Notes of Metric Space

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❖ METRIC SPACE:-

Let X be a non-empty set and R denotes the set of real numbers. A function $d: X \times X \to R$ is said to be metric if it satisfies the following axioms $\forall\ x, y, z \in X$.

$[M_1]$ $d(x,y) \geq 0$ i.e, d is finite and non-negative real valued function.
$[M_2]$ $d(x,y) = 0 \iff x = y$
$[M_3]$ $d(x,y) = d(y,x)$ (Symmetric property)
$[M_4]$ $d(x,z) \leq d(x,y) + d(y,z)$ (Triangular property)

❖ IMPORTANT POINTS:-

Let "*d*" be a metric on a set "X" , then

- (X,d) is called metric space with metric d.
- X is called underlying or ground set.
- d is called the metric or distance function on X.
- Elements of a metric space are called its points.

❖ OPEN BALL:-

Let $B(y,r)$ be a subset of a metric space$X$, then

$$B(y,r) = \{x \in X: d(x,y) < r\}$$

is said to be an open ball with radius "r" centered at "y".

❖ REMEMBER THAT:

If any point "$a \in X$" of metric space belongs to open ball, that is

$$a \in B(x,r)$$

This will be possible only if $d(x,a) < r$

Theorem # 1:- Open ball in a metric space is an open interval.

Proof:-

The metric "d" on real line "R" is defined by

$$d(x,y) = |x - y|$$

Let $B(a,r)$ be an open ball of a metric space (X,d). Then,

$B(a,r) = \{x \in X: d(a,x) < r\}$

$\Rightarrow B(a,r) = \{x \in X: |a - x| < r\}$

$\Rightarrow B(a,r) = \{x \in X: |x - a| < r\}$

$\Rightarrow B(a,r) = \{x \in X: -r < x - a < r\}$

$\Rightarrow B(a,r) = \{x \in X: a - r < x < a + r\}$

$\Rightarrow B(a,r) =]a - r, a + r[$

This is an open interval having length "$2r$".

❖ OPEN SET:-

Let (X, d) be a metric space and set G is called open in X if for every $x \in G$, there exists an open ball $B(x,r) \subseteq A$.

❖ SCHEME TO PROVE ANY SET TO BE AN OPEN SET:-

To prove any set "G" to be an open in X, we have to follow the following steps:-

- First, we will take an arbitrary element $x \in A$
- Secondly, we consider an open ball $B(x,r)$.
- Thirdly, we have to prove$B(x,r) \subseteq A$

Theorem # 2:- Open Ball in a metric space is an open set.

Proof:-

Let $B(x,r)$ be an open ball in (X,d).
We want to show that "$B(x,r)$" is open.
Let $y \in B(x,r)$ then $d(x,y) < r$
Let $d(x,y) = r_1 \Rightarrow r_1 < r$
Put $\varepsilon = r - r_1$ and consider an open ball $B(y,\varepsilon)$.
We prove that $B(y,\varepsilon) \subseteq B(x,r)$
For this let $z \in B(y,\varepsilon)$ then$d(y,z) < \varepsilon$
By triangular property, we have
$d(x,z) \leq d(x,y) + d(y,z)$
$\Rightarrow d(x,z) < r_1 + \varepsilon$
$\Rightarrow d(x,z) < r - \varepsilon + \varepsilon$
$\Rightarrow d(x,z) < r$
Hence $z \in B(x,r)$ so that $B(y,\varepsilon) \subseteq B(x,r)$

Thus $B(x, r)$ is an open set.

❖ **Theorem # 3: Let $(\mathbf{X}, \mathbf{d})$ be a metric space, then**

(i) Both φ and X are open sets.

(ii) Intersection of finite collection of open sets is open

(iii) Union of any collection of open sets is open

PROOF: - (i)

Let us suppose φ is open.
Then for each $x \in \varphi$, there exist an open ball $B(x, r)$ such that

$$B(x, r) \subseteq \varphi$$

But since φ has no element.
Hence the condition is automatically satisfied. Hence φ is open.

