vec{b} \times \vec{a})$

$+ 2\cos 2t \sin 2t\,(\vec{b} \times \vec{b})$ $\therefore \vec{a} \times \vec{a} = 0$ & $\vec{b} \times \vec{b} = 0$

$= 0 + 2\cos^2 2t(\vec{a} \times \vec{b}) + 2\sin^2 2t(\vec{a} \times \vec{b}) + 0$ $\therefore \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$= 2\,(\vec{a} \times \vec{b})[\cos^2 2t + \sin^2 2t]$

$\vec{r} \times \frac{d\vec{r}}{dt} = 2\,(\vec{a} \times \vec{b})$ ***Hence proved .***

Q#10: If $\vec{r} = \vec{a}\,e^{3t} + \vec{b}\,e^{-3t}$***. Where*** $\vec{a}$ & $\vec{b}$ ***are constant vectors. Show that*** $\frac{d^2\vec{r}}{dt^2} - 9\vec{r} = 0$

Solution: Given vector function $\vec{r} = \vec{a}\,e^{3t} + \vec{b}\,e^{-3t}$---------***(i)***

Then $\frac{d\vec{r}}{dt} = 3\vec{a}\,e^{3t} - 3\vec{b}\,e^{-3t}$

& $\frac{d^2\vec{r}}{dt^2} = 9\vec{a}\,e^{3t} + 9\,\vec{b}\,e^{-3t}$

$\frac{d^2\vec{r}}{dt^2} = 9[\,\vec{a}\,e^{3t} + \vec{b}\,e^{-3t}\,]$

$\frac{d^2\vec{r}}{dt^2} = 9\,\vec{r}$ $\therefore$ ***From (i)***

$\frac{d^2\vec{r}}{dt^2} - 9\vec{r} = 0$ ***Hence proved .***

***Q#11:** If* $\vec{r} = \vec{a}\, e^{2t} + \vec{b}\, e^{3t}$. *Where* $\vec{a}$ & $\vec{b}$ *are constant vectors. Show that* $\frac{d^2\vec{r}}{dt^2} - 5\frac{d\vec{r}}{dt} + 6\vec{r} = 0$

***Solution:** Given vector function* $\vec{r} = \vec{a}\, e^{2t} + \vec{b}\, e^{3t}$---------*(i)*

Then $\frac{d\vec{r}}{dt} = 2\vec{a}\, e^{2t} + 3\vec{b}\, e^{3t}$

& $\frac{d^2\vec{r}}{dt^2} = 4\vec{a}\, e^{2t} + 9\,\vec{b}\, e^{3t}$

Now $\frac{d^2\vec{r}}{dt^2} - 5\frac{d\vec{r}}{dt} + 6\vec{r} = 4\vec{a}\, e^{2t} + 9\,\vec{b}\, e^{3t} - 5[2\vec{a}\, e^{2t} + 3\vec{b}\, e^{3t}] + 6[\vec{a}\, e^{2t} + \vec{b}\, e^{3t}]$

$= 4\vec{a}\, e^{2t} + 9\,\vec{b}\, e^{3t} - 10\vec{a}\, e^{2t} - 15\vec{b}\, e^{3t} + 6\vec{a}\, e^{2t} + 6\,\vec{b}\, e^{3t}$

$\frac{d^2\vec{r}}{dt^2} - 5\frac{d\vec{r}}{dt} + 6\vec{r} = 0$

Hence proved .

Q#12: (i) If $\vec{r} = \boldsymbol{a}\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + b\,t\hat{k}$ ***then show that***

(a) $\left|\frac{d\vec{r}}{dt}\right|^2 = a^2 + b^2$ ***(b)*** $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2 = a^2\,(a^2 + b^2)$ ***(c)*** $\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right] = a^2\,\boldsymbol{b}$

(ii) If $\vec{r} = \vec{c_1}\, e^{\omega t} + \vec{c_2}\, e^{-\omega t}$. ***Where*** $\vec{c_1}$ & $\vec{c_2}$ ***are constant vectors . Show that*** $\frac{d^2\vec{r}}{dt^2} - \omega^2\vec{r} = 0$

(i) If $\vec{r} = \boldsymbol{a}\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + b\,t\hat{k}$ ***then show that***

Solution: Given vector function $\vec{r} = \boldsymbol{a}\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + b\,t\hat{k}$

Then $\frac{d\vec{r}}{dt} = -\boldsymbol{a}\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} + b\hat{k}$

& $\frac{d^2\vec{r}}{dt^2} = -\boldsymbol{a}\cos t\,\hat{\imath} - a\sin t\,\hat{\jmath} + 0\,\hat{k}$

$\frac{d^3\vec{r}}{dt^3} = \boldsymbol{a}\sin t\,\hat{\imath} - a\cos t\,\hat{\jmath} + 0\,\hat{k}$

(a) $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(a\sin t\,)^2 + (-a\cos t\,)^2 + (b)^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2\,[\sin^2 t + \cos^2 t] + b^2}$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{a^2 + b^2}$

Taking square on both sides

$\left|\frac{d\vec{r}}{dt}\right|^2 = a^2 + b^2$ ***Hence proved***

(b) $\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -a\sin t & a\cos t & b\\ -a\cos t & -a\sin t & 0\end{vmatrix}=\hat{\imath}\begin{vmatrix}a\cos t & b\\ -a\sin t & 0\end{vmatrix}-\hat{\jmath}\begin{vmatrix}-a\sin t & b\\ -a\cos t & 0\end{vmatrix}+\hat{k}\begin{vmatrix}-a\sin t & a\cos t\\ -a\cos t & -a\sin t\end{vmatrix}$

$$=\hat{\imath}\,[\,(a\cos t)(0)-(-a\sin t)(b)]-\hat{\jmath}[(-a\sin t)(0)-(-a\cos t)(b)]$$

$$+\hat{k}[(-a\sin t)(-a\sin t)-(-a\cos t)(a\cos t)]$$

$$=\hat{\imath}\,[0+ab\sin t\,]-\hat{\jmath}[0+ab\cos t\,]+\hat{k}[a^2\sin^2 t+a^2\cos^2 t]$$

$$=\hat{\imath}\,[ab\sin t\,]-\hat{\jmath}[ab\cos t\,]+\hat{k}a^2\,[\sin^2 t+\cos^2 t]$$

$$\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}=[ab\sin t\,]\hat{\imath}-\hat{\jmath}[ab\cos t\,]\hat{\jmath}+a^2\hat{k}$$

Taking magnitude $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|=\sqrt{(ab\sin t\,)^2+(ab\cos t\,)^2+(a^2)^2}=\sqrt{[a^2b^2]\,[\sin^2 t\,+\cos^2 t\,]\;+a^4}$

$$\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|=\sqrt{a^2b^2\;+a^4}=\sqrt{a^2(b^2\;+a^2)}$$

Taking square on both sides $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|^2=a^2\,(a^2+b^2\,)$

(c) $\left[\frac{d\vec{r}}{dt}\;\;\frac{d^2\vec{r}}{dt^2}\;\;\frac{d^3\vec{r}}{dt^3}\right]=\frac{d\vec{r}}{dt}\cdot\left(\frac{d^2\vec{r}}{dt^2}\times\frac{d^3\vec{r}}{dt^3}\right)=\begin{vmatrix}-a\sin t & a\cos t & b\\ -a\cos t & -a\sin t & 0\\ a\sin t & -a\cos t & 0\end{vmatrix}=b\begin{vmatrix}-a\cos t & -a\sin t\\ a\sin t & -a\cos t\end{vmatrix}$ ∴ ***Expanding by*** C_3

$$=b[(-a\cos t)(-a\;\cos t)-(a\sin t)(-\sin t)]=b[a^2\cos^2 t+a^2\sin^2 t]$$

$\left[\frac{d\vec{r}}{dt}\;\;\frac{d^2\vec{r}}{dt^2}\;\;\frac{d^3\vec{r}}{dt^3}\right]=a^2b$ ***Hence proved.***

(ii) If $\vec{r}=\overrightarrow{c_1}\,e^{\omega t}+\overrightarrow{c_2}\,e^{-\omega t}$. ***Where*** $\overrightarrow{c_1}$ & $\overrightarrow{c_2}$ ***are constant vectors . Show that*** $\frac{d^2\vec{r}}{dt^2}-\omega^2\vec{r}=0$

Solution: Given vector function $\vec{r}=\overrightarrow{c_1}\,e^{\omega t}+\overrightarrow{c_2}\,e^{-\omega t}$---------***(i)***

Then $\frac{d\vec{r}}{dt}=\omega\overrightarrow{c_1}\,e^{\omega t}-\omega\overrightarrow{c_2}\,e^{-\omega t}$

& $\frac{d^2\vec{r}}{dt^2}=\omega^2\overrightarrow{c_1}\,e^{\omega t}+\omega^2\,\overrightarrow{c_2}\,e^{-\omega t}$

$$\frac{d^2\vec{r}}{dt^2}=\omega^2[\,\overrightarrow{c_1}\,e^{\omega t}+\overrightarrow{c_2}\,e^{-\omega t}\,]$$

$\frac{d^2\vec{r}}{dt^2}=\omega^2\,\vec{r}$ ∴***From (i)***

$$\frac{d^2\vec{r}}{dt^2}-\omega^2\vec{r}=0$$

Hence proved.

Q#13: If $\vec{f}$ (t) is a vector function. Show that $\frac{d}{dt}(\vec{f}\times\vec{f}\,') = \vec{f}\times\vec{f}\,''$

Solution: If $\vec{f}$ (t) is a vector function. Then

$$\frac{d}{dt}(\vec{f}\times\vec{f}\,') = \frac{d}{dt}\vec{f}\times\vec{f}\,' + \vec{f}\times\frac{d}{dt}\vec{f}\,'$$

$$= \vec{f}\,'\times\vec{f}\,' + \vec{f}\times\vec{f}\,''$$

$$= \mathbf{0} + \vec{f}\times\vec{f}\,'' \qquad \therefore \vec{f}\,'\times\vec{f}\,' = 0$$

$$\frac{d}{dt}(\vec{f}\times\vec{f}\,') = \vec{f}\times\vec{f}\,''$$ ***Hence proved.***

Q#14: If $\vec{r} = t^3\hat{\imath} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{\jmath}$ ***. Where n is a constant . show that*** $\vec{r}\times\frac{d\vec{r}}{dt} = \hat{k}$.

Solution: Given vector function is

$$\vec{r} = t^3\hat{\imath} + \left(2t^3 - \frac{1}{5t^2}\right)\hat{\jmath}$$

Differentiate w. r. t t

$$\frac{d\vec{r}}{dt} = 3t^2\hat{\imath} + \left(6t^2 - \frac{-2}{5t^3}\right)\hat{\jmath} = 3t^2\hat{\imath} + \left(6t^2 + \frac{2}{5t^3}\right)\hat{\jmath}$$

Now $$\vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t^3 & 2 - \frac{1}{5t^2} & 0 \\ 3t^2 & 6t^2 + \frac{2}{5t^3} & 0 \end{vmatrix} = \mathbf{0}\,\hat{\imath} - 0\hat{\jmath} + \hat{k}\begin{vmatrix} t^3 & 2t^3 - \frac{1}{5t^2} \\ 3t^2 & 6t^2 + \frac{2}{5t^3} \end{vmatrix}$$

$$= \hat{k}\left[(t^3)\left(6t^2 + \frac{2}{5t^3}\right) - (3t^2)\left(2t^3 - \frac{1}{5t^2}\right)\right]$$

$$= \hat{k}\left[6t^5 + \frac{2}{5} - 6t^3 + \frac{3}{5}\right]$$

$$= \hat{k}\left[\frac{2}{5} + \frac{3}{5}\right] = \hat{k}\left[\frac{2+3}{5}\right] = \hat{k}\left[\frac{5}{5}\right]$$

$$= \hat{k}[1]$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \hat{k}$$ ***Hence proved.***

Q#15:If $\vec{a} = 5t^2\hat{\imath} + t\hat{\jmath} - t^3\hat{k}$ ***&*** $\vec{b} = \sin t\,\hat{\imath} - \cos t\,\hat{\jmath}$ ***(i)*** $\frac{d}{dt}(\vec{a}.\vec{b})$ ***(ii)*** $\frac{d}{dt}(\vec{a}.\vec{a})$ ***(iii)*** $\frac{d}{dt}(\vec{a}\times\vec{b})$

Solution: Given vectors $\vec{a} = 5t^2\hat{\imath} + t\hat{\jmath} - t^3\hat{k}$ ***&*** $\vec{b} = \sin t\,\hat{\imath} - \cos t\,\hat{\jmath}$

(i) $\vec{a}.\vec{b} = (5t^2\hat{\imath} + t\hat{\jmath} - t^3\hat{k}).(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + 0\,\hat{k}) = 5t^2\sin t - t\cos t - t^3(0)$

$\vec{a}.\vec{b} = 5t^2\sin t - t\cos t$

Now $\frac{d}{dt}(\vec{a}.\vec{b}) = 5[t^2\cos t + 2t\sin t] - [-t\sin t + \cos t] = 5t^2\cos t + 10t\sin t + t\sin t - \cos t$

$= (5t^2 - 1)\cos t + 11\,t\sin t$

(ii) $\vec{a}.\vec{a} = (5t^2\hat{\imath} + t\hat{\jmath} - t^3\hat{k}).(5t^2\hat{\imath} + t\hat{\jmath} - t^3\hat{k}) = (5t^2)(5t^2) + (t)(t) + (-t^3)(-t^3) = 25t^4 + t^2 + t^6$

$\vec{a}.\vec{a} = t^6 + 25t^4 + t^2$

Differentiate w.r.t t t

$\frac{d}{dt}(\vec{a}.\vec{a}) = 6t^5 + 100t^3 + 2t$

(iii) $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} t & -t^3 \\ -\cos t & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 5t^2 & -t^3 \\ \sin t & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 5t^2 & t \\ \sin t & -\cos t \end{vmatrix}$

$= \hat{\imath}[(t)(0) - (\cos t)(-t^3)] - \hat{\jmath}[(5t^2)(0) - (\sin t)(-t^3)] + \hat{k}[(5t^2)(-\cos t) - (\sin t)(t)]$

$= \hat{\imath}[0 - t^3\cos t] - \hat{\jmath}[0 + t^3\sin t] + \hat{k}[-5t^2\cos t - t\sin t]$

$= -\hat{\imath}[t^3\cos t] - \hat{\jmath}[t^3\sin t] + \hat{k}[-5t^2\cos t - t\sin t]$

Differentiate w.r.t t t

$\frac{d}{dt}(\vec{a}\times\vec{b}) = -\hat{\imath}[-t^3\sin t + 3t^2\cos t] - \hat{\jmath}[t^3\cos t + 3t^2\sin t] + \hat{k}[-5(-t^2\sin t + 2t\cos t) - (t\cos t + \sin t)]$

$= \hat{\imath}[t^3\sin t - 3t^2\cos t] - \hat{\jmath}[t^3\cos t + 3t^2\sin t] + \hat{k}[5t^2\sin t - 10t\cos t - t\cos t - \sin t]$

$= \hat{\imath}[t^3\sin t - 3t^2\cos t] - \hat{\jmath}[t^3\cos t + 3t^2\sin t] + \hat{k}[(5t^2 - 1)\sin t - 11\,t\cos t]$

Q#16: If $\vec{f}(t)$ is a vector function. Show that $\frac{d}{dt}[\vec{f}.(\vec{f}'\times\vec{f}'')] = \vec{f}.(\vec{f}''\times\vec{f}''')$

