Superior College Sargodha

Mechanics
Centre of Mass
Chapter # 4

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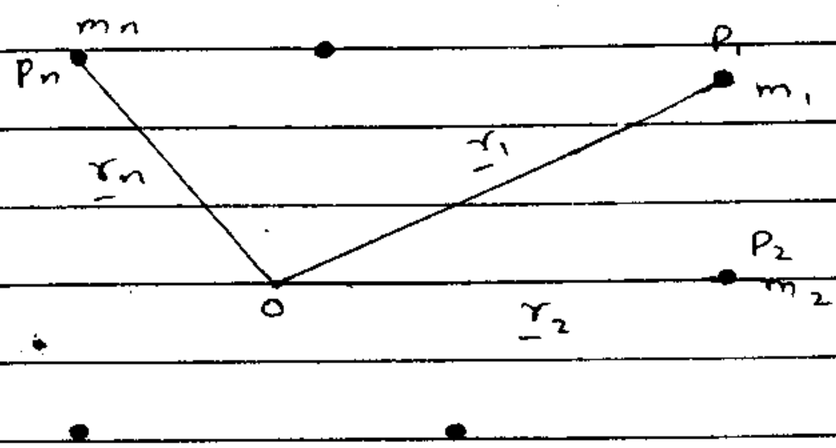
CHAPTER # 4:-

Centres of mass and Gravity:-

-: Centre of mass:-

Linear Moment:-

Consider a set of particles of masses $m_1, m_2, \ldots, m_n$ place at $P_1, P_2, \ldots, P_n$ with position vectors $r_1, r_2, r_3, \ldots, r_n$ resp.



Then the linear moment of set of particles is

$$\sum_{i=1}^{n} m_i \underline{r}_i = m_1 \underline{r}_1 + m_2 \underline{r}_2 + \cdots + m_n \underline{r}_n$$

Example # 1:-

$m_1 = 1\,lb \quad , \quad m_2 = 2\,lb$

$\underline{r}_1 = \underline{i} - 2\underline{j} \quad , \quad \underline{r}_2 = 3\underline{i}$

$$\sum_{i=1}^{n} m_i r_i = m_1 r_1 + m_2 \underline{r}_2$$

$$= (\underline{i} - 2\underline{j}) + 2(3\underline{i})$$

$= \underline{i} - 2\underline{j} + 6\underline{i}$

$= 7\underline{i} - 2\underline{j}$

Centre mass: (C.m)

C.m is the point w.r.t which the linear moment of the set of particles is zero.

Theorem:-

1:- Every set of particles has one and only one c.m?

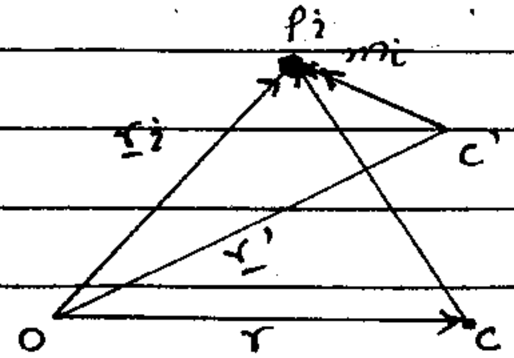
OR

2:- Find centre of mass of set of particles?

Solution:-

Let c be the centre of mass of set of particles of masses $m_1, m_2, \ldots, m_n$ placed at $P_1; P_2; \ldots; P_n$ resp.

Consider a particle of mass m_i at P_i with Position vector r_i where $\underline{r}$ is a Position vector of c.



$\overrightarrow{OC'} + \overrightarrow{CP_i} = \overrightarrow{OP_i}$

$r + \overrightarrow{CP_i} = r_i$

$\overrightarrow{CP_i} = \underline{r}_i - \underline{r}$

linear moment of m_i about C(c.m) $= 0$

By definition.

$$\sum_{i=1}^{n} m_i \overrightarrow{CP_i} = 0$$

where $i = 1, 2, 3, \ldots\ldots, n$

$$\sum_{i=1}^{n} m_i (\underline{r}_i - \underline{r}) = 0$$

$$\sum_{i=1}^{n} m_i \underline{r}_i - \sum_{i=1}^{n} m_i \underline{r} = 0$$

$$\sum_{i=1}^{n} m_i \underline{r}_i = \sum_{i=1}^{n} m_i \underline{r}$$

$$\underline{r} = \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i} \quad \text{c.m.}$$

$\boxed{\text{Q : 2 : complete.}}$

$$\underline{r} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2 + \cdots\cdots + m_n r_n}{m_1 + m_2 + \cdots\cdots + m_n}$$

For uniqueness of c.m :-

Let c' be an other c.m with p.v $\underline{r}'$. Then fellow As above. we get

$$\underline{r}' = \frac{\sum_{i=1}^{n} m_i \underline{r}_i}{\sum_{i=1}^{n} m_i}$$

$$\Rightarrow \quad \underline{r} = \underline{r}'$$

$$\Rightarrow \quad \underset{\substack{\downarrow \\ \text{congracy}}}{C \equiv C'} \quad \text{i.e}$$

C and C' lies at same position.

$\Rightarrow$ C.m is unique.

Cartesian Component of c.m:-

$$\underline{r}_i = (x_i, y_i, z_i)$$

$$\underline{r} = (\bar{x}, \bar{y}, \bar{z})$$

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

$$\bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

We use these co-ordinates according to object which deal.

i.e object is one dimension, or two, or three dimension.

Define: Centroid:-

The point $\dfrac{\sum \underline{r}_i}{n} = \left(\dfrac{\sum x_i}{n}, \dfrac{\sum y_i}{n}, \dfrac{\sum z_i}{n}\right)$

is called the centroid of set of n particles of same masses.