PROOF:- (ii)

Suppose $\{A_i : i = 1,2, \dots \dots, n\}$ be a finite number of open sets in X.

We show that $\bigcap_{i=1}^{n} A_i$ is open.

Case-1:-

If A_i is empty for some $i = 1,2, \dots \dots, n$ then $\bigcap_{i=1}^{n} A_i = \varphi$ is an open set in X.

Case-2:-

If $\bigcap_{i=1}^{n} A_i \neq \varphi$.

Let $x \in \bigcap_{i=1}^{n} A_i$ then $x \in A_i \quad \forall i = 1,2, \dots \dots, n$

Since each A_i is an open set.

Then there exist an open ball $B(x, r_i)$ such that

$\mathrm{B}(x, \mathrm{r_i}) \subseteq \mathrm{A_i} \quad \forall \quad \mathrm{i} = 1,2, \dots \dots, \mathrm{n}$

$$\Rightarrow B(x, r_i) \subseteq \bigcap_{i=1}^{n} A_i \quad \forall \; i$$

We choose $r_i = min\{r_1, r_2, \dots \dots, r_n\}, then$

$B(x, r) \subseteq B(x, r_i) \;\; \forall i = 1,2, \dots \dots, n$

Hence

$$B(x,r) \subseteq \bigcap_{i=1}^{n} A_i$$

Thus $\bigcap_{i=1}^{n} A_i$ is an open set.

PROOF:- (iii)

Suppose A_i be a class of open sets in X.
We show that $\bigcup_i A_i$ is open in X.

If A_i is empty $\forall\, i$, then $\bigcup_i A_i = \varphi$.

Hence $\bigcup_i A_i$ is open in X.

But if $A_i \neq \varphi$, then

Let $x \in \bigcup_i A_i$ then $x \in A_i$

Since each A_i is open. Then there exist an open ball $B(x,r)$ such that

$B(x,r) \subseteq A_i \subseteq \bigcup_i A_i$

$$\Longrightarrow B(x,r) \subseteq \bigcup_i A_i$$

Hence $\bigcup_i A_i$ is open.

Theorem # 4: Let (X, d) be a metric space. A subset A of X is open if and only if it is the union of open balls.

PROOF:-

Suppose A be any subset of metric space (X,d).
Firstly, we will suppose that "A" is an open set.Then we show that A open balls.

Case-1:

If $A = \emptyset$ then A is regarded as union of is empty class of open balls.

Case-2:

If $A \neq \emptyset$ then let $x \in A$.

Since A is open. Then there exist an open ball $B(x,r)$ such that

$B(x,r) \subseteq A$ then $A = \cup\, B(x,r)$

Conversely, we suppose that A is union of open balls.

If union of open ball is an empty set, then A is trivially open.

If union of open ball is not empty.

Then for $x_o \in A$, there exist an open ball $B(x_o, r)$ such that

$$A = \cup B(x_o, r)$$

Since every open ball is an open set. That is, $\exists\, r_1 > 0$ such that

$B(x_o, r_1) \subseteq B(x_o, r) \subseteq A$

$\Longrightarrow B(x_o, r_1) \subseteq A$

It follows that A is open.

Theorem # 5: The complement of a singleton set is open.

PROOF:

Let (X, d) be a metric space.

Let $F = \{x\}$ be a singleton set.

We want to show that "$F^c$is open".

Let $y \in F^c$ then $y \notin F$

Hence $y \neq x$. Then $d(x, y) \neq 0$ $\because$ By definition

Let $d(x, y) = r$ and consider an open ball $B(y, r)$. Then,

$B(y, r) \cap F = \emptyset$

$\Longrightarrow B(y, r) \subseteq F^c$

Thus F^c is open.