Solution: If $\vec{f}(t)$ is a vector function. Then

$\frac{d}{dt}[\vec{f}.(\vec{f}'\times\vec{f}'')] = \frac{d}{dt}\vec{f}.(\vec{f}'\times\vec{f}'') + [\vec{f}.(\frac{d}{dt}\vec{f}'\times\vec{f}'') + \vec{f}.(\vec{f}'\times\frac{d}{dt}\vec{f}'')$

$= \vec{f}'.(\vec{f}'\times\vec{f}'') + \vec{f}\times(\vec{f}''\times\vec{f}'') + \vec{f}\times(\vec{f}''\times\vec{f}''')$

$= 0 + 0 + \vec{f}.(\vec{f}''\times\vec{f}''')$

$\frac{d}{dt}[\vec{f}.(\vec{f}'\times\vec{f}'')] = \vec{f}.(\vec{f}''\times\vec{f}''')$ ***Hence proved.***

Q#17: Show that $\frac{d}{dt}\left[\vec{a}\times\frac{d\vec{b}}{dt}-\frac{d\vec{a}}{dt}\times\vec{b}\right]=\vec{a}\times\frac{d^2\vec{b}}{dt^2}-\frac{d^2\vec{a}}{dt^2}\times\vec{b}$

Solution: Let $\frac{d}{dt}\left[\vec{a}\times\frac{d\vec{b}}{dt}-\frac{d\vec{a}}{dt}\times\vec{b}\right]=\frac{d}{dt}\left[\vec{a}\times\frac{d\vec{b}}{dt}\right]-\frac{d}{dt}\left[\frac{d\vec{a}}{dt}\times\vec{b}\right]$

$$=\left[\frac{d\vec{a}}{dt}\times\frac{d\vec{b}}{dt}+\vec{a}\times\frac{d}{dt}\left(\frac{d\vec{b}}{dt}\right)\right]-\left[\frac{d}{dt}\left(\frac{d\vec{a}}{dt}\right)\times\vec{b}+\frac{d\vec{a}}{dt}\times\frac{d\vec{b}}{dt}\right]$$

$$=\frac{d\vec{a}}{dt}\times\frac{d\vec{b}}{dt}+\vec{a}\times\frac{d^2\vec{b}}{dt^2}-\frac{d^2\vec{a}}{dt^2}\times\vec{b}-\frac{d\vec{a}}{dt}\times\frac{d\vec{b}}{dt}$$

$$\frac{d}{dt}\left[\vec{a}\times\frac{d\vec{b}}{dt}-\frac{d\vec{a}}{dt}\times\vec{b}\right]=\vec{a}\times\frac{d^2\vec{b}}{dt^2}-\frac{d^2\vec{a}}{dt^2}\times\vec{b}$$ ***Hence proved.***

Q#18: If $\hat{r}(t)$ ***is a unit vector then show that*** $\hat{r}.\left(\hat{r}\times\frac{d^2\hat{r}}{dt^2}\right)+\left(\frac{d\hat{r}}{dt}\right)^2=1.$

Solution: If $\hat{r}(t)$ ***is a unit vector. Then***

$$\hat{r}.\hat{r}=|\hat{r}|^2$$

$$\hat{r}.\hat{r}=1 \quad \text{----------}(i) \qquad \therefore |\hat{r}|=1$$

Differentiate (i) w. r .t t $\frac{d}{dt}(\hat{r}.\hat{r})=\frac{d}{dt}(1)$

$$\frac{d\hat{r}}{dt}.\hat{r}+\hat{r}.\frac{d\hat{r}}{dt}=0 \qquad \therefore \frac{d\hat{r}}{dt}.\hat{r}=\hat{r}.\frac{d\hat{r}}{dt}$$

$$2\,\hat{r}.\frac{d\hat{r}}{dt}=0$$

$$\hat{r}.\frac{d\hat{r}}{dt}=0$$

Again differentiate w.r. t t $\frac{d}{dt}\left(\hat{r}.\frac{d\hat{r}}{dt}\right)=\frac{d}{dt}(0)$

$$\hat{r}.\frac{d^2\hat{r}}{dt^2}+\frac{d\hat{r}}{dt}.\frac{d\hat{r}}{dt}=0 \qquad \therefore \frac{d\hat{r}}{dt}.\frac{d\hat{r}}{dt}=\left(\frac{d\hat{r}}{dt}\right)^2$$

$$\hat{r}.\frac{d^2\hat{r}}{dt^2}+\left(\frac{d\hat{r}}{dt}\right)^2=0 \text{------------}(ii)$$

Adding (i) & (ii)

$$\hat{r}.\hat{r}+\hat{r}.\frac{d^2\hat{r}}{dt^2}+\left(\frac{d\hat{r}}{dt}\right)^2=1+0$$

$$\hat{r}.\left(\hat{r}.\frac{d^2\hat{r}}{dt^2}\right)+\left(\frac{d\hat{r}}{dt}\right)^2=1$$ ***Hence proved.***

Q#19: If $\hat{a}$ is a unit vector in the direction of $\vec{a}$. Then prove that $\hat{a} \times \frac{d\hat{a}}{dt} = \frac{1}{|\vec{a}|^2}\ \vec{a} \times \frac{d\vec{a}}{dt}$

Solution: If $\hat{a}$ is a unit vector in the direction of $\vec{a}$. Then

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \text{----------------(i)}$$

Differentiate (i) w. r .t t. $\frac{d\hat{a}}{dt} = \frac{1}{|\vec{a}|}\ \frac{d\vec{a}}{dt}$ ------------**(ii)**

Taking cross product of (i) & (ii)

$$\hat{a} \times \frac{d\hat{a}}{dt} = \frac{\vec{a}}{|\vec{a}|} \times \frac{1}{|\vec{a}|}\ \frac{d\vec{a}}{dt}$$

$$\hat{a} \times \frac{d\hat{a}}{dt} = \frac{1}{|\vec{a}|^2}\ \vec{a} \times \frac{d\vec{a}}{dt}$$ ***Hence proved.***

Q#20: If $\vec{r}(t)$ is a vector of magnitude 2. then show that $\vec{r}.\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) + \left(\frac{d\vec{r}}{dt}\right)^2 = 1.$

Solution: If $\vec{r}$ (t) is a vector of magnitude 2 . $\{\ |\vec{r}| = 2\ \}$

. Then $\vec{r}.\vec{r} = |\vec{r}|^2$

$\vec{r}.\vec{r} = 4$ -----------**(i)** $\therefore |\vec{r}|^2 = 4$

Differentiate (i) w. r .t t $\frac{d}{dt}\ (\vec{r}.\vec{r}) = \frac{d}{dt}(4)$

$\frac{d\vec{r}}{dt}.\vec{r} + \vec{r}.\frac{d\vec{r}}{dt} = 0$ $\therefore \frac{d\vec{r}}{dt}.\vec{r} = \vec{r}.\frac{d\vec{r}}{dt}$

$$2\,\vec{r}.\frac{d\vec{r}}{dt} = 0$$

$$\vec{r}.\frac{d\vec{r}}{dt} = 0$$

Again differentiate w.r. t t $\frac{d}{dt}\left(\vec{r}.\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\ (0)$

$\vec{r}.\frac{d^2\vec{r}}{dt^2} + \frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt} = 0$ $\therefore \frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)^2$

$$\vec{r}.\frac{d^2\vec{r}}{dt^2} + \left(\frac{d\vec{r}}{dt}\right)^2 = 0 \text{-------(ii)}$$

Adding (i) & (ii)

$$\vec{r}.\vec{r} + \vec{r}.\frac{d^2\vec{r}}{dt^2} + \left(\frac{d\vec{r}}{dt}\right)^2 = 4 + 0$$

$$\vec{r}.\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) + \left(\frac{d\vec{r}}{dt}\right)^2 = 4$$ ***Hence proved.***

Q#21: If $\vec{f}, \vec{g}$ & $\vec{h}$ ***are vector function of scalar variable t. then show that***

(i) $\frac{d}{dt}[\vec{f}\,.\,(\vec{g}\times\vec{h}\,]) = \vec{f}\,'.(\vec{g}\times\vec{h}) + \vec{f}\,.(\vec{g}\,'\times\vec{h}) + \vec{f}\,.(\vec{g}\times\vec{h}\,')$

(ii) $\frac{d}{dt}[\vec{f}\times(\vec{g}\times\vec{h}\,]) = \vec{f}\,'\times(\vec{g}\times\vec{h}) + \vec{f}\times(\vec{g}\,'\times\vec{h}) + \vec{f}\times(\vec{g}\times\vec{h}\,')$

(i) $\frac{d}{dt}[\vec{f}\,.\,(\vec{g}\times\vec{h}\,]) = \vec{f}\,'.(\vec{g}\times\vec{h}) + \vec{f}\,.(\vec{g}\,'\times\vec{h}) + \vec{f}\,.(\vec{g}\times\vec{h}\,')$

Solution: Let

$$\frac{d}{dt}[\vec{f}\,.\,(\vec{g}\times\vec{h}\,]) = \frac{d}{dt}\vec{f}.(\vec{g}\times\vec{h}) + \vec{f}\,.(\frac{d}{dt}\vec{g}\times\vec{h}) + \vec{f}\,.(\vec{g}\times\frac{d}{dt}\vec{h})$$

$$\frac{d}{dt}[\vec{f}\,.\,(\vec{g}\times\vec{h}\,]) = \vec{f}\,'.(\vec{g}\times\vec{h}) + \vec{f}\,.(\vec{g}\,'\times\vec{h}) + \vec{f}\,.(\vec{g}\times\vec{h}\,')$$ ***Hence proved.***

(ii) $\frac{d}{dt}[\vec{f}\times(\vec{g}\times\vec{h}\,]) = \vec{f}\,'\times(\vec{g}\times\vec{h}) + \vec{f}\times(\vec{g}\,'\times\vec{h}) + \vec{f}\times(\vec{g}\times\vec{h}\,')$

Solution: Let

$$\frac{d}{dt}[\vec{f}\times(\vec{g}\times\vec{h})] = \frac{d}{dt}\vec{f}\times(\vec{g}\times\vec{h}) + \vec{f}\times(\frac{d}{dt}\vec{g}\times\vec{h}) + \vec{f}\times(\vec{g}\times\frac{d}{dt}\vec{h})$$

$$\frac{d}{dt}[\vec{f}\times(\vec{g}\times\vec{h})] = \vec{f}\,'\times(\vec{g}\times\vec{h}) + \vec{f}\times(\vec{g}\,'\times\vec{h}) + \vec{f}\times(\vec{g}\times\vec{h}\,')$$ ***Hence proved.***

Q#22: If $\vec{f}, \vec{g}$ & $\vec{h}$ ***are vector functions of scalar variable t and if***

$\vec{f}\,' = \vec{h}\times\vec{f}$ & $\vec{g}\,' = \vec{h}\times\vec{g}$ ***Then show that*** $\frac{d}{dt}(\vec{f}\times\vec{g}) = \vec{h}\times(\vec{f}\times\vec{g})$.

Solution: Taking L.H.S

$$\frac{d}{dt}(\vec{f}\times\vec{g}) = \frac{d\vec{f}}{dt}\times\vec{g} + \vec{f}\times\frac{d\vec{g}}{dt} = \vec{f}\,'\times\vec{g} + \vec{f}\times\vec{g}\,'$$

Using given values

$$\vec{f}\,' = \vec{h}\times\vec{f} \quad \& \quad \vec{g}\,' = \vec{h}\times\vec{g}$$

$$\frac{d}{dt}(\vec{f}\times\vec{g}) = (\vec{h}\times\vec{f})\times\vec{g} + \vec{f}\times(\vec{h}\times\vec{g})$$

$$= (\vec{h}.\vec{g})\vec{f} - (\vec{f}.\vec{g})\vec{h} + (\vec{f}.\vec{g})\vec{h} - (\vec{f}.\vec{h})\vec{g}$$

$$\frac{d}{dt}(\vec{f}\times\vec{g}) = (\vec{h}.\vec{g})\vec{f} - (\vec{f}.\vec{h})\vec{g} \text{ ---------------}(i)$$

Now taking R.H.S

$$\vec{h}\times(\vec{g}\times\vec{f}) = (\vec{h}.\vec{g})\vec{f} - (\vec{f}.\vec{h})\vec{g} \text{---------------}(ii)$$

From (i) & (ii) Hence proved that

$$\frac{d}{dt}(\vec{f}\times\vec{g}) = \vec{h}\times(\vec{f}\times\vec{g})$$

Q#23: If $\hat{u}(t)$ is a unit vector then show that $\hat{u}.\left(\hat{u}\times\frac{d^2\hat{u}}{dt^2}\right)+\left(\frac{d\hat{u}}{dt}\right)^2 = 1.$

Solution: If $\hat{u}(t)$ is a unit vector. Then

$$\hat{u}.\hat{u} = |\hat{u}|^2$$

$$\hat{u}.\hat{u} = 1 \quad \text{----------}(i) \qquad \therefore |\hat{u}| = 1$$

Differentiate (i) w. r .t t $\frac{d}{dt}(\hat{u}.\hat{u}) = \frac{d}{dt}(1)$

$$\frac{d\hat{u}}{dt}.\hat{u} + \hat{u}.\frac{d\hat{u}}{dt} = 0 \qquad \therefore \frac{d\hat{u}}{dt}.\hat{u} = \hat{u}.\frac{d\hat{u}}{dt}$$

$$2\,\hat{u}.\frac{d\hat{u}}{dt} = 0$$

$$\hat{u}.\frac{d\hat{u}}{dt} = 0$$

Again differentiate w.r. t t $\frac{d}{dt}\left(\hat{u}.\frac{d\hat{u}}{dt}\right) = \frac{d}{dt}(0)$

$$\hat{u}.\frac{d^2\hat{u}}{dt^2} + \frac{d\hat{u}}{dt}.\frac{d\hat{u}}{dt} = 0 \qquad \therefore \frac{d\hat{u}}{dt}.\frac{d\hat{u}}{dt} = \left(\frac{d\hat{u}}{dt}\right)^2$$

$$\hat{u}.\frac{d^2\hat{u}}{dt^2} + \left(\frac{d\hat{u}}{dt}\right)^2 = 0 \text{-------------}(ii)$$

Adding (i) & (ii)

$$\hat{u}.\hat{u} + \hat{u}.\frac{d^2\hat{u}}{dt^2} + \left(\frac{d\hat{u}}{dt}\right)^2 = 1 + 0$$

$$\hat{u}.\left(\hat{u}.\frac{d^2\hat{u}}{dt^2}\right) + \left(\frac{d\hat{u}}{dt}\right)^2 = 1$$

Hence proved.