(5)

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Note:-

If particles have same masses then c.m is same as centroid.

$$c.m = \frac{\sum m_i r_i}{\sum m} \quad \text{If } m_1 = m_2 = \cdots = m_n = m$$

$$= \frac{\sum m r_i}{\sum m} \quad \because \sum_{i=1}^{n} 1 = n$$

$$= \frac{\sum r_i}{n}$$

$$= \text{Centroid}$$

Example#1:- Page: 67:-

Find the centroid of the points $\underline{i}$, $2\underline{i}-\underline{j}$ and $3\underline{i}+\underline{j}-4\underline{k}$. If particles of mass 2, 4, 3 grams are placed respectively at these points, what will be their c.m?

Solution:-

$$\underline{r}_1 = \underline{i} \quad ; \quad m_1 = 2g$$

$$\underline{r}_2 = 2\underline{i} - \underline{j} \quad ; \quad m_2 = 4g$$

$$\underline{r}_3 = 3\underline{i} + \underline{j} - 4\underline{k} \quad ; \quad m_3 = 3g$$

$$\text{Centroid} = \frac{\sum r_i}{n} = \frac{\underline{r}_1 + \underline{r}_2 + \underline{r}_3}{3}$$

$$= \frac{\underline{i} + 2\underline{i} - \cancel{\underline{j}} + 3\underline{i} + \cancel{\underline{j}} - 4\underline{k}}{3}$$

$$= \frac{6\underline{i} - 4\underline{k}}{3}$$

$$= 2\underline{i} - \frac{4}{3}\underline{k}$$

Centroid $= (2, 0, -4/3)$.

$$C.m = \frac{\sum m_i r_i}{\sum m_i} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2 + m_3\underline{r}_3}{m_1 + m_2 + m_3}$$

$$= \frac{2(\underline{i}) + 4(2\underline{i} - \underline{j}) + 3(3\underline{i} + \underline{j} - 4\underline{k})}{2 + 4 + 3}$$

$$= \frac{2\underline{i} + 8\underline{i} - 4\underline{j} + 9\underline{i} + 3\underline{j} - 12\underline{k}}{9}$$

$$= \frac{19\underline{i} - \underline{j} - 12\underline{k}}{9}$$

$$C.m = \left(\frac{19}{9}, -\frac{1}{9}, -\frac{4}{3}\right) \quad \text{Ans}$$

1:- linear Moment $= \sum_{i=1}^{n} m_i r_i$

2:- $C.M = \dfrac{\sum_{i=1}^{n} m_i \underline{r}_i}{\sum_{i=1}^{n} m_i}$

3:- Cartesian components as c.m.

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

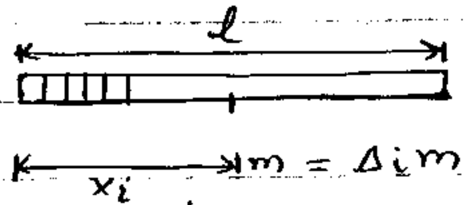
$$\bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

4:- Centroid $= \dfrac{\sum r_i}{n} = \left(\dfrac{\sum x_i}{n}, \dfrac{\sum y_i}{n}, \dfrac{\sum z_i}{n}\right)$.

Define: thin rod by a thin rod we mean a rigid body whose width or breath and thickness is negligible:.

Centre of Mass of a thin rod:-

=> Consider a thin rod of mass 'm' and length 'l'.



Sub divide the rod into "n" Parts and lebel then $\Delta_i m$ be the mass of ith part x_i be the distance as a point m ith part from A (One and point as rod).

$$C.M = \frac{\sum \Delta_i m\, x_i}{\sum \Delta_i m}$$

$$= \lim_{\substack{n \to \infty \\ \Delta_i m \to 0}} \frac{\sum \Delta_i m\, x_i}{\sum \Delta_i m}$$

$$= \frac{\int_0^l x\, dm}{\int_0^l dm} \qquad \because \rho = \frac{dm}{dx}$$

$$\rho\, dx = dm$$

$$= \frac{\int_0^l x \rho\, dx}{\int_0^l \rho\, dx}$$

If rod is uniform.

$$c.m = \frac{\rho \int_0^{l} x\,dx}{\rho \int_0^{l} dx}$$

$$= \frac{\left[\frac{x^2}{2}\right]_0^{l}}{\left[x\right]_0^{l}}$$

$$= \frac{\frac{l^2}{2}}{\frac{l}{1}}$$

$$= \frac{l}{2}$$

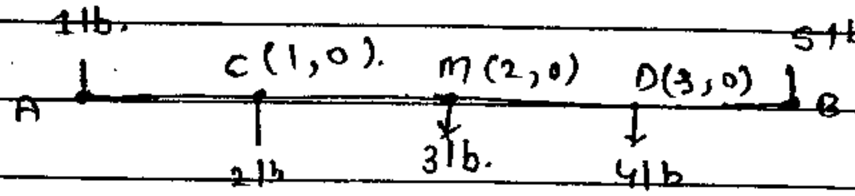
If rod is uniform c.m is some as Centroid.

EXERCISE # 4:-

:Q: No: 1:

A uniform rod AB is 4ft. long and weighs 6 lb. and weights are attached to its as follows: 1 lb. at A, 2 lb. at 1ft. from A, 3lb. at 2ft. from A, 4lb. at 3ft. from A and 5lb. at B. Find the distance from A of the centre of gravity of the system?

Solution:-



Let "A" be the origin and rod AB is along x-axis.

Given $AB = 4ft$

weight $= 6