❖ DISCRETE METRIC SPACE:-

Let X be a non-empty set. Let R denote the set of real numbers. We define $\mathrm{d}: \mathrm{X} \times \mathrm{X} \rightarrow \mathrm{R}$ such that

$$d(x, y) = 0 \quad \Leftrightarrow x = y$$

$$d(x, y) = 1 \quad \Leftrightarrow x \neq y$$

This can also be written as

$$d(x, y) = \begin{cases} 0 & if\ x \neq y \\ 1 & if\ x \neq y \end{cases}$$

Then X is called discrete metric space or trivial metric space.

Theorem # 6: An open ball of radius $0 < r < 1$ in a discrete metric space contain only its Centre.

Proof:

Let (X, d) be a discrete metric space and let $0 < r < 1$.

Suppose $B(x, r)$ be an open ball.

Then we shall prove that

"$B(x, r)$ contains only its centre".

Let $x_o \in B(x, r)$ & $x \neq x_o$

$\Longrightarrow d(x, x_o) < r$

$\Longrightarrow d(x, x_o) < 1 \quad \because 0 < r < 1$

$\Longrightarrow d(x, x_o) = 0$

$\Longrightarrow x = x_o \quad \because X$ is discrete metric space

This contradicts our supposition. Hence our supposition$x_o \in B(x, r)$ is wrong.
Therefore, $B(x, r)$ contains only its Centre.

Theorem # 7: An open ball of radius $r > 1$ in a discrete metric space is a whole space.

Proof:

Let (X, d) be a discrete metric space and let $r > 1$.

Suppose $B(x, r)$ be an open ball.

Then we shall prove that

$$B(x, r) = X$$

Let $x_o \in X --- (a)$ & $x \neq x_o$

$\Longrightarrow d(x, x_o) = 1 \quad \because$ X is discrete metric space

$\Longrightarrow d(x, x_o) < r \quad \because r > 1$

$\Longrightarrow x_o \in B(x, r) --- (b)$

Therefore, from (a) & (b)

$$X \subseteq B(x, r) --- (A)$$

$\because$ By definition, we know

$$B(x, r) \subseteq X --- (B)$$

From (A) & (B), we have

$$B(x, r) = X$$

This completes the proof.

Theorem # 8: Every non-empty subset of a discrete metric space is open.

PROOF:

Let (X, d) be a discrete metric space.

Let $U \subseteq X$ such that $U \neq \emptyset$.

We shall prove that U is an open set.

Suppose that $x_o \in U$

Since every open ball $B(x_o, r)$ of radius $0 < r < 1$ in a discrete metric space contain only its Centre. That is,

$B(x_o, r) = \{x_o\}$ then U is open.

Since every open ball $\mathrm{B}(\mathrm{x}_o, \mathrm{r})$ of radius $r > 1$in a discrete metric space is a whole space. That is,

$B(x_o, r) = X$

Since X is open. Therefore, $B(x_o, r)$ is also open.

It follows that U is open.

❖ CLOSE BALL

Let (X, d) be a metric space. If "a" is a point of X and "r" is positive real number, that is, $r > 0$ then the subset

$$\overline{B}(a, r) = \{x \in X: d(a, x) \leq r\}$$

is called close ball centered at "a" and radius "r".

Theorem # 9:- Close ball in a metric space is an open interval.

Proof:-

Let (X, d) be a metric space.

Let $\overline{B}(a, r)$ be a close ball.

The close ball is defined as follow:-

$$\overline{B}(a, r) = \{x \in X: d(a, x) \leq r\} - - - (i)$$

The metric "d" on real line "R" is defined by

$$d(x, y) = |x - y|$$

$$\Longrightarrow d(a, x) = |a - x|$$

Thus equation (i) will become

$$\overline{B}(a, r) = \{x \in X: |a - x| \leq r\}$$

$$\Longrightarrow \overline{B}(a, r) = \{x \in X: |x - a| \leq r\}$$

$$\Longrightarrow \overline{B}(a, r) = \{x \in X: -r \leq x - a \leq r\}$$

$$\Longrightarrow \overline{B}(a, r) = \{x \in X: a - r \leq x \leq a + r\}$$

$$\Longrightarrow \overline{B}(a, r) = [a - r, a + r]$$

This is a closed interval having length "2r".