Q#24(i) Show that $\frac{d}{dt}\left[\vec{f}\times\vec{g}\,' - \vec{f}\,'\times\vec{g}\right] = \vec{f}\times\vec{g}\,'' - \vec{f}\,''\times\vec{g}$

Solution: Let

$$\frac{d}{dt}\left[\vec{f}\times\vec{g}\,' - \vec{f}\,'\times\vec{g}\right] = \frac{d}{dt}\left[\vec{f}\times\vec{g}\,'\right] - \frac{d}{dt}\left[\vec{f}\,'\times\vec{g}\right]$$

$$= \left[\frac{d\vec{f}}{dt}\times\vec{g}\,' + \vec{f}\times\frac{d}{dt}(\vec{g}\,')\right] - \left[\frac{d}{dt}(\vec{f}\,')\times\vec{g} + \vec{f}\,'\times\frac{d\vec{g}}{dt}\right]$$

$$= \vec{f}\,'\times\vec{g}\,' + \vec{f}\times\vec{g}\,'' - \vec{f}\,''\times\vec{g} - \vec{f}\,'\times\vec{g}\,'$$

$$\frac{d}{dt}\left[\vec{f}\times\vec{g}\,' - \vec{f}\,'\times\vec{g}\right] = \vec{f}\times\vec{g}\,'' - \vec{f}\,''\times\vec{g}$$

Hence proved

Q#24 (ii)If $\hat{f}(t)$ ***is a unit vector in the direction of vector*** $\vec{g}(t)$ ***. Then show that*** $\hat{f}\times\hat{f}' = \frac{\vec{g}\times\vec{g}'}{\vec{g}.\vec{g}}$ **.**

Solution: If $\hat{f}(t)$ ***is a unit vector in the direction of vector*** $\vec{g}(t)$***. Then***

$$\hat{f} = \frac{\vec{g}}{g} \qquad \therefore |\vec{g}| = g$$

$$g\,\hat{f} = \vec{g}$$

$$\vec{g} = g\,\hat{f} \text{------------------------------------}(i)$$

Differentiate w. r. t t

$$\vec{g}' = g\hat{f}' + g'\,\hat{f} \text{--------------------}(ii)$$

Taking cross product of equation (i) & (ii)

$$\vec{g}\times\vec{g}' = g\,\hat{f}\times[g\hat{f}' + g'\hat{f}]$$

$$\vec{g}\times\vec{g}' = g^2(\hat{f}\times\hat{f}') + g\,g'(\hat{f}\times\hat{f})$$

$$\vec{g}\times\vec{g}' = (\vec{g}.\vec{g})(\hat{f}\times\hat{f}') + g\,g'(0) \qquad \therefore g^2 = \vec{g}.\vec{g} \;\;\&\; \hat{f}\times\hat{f} = 0$$

$$\vec{g}\times\vec{g}' = (\vec{g}.\vec{g})(\hat{f}\times\hat{f}') + 0$$

$$\vec{g}\times\vec{g}' = (\vec{g}.\vec{g})(\hat{f}\times\hat{f}')$$

$$\frac{\vec{g}\times\vec{g}'}{\vec{g}.\vec{g}} = \hat{f}\times\hat{f}'$$

Hence proved

$$\hat{f}\times\hat{f}' = \frac{\vec{g}\times\vec{g}'}{\vec{g}.\vec{g}}$$

Q#25: If $\vec{r}\,(t) = 2t\,\hat{i} + t^2\hat{j} + \frac{1}{3}t^3\hat{k}$ ***then show that***

(i) $\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = 2t^2\,\hat{i} - 4t\,\hat{j} + 4\hat{k}$ ***(ii)*** $\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right] = 8$ ***(iii)*** $\vec{r}\,.\,\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$ ***here*** $|\vec{r}| = r$ **.**

Solution: Given vector function $\vec{r} = 2t\,\hat{i} + t^2\hat{j} + \frac{1}{3}t^3\hat{k}$

Then $\frac{d\vec{r}}{dt} = 2\,\hat{i} + 2t\,\hat{j} + \frac{1}{3}3t^2\hat{k} = 2\,\hat{i} + 2t\,\hat{j} + t^2\hat{k}$

$$\frac{d^2\vec{r}}{dt^2} = 0\,\hat{i} + 2\,\hat{j} + 2t\,\hat{k} \quad \& \quad \frac{d^3\vec{r}}{dt^3} = 0\,\hat{i} + 0\,\hat{j} + 2\,\hat{k}$$

(i) $\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2t & t^2 \\ 0 & 2 & 2t \end{vmatrix} = \hat{i}\begin{vmatrix} 2t & t^2 \\ 2 & 2t \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & t^2 \\ 0 & 2t \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 2t \\ 0 & 2 \end{vmatrix}$

$$= \hat{i}\,[\,(2t)(2t) - (2)(t^2)] - \hat{j}[(2)(2t) - (0)(t^2)] + \hat{k}[(2)(2) - (0)(2t)]$$

$$= \hat{i}\,[4t^2 - 2t^2] - \hat{j}[4t - 0\,] + \hat{k}[4 - 0] = 2t^2\,\hat{i} - 4t\hat{j} + 4\hat{k}$$

(ii) $\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt}\,.\left(\frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3}\right) = \begin{vmatrix} 2 & 2t & t^2 \\ 0 & 2 & 2t \\ 0 & 0 & 2 \end{vmatrix}$ $\therefore$ ***Expanding by R3***

$$= \mathbf{0 + 0 + 2}\begin{vmatrix} 2 & 2t \\ 0 & 2 \end{vmatrix} = 2\,[(\,2)\,(2) - (0)(2t)] = 2(4 - 0) = 2(4)$$

$\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right] = 8$ ***Hence proved.***

(iii) Given $\vec{r} = 2t\,\hat{i} + t^2\hat{j} + \frac{1}{3}t^3\hat{k}$

Now $\vec{r}\,.\,\frac{d\vec{r}}{dt} = (2t\,\hat{i} + t^2\hat{j} + \frac{1}{3}t^3\hat{k})\,.\,(2\,\hat{i} + 2t\,\hat{j} + t^2\hat{k}) = 2t\,(2) + t^2(2t\,) + \left(\frac{1}{3}t^3\right)(t^2)$

$\vec{r}\,.\,\frac{d\vec{r}}{dt} = 4t + 2t^3 + \frac{1}{3}t^5$ --***(i)***

Then $|\vec{r}| = \sqrt{(2t)^2 + (t^2)^2 + \left(\frac{1}{3}t^3\right)^2} = \sqrt{4\,t^2 + t^4 + \frac{1}{9}t^6}$

Taking square on both sides

$$r^2 = 4\,t^2 + t^4 + \frac{1}{9}t^6$$

Differentiate w.r.t t $2\,r\frac{dr}{dt} = 8t + 4t^3 + \frac{1}{9}(\,6\,t^5) = 8t + 4t^3 + \frac{2}{3}t^5$

Dividing by $r\frac{dr}{dt} = 4t + 2t^3 + \frac{1}{3}t^5$--***(ii)***

From (i) & (ii) hence proved $\vec{r}\,.\,\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$

Q#26: If $\vec{f}$ (t)is a vector function then show that $\frac{d}{dt}\left(\frac{\vec{f}}{|\vec{f}|}\right) = \frac{\vec{f}'(\vec{f}.\vec{f}) - \vec{f}(\vec{f}.\vec{f}')}{(\vec{f}.\vec{f})^{3/2}}$.

***Solution:* Let** $\vec{f} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}$

Taking product with itself

$$\vec{f}.\vec{f} = (f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}).(f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}) = f_1^2 + f_2^2 + f_3^2 \text{ ------}(i)$$

Taking magnitude of given vector.

$$|\vec{f}| = \sqrt{f_1^2 + f_2^2 + f_3^2}$$

$$|\vec{f}| = (\vec{f}.\vec{f})^{1/2} \text{ --}(ii) \therefore \text{using equ.}(i)$$

If $\vec{f}' = f_1'\hat{\imath} + f_2'\hat{\jmath} + f_3'\hat{k}$

Then $\vec{f}.\vec{f}' = |\vec{f}||\vec{f}'|$ --***(iii)***

Now taking

$$\textbf{L.H.S} = \frac{d}{dt}\left(\frac{\vec{f}}{|\vec{f}|}\right) = \frac{|\vec{f}|\,\vec{f}' - \vec{f}\,|\vec{f}'|}{(|\vec{f}|)^2} \qquad \therefore \text{By applyiong quotient rule}$$

$$= \frac{|\vec{f}|\left[|\vec{f}|\,\vec{f}' - \vec{f}\,|\vec{f}'|\right]}{(|\vec{f}|)^3} \qquad \therefore \textit{Multiplying and dividing by } |\vec{f}|$$

$$= \frac{|\vec{f}||\vec{f}|\,\vec{f}' - \vec{f}\,|\vec{f}|\,|\vec{f}'|}{\left[(\vec{f}.\vec{f})^{1/2}\right]^3} \qquad \therefore \textit{From(ii)} \ |\vec{f}| = (\vec{f}.\vec{f})^{1/2}$$

$$= \frac{(|\vec{f}|)^2\,\vec{f}' - \vec{f}\,(\vec{f}.\vec{f}')}{(\vec{f}.\vec{f})^{3/2}} \qquad \therefore \textit{From(iii)} \ \vec{f}.\vec{f}' = |\vec{f}||\vec{f}'|$$

$$= \frac{(\vec{f}.\vec{f})\,\vec{f}' - \vec{f}\,(\vec{f}.\vec{f}')}{(\vec{f}.\vec{f})^{3/2}} \qquad \therefore \vec{f}.\vec{f} = (|\vec{f}|)^2$$

=R.H.S

Hence proved That L.H.S =R.H.S

Q#27: Show that

(i) Necessary and sufficient condition for a vector $\vec{f}$ of scalar variable t to be a constant is $\frac{d\vec{f}}{dt} = 0$

***Proof:** By given condition. That $\vec{f}$ be constant vector. Then $\vec{f}$ = constant*

Differentiate w .r .t t $\frac{d\vec{f}}{dt} = \frac{d}{dt}(\text{constant}) \Rightarrow \frac{d\vec{f}}{dt} = 0$

Conversely, suppose that $\frac{d\vec{f}}{dt} = 0 \Rightarrow d\vec{f} = 0\, dt$

On integrating both sides $\int d\vec{f} = \int 0\, dt$

$$\vec{f} = 0.t + \text{constant} \Rightarrow \vec{f} = \text{constant}$$

Hence prove that

The Necessary and sufficient condition for a vector $\vec{f}$ of scalar variable t to be a constant is $\frac{d\vec{f}}{dt} = 0$.

(ii)Necessary and sufficient condition for a vector $\vec{f}$ of scalar variable t to have a constant magnitude is $\vec{f}.\frac{d\vec{f}}{dt} = 0$.

***Proof:** By given condition. That vector $\vec{f}$ have a constant magnitude. Then $|\vec{f}|$ = constant*

Taking square on both sides $|\vec{f}|^2 = (\text{constant})2 = \text{constant}$

We know that $\vec{f}.\vec{f} = |\vec{f}|^2$ *then* $\vec{f}.\vec{f} = \text{constant}$

Differentiate w.r.t t $\frac{d}{dt}(\vec{f}.\vec{f}) = \frac{d}{dt}(\text{constant})$

$$\frac{d\vec{f}}{dt}.\vec{f} + \vec{f}.\frac{d\vec{f}}{dt} = 0 \qquad \therefore \vec{f}.\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{dt}.\vec{f}$$

$$2\,\vec{f}.\frac{d\vec{f}}{dt} = 0 \Rightarrow \vec{f}.\frac{d\vec{f}}{dt} = 0$$

Conversely , suppose that $\vec{f}.\frac{d\vec{f}}{dt} = 0 \qquad \therefore \vec{f}.\frac{d\vec{f}}{dt} = f\frac{d f}{dt}$

$$f\frac{d f}{dt} = 0 \Rightarrow f\, df = 0\, dt$$

on integrating both sides $\int f\, df = \int 0\, dt$

$$\Rightarrow \frac{|\vec{f}|^2}{2} = 0.t + \text{constant} \Rightarrow |\vec{f}|^2 = 2(\text{constant})$$

Taking square-root on both sides $|\vec{f}| = \sqrt{2(\text{constant})} \Rightarrow |\vec{f}| = \text{constant}$

Hence prove that Necessary and sufficient condition for a vector $\vec{f}$ of scalar variable t to have a constant magnitude is $\vec{f}.\frac{d\vec{f}}{dt} = 0$

(iii) Necessary and sufficient condition for a vector $\vec{f}$ of scalar variable t to have a constant direction is $\vec{f} \times \frac{d\vec{f}}{dt} = 0$

Proof: *Let $\hat{r}$ be unit vector in the direction of vector $\vec{f}$. By given condition, that direction is constant.*

$\hat{r}$ = *constant*   *Then* $\frac{d\hat{r}}{dt} = 0$ --------------------(i)

As we know that $\hat{r} = \frac{\vec{f}}{f} \Rightarrow \vec{f} = f\,\hat{r}$ ------------------(ii) $\because |\vec{f}| = f$

Differentiate w.r.t t $\frac{d\vec{f}}{dt} = \frac{d}{dt}(f\,\hat{r})$

$$\frac{d\vec{f}}{dt} = \frac{df}{dt}\hat{r} + f\frac{d\hat{r}}{dt} \text{ ----------------------(iii)}$$

Takin cross product of equation (ii) & (iii)

$$\vec{f} \times \frac{d\vec{f}}{dt} = f\,\hat{r} \times \left(\frac{df}{dt}\hat{r} + f\frac{d\hat{r}}{dt}\right)$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = f\frac{df}{dt}(\hat{r} \times \hat{r}) + f^2\left(\hat{r} \times \frac{d\hat{r}}{dt}\right)$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = f\frac{df}{dt}(0) + f^2\left(\hat{r} \times \frac{d\hat{r}}{dt}\right) \qquad \because \hat{r} \times \hat{r} = 0$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = 0 + f^2\left(\hat{r} \times \frac{d\hat{r}}{dt}\right)$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = f^2\left(\hat{r} \times \frac{d\hat{r}}{dt}\right) \text{ ------------------------------(iv)}$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = f^2(\hat{r} \times 0) \qquad \because \textit{From (i)} \quad \frac{d\hat{r}}{dt} = 0$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = 0$$

Conversely, suppose that

$$\vec{f} \times \frac{d\vec{f}}{dt} = 0$$

Then equation (iv) will become

$$f^2\left(\hat{r} \times \frac{d\hat{r}}{dt}\right) = 0 \quad \Rightarrow \quad \hat{r} \times \frac{d\hat{r}}{dt} = 0$$

Here $\hat{r} \neq 0$ *but* $\frac{d\hat{r}}{dt} = 0$ *Therefore* $\hat{r}$ = *constant*

Hence prove that Necessary and sufficient condition for a vector $\vec{f}$ of scalar variable t to have a constant direction is $\vec{f} \times \frac{d\vec{f}}{dt} = 0$.

Q#28: A particle that move along a curve . $x = 2t^2, y = t^2 - 4t, z = 3t - 5$ ***. Where t is time Find component of velocity and acceleration at t =1 in the direction of*** $\hat{i} + 3\hat{j} + 3\hat{k}$ ***.***

Solution: Let $\vec{r}(t)$ be a position vector. Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Putting $x = 2t^2, y = t^2 - 4t, z = 3t - 5$

$$\vec{r} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$$

Velocity***: Differentiate w. r. t t.*** $\vec{v} = \frac{d\vec{r}}{dt} = 4t\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$

At t = 1: $\vec{v} = 4\hat{i} + [2(2) - 4]\hat{j} + 3\hat{k} = 4\hat{i} - 2\hat{j} + 3\hat{k} \implies \vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

Acceleration: ***Differentiate w. r. t t.*** $\vec{a} = \frac{d\vec{v}}{dt} = 0\hat{i} + 2\hat{j} + 0\hat{k}$

At t = 1: $\vec{a} = 4\hat{i} + 2\hat{j} + 0\hat{k} \implies \vec{a} = 4\hat{i} + 2\hat{j}$

Let $\vec{u} = \hat{i} + 3\hat{j} + 3\hat{k}$ ***Then*** $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\hat{i}+3\hat{j}+3\hat{k}}{\sqrt{(1)^2+(3)^2+(3)^2}} = \frac{\hat{i}+3\hat{j}+3\hat{k}}{\sqrt{1+9+9}} = \frac{\hat{i}+3\hat{j}+3\hat{k}}{\sqrt{19}}$

Now

Component of $\vec{v}$ ***along*** $\vec{u} = \vec{v}.\hat{u} = (4\hat{i} - 2\hat{j} + 3\hat{k}).\left(\frac{\hat{i}+3\hat{j}+3\hat{k}}{\sqrt{19}}\right) = \frac{(4\hat{i}-2\hat{j}+3\hat{k}).(\hat{i}+3\hat{j}+3\hat{k})}{\sqrt{19}} = \frac{4-6+9}{\sqrt{19}} = \frac{7}{\sqrt{19}}$

Component of $\vec{a}$ ***along*** $\vec{u} = \vec{a}.\hat{u} = (4\hat{i} + 2\hat{j} + 0\hat{k}).\left(\frac{\hat{i}+3\hat{j}+3\hat{k}}{\sqrt{19}}\right) = \frac{(4\hat{i}+2\hat{j}+0\hat{k}).(\hat{i}+3\hat{j}+3\hat{k})}{\sqrt{19}} = \frac{4+6+0}{\sqrt{19}} = \frac{10}{\sqrt{19}}$

Q#29: A particle moves , so that its position vector is given by $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ ***. Where ω is constant. Show that (i) the velocity*** $\vec{v}$ ***of a particle is perpendicular to*** $\vec{r}$ ***(ii) The acceleration*** $\vec{a}$ ***is directed toward the origin and has magnitude proportional to the displacement*** $\vec{r}$ ***from the origin. (iii)*** $\vec{r} \times \vec{v} = \vec{c}$ ***.*** ***(***$\vec{c}$ is constant vector)

Solution: Given position vector $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ ------------------------***(i)***

Velocity: Differentiate w. r. t t. $\vec{v} = \frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$-----------***(ii)***

Acceleration: Differentiate w. r. t t . $\vec{a} = \frac{d\vec{v}}{dt} = -\omega^2\cos\omega t\,\hat{i} - \omega^2\sin\omega t\,\hat{j}$-------***(iii)***

(i) we have to prove $\vec{v} \perp \vec{r}$ ***for this*** $\vec{v}.\vec{r} = 0$

$$\vec{v}.\vec{r} = (-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}).(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}) = -\omega\sin\omega t\cos\omega t + \omega\sin\omega t\cos\omega t$$

$\vec{v}.\vec{r} = 0$

Hence prove $\vec{v} \perp \vec{r}$

(ii) We have to prove $\vec{a} \propto -\vec{r}$.

For this using (iii) $\vec{a} = -\omega^2 \cos\omega t\,\hat{i} - \omega^2 \sin\omega t\,\hat{j} = -\omega^2[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\,]$

$\vec{a} = -\omega^2\,\vec{r}$ ∴*From (i)*

This shows that $\vec{a} \propto -\vec{r}$ *. Negative sign indicate the acceleration* $\vec{a}$ *is directed toward the origin .*

(iii) We have to prove $\vec{r} \times \vec{v} = \vec{c}$ *.* ($\vec{c}$ is constant vector)

$$\vec{r} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix} = \hat{k}\begin{vmatrix} \cos\omega t & \sin\omega t \\ -\omega\sin\omega t & \omega\cos\omega t \end{vmatrix} \quad \therefore \textit{Expanding by } C3$$

$$= \hat{k}[(\cos\omega t)(\omega\cos\omega t) - (-\omega\sin\omega t)(\sin\omega t)] = \hat{k}[\omega^2\cos^2\omega t + \omega^2\sin^2\omega t]$$

$$= \hat{k}[\omega^2(\cos^2\omega t + \sin^2\omega t)\,]$$

$\vec{r} \times \vec{v} = \omega^2\hat{k}$ ***Hence proved*** $\vec{r} \times \vec{v} = \vec{c}$ ***Here*** $\vec{c} = \omega^2\hat{k}$ *(*$\vec{c}$ is constant vector)

Q#30: A particle moves along a curve whose parametric equation are $x = e^{-t}, y = 2\cos 3t\,, z = 2\sin 3t\,,$

where t is time

(a) Determine its velocity and acceleration at any time t (b) Find magnitudes of velocity and acceleration at t = 0.

Solution: Let $\vec{r}$ (t) be a position vector. Then $\vec{r} = x\hat{i} + y\hat{j} + z\,\hat{k}$

Putting $x = e^{-t}, y = 2\cos 3t$ and $z = 2\sin 3t$

$$\vec{r} = e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

(a) <u>Velocity</u>: Differentiate w. r. t t. $\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$

<u>Acceleration:</u> Differentiate w. r. t t. $\vec{a} = \frac{d\vec{v}}{dt} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$

(b) <u>Magnitude of Velocity:</u> at $t = 0$

$$\vec{v} = -e^{-0}\hat{i} - 6\sin 3(0)\,\hat{j} + 6\cos 3(0)\,\hat{k} = -1\hat{i} - 0\hat{j} + 6\,\hat{k}$$

$$|\vec{v}| = \sqrt{(-1)^2 + (0)^2 + (6)^2} = \sqrt{1 + 0 + 36} = \sqrt{37}$$

<u>Magnitude of Acceleration</u>: at t $= 0$

$$\vec{a} = \frac{d\vec{v}}{dt} = e^{-0}\hat{i} - 18\cos 3(0)\,\hat{j} - 18\sin 3(0)\,\hat{k} = 1\hat{i} - 18\hat{j} + 0\hat{k}$$

$$|\vec{a}| = \sqrt{(1)^2 + (18)^2 + (0)^2} = \sqrt{1 + 324 + 0} = \sqrt{325}$$

Q#31: Find the velocity and acceleration of a particle moves along a curve whose parametric equation are

$x = 2\sin 3t$, $y = 2\cos 3t$ and $z = 8t$ ***at any time. Find the magnitude of velocity and acceleration.***

Solution: ***Let*** $\vec{r}\,(t)$ be a position vector. Then $\vec{r} = x\hat{i} + y\hat{j} + z\,\hat{k}$

Putting $x = 2\sin 3t$, $y = 2\cos 3t$ and $z = 8t$ ***Then*** $\vec{r} = 2\sin 3t\,\hat{i} + 2\cos 3t\,\hat{j} + 8t\hat{k}$

Velocity: ***Differentiate w. r. t t.*** $\vec{v} = \frac{d\vec{r}}{dt} = 6\cos 3t\,\hat{i} - 6\sin 3t\,\hat{j} + 8\hat{k}$

$|\vec{v}| = \sqrt{(6\cos 3t)^2 + (-6\sin 3t)^2 + (8)^2} = \sqrt{36\cos^2 3t + 36\sin^2 3t + 64} = \sqrt{36[\cos^2 3t + \sin^2 3t] + 64} = \sqrt{100} = \boldsymbol{10}$

Acceleration: ***Differentiate w. r. t t.*** $\vec{a} = \frac{d\vec{v}}{dt} = -18\sin 3t\,\hat{i} - 18\cos 3t\,\hat{j} + 0\hat{k}$

$|\vec{a}| = \sqrt{(-18\sin 3t)^2 + (-18\cos 3t)^2 + (0)^2} = \sqrt{324\sin^2 3t + 324\cos^2 3t + 0} = \sqrt{324[\sin^2 3t + \cos^2 3t]} = \sqrt{324} = \boldsymbol{18}$

Q#32: A particle that move along a curve . $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$. ***Where t is time Find component of velocity and acceleration at t=1 in the direction of*** $\vec{b} = \hat{i} - 3\hat{j} + 2\,\hat{k}$ **.**

Solution: Let $\vec{r}\,(t)$ be a position vector. Then $\vec{r} = x\hat{i} + y\hat{j} + z\,\hat{k}$

Putting $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ ***Then*** $\vec{r} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\,\hat{k}$

Velocity: ***Differentiate w. r. t t.*** $\vec{v} = \frac{d\vec{r}}{dt} = 4\,t\,\hat{i} + (2t - 4)\hat{j} + 3\,\hat{k}$

At t = 1: $\vec{v} = 4\hat{i} + [2(2) - 4]\hat{j} + 3\,\hat{k} = 4\hat{i} - 2\hat{j} + 3\,\hat{k} \implies \vec{v} = 4\hat{i} - 2\hat{j} + 3\,\hat{k}$

Acceleration: ***Differentiate w. r. t t*** $\vec{a} = \frac{d\vec{v}}{dt} = 0\hat{i} + 2\hat{j} + 0\,\hat{k}$

At t = 1 $\vec{a} = 4\hat{i} + 2\hat{j} + 0\,\hat{k} \implies \vec{a} = 4\hat{i} + 2\hat{j}$

Let $\vec{b} = \hat{i} - 3\hat{j} + 2\hat{k}$ ***Then*** $\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-3)^2 + (2)^2}} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1 + 9 + 4}} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}}$

Now

Component of $\vec{v}$ ***along*** $\vec{b} = \vec{v}.\,\hat{b} = (4\hat{i} - 2\hat{j} + 3\,\hat{k}).\left(\frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}}\right) = \frac{(4\hat{i} - 2\hat{j} + 3\,\hat{k}).\,(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}}$

$$= \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}} = \frac{16\sqrt{14}}{\sqrt{14}.\sqrt{14}} = \frac{16\sqrt{14}}{14} = \frac{8\sqrt{14}}{7}$$

Component of $\vec{a}$ ***along*** $\vec{b} = \vec{a}.\,\hat{b} = (4\hat{i} + 2\hat{j} + 0\,\hat{k}).\left(\frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}}\right) = \frac{(4\hat{i} + 2\hat{j} + 0\,\hat{k}).\,(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}}$

$$= \frac{4 - 6 + 0}{\sqrt{14}} = \frac{-2\sqrt{14}}{\sqrt{14}.\sqrt{14}} = \frac{-2\sqrt{14}}{14} = \frac{-\sqrt{14}}{7}$$

INTEGRATION OF A VECTOR FUNCTION :

Integration of a vector function is define as the inverse or reverse process of differentiation.

Let $\vec{f}(t)$ & $\vec{g}(t)$ ***are two vector function . such that*** $\frac{d}{dt}[\vec{g}(t)] = \vec{f}(t)$ ***Then*** $\int \vec{f}(t)\, dt = \vec{g}(t) + c$

{ c is a constant of integration}. This is called indefinite integral of a vector function.

Definite integral is define on the interval [a, b] as $\int_a^b \vec{f}(t) = |\vec{g}(t)|_a^b = \vec{g}(b) - \vec{g}(a)$.

Theorem # I: if $\vec{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$ ***Then prove that***

$\int \vec{f}(t)\, dt = \hat{i}\int f_1(t)\, dt + \hat{j}\int f_2(t)\, dt + \hat{k}\int f_1(t)\, dt$

Proof: Let $\frac{d}{dt}[\vec{F}(t)] = \vec{f}(t)$ --------------------------***(i)***

Then $\int \vec{f}(t)\, dt = \vec{F}(t)$ ------------------------***(ii)***

Let $\vec{F}(t) = F_1(t)\,\hat{i} + F_2(t)\,\hat{j} + F_3(t)\,\hat{k}$ ------***(iii)***

Put in equation (i) $\frac{d}{dt}\left[F_1(t)\,\hat{i} + F_2(t)\,\hat{j} + F_3(t)\,\hat{k}\right] = \vec{f}(t)$

$$\frac{d}{dt}[F_1(t)]\,\hat{i} + \frac{d}{dt}[F_2(t)]\,\hat{j} + \frac{d}{dt}[F_3(t)]\,\hat{k} = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$$

Equating coefficients of $\hat{i}, \hat{j}$ & $\hat{k}$

$$\frac{d}{dt}[F_1(t)] = f_1(t) \quad \Rightarrow \quad \int f_1(t)\, dt = F_1(t)$$

$$\frac{d}{dt}[F_2(t)] = f_2(t) \quad \Rightarrow \quad \int f_2(t)\, dt = F_2(t)$$

$$\frac{d}{dt}[F_3(t)] = f_3(t) \quad \Rightarrow \quad \int f_3(t)\, dt = F_3(t)$$

Using values in equation (iii)

$$\vec{F}(t) = \left[\int f_1(t)\, dt\right]\hat{i} + \left[\int f_2(t)\, dt\right]\hat{j} + \left[\int f_1(t)\, dt\right]\hat{k}$$

Equation (ii) will become

$$\int \vec{f}(t)\, dt = \hat{i}\int f_1(t)\, dt + \hat{j}\int f_2(t)\, dt + \hat{k}\int f_1(t)\, dt$$

Hence proved.

Example#01: If $\vec{f}(t) = (t - t^2)\hat{i} + 2t^3\,\hat{j} - 3\,\hat{k}$ Find ***(i)*** $\int \vec{f}(t)\ dt$ ***(ii)*** $\int_1^2 \vec{f}(t)\ dt$

Solution: Given $\vec{f}(t) = (t - t^2)\hat{i} + 2t^3\,\hat{j} - 3\,\hat{k}$

$$\text{(i) } \int \vec{f}(t)\ dt = \int\left[(t - t^2)\hat{i} + 2t^3\,\hat{j} - 3\,\hat{k}\right] d = \hat{\imath}\left[\int (t - t^2)\ dt\right] + \hat{\jmath}\left[\,2\int t^3\ dt\,\right] + \hat{k}\left[-\int 3\ dt\right]$$

$$= \hat{\imath}\left[\frac{t^2}{2} - \frac{t^3}{3}\right] + \hat{\jmath}\left[\,2\left(\frac{t^4}{4}\right)\right] + \hat{k}[-3t]$$

$$\int \vec{f}(t)\ dt = \hat{\imath}\left[\frac{t^2}{2} - \frac{t^3}{3}\right] + \hat{\jmath}\left[\frac{t^4}{2}\right] - 3\hat{k}[t] + c$$ ***{c is constant of integration}***

$$\textbf{(ii) } \int_1^2 \vec{f}(t)\ dt = \hat{\imath}\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_1^2 + \hat{\jmath}\left[\frac{t^4}{2}\right]_1^2 - 3\hat{k}[t]_1^2 = \hat{\imath}\left[\left(\frac{2^2}{2} - \frac{2^3}{3}\right) - \left(\frac{1^2}{2} - \frac{1^3}{3}\right)\right] + \hat{\jmath}\left[\frac{2^4}{2} - \frac{1^4}{2}\right] - 3\hat{k}[2 - 1]$$

$$= \hat{\imath}\left[2 - \frac{8}{3} - \frac{1}{2} + \frac{1}{3}\right] + \hat{\jmath}\left[\,8 - \frac{1}{2}\right] - 3\hat{k}[1]$$

$$\int_1^2 \vec{f}(t)\ dt = \frac{-5}{6}\hat{\imath} + \frac{15}{2}\hat{\jmath} - 3\hat{k}$$

Example# 02:Solve $\vec{a} \times \frac{d^2\vec{v}}{dt^2} = \vec{b}$ **.** $\vec{a}$ ***&*** $\vec{b}$ ***are constant vectors and*** $\vec{v}$ ***is a vector function of t.***

Solution: Given equation is $\vec{a} \times \frac{d^2\vec{v}}{dt^2} = \vec{b}$

On Integrating both sides $\int\left(\vec{a} \times \frac{d^2\vec{v}}{dt^2}\right) dt = \int \vec{b}\ dt$ ---------***(i)***

Let $\frac{d}{dt}\left(\vec{a} \times \frac{d\vec{v}}{dt}\right) = \frac{d\vec{a}}{dt} \times \frac{d\vec{v}}{dt} + \vec{a} \times \frac{d^2\vec{v}}{dt^2} = \vec{a} \times \frac{d^2\vec{v}}{dt^2}$ $\quad\therefore \frac{d\vec{a}}{dt} = 0$

$$d\left(\vec{a} \times \frac{d\vec{v}}{dt}\right) = \left(\vec{a} \times \frac{d^2\vec{v}}{dt^2}\right) dt$$