❖ CLOSE SET:

A subset “A” of metric space is said to be closed if and only if its complement is open.

Theorem # 10: Close ball in a metric space is close.

PROOF:

Let (X, d) be a metric space and $\overline{B}(a, r)$ be a closed ball.

Then we show $\overline{B}(a, r)$ is close.

For this we shall prove $\overline{B}^c(a, r) = X - \overline{B}(a, r)$ is open.

Let $y \in \overline{B}^c(a, r)$ then $y \notin \overline{B}(a, r) \Longrightarrow d(a, y) > r$

Put $\varepsilon = d(a, y) - r$ and consider an open ball $B(y, \varepsilon)$.

Now we prove $B(y, \varepsilon) \subseteq \overline{B}^c(a, r)$

For this let $z \in B(y, \varepsilon)$ then $d(y, z) < \varepsilon$

By triangular property, we have

$d(a, y) \leq d(a, z) + d(y, z)$

$\Longrightarrow \varepsilon + r < d(a, z) + \varepsilon$

$\Longrightarrow r < d(a, z)$ *Or* $d(a, z) < r$

$\Longrightarrow z \in \overline{B}^c(a, r)$

Hence $B(y, \varepsilon) \subseteq \overline{B}^c(a, r)$. It follows that $\overline{B}^c(a, r)$ is open.

Thus $\overline{B}(a, r)$ is closed.

❖ LIMIT POINT:

Let $(\boldsymbol{X}, \boldsymbol{d})$ be a metric space. Then a point $\mathrm{x} \in \mathrm{X}$ is called limit point of $\boldsymbol{A}$ if every open ball $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r})$ with center "x" contains a point of $\boldsymbol{A}$ other than "x" . that is,

$$B(x, r) \cap A - \{x\} \neq \emptyset$$

❖ ALTERNATIVE DEFINITION OF CLOSED SET:

A subset "***A***" of metric space is said to be closed if it contains all of its limit point.

Theorem # 11: A subset of metric space is close if and only if its complement is open.

PROOF:

Let (X, d) be a metric space and let $A \subseteq X$.

Suppose A is close and we wanted to show " A^c is close".

Let $y \in A^c$ then $y \notin A$

$\Longrightarrow y$ is not limit point of A.

Then by definition of a limit point there exists an open ball $B(y, r)$ such that

$B(y, r) \cap A - \{y\} = \emptyset$

$\Longrightarrow B(y, r) \cap A = \emptyset \quad \because A - \{y\} \subseteq A$

$\Longrightarrow B(y, r) \subseteq A^c$

$\Longrightarrow A^c$ is open.

Conversely, we suppose that A^c is open.

Then we show A is close.

Let x be the limit point of A which does not belong to A. That is,

$x \notin A$ then $x \in A^c$

Then there exists an open ball $B(x, r)$ such that $B(x, r) \subseteq A^c$ and

$B(x, r) \cap A = \emptyset$

This implies x is not limit point of A which is contradiction to our supposition.

Hence $x \in A$. Accordingly A is close.

Theorem 12: Let (X, d) be a metric space, then

(i) Both $\emptyset$ and X are close.
(ii) Union of finite collection of close sets is a close.
(iii) Intersection of any collection of close sets is close.

PROOF:

(i)

As $\emptyset^c = X - \emptyset$ then $\emptyset^c = X$

Since X is open then $\emptyset^c$ is open. It follows that $\emptyset$ is close.

Again, since $X^c = X - X$ then $X^c = \emptyset$

Since $\emptyset$ is open then X^c is open. It follows that X is close.