On Integrating both side $\int d\left(\vec{a} \times \frac{d\vec{v}}{dt}\right) = \int\left(\vec{a} \times \frac{d^2\vec{v}}{dt^2}\right) dt$

$\vec{a} \times \frac{d\vec{v}}{dt} = \int \vec{b}\ dt$ $\quad$ ***∴From (i)***

$\vec{a} \times \frac{d\vec{v}}{dt} = \vec{b}\,t + \vec{c}$ ------------***(ii)***

Let $\frac{d}{dt}(\vec{a} \times \vec{v}) = \frac{d\vec{a}}{dt} \times \frac{d\vec{v}}{dt} + \vec{a} \times \frac{d\vec{v}}{dt} = \vec{a} \times \frac{d\vec{v}}{dt}$ $\quad\therefore \frac{d\vec{a}}{dt} = 0$

$$d(\vec{a} \times \vec{v}) = \left(\vec{a} \times \frac{d\vec{v}}{dt}\right) dt$$

On Integrating both sides $\int d(\vec{a} \times \vec{v}) = \int\left(\vec{a} \times \frac{d\vec{v}}{dt}\right) dt$

$$\vec{a} \times \vec{v} = \int\left(\vec{b}\,t + \vec{c}\right) dt$$

$\vec{a} \times \vec{v} = \vec{b}\,\frac{t^2}{2} + \vec{c}\,t + \vec{d}$ $\quad$ ***{Where*** $\vec{c}$ & $\vec{d}$ ***are constant of integration}***

Example#03:Find the value of $\vec{r}$ ***satisfying the equation*** $\frac{d^2\vec{r}}{dt^2} = \vec{a}$ ***. Where*** $\vec{a}$ ***is a constant of vector , also it is given that when t= 0,*** $\vec{r} = 0$ ***and*** $\frac{d\vec{r}}{dt} = \vec{u}$***.***

Solution: Given equation $\frac{d^2\vec{r}}{dt^2} = \vec{a}$

On integrating both sides $\frac{d\vec{r}}{dt} = \vec{a}\int 1\ dt = \vec{a}\,t + A$ ------------------------***(i)***

When t= 0 & $\frac{d\vec{r}}{dt} = \vec{u}$ ***then*** $\vec{u} = \vec{a}\,(0) + A \quad \Longrightarrow \quad$ ***A***$= \vec{u}$

Using in equation (i) $\frac{d\vec{r}}{dt} = \vec{a}\,t + \vec{u}$

On integrating both sides $\vec{r} = \int(\vec{a}\,t + \vec{u}\,)\ dt = \vec{a}\frac{t^2}{2} + \vec{u}\,t + B$---------***(ii)***

When t= 0 & $\vec{r} = 0$ ***then*** ***0*** $= \vec{a}\frac{(0)^2}{2} + \vec{u}\,(0) + B \quad \Longrightarrow \quad$ ***B= 0***

Using in equation (ii) $\vec{r} = \vec{a}\frac{t^2}{2} + \vec{u}\,t$

Exercise # 3.3

Q#01: Integrate the following w. r. t t.

(i) $(t^2+1)\hat{\imath}+(t^3+t^2+3)\hat{\jmath}+(2-t)\,\hat{k}$ ***(ii)*** $\cos t\ \hat{\imath}+(t\sec^2 t+\tan t)\,\hat{\jmath}+\sin t\,\hat{k}$

Solution: (i) Let $\vec{f}(t)=(t^2+1)\hat{\imath}+(t^3+t^2+3)\hat{\jmath}+(2-t)\,\hat{k}$

On integrating both sides

$$\int \vec{f}(t)\,dt=\int\left[(t^2+1)\hat{\imath}+(t^3+t^2+3)\hat{\jmath}+(2-t)\,\hat{k}\ \right]dt$$

$$=\hat{\imath}\left[\int (t^2+1)dt\right]+\hat{\jmath}\left[\int (t^3+t^2+3)\ dt\right]+\hat{k}\left[\int (2-t)\ dt\right]$$

$$\int \vec{f}(t)\,dt=\hat{\imath}\left[\frac{t^3}{3}+t\right]+\hat{\jmath}\left[\frac{t^4}{4}+\frac{t^3}{3}+3t\right]+\hat{k}\left[2t-\frac{t^2}{2}\right]$$

(ii) Let $\vec{f}(t)=\cos t\ \hat{\imath}+(t\sec^2 t+\tan t)\,\hat{\jmath}+\sin t\,\hat{k}$

On integrating both sides

$$\int \vec{f}(t)\,dt=\int\left[\cos t\ \hat{\imath}+(t\sec^2 t+\tan t)\,\hat{\jmath}+\sin t\,\hat{k}\ \right]dt$$

$$=\hat{\imath}\left[\int \cos t\,dt\right]+\hat{\jmath}\left[\int (t\sec^2 t+\tan t)\ dt\right]+\hat{k}\left[\int \sin t\ dt\right]$$

$$=\hat{\imath}\,[\sin t]+\hat{\jmath}\,\left[t\tan t-\int \tan t\,dt+\int \tan t\,dt\right]+\hat{k}[-\cos t]$$

$$\int \vec{f}(t)\,dt=\sin t\ \hat{\imath}+t\tan t\,\hat{\jmath}-\cos t\ \hat{k}$$

Q#02: If $\vec{r}=5t^2\hat{\imath}+t\,\hat{\jmath}-t^3\,\hat{k}$ ***. Prove that*** $\int_1^3\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=-52\,\hat{\imath}+400\,\hat{\jmath}-40\,\hat{k}$

Solution: Given vector $\vec{r}=5t^2\hat{\imath}+t\,\hat{\jmath}-t^3\,\hat{k}$ ***Then*** $\frac{d\vec{r}}{dt}=10t\,\hat{\imath}+1\,\hat{\jmath}-3t^2\,\hat{k}$ ***&*** $\frac{d^2\vec{r}}{dt^2}=10\ \hat{\imath}+0\,\hat{\jmath}-6t\,\hat{k}$

Now
$$\vec{r}\times\frac{d^2\vec{r}}{dt^2}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\5t^2&t&-t^3\\10&0&-6t\end{vmatrix}=\hat{\imath}\begin{vmatrix}t&-t^3\\0&-6t\end{vmatrix}-\hat{\jmath}\begin{vmatrix}5t^2&-t^3\\10&-6t\end{vmatrix}+\hat{k}\begin{vmatrix}5t^2&t\\10&0\end{vmatrix}$$

$$=[-6t^2-0]\hat{\imath}-[-30t^3+10t^3]\,\hat{\jmath}+[0-10t]\hat{k}=-6t^2\hat{\imath}-[-20t^3]\,\hat{\jmath}+[-10t]\,\hat{k}$$

$$\vec{r}\times\frac{d^2\vec{r}}{dt^2}=-6t^2\hat{\imath}+20t^3\,\hat{\jmath}-10t\,\hat{k}$$

On Integrating.

$$\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=\int\left[-6t^2\hat{\imath}+20t^3\,\hat{\jmath}-10t\,\hat{k}\right]dt=\hat{\imath}\left[-6\int t^2dt\right]+\hat{\jmath}\left[20\int t^3\ dt\right]+\hat{k}\left[-10\int t\ dt\right]$$

$$\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=\hat{\imath}\left[-6\left(\frac{t^3}{3}\right)\right]+\hat{\jmath}\left[20\left(\frac{t^4}{4}\right)\right]+\hat{k}\left[-10\left(\frac{t^2}{2}\right)\right]=\hat{\imath}[-2t^3]+\hat{\jmath}\,[5t^4]+\hat{k}[-5t^2]$$

Now applying limits

$$\int_1^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = -2\,\hat{\imath}[t^3]_1^3 + 5\,\hat{\jmath}\,[\,t^4\,]_1^3 - 5\hat{k}[t^2]_1^3 = -2\,\hat{\imath}\,[3^3 - 1^3] + 5\,\hat{\jmath}\,[3^4 - 1^4] - 5\hat{k}[3^2 - 1^2]$$

$$= -2\,\hat{\imath}\,[27 - 1] + 5\,\hat{\jmath}\,[81 - 1] - 5\hat{k}[9 - 1] = -2\,\hat{\imath}\,[26] + 5\,\hat{\jmath}\,[80] - 5\hat{k}[8]$$

$$\int_1^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = -52\,\hat{\imath} + 400\,\hat{\jmath} - 40\,\hat{k}$$ ***Hence proved.***

Q#03:Determine a vector function which has $2\cos 2t\,\hat{\imath} + 2\sin 2t\,\hat{\jmath} + 4\,\hat{k}$ ***as its derivative***

and $\hat{\imath} + \hat{\jmath} + \hat{k}$ as its value at $t = 0$.

Solution: Let $\vec{f}(t)$ ***be a required vector which has derivative***

$$\vec{f}\,'(t) = 2\cos 2t\,\hat{\imath} + 2\sin 2t\,\hat{\jmath} + 4\,\hat{k}$$

On integrating both sides

$$\vec{f}(t) = \int\left[2\cos 2t\,\hat{\imath} + 2\sin 2t\,\hat{\jmath} + 4\,\hat{k}\right]dt = \hat{\imath}\,[2\int \cos 2t\;dt] + \hat{\jmath}\,[\,2\int \sin 2t\;dt\,] + \hat{k}[\int 4\;dt]$$

$$= \hat{\imath}\left[2\left(\frac{\sin 2t}{2}\right)\right] + \hat{\jmath}\left[\,2\left(\frac{-\cos 2t}{2}\right)\right] + \hat{k}[4t]$$

$$\vec{f}(t) = \sin 2t\,\hat{\imath} - \cos 2t\,\hat{\jmath} + 4t\,\hat{k} + \vec{A}$$ ---------------***(i)***

Given initial values ***t = 0 &*** $\vec{f}(t) = \hat{\imath} + \hat{\jmath} + \hat{k}$

$$\hat{\imath} + \hat{\jmath} + \hat{k} = \sin 2(0)\,\hat{\imath} - \cos 2(0)\,\hat{\jmath} + 4(0)\,\hat{k} + \vec{A}$$

$$\hat{\imath} + \hat{\jmath} + \hat{k} = 0\hat{\imath} - 1\,\hat{\jmath} + 0\,\hat{k} + \vec{A} \quad\Longrightarrow\quad \vec{A} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$$

Using in equation (i)

$$\vec{f}(t) = \sin 2t\,\hat{\imath} - \cos 2t\,\hat{\jmath} + 4t\,\hat{k} + \hat{\imath} + 2\hat{\jmath} + \hat{k}$$

$$\vec{f}(t) = (1 + \sin 2t)\,\hat{\imath} + (2 - \cos 2t)\hat{\jmath} + (4t + 1)\,\hat{k}$$

Q#04: Solve $\frac{d^2\vec{r}}{dt^2} = \vec{a}$ ***where*** $\vec{a}$ ***is a constant of vector, given that*** $\vec{r} = 0$ ***and*** $\frac{d\vec{r}}{dt} = 0$ ***at t=0.***

Solution: Given equation $\frac{d^2\vec{r}}{dt^2} = \vec{a}$

On integrating both sides $\frac{d\vec{r}}{dt} = \vec{a}\int 1\ dt = \vec{a}\,t + A$ ------------------------***(i)***

When t= 0 & $\frac{d\vec{r}}{dt} = 0$ ***then*** $0 = \vec{a}\,(0) + A \implies$ ***A= 0***

Using in equation (i) $\frac{d\vec{r}}{dt} = \vec{a}\,t$

On integrating both sides $\vec{r} = \int(\vec{a}\,t)\ dt = \vec{a}\,\frac{t^2}{2} + B$--------***(ii)***

When t= 0 & $\vec{r} = 0$ ***then*** $0 = \vec{a}\,\frac{(0)^2}{2} + B \implies$ ***B= 0***

Using in equation (ii) $\vec{r} = \vec{a}\,\frac{t^2}{2}$

Q#05: If $\vec{f}''(t) = 4\,\hat{\imath}$ ***and*** $\vec{f}(t) = 0$ when $t = 0$ ***and*** $\vec{f}'(t) = 4\,\hat{\jmath}$ ***when t= 0 show that the tip of position vector*** $\vec{f}(t)$ describes ***a parabola.***

Solution: Given $\vec{f}''(t) = 4\,\hat{\imath}$

On integrating both sides $\vec{f}'(t) = \hat{\imath}\int 4\,dt \implies \vec{f}'(t) = 4t\,\hat{\imath} + \vec{A}$ ----------------***(i)***

Given initial values at t = 0 & $\vec{f}'(t) = 4\,\hat{\jmath} \implies 4\,\hat{\jmath} = 4(0)\,\hat{\imath} + \vec{A} \implies \vec{A} = 4\hat{\jmath}$

Using $\vec{A} = 4\hat{\jmath}$ ***in Equation (i)*** $\vec{f}'(t) = 4t\,\hat{\imath} + 4\hat{\jmath}$

On integrating both sides $\vec{f}(t) = \int[4t\,\hat{\imath} + 4\hat{\jmath}]\,dt = \hat{\imath}\,[4\int t\,dt] + \hat{\jmath}\int 4\,dt = \hat{\imath}\left[4\left(\frac{t^2}{2}\right)\right] + \hat{\jmath}\,[4t] + \vec{B}$

$\vec{f}(t) = 2t^2\,\hat{\imath} + 4t\,\hat{\jmath} + \vec{B}$ -------***(ii)***

Given initial values at t = 0 & $\vec{f}(t) = 0 \implies 0 = 2(0)^2\,\hat{\imath} + 4(0)\,\hat{\jmath} + \vec{B}$

$\implies \vec{B} = 0$

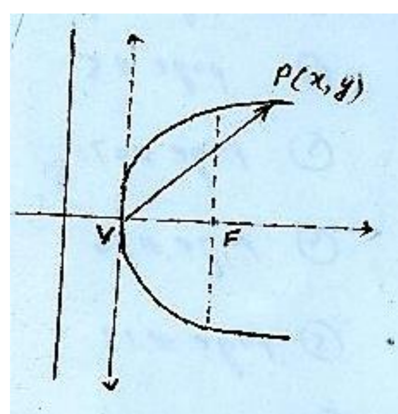
Using $\vec{B} = 0$ ***in Equation (ii)*** $\vec{f}(t) = 2t^2\,\hat{\imath} + 4t\,\hat{\jmath}$--------------***(iii)***

Comparing equation (iii) with $\vec{f}(t) = x\,\hat{\imath} + y\,\hat{\jmath}$

$x = 2t^2 \quad --- (a) \qquad \& \qquad y = 4t \implies t = \frac{y}{4}$

Using $t = \frac{y}{4}$ in equation (a) $x = 2\left(\frac{y}{4}\right)^2 \implies x = 2\,.\,\frac{y^2}{16} \implies x = \frac{y^2}{8}$

$\implies y^2 = 8x$



This is an equation of parabola. Hence proved that the tip of position vector $\vec{f}(t)$ describes ***a parabola.***

Q#06: Solve the equation $\frac{d^2\vec{v}}{dt^2} + 2\frac{d\vec{v}}{dt} + 4\vec{v} = 0$. Where $\vec{v}$ is a vector function of t.

Solution: Given equation $\frac{d^2\vec{v}}{dt^2} + 2\frac{d\vec{v}}{dt} + 4\vec{v} = 0$

This is higher order differential equation we can solve it by the following method.

Put $\frac{d^2\vec{v}}{dt^2} = D^2\vec{v}$ ***&*** $\frac{d\vec{v}}{dt} = D\vec{v}$ ***in given equation***

$$D^2\vec{v} + 2D\vec{v} + 4\vec{v} = 0 \quad \Longrightarrow \quad [D^2 + 2D + 4]\vec{v} = 0$$

Characteristic equation*:*

$D^2 + 2D + 4 = 0$ ***{This is a quadratic equation in D }***

By using quadratic formula D = $\frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)} = \frac{-2\pm\sqrt{4-16}}{2} = \frac{-2\pm\sqrt{-12}}{2} = \frac{-2\pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}\,i$

Characteristic Solution*:*

$$\vec{v}(t) = e^{-t}\{\vec{a}\cos\sqrt{3}\,t + \vec{b}\sin\sqrt{3}\,t\}$$

Q#07 : Solve the equation $\frac{d^2\vec{v}}{dt^2} = \pm\omega^2\vec{v}$ Where $\vec{v}$ is a vector function of t & ω is a constant .

Solution: Given equation $\frac{d^2\vec{v}}{dt^2} = \pm\omega^2\vec{v} \quad \Longrightarrow \quad \frac{d^2\vec{v}}{dt^2} \mp \omega^2\vec{v} = 0$

This is Higher order differential equation we can solve it by the following method.

Put $\frac{d^2\vec{v}}{dt^2} = D^2\vec{v}$

$$D^2\vec{v} \mp \omega^2\vec{v} = 0 \quad \Longrightarrow \quad [D^2 \mp \omega^2]\vec{v} = 0$$

Characteristic equation ***:***

$$D^2 \mp \omega^2 = 0$$

$$D^2 = \pm\omega^2$$

$D^2 = \omega^2$ ***&*** $D^2 = -\omega^2$

$D = \pm\,\omega$ ***&*** $D = \pm\, i\,\omega$ ∴***Taking square-root***

Characteristic Solution:

$\vec{v}(t) = \vec{a}\,e^{\omega t} + \vec{b}\,e^{-\omega t}$ ***&*** $\vec{v}(t) = \vec{c}\cos\omega t + \vec{d}\sin\omega t$

Q#08:If $\vec{v}(t)$ ***is a vector function , Solve the equation*** $\frac{d^2\vec{v}}{dt^2} = \vec{a}\,t + \vec{b}$ ***where*** $\vec{a}$ ***and*** $\vec{b}$ ***are constants and both*** $\vec{v}(t)$ & $\vec{v}\,'(t)$ ***both vanishes at t= 0.***

Solution: Given Equation is $\frac{d^2\vec{v}}{dt^2} = \vec{a}\,t + \vec{b}$ ***O r*** $\vec{v}\,''(t) = \vec{a}\,t + \vec{b}$

On integrating both sides $\vec{v}\,'(t) = \int(\vec{a}\,t + \vec{b})\,dt$

$\vec{v}\,'(t) = \vec{a}\,\frac{t^2}{2} + \vec{b}\,t + \vec{A}$----------------***(i)***

Given initial values at t = 0 & $\vec{v}'(t) = 0$ $\Longrightarrow$ $0 = \vec{a}\,\frac{(0)^2}{2} + \vec{b}\,(0) + \vec{A}$ $\Longrightarrow$ $\vec{A} = 0$

Using in Equation (i) $\vec{v}\,'(t) = \vec{a}\,\frac{t^2}{2} + \vec{b}\,t$

On integrating both sides $\vec{v}(t) = \int\left[\vec{a}\,\frac{t^2}{2} + \vec{b}\,t\right]dt = \hat{i}\,[4\int t\,dt] + \hat{j}\int 4\,dt = \vec{a}\,\frac{t^3}{2.3} + \vec{b}\,\frac{t^2}{2} + \vec{B}$

$\vec{v}(t) = \vec{a}\,\frac{t^3}{6} + \vec{b}\,\frac{t^2}{2} + \vec{B}$ -------***(ii)***

Given initial values at t = 0 & $\vec{v}(t) = 0$ $\Longrightarrow$ $0 = \vec{a}\,\frac{(0)^3}{6} + \vec{b}\,\frac{(0)^2}{2} + \vec{B}$ $\Longrightarrow$ $\vec{B} = 0$

$\vec{v}(t) = \vec{a}\,\frac{t^3}{6} + \vec{b}\,\frac{t^2}{2}$

Q#09: Solve the equation $\frac{d^2\vec{v}}{dt^2} - \frac{d^2\vec{v}}{dt^2} - 2\frac{d\vec{v}}{dt} = 0$. Where $\vec{v}$ is a vector function of t. ***Such that*** $\vec{v} = 0$; $\frac{d\vec{v}}{dt} = 0$ & $\frac{d^2\vec{v}}{dt^2} = 0$ at $t = 0$.

Solution: Given equation

$$\frac{d^2\vec{v}}{dt^2} - \frac{d^2\vec{v}}{dt^2} - 2\frac{d\vec{v}}{dt} = 0$$

This is higher order differential equation we can solve it by using the following method.

Put $\frac{d^3\vec{v}}{dt^3} = D^3\vec{v}$ ***;*** $\frac{d^2\vec{v}}{dt^2} = D^2\vec{v}$ ***&*** $\frac{d\vec{v}}{dt} = D\vec{v}$ ***in given equation***

$$D^3\vec{v} - D^2\vec{v} - 2D\,\vec{v} = 0$$

$$[\,D^3 - D^2 - 2\,D]\,\vec{v} = 0$$

Characteristic equation*:*

$$D^3 - D^2 - 2\,D = 0$$

$$D\,[\,D^2 - D - 2\,] = 0$$

Either $D = 0$ ***or*** $D^2 - D - 2 = 0$ ***{This is a quadratic equation in D }***

By using quadratic formula $D = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}$

$D = \frac{1+3}{2} = \frac{4}{2} = 2$ ***or*** $D = \frac{1-3}{2} = \frac{-2}{2} = -1$ ***Hence*** $D = -1, 0, 2$

Characteristic Solution :

$$\vec{v}(t) = c_1 e^{-t} + c_2 e^{0t} + c_3 e^{2t}$$

$$\vec{v}(t) = c_1 e^{-t} + c_2 + c_3 e^{2t} \text{ ---------}(A)$$

At $t = 0$ & $\vec{v}(t) = \hat{i}$ $\Longrightarrow c_1 e^{-(0)} + c_2 + c_3 e^{2(0)} = \hat{i}$ $\Longrightarrow c_1 + c_2 + c_3 = \hat{i}$ ---------------***(i)***

$$\vec{v}\,'(t) = -c_1 e^{-t} + 0 + 2c_3 e^{2t}$$

At $t = 0$ & $\vec{v}\,'(t) = \hat{j}$ $\Longrightarrow -c_1 e^{-(0)} + 2c_3 e^{2(0)} = \hat{j}$ $\Longrightarrow -c_1 + 2c_3 = \hat{j}$------------------***(ii)***

$$\vec{v}\,''(t) = c_1 e^{-t} + 4c_3 e^{2t}$$

At $t = 0$ & $\vec{v}\,''(t) = \hat{k}$ $\Longrightarrow c_1 e^{-(0)} + 4c_3 e^{2(0)} = \hat{k}$ $\Longrightarrow c_1 + 4c_3 = \hat{k}$----------------***(iii)***

Adding (ii) & (iii) $6c_3 = \hat{j} + \hat{k}$ $\Longrightarrow c_3 = \frac{1}{6}[\hat{j} + \hat{k}]$

Using c_3 ***in equation (iii)*** $c_1 + 4\left(\frac{1}{6}[\hat{j} + \hat{k}]\right) = \hat{k}$ $\Longrightarrow c_1 + \frac{2}{3}[\hat{j} + \hat{k}] = \hat{k}$

$\Longrightarrow c_1 = \hat{k} - \frac{2}{3}[\hat{j} + \hat{k}] = \hat{k} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$ $\Longrightarrow c_1 = -\frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$

Using values of c_1 & c_3 ***in equation (i)*** $-\frac{2}{3}\hat{j} + \frac{1}{3}\hat{k} + c_2 + \frac{1}{6}[\hat{j} + \hat{k}] = \hat{i}$

$\Longrightarrow c_2 = \hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k} + c_2 + \frac{1}{6}\hat{j} + \frac{1}{6}\hat{k} = \hat{i} + \left(\frac{-4+1}{6}\right)\hat{j} + \left(\frac{2+1}{6}\right)\hat{k} = \hat{i} + \left(\frac{-3}{6}\right)\hat{j} + \left(\frac{3}{6}\right)\hat{k}$

$\Longrightarrow c_2 = \hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$

Using values 0f c_1, c_2 & c_3 ***in equation (i)***

$$\vec{v}(t) = \left(-\frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)e^{-t} + \left(\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) + \left(\frac{1}{6}\hat{j} + \frac{1}{6}\hat{k}\right)e^{2t}$$

$$\vec{v}(t) = \hat{i} + \left(-\frac{2}{3}e^{-t} - \frac{1}{2} + \frac{1}{6}e^{2t}\right)\hat{j} + \left(\frac{1}{3}e^{-t} + \frac{1}{2} + \frac{1}{6}e^{2t}\right)$$

Q#10: Prove that ***(i)*** $\int \vec{a}\,.\vec{f}(t)dt = \vec{a}\,.\int \vec{f}(t)dt$ ***(ii)*** $\int \vec{a} \times \vec{f}(t)dt = \vec{a} \times \int \vec{f}(t)dt$

(i) $\int \vec{\mathbf{a}}\,.\vec{\mathbf{f}}(\mathbf{t})\mathbf{dt} = \vec{\mathbf{a}}\,.\int \vec{\mathbf{f}}(\mathbf{t})\mathbf{dt}$

Proof: Let $\frac{d}{dt}\left[\vec{a}\,.\int \vec{f}(t)dt\right] = \frac{d\vec{a}}{dt}.\int \vec{f}(t)dt + \vec{a}.\ \frac{d}{dt}\left[\int \vec{f}(t)dt\right]$

$= (0).\int \vec{f}(t)dt + \vec{a}.\ \frac{d}{dt}\left[\int \vec{f}(t)dt\right]$ $\quad \therefore \frac{d\vec{a}}{dt} = 0$

$$\frac{d}{dt}\left[\vec{a}\,.\int \vec{f}(t)dt\right] = \vec{a}.\ \vec{f}(t)$$

$$d\left(\vec{a}\,.\int \vec{f}(t)dt\right) = \vec{a}.\ \vec{f}(t)\ dt$$

On integrating both sides

$$\int d\left(\vec{a}\,.\int \vec{f}(t)dt\right) = \int \vec{a}\,.\vec{f}(t)dt$$

$$\vec{a}\,.\int \vec{f}(t)dt = \int \vec{a}\,.\vec{f}(t)dt$$

Hence proved that $\int \vec{a}\,.\vec{f}(t)dt = \vec{a}\,.\int \vec{f}(t)dt$

(ii) $\int \vec{\mathbf{a}} \times \vec{\mathbf{f}}(\mathbf{t})\mathbf{dt} = \vec{\mathbf{a}} \times \int \vec{\mathbf{f}}(\mathbf{t})\mathbf{dt}$

Proof: Let $\frac{d}{dt}\left[\vec{a} \times \int \vec{f}(t)dt\right] = \frac{d\vec{a}}{dt} \times \int \vec{f}(t)dt + \vec{a} \times \frac{d}{dt}\left[\int \vec{f}(t)dt\right]$

$= (0) \times \int \vec{f}(t)dt + \vec{a} \times \frac{d}{dt}\left[\int \vec{f}(t)dt\right]$ $\quad \therefore \frac{d\vec{a}}{dt} = 0$

$$\frac{d}{dt}\left[\vec{a} \times \int \vec{f}(t)dt\right] = \vec{a} \times \vec{f}(t)$$

$$d\left(\vec{a} \times \int \vec{f}(t)dt\right) = \vec{a} \times \vec{f}(t)\ dt$$

On integrating both sides

$$\int d\left(\vec{a} \times \int \vec{f}(t)dt\right) = \int \vec{a} \times \vec{f}(t)dt$$

$$\vec{a} \times \int \vec{f}(t)dt = \int \vec{a} \times \vec{f}(t)dt$$

Hence proved that $\int \vec{a} \times \vec{f}(t)dt = \vec{a} \times \int \vec{f}(t)dt$

Q#11: Evaluate $\int_2^3 \vec{r}.\frac{d\vec{r}}{dt}\,dt$ ***if*** $\vec{r}(2) = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ **&** $\vec{r}(3) = 4\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

Solution: We know that $\vec{r}.\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$ ***Then***

Let $I = \int_2^3 \vec{r}.\frac{d\vec{r}}{dt}\,dt = \int_2^3 r\frac{dr}{dt}\,dt = \int_2^3 r\,dr = \left|\frac{r^2}{2}\right|_2^3 = \frac{1}{2}[r^2]_2^3 = \frac{1}{2}[r^2(3) - r^2(2)]$

$\Rightarrow$ $I = \frac{1}{2}[\,|\vec{r}(3)|^2 - |\vec{r}(2)|^2]$ ---------- ***(i)***

Given that

$\vec{r}(2) = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ **&** $\vec{r}(3) = 4\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

Then $|\vec{r}(2)|^2 = (2)^2 + (-1)^2 + (2)^2$ & $|\vec{r}(3)|^2 = (4)^2 + (-2)^2 + (3)^2$

$= 4 + 1 + 4$ $= 16 + 4 + 9$

$|\vec{r}(2)|^2 = 9$ & $|\vec{r}(3)|^2 = 29$

Using values in equation (i)

$$I = \frac{1}{2}[\,|\vec{r}(3)|^2 - |\vec{r}(2)|^2] = \frac{1}{2}[29 - 9] = \frac{1}{2}[20] = \mathbf{10}$$

Hence $\int_2^3 \vec{r}.\frac{d\vec{r}}{dt}\,dt = \mathbf{10}$

Q#12: Find $\vec{f}(t)$ ***when*** $\vec{f}'(t) = e^t\,\hat{\imath} + 2t\,\hat{\jmath} - \sin t\,\hat{k}$ ***and*** $\vec{f}(0) = 2\,\hat{\imath} + 3\,\hat{\jmath} + 4\hat{k}$.