(ii)

Let $\{O_\alpha : \alpha = 1,2, \dots \dots, n\}$ be a finite class of close sets.

Then we show $\bigcup_{\alpha=1}^{n} O_\alpha$ is close.

Suppose that,

$$F = \bigcup_{\alpha=1}^{n} O_\alpha \text{ then } F^c = \left[\bigcup_{\alpha=1}^{n} O_\alpha\right]^c$$

$$\Rightarrow F^c = \bigcap_{\alpha=1}^{n} {O_\alpha}^c \quad \because \text{ De} - \text{Morgan theorem}$$

Since ${O_\alpha}^c$ being the complement of a finite class of close sets will be open and consequently $\bigcap_{\alpha=1}^{n} {O_\alpha}^c$ is open. Therefore, F^c is open.

This implies that F is closed

This complete the proof.

(iii)

Let $\{O_\alpha : \alpha \in I\}$ be a class of infinite collection of close sets.

Then we prove $\bigcap_{\alpha \in I} O_\alpha$ is close.

Suppose that $F = \bigcap_{\alpha \in I} O_\alpha$, then

$$F^c = \left(\bigcap_{\alpha \in I} O_\alpha\right)^c$$

$$\Rightarrow F^c = \bigcup_{\alpha \in I} {O_\alpha}^c \quad \because \text{by De} - \text{Margan law}$$

$\because$ O_α be the collection of close sets. Then, ${O_\alpha}^c$ is open and consequently $\bigcup_{\alpha\in I} {O_\alpha}^c$ is open set. Therefore, F^c is open.

this implies F is close.

This completes the proof.

❖ DIAMETER OF A SET:

Let $(\boldsymbol{X}, \boldsymbol{d})$ be a metric space. The diameter of a non-empty subset $\boldsymbol{A}$ of metric space is defined as

$$\delta(A) = sup\,\{d(x, y)\}$$

❖ Theorem # 13: The diameter of a close ball in a metric space is $\leq \boldsymbol{2r}$.

PROOF:-

Let (X, d) be a metric space.

Let $\overline{B}(a, r)$ denote a close ball.

Let $x, y \in \overline{B}(a, r)$ then $d(a, x) \leq r$ & $d(a, y) \leq r$

By triangular inequality, we have

$d(x, y) \leq d(x, a) + d(a, y)$

$\Longrightarrow d(x, y) \leq r + r$

$\Longrightarrow d(x, y) \leq 2r$

This completes the proof.

❖ *NEIGHBORHOOD:-*

Let (X, d) be a metric space and "a" be any point of X. A subset N of X is called neighborhood of "a" if there exist an open set G such that $a \in G$ and G is contained in N. That is,

$x \in G \subseteq N$

Theorem # 14: Let (X, d) be a metric space and $A \subseteq X$. Then A is open iff A is nhd of each of its point.

PROOF:

Let (X, d) be a metric space and $A \subseteq X$.

Suppose "A" is open.

We show "A is nhd of each of its point".

Let $x \in A$. Then,

$x \in A \subseteq A$ $\quad\because$ *every set is subset of itself*

$\Longrightarrow$ *A is nhd of x.*

As "x" be an arbitrary point of A. So, A is nhb of each of its point.

Conversely, we suppose that "A is nhb of each of its point."

Let x be any point of A.

Since A is nhd of x. Therefore, $\exists$ an open set G such that

$x \in G \subseteq A$

and

$A = \cup \{\{x\}: x \in A\} \subseteq \cup \{G_x: x \in A\} \subseteq A$

Thus A is open.

- ***INHERIDITY PROPERTY:***

If $A \subseteq B$ then we read A is a subset of B and B is superset of A. If any property contains in superset then it must also be contained in subset. This property is said to be inherdity property.