Solution: Given $\vec{f}'(t) = e^t\,\hat{\imath} + 2t\,\hat{\jmath} - \sin t\,\hat{k}$

On integrating both sides

$$\vec{f}(t) = \int\left[e^t\,\hat{\imath} + 2t\,\hat{\jmath} - \sin t\,\hat{k}\right]dt = \hat{\imath}\left[\int e^t\,dt\right] + \hat{\jmath}\left[2\int t\,dt\right] + \hat{k}\left[-\int \sin t\,dt\right]$$

$$= \hat{\imath}[e^t] + \hat{\jmath}\left[2\left(\frac{t^2}{2}\right)\right] + \hat{k}[-(-\cos t)]$$

$\vec{f}(t) = e^t\,\hat{\imath} + t^2\hat{\jmath} + \cos t\,\hat{k} + \vec{A}$ ---------------***(i)***

Given that $\vec{f}(0) = 2\hat{\imath} + 3\,\hat{\jmath} + 4\hat{k}$

$2\hat{\imath} + 3\,\hat{\jmath} + 4\hat{k} = e^0\,\hat{\imath} + (0)^2\hat{\jmath} + \cos(0)\,\hat{k} + \vec{A}$

$2\hat{\imath} + 3\,\hat{\jmath} + 4\hat{k} = 1\hat{\imath} + 0\,\hat{\jmath} + 1\,\hat{k} + \vec{A}$ $\Rightarrow$ $\vec{A} = 2\,\hat{\imath} + 3\,\hat{\jmath} + 4\hat{k} - \hat{\imath} - \hat{k}$ $\Rightarrow$ $\vec{A} = \hat{\imath} + 3\,\hat{\jmath} + 3\hat{k}$

Using in equation (i) $\vec{f}(t) = e^t\,\hat{\imath} + t^2\hat{\jmath} + \cos t\,\hat{k} + \hat{\imath} + 3\,\hat{\jmath} + 3\hat{k}$

$$\vec{f}(t) = (1 + e^t)\,\hat{\imath} + (3 + t^2)\hat{\jmath} + (3 + \cos t)\,\hat{k}$$

Q#13: (a) If $\vec{f}(t) = (3t^2 - t)\,\hat{\imath} + (2 - 6t)\hat{\jmath} - 4t\,\hat{k}$ **. Find (i)** $\int \vec{f}(t)\ dt$ (ii) $\int_2^4 \vec{f}(t)\ dt$

(b) $\int_0^{\pi/2} [3\sin t\ \hat{\imath} + 2\cos t\ \hat{\jmath}]\ dt$

(a) If $\vec{\mathbf{f}}(\mathbf{t}) = (\mathbf{3t^2 - t})\,\hat{\imath} + (\mathbf{2 - 6t})\hat{\jmath} - \mathbf{4t}\,\hat{\mathbf{k}}$ ***. Find (i)*** $\int \vec{\mathbf{f}}(\mathbf{t})\ \mathbf{dt}$ (ii) $\int_2^4 \vec{\mathbf{f}}(\mathbf{t})\ \mathbf{dt}$

Solution: Given $\vec{f}(t) = (3t^2 - t)\,\hat{\imath} + (2 - 6t)\hat{\jmath} - 4t\,\hat{k}$

(i) $\int \vec{f}(t)\ dt = \int\left[(3t^2 - t)\,\hat{\imath} + (2 - 6t)\hat{\jmath} - 4t\,\hat{k}\right] dt$

$$= \hat{\imath}\left[\int (3t^2 - t)\ dt\right] + \hat{\jmath}\left[\int (2 - 6t)\ dt\right] + \hat{k}\left[-4\int t\ dt\right]$$

$$= \hat{\imath}\left[\frac{3t^3}{3} - \frac{t^2}{2}\right] + \hat{\jmath}\left[2t - \frac{6t^2}{2}\right] + \hat{k}\left[\frac{-4t^2}{2}\right]$$

$$\int \vec{f}(t)\ dt = \hat{\imath}\left[t^3 - \frac{t^2}{2}\right] + \hat{\jmath}\,[2t - 3t^2] - 2\hat{k}[t^2] + c$$ ***{c is constant of integration}***

(ii) $\int_2^4 \vec{f}(t)\ dt = \hat{\imath}\left[t^3 - \frac{t^2}{2}\right]_2^4 + \hat{\jmath}\,[2t - 3t^2]_2^4 - 2\hat{k}[t^2]_2^4$

$$= \hat{\imath}\left[\left(4^3 - \frac{4^2}{2}\right) - \left(2^2 - \frac{2^2}{2}\right)\right] + \hat{\jmath}\left[\left(2\{4\} - \frac{6(4)^2}{2}\right) - \left(2\{2\} - \frac{6(2)^2}{2}\right)\right] - 2\hat{k}[4^2 - 2^2]$$

$$= \hat{\imath}\,[64 - 8 - 4 + 2] + \hat{\jmath}\,[8 - 48 - 4 + 12] - 2\hat{k}[16 - 4]$$

$$\int_2^4 \vec{f}(t)\ dt = 54\hat{\imath} - 32\hat{\jmath} - 24\hat{k}$$

(b) $\int_0^{\frac{\pi}{2}} [\mathbf{3\sin t}\ \hat{\imath} + \mathbf{2\cos t}\ \hat{\jmath}]\ \mathbf{dt}$

Solution: Let $I = \int_0^{\frac{\pi}{2}} [3\sin t\ \hat{\imath} + 2\cos t\ \hat{\jmath}]\ dt$

$$I = \hat{\imath}\left[3\int_0^{\frac{\pi}{2}} \sin t\ dt\right] + \hat{\jmath}\left[2\int_0^{\frac{\pi}{2}} \cos t\ dt\right]$$

$$I = \hat{\imath}\left[3\left(\frac{-\cos t}{2}\right)\right]_0^{\frac{\pi}{2}} + \hat{\jmath}\left[2\left(\frac{\sin t}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$I = \frac{-3}{2}\,\hat{\imath}\,[\cos t]_0^{\frac{\pi}{2}} + \hat{\jmath}\,[\sin t]_0^{\frac{\pi}{2}}$$

$$I = \frac{-3}{2}\,\hat{\imath}\left[\cos\frac{\pi}{2} - \cos 0\right] + \hat{\jmath}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$I = \frac{-3}{2}\,\hat{\imath}\,[0 - 1] + \hat{\jmath}\,[1 - 0]$$

$$I = \frac{3}{2}\,\hat{\imath} + \hat{\jmath}$$

Q#15: Evaluate $\int_0^7 \vec{r}.\frac{d\vec{r}}{dt}\ dt$ ***if*** $\vec{r}(0) = 5\hat{\imath} - 3\,\hat{\jmath} + 2\,\hat{k}$ **&** $\vec{r}(7) = \hat{\imath} + 8\hat{\jmath} + 9\,\hat{k}$**.**

Solution: We know that $\vec{r}.\frac{d\vec{r}}{dt} = r\,\frac{d\,r}{dt}$ ***Then***

$$I = \int_0^7 \vec{r}.\frac{d\vec{r}}{dt}\ dt = \int_0^7 r\,\frac{d\,r}{dt}\ dt = \int_0^7 r\ \ dr = \left|\frac{r^2}{2}\right|_0^7 = \frac{1}{2}[r^2]_0^7 = \frac{1}{2}[r^2(7) - r^2(0)]$$

$$I = \frac{1}{2}\,[\,|\vec{r}(7)|^2 - |\vec{r}(0)|^2]\ \text{----------}\ (i)$$

Given that $\vec{r}(0) = 5\hat{\imath} - 3\,\hat{\jmath} + 2\,\hat{k}$ ***&*** $\vec{r}(7) = \hat{\imath} + 8\,\hat{\jmath} + 9\hat{k}$**.**

Then $|\vec{r}(0)|^2 = (5)^2 + (-3)^2 + (2)^2$ & $|\vec{r}(7)|^2 = (1)^2 + (8)^2 + (9)^2$

$|\vec{r}(0)|^2 = 25 + 9 + 4$ & $|\vec{r}(7)|^2 = 1 + 64 + 81$

$|\vec{r}(0)|^2 = 38$ & $|\vec{r}(7)|^2 = 146$

Using values in equation (i)

$$I = \frac{1}{2}[\,|\vec{r}(3)\,|^2 - |\vec{r}(2)\,|^2] = \frac{1}{2}[\,146 - 38] = \frac{1}{2}[\,108] = \mathbf{54}$$

Hence $\int_2^3 \vec{r}.\frac{d\vec{r}}{dt}\ dt\ = \ \mathbf{54}$

Q#16: Example#05:If $\vec{r} = 5\,t^2\hat{\imath} + t\,\hat{\jmath} - t^3\,\hat{k}$ ***. Prove that*** $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = -14\hat{\imath} + 75\,\hat{\jmath} - 15\,\hat{k}$

Solution: Given vector $\vec{r} = 5\,t^2\hat{\imath} + t\,\hat{\jmath} - t^3\,\hat{k}$ ***Then*** $\frac{d\vec{r}}{dt} = 10\,t\,\hat{\imath} + 1\,\hat{\jmath} - 3t^2\,\hat{k}$ ***&*** $\frac{d^2\vec{r}}{dt^2} = 10\,\hat{\imath} + 0\,\hat{\jmath} - 6t\,\hat{k}$

Now $$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5\,t^2 & t & -t^3 \\ 10 & 0 & -6t \end{vmatrix} = \hat{\imath}\begin{vmatrix} t & -t^3 \\ 0 & -6t \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 5\,t^2 & -t^3 \\ 10 & -6t \end{vmatrix} + \hat{k}\begin{vmatrix} 5\,t^2 & t \\ 10 & 0 \end{vmatrix}$$

$$= [-\,6t^2 - 0]\hat{\imath} - [-30t^3 + 10t^3]\,\hat{\jmath} + [0 - 10t]\hat{k} = -\,6t^2\hat{\imath} - [-20t^3]\,\hat{\jmath} + [-10t]\,\hat{k}$$

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = -\,6t^2\hat{\imath} + 20t^3\,\hat{\jmath} - 10t\,\hat{k}$$

On Integrating $\int\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \int[-\,6t^2\hat{\imath} + 20t^3\,\hat{\jmath} - 10t\,\hat{k}\,]\,dt = \hat{\imath}\,[-6\int\ t^2dt] + \hat{\jmath}\,[\,20\int t^3\ dt\,] + \hat{k}[-10\int t\ dt]$

$$\int\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \hat{\imath}\left[-6\left(\frac{t^3}{3}\right)\right] + \hat{\jmath}\left[\,20\left(\frac{t^4}{4}\right)\right] + \hat{k}\left[-10\left(\frac{t^2}{2}\right)\right] = \hat{\imath}\,[-2t^3] + \hat{\jmath}\,[\,5\,t^4\,] + \hat{k}[-5t^2]$$

Now $$\int_1^2\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = -2\,\hat{\imath}\,[t^3]_1^2 + 5\,\hat{\jmath}\,[\,t^4\,]_1^2 - 5\hat{k}[t^2]_1^2 = -2\,\hat{\imath}\,[2^3 - 1^3] + 5\,\hat{\jmath}\,[2^4 - 1^4] - 5\hat{k}[2^2 - 1^2]$$

$$= -2\,\hat{\imath}\,[8 - 1] + 5\,\hat{\jmath}\,[16 - 1] - 5\hat{k}[4 - 1] = -2\,\hat{\imath}\,[7] + 5\,\hat{\jmath}\,[15] - 5\hat{k}[3]$$

$$\int_1^2\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = -14\hat{\imath} + 75\,\hat{\jmath} - 15\,\hat{k}$$ ***Hence proved.***

Q#17: If $\vec{f} = \hat{\imath} - 3\hat{\jmath} + 2t\hat{k}$; $\vec{g} = t\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ & $\vec{h} = 3\hat{\imath} + t\hat{\jmath} - \hat{k}$. ***Prove that*** $\int_1^2 \vec{f}.(\vec{g} \times \vec{h})\,dt = 1$

Solution: Given vectors $\vec{f} = \hat{\imath} - 3\hat{\jmath} + 2t\hat{k}$: $\vec{g} = t\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ & $\vec{h} = 3\hat{\imath} + t\hat{\jmath} - \hat{k}$

Then $\vec{f}.(\vec{g} \times \vec{h}) = \begin{vmatrix} 1 & -3 & 2t \\ t & -2 & 2 \\ 3 & t & -1 \end{vmatrix} = t\begin{vmatrix} -2 & 2 \\ t & -1 \end{vmatrix} - (-3)\begin{vmatrix} t & 2 \\ 3 & -1 \end{vmatrix} + 2t\begin{vmatrix} t & -2 \\ 3 & t \end{vmatrix}$

$$= 1[(-2)(-1) - (t)(2)] + 3[(t)(-1) - (3)(2)] + 2t[(t)(t) - (3)(-2)]$$

$$= 1[2 - 2t] + 3[-t - 6] + 2t[t^2 + 6] = 2 - 2t - 3t - 18 + 2t^3 + 12t$$

$$\vec{f}.(\vec{g} \times \vec{h}) = 2t^3 + 7t - 17$$

Now $I = \int_1^2 \vec{f}.(\vec{g} \times \vec{h})\,dt = \int_1^2 [2t^3 + 7t - 17]\,dt = \left[2\left(\frac{t^4}{4}\right) + 7\left(\frac{t^2}{2}\right) - 17t\right]_1^2 = \left[\frac{t^4}{2} + \frac{7t^2}{2} - 17t\right]_1^2$

$$I = \left[\frac{t^4 + 7t^2 - 34t}{2}\right]_1^2 = \left[\frac{2^4 + 7(2)^2 - 34(2)}{2} - \frac{1^4 + 7(1)^2 - 34(1)}{2}\right] = \left[\frac{16 + 28 - 68}{2} - \frac{1 + 7 - 34}{2}\right]$$

$$I = \left[\frac{-24}{2} - \frac{-26}{2}\right] = \left[\frac{-24}{2} + \frac{26}{2}\right] = \left[\frac{-24 + 26}{2}\right] = \frac{2}{2} = \mathbf{1}$$

Hence proved that

$$\int_1^2 \vec{f}.(\vec{g} \times \vec{h})\,dt = 1$$

Q# 18: Evaluate $\int_1^7 \vec{r}.\frac{d\vec{r}}{dt}\,dt$ ***if*** $\vec{r}(1) = 5\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ ***&*** $\vec{r}(7) = \hat{\imath} + 8\hat{\jmath} + 9\hat{k}$.

Solution: We know that $\vec{r}.\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$ ***Then***

$$I = \int_1^7 \vec{r}.\frac{d\vec{r}}{dt}\,dt = \int_1^7 r\frac{dr}{dt}\,dt = \int_1^7 r\,dr = \left|\frac{r^2}{2}\right|_1^7 = \frac{1}{2}[r^2]_1^7 = \frac{1}{2}[r^2(7) - r^2(1)]$$

$I = \frac{1}{2}[\,|\vec{r}(7)|^2 - |\vec{r}(0)|^2]$ ---------- ***(i)***

Given that $\vec{r}(1) = 5\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ ***&*** $\vec{r}(7) = \hat{\imath} + 8\hat{\jmath} + 9\hat{k}$.

Then $|\vec{r}(1)|^2 = (5)^2 + (-3)^2 + (2)^2$ & $|\vec{r}(7)|^2 = (1)^2 + (8)^2 + (9)^2$

$|\vec{r}(1)|^2 = 25 + 9 + 4$ & $|\vec{r}(7)|^2 = 1 + 64 + 81$

$|\vec{r}(1)|^2 = 38$ & $|\vec{r}(7)|^2 = 146$

Using values in equation (i)

$$I = \frac{1}{2}[\,|\vec{r}(7)|^2 - |\vec{r}(1)|^2] = \frac{1}{2}[146 - 38] = \frac{1}{2}[108] = \mathbf{54}$$

Hence $\int_2^3 \vec{r}.\frac{d\vec{r}}{dt}\,dt = \mathbf{54}$

Q# 19: (i) Example#04: Integrate the equation $\frac{d^2\vec{r}}{dt^2} = -n^2\,\vec{r}$.

Solution : Given equation is

$$\frac{d^2\vec{r}}{dt^2} = -n^2\,\vec{r}$$

Multiplying both sides of equation by $\left(\frac{d\vec{r}}{dt}\right)$:

$$\left(\frac{d\vec{r}}{dt}\right)\frac{d^2\vec{r}}{dt^2} = -n^2\,\vec{r}\left(\frac{d\vec{r}}{dt}\right)$$

On integrating both sides $\int\left(\frac{d\vec{r}}{dt}\right)^1\left(\frac{d^2\vec{r}}{dt^2}\right)\,dt = -n^2\int\vec{r}\left(\frac{d\vec{r}}{dt}\right)\,dt$

{Power rule of integration} $\frac{\left(\frac{d\vec{r}}{dt}\right)^{1+1}}{1+1} = -n^2\,\frac{\vec{r}^{\;1+1}}{1+1} + \boldsymbol{A}$

$$\frac{\left(\frac{d\vec{r}}{dt}\right)^{2}}{2} = -n^2\,\frac{\vec{r}^{\;2}}{2} + \boldsymbol{A}$$

Multiplying both sides by 2 $\left(\frac{d\vec{r}}{dt}\right)^2 = -n^2\,\vec{r}^{\;2} + 2A$

$$\left(\frac{d\vec{r}}{dt}\right)^2 = -n^2\,\vec{r}^{\;2} + c \qquad \therefore 2A = c$$

Q# 19: (ii)If $\frac{d^2\vec{r}}{dt^2} = -\mu\,\vec{r}$. ***then show that*** $\left(\frac{d\vec{r}}{dt}\right)^2 = -\mu\,\vec{r}^{\;2} + c$ where c is constant.