- **Theorem # 15:Let (X, d) be a metric space and "A" is infinite subset of X. If $x \in X$ is a limit point of A, then every nhd of x contains infinite points of A.**

PROOF:

Let (X, d) be a metric space and A is infinite subset of X.

Let $x \in X$ is a limit point of A and let N_x be the neighborhood of x.

Then by definition of neighborhood $\exists$ an open set "O" such that

$$x \in O \subseteq N_x$$

Then we will show "N_x contains infinite point of A".

To prove this, we shall prove

"O contains infinite points of A".

Since "O" is open. So by definition, , $\exists$ an open ball $B(x, r)$ such that

$$B(x, r) \subseteq O$$

Suppose to contrary that O contain only finite number of points of A.

It follows that $B(x, r)$ contains only finite number of points of A, say $a_1, a_2, \dots \dots, a_n$ only, that is, ${}_{i=1}^{n}a_i \in B(x, r)$ then $d({}_{i=1}^{n}a_i, x) < r$

Let,

$$d(x, a_1) = r_1$$

$$d(x, a_2) = r_2$$

....

....

....

$$d(x, a_n) = r_n$$

And,

$$r' = min\,\{r_1, r_2, \dots \dots \dots, r_n\}$$

Then the open ball $B(x, r')$ does not contain any point of A other than x.

This is contradiction, since x is a limit point of A.

Hence $B(x, r')$ contains at least one point of A other than x.

Hence every neighborhood of x contains infinite point of X.

❖ CLOSURE OF A SET:

Let A be a subset of a metric space (X, d). Then we define closure of A as "the union of A and all limit points of A". It is denoted as $\overline{A}$. Mathematically, it is defined as

$$\overline{A} = A \cup A^d,$$

where A^d denotes set of limit points of A.

❖ *Adharent Point:*

Let (X, d) be a metric space and $A \subseteq X$, then $x \in X$ is said to be adharent to A if

$$A \cap N(x) \neq \emptyset$$

❖ *Theorem # 16: Let (X, d) be a metric space and $A \subseteq X$. Then $\overline{A} = A \cup A^d$.*

PROOF:

Let (X, d) be a metric space and $A \subseteq X$.

Let A^d be the closed set consisting of all of its limit point.

Let $x \in A^d - - - (a)$

$\Longrightarrow x$ is limit point of A. Therefore,

$A - \{x\} \cap N(x) \neq \emptyset$

$\Longrightarrow A \cap N(x) \neq \emptyset \qquad \because \ A - \{x\} \subseteq A$

$\Longrightarrow x$ is adharent point to A. Therefore,

$x \in \overline{A} - - - (b)$

From (a) & (b), we have

$A^d \subseteq \overline{A} - - - (c)$

Since by definition $A \subseteq \overline{A} - - - (d)$

From (c) & (d), we have

$A \cup A^d \subseteq \overline{A} - - - (i)$

Now we show that $\overline{A} \subseteq A \cup A^d$.

Let $x \in \overline{A} - - - (e)$

$\Longrightarrow\ x \in A \cup A^d$

$\Longrightarrow x \in A\ \ or\ \ x \in A^d$

$\because x \in \overline{A}$. *then by definition*

$A \cap N(x) \neq \emptyset \qquad \because x\ is\ adharent$

$\Longrightarrow A - \{x\} \cap N(x) \neq \emptyset$

$\Longrightarrow\ x \in A^d$ $\quad\because$ *x is limit point*

$\Longrightarrow\ x \in A \cup A^d - - - (f)$ $\quad\because$ *x is limit point*

From (e) & (f), *we have*

$\overline{A} \subseteq A \cup A^d - - - (ii)$

Now $(i), (ii) \Longrightarrow$

$\overline{A} = A \cup A^d$

Theorem # 17: In any metric space (X, d), show that A is closed if and only if $\overline{A}$ = A

PROOF:-

Let (X, d) *be a metric space and* $A \subseteq X$.