Solution : Given equation is $\frac{d^2\vec{r}}{dt^2} = -\mu\,\vec{r}$

Multiplying both sides of equation by $\left(\frac{d\vec{r}}{dt}\right)$:

$$\left(\frac{d\vec{r}}{dt}\right)\frac{d^2\vec{r}}{dt^2} = -\mu\,\vec{r}\left(\frac{d\vec{r}}{dt}\right)$$

On integrating both sides $\int\left(\frac{d\vec{r}}{dt}\right)^1\left(\frac{d^2\vec{r}}{dt^2}\right)\,dt = -\mu\int\vec{r}\left(\frac{d\vec{r}}{dt}\right)\,dt$

{Power rule of integration} $\frac{\left(\frac{d\vec{r}}{dt}\right)^{1+1}}{1+1} = -\mu\,\frac{\vec{r}^{\;1+1}}{1+1} + \boldsymbol{A}$

$$\frac{\left(\frac{d\vec{r}}{dt}\right)^{2}}{2} = -\mu\,\frac{\vec{r}^{\;2}}{2} + \boldsymbol{A}$$

Multiplying both sides by 2 $\left(\frac{d\vec{r}}{dt}\right)^2 = -\mu\,\vec{r}^{\;2} + 2A$

Hence proved $\left(\frac{d\vec{r}}{dt}\right)^2 = -\mu\,\vec{r}^{\;2} + c \qquad \therefore 2A = c$

Q#20: If $\frac{d^2\vec{r}}{dt^2} = 6t\,\hat{\imath} - 24t^2\,\hat{\jmath} + 4\sin t\;\hat{k}$. ***Find*** $\vec{r}(t)$, ***when*** $t = 0$, $\vec{r} = 2\hat{\imath} + \hat{\jmath}$ ***and*** $\frac{d\vec{r}}{dt} = -\hat{\imath} - 3\,\hat{k}$

Solution: Given equation $\frac{d^2\vec{r}}{dt^2} = 6t\,\hat{\imath} - 24t^2\,\hat{\jmath} + 4\sin t\;\hat{k}$

On integrating both sides $\frac{d\vec{r}}{dt} = \int\left[6t\,\hat{\imath} - 24t^2\,\hat{\jmath} + 4\sin t\;\hat{k}\right]\,dt$

$$= \hat{\imath}\,[6\int t\,dt] + \hat{\jmath}\,[-24\int t^2\,dt] + \hat{k}[4\int \sin t\;dt]$$

$$= \hat{\imath}\left[6\left(\frac{t^2}{2}\right)\right] + \hat{\jmath}\left[-24\left(\frac{t^3}{3}\right)\right] + \hat{k}[4(-\cos t)] + \vec{A}$$

$$= \hat{\imath}\,[3t^2] + \hat{\jmath}\,[-8\,t^3] + \hat{k}[-4\cos t] + \vec{A}$$

$$\frac{d\vec{r}}{dt} = 3t^2\,\hat{\imath} - 8\,t^3\hat{\jmath} - 4\cos t\,\hat{k} + \vec{A} \text{-------------}(i)$$

When t= 0 & $\frac{d\vec{r}}{dt} = -\hat{\imath} - 3\,\hat{k}$ then $3(0)^2\,\hat{\imath} - 8\,(0)^3\hat{\jmath} - 4\cos(0)\,\hat{k} + \vec{A} = -\hat{\imath} - 3\,\hat{k}$

$$0\,\hat{\imath} + 0\hat{\jmath} - 4\hat{k} + \vec{A} = -\hat{\imath} - 3\,\hat{k} \quad\Longrightarrow\quad \vec{A} = -\hat{\imath} - 3\,\hat{k} + 4\hat{k} \quad\Longrightarrow\quad \vec{A} = -\hat{\imath} + \hat{k}$$

Using in equation (i) $\frac{d\vec{r}}{dt} = 3t^2\,\hat{\imath} - 8\,t^3\hat{\jmath} - 4\cos t\,\hat{k} + -\hat{\imath} + \hat{k}$

$$\frac{d\vec{r}}{dt} = (3t^2 - 1)\,\hat{\imath} - 8\,t^3\hat{\jmath} + (1 - 4\cos t)\hat{k}$$

On integrating both sides

$$\vec{r} = \int\left[(3t^2 - 1)\,\hat{\imath} - 8\,t^3\hat{\jmath} + (1 - 4\cos t)\hat{k}\right]\,dt$$

$$= \hat{\imath}[\int (3t^2 - 1)\,dt] + \hat{\jmath}\,[-8\int t^3\,dt] + \hat{k}[\int(1 - 4\cos t)\,dt]$$

$$= \hat{\imath}\left[3\left(\frac{t^3}{3}\right) - t\right] + \hat{\jmath}\left[-8\left(\frac{t^4}{4}\right)\right] + \hat{k}[(t - 4\sin t)] + \vec{B}$$

$$\vec{r} = \hat{\imath}\,[t^3 - t] + \hat{\jmath}\,[-2\,t^4] + \hat{k}[t - 4\sin t] + \vec{A}$$

$$\vec{r} = [t^3 - t]\,\hat{\imath} - 2\,t^4\hat{\jmath} + [t - 4\sin t]\hat{k} + \vec{B} \text{-------------}(ii)$$

When t= 0 & $\vec{r} = 2\,\hat{\imath} + \hat{\jmath}$ ***then*** $[(0)^3 - (0)]\,\hat{\imath} - 2\,(0)^4\hat{\jmath} + [(0) - 4\sin(0)]\hat{k} + \vec{B} = 2\,\hat{\imath} + \hat{\jmath}$

$$\Longrightarrow 0\,\hat{\imath} + 0\hat{\jmath} + 0\hat{k} + \vec{B} = 2\,\hat{\imath} + \hat{\jmath} \quad \Longrightarrow \vec{B} = 2\,\hat{\imath} + \hat{\jmath}$$

Using in equation (ii)

$$\vec{r} = [t^3 - t]\,\hat{\imath} - 2\,t^4\hat{\jmath} + [t - 4\sin t]\hat{k} + 2\,\hat{\imath} + \hat{\jmath}$$

$$\vec{r} = [t^3 - t + 2]\,\hat{\imath}\,(1 - 2\,t^4)\,\hat{\jmath} + [t - 4\sin t]$$

Q#21: If $\vec{a} = t\,\hat{\imath} - 3\,\hat{\jmath} + 2t\,\hat{k}$ **:** $\vec{b} = \hat{\imath} - 2\,\hat{\jmath} + 2\,\hat{k}$ & $\vec{c} = 3\hat{\imath} + t\,\hat{\jmath} - \hat{k}$. ***Find*** $\int_1^2 \vec{f} \times (\vec{g} \times \vec{h})\,dt$

Solution: Given vectors

$$\vec{a} = t\,\hat{\imath} - 3\,\hat{\jmath} + 2t\,\hat{k} \quad : \vec{b} = \hat{\imath} - 2\,\hat{\jmath} + 2\,\hat{k} \quad \& \quad \vec{c} = 3\hat{\imath} + t\,\hat{\jmath} - \hat{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$= [(t\hat{\imath} - 3\,\hat{\jmath} + 2t\,\hat{k}).(3\hat{\imath} + t\,\hat{\jmath} - \hat{k})]\,\vec{b} - [(t\hat{\imath} - 3\,\hat{\jmath} + 2t\,\hat{k}).(\hat{\imath} - 2\,\hat{\jmath} + 2\,\hat{k})]\vec{c}$$

$$= \{3t - 3t - 2t\}\vec{b} - \{t + 6 + 4t\}\vec{c}$$

$$= (-2t)\,(\hat{\imath} - 2\,\hat{\jmath} + 2\,\hat{k}) - (6+5t)\,(3\,\hat{\imath} + t\,\hat{\jmath} - \hat{k})$$

$$= (-2t)\hat{\imath} + (4t)\hat{\jmath} + (-4t)\hat{k} - 3(6 + 5t)\,\hat{\imath} - t(6 + 5t)\,\hat{\jmath} + (6 + 5t)\,\hat{k}$$

$$= (-2t - 18 - 15t)\hat{\imath} + (4t - 6t - 5t^2)\hat{\jmath} + (-4t + 6 + 5t)\,\hat{k}$$

$$= (-17t - 18)\hat{\imath} + (-5t^2 - 2t)\hat{\jmath} + (t + 6)\,\hat{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = -(17t + 18)\hat{\imath} - (5t^2 + 2t)\hat{\jmath} + (7t + 6)\hat{k}$$

Now

$$\boldsymbol{I} = \int_1^2 \vec{a} \times (\vec{b} \times \vec{c})\,dt = \int_1^2 [-(17t + 18)\hat{\imath} - (5t^2 + 2t)\hat{\jmath} + (7t + 6)\hat{k}]\,dt$$

$$= \left[-\hat{\imath}[\textstyle\int(17t + 18)\,dt] - \hat{\jmath}[\int(5t^2 + 2t)\,dt] + \hat{k}[\int(7t + 6)\,dt]\right]$$

$$= -\hat{\imath}\left[17\left(\frac{t^2}{2}\right) + 18t\right]_1^2 - \hat{\jmath}\left[5\left(\frac{t^3}{3}\right) + 2\left(\frac{t^2}{2}\right)\right]_1^2 + \hat{k}\left[7\left(\frac{t^2}{2}\right) + 6t\right]_1^2$$

$$= -\hat{\imath}\left[\left\{17\left(\frac{2^2}{2}\right) + 18(2)\right\} - \left\{17\left(\frac{1^2}{2}\right) + 18(1)\right\}\right] - \hat{\jmath}\left[\left\{5\left(\frac{2^3}{3}\right) + 2\left(\frac{2^2}{2}\right)\right\} - \left\{5\left(\frac{1^3}{3}\right) + 2\left(\frac{1^2}{2}\right)\right\}\right]$$

$$-\hat{k}\left[\left\{\left(\frac{2^2}{2}\right) + 6(2)\right\} - \left\{\left(\frac{1^2}{2}\right) + 6(1)\right\}\right]$$

$$= -\hat{\imath}\left[34 + 36 - \frac{17}{2} - 18\right] - \hat{\jmath}\left[\frac{40}{3} + 4 - \frac{5}{3} - 1\right] + \hat{k}\left[2 + 12 - \frac{7}{2} - 6\right]$$

$$= -\hat{\imath}\left[52 - \frac{17}{2}\right] - \hat{\jmath}\left[\frac{40}{3} - \frac{5}{3} - 3\right] + \left[8 - \frac{1}{2}\right]\hat{k}$$

$$= -\hat{\imath}\left[\frac{104 - 17}{2}\right] - \hat{\jmath}\left[\frac{40 - 5 - 9}{3}\right] + \left[\frac{16 - 1}{2}\right]\hat{k}$$

$$= -\hat{\imath}\left[\frac{87}{2}\right] - \hat{\jmath}\left[\frac{44}{3}\right] + \left[\frac{15}{2}\right]\hat{k}$$

$$\boldsymbol{I} = -\frac{87}{2}\,\hat{\imath} - \frac{44}{3}\,\hat{\jmath} + \frac{15}{2}\hat{k}$$

Q#22: The acceleration of a particle at any time t is given by $\vec{a} = 12\cos 2t\,\hat{\imath} - 8\sin 2t\,\hat{\jmath} + 6t\,\hat{k}$. ***If velocity*** $\vec{v}$ ***& Displacement*** $\vec{r}$ ***are zero at t=0 . then find*** $\vec{v}$ ***&*** $\vec{r}$ ***at any time.***

Solution: Given that $\vec{a} = \frac{d\vec{v}}{dt} = 12\cos 2t\,\hat{\imath} - 8\sin 2t\,\hat{\jmath} + 6t\,\hat{k}$

On integrating both sides $\vec{v} = \int\left[12\cos 2t\,\hat{\imath} - 8\sin 2t\,\hat{\jmath} + 6t\,\hat{k}\right]dt$

$$= \hat{\imath}[12\int \cos 2t\,dt] + \hat{\jmath}[-8\int \sin 2t\,dt] + \hat{k}[6\int t\,dt]$$

$$= \hat{\imath}\left[12\left(\frac{\sin 2t}{2}\right)\right] + \hat{\jmath}\left[-8\left(\frac{-\cos 2t}{2}\right)\right] + \hat{k}[6\left(\frac{t^2}{2}\right)] + \vec{A}$$

$$\vec{v} = 6\sin 2t\,\hat{\imath} + 4\cos 2t\,\hat{\jmath} + 3t^2\hat{k} + \vec{A} \text{-------------}(i)$$

When t = 0 & $\vec{v} = 0$ then

$$6\sin 2(0)\,\hat{\imath} + 4\cos 2(0)\,\hat{\jmath} + 3(0)^2\hat{k} + \vec{A} = 0$$

$$0\,\hat{\imath} + 4\hat{\jmath} + 0\hat{k} + \vec{A} = 0 \quad \Longrightarrow \quad \vec{A} = -4\hat{\jmath}$$

Using in equation (i) $\vec{v} = 6\sin 2t\,\hat{\imath} + 4\cos 2t\,\hat{\jmath} + 3t^2\hat{k} - 4\hat{\jmath}$

$$\vec{v} = \frac{d\vec{r}}{dt} = 6\sin 2t\,\hat{\imath} + (4\cos 2t - 4)\hat{\jmath} + 3t^2\hat{k}$$

On integrating both sides $\vec{r} = \int\left[6\sin 2t\,\hat{\imath} + (4\cos 2t - 4)\hat{\jmath} + 3t^2\hat{k}\right]dt$

$$= \hat{\imath}[6\int \sin 2t\,dt] + \hat{\jmath}[\int(4\cos 2t - 4)\,dt] + \hat{k}[3\int t^2\,dt]$$

$$= \hat{\imath}\left[6\left(\frac{-\cos 2t}{2}\right)\right] + \hat{\jmath}\left[4\left(\frac{\sin 2t}{2}\right) - 4t\right] + 3\hat{k}\left[\frac{t^3}{3}\right] + \vec{B}$$

$$\vec{r} = -3\cos 2t\,\hat{\imath} + [2\sin 2t - 4t]\hat{\jmath} + t^3\hat{k} + \vec{B} \text{-------------}(ii)$$

When t = 0 & $\vec{r} = 0$ ***then*** $-3\cos 2(0)\,\hat{\imath} + [2\sin 2(0) - 4(0)]\hat{\jmath} + (0)^3\hat{k} + \vec{B} = 0$

$$-3\,\hat{\imath} + 0\hat{\jmath} + 0\hat{k} + \vec{B} = 0 \quad \Longrightarrow \quad \vec{B} = 3\,\hat{\imath}$$

Using in equation (ii)

$$\vec{r} = -3\cos 2t\,\hat{\imath} + [2\sin 2t - 4t]\hat{\jmath} + t^3\hat{k} + 3\,\hat{\imath}$$

$$\vec{r} = 3(1 - \cos 2t)\hat{\imath} + 2(\sin 2t - 2t)\hat{\jmath} + t^3\hat{k}$$

The end of chapter #3