First we suppose $\overline{A}\ =\ A$

We show that "A is closed".

As $\overline{A} = A \cup A^d$

$\Longrightarrow A = A \cup A^d\ \ \because\ \overline{A}\ =\ A$

$\Longrightarrow A^d \subseteq A$

$\Longrightarrow A$ *is close.*

Conversely, we suppose " A is close".

$\Longrightarrow A^d \subseteq A$

$\Longrightarrow A \cup A^d = A$

$\Longrightarrow \overline{A} = A$

This completes the proof.

Theorem # 18: Let (X, d) be a metric space and let A & B be two arbitrary subsets of X. Then

(i) $\emptyset = \overline{\emptyset}$

(ii) $\boldsymbol{X = \overline{X}}$

(iii) $\boldsymbol{\overline{A \cup B} = \overline{A} \cup \overline{B}}$

(iv) $\boldsymbol{\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}}$

(v) $\boldsymbol{\overline{\overline{A}} = \overline{A}}$

(vi) ***If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$***

PROOF:- ***(i)***

Since $\emptyset$ is close. Therefore,

$\emptyset^d \subseteq \emptyset$

$\Longrightarrow \emptyset \cup \emptyset^d = \emptyset \quad \because$ *if $A \subseteq B$ then $A \cup B = B$*

$\Longrightarrow \overline{\emptyset} = \emptyset$

This completes the proof.

PROOF:- ***(ii)***

Since X is close. Therefore,

$X^d \subseteq X \quad \because$ *by definition*

$\Longrightarrow X \cup X^d = X \quad \because$ *if $A \subseteq B$ then $A \cup B = B$*

$\Longrightarrow \overline{X} = X$

This completes the proof.

PROOF:- ***(iii)***

As we know

$A \subseteq A \cup B \;\;\&\;\; B \subseteq A \cup B$

$\Longrightarrow \overline{A} \subseteq \overline{A \cup B} \;\&\; \overline{B} \subseteq \overline{A \cup B}$

$\Longrightarrow \overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \;---\; (a)$

As by definition, we have

$A \subseteq \overline{A} \;\;\&\;\; B \subseteq \overline{B}$

$\Longrightarrow A \cup B \subseteq \overline{A} \cup \overline{B} \;---\; (i)$

$\Longrightarrow$ $\overline{A} \cup \overline{B}$ *is the superset of* $A \cup B$.

But we know the smallest superset of $A \cup B$ *is* $\overline{A \cup B}$ *. hence,*

$A \cup B \subseteq \overline{A \cup B}$

$(i) \Longrightarrow \overline{A \cup B} \;\subseteq \overline{A} \cup \overline{B} \;---\; (b)$

from (a) & (b), *we have*

$\overline{A \cup B} \;= \overline{A} \cup \overline{B}$

This completes the proof.

PROOF:- ***(iv)***

As we know

$A \cap B \subseteq A \;\;\&\;\; A \cap B \subseteq B$

$\Longrightarrow \;\overline{A \cap B} \subseteq \overline{A} \;\;\&\;\; \overline{A \cap B} \subseteq \overline{B}$

$\Longrightarrow \;\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

PROOF:-   (v)

If A is close. Then

$\overline{A} = A$

$\Longrightarrow$ $\overline{A}$ *is close.*

If $\overline{A}$ *is close, then by definition*

$\overline{\overline{A}} = \overline{A}$

This completes the proof.

PROOF:-   (vi)

As it is given

$A \subseteq B \;\; \& \;\; B \subseteq \overline{B}$

$\Longrightarrow A \subseteq B \subseteq \overline{B}$

$\Longrightarrow A \subseteq \overline{B}$

As $\overline{A}$ *is the smallest superset of A. so,*

$A \subseteq \overline{A}$

$\Longrightarrow A \subseteq \overline{A} \subseteq \overline{B}$

$\Longrightarrow \overline{A} \subseteq \overline{B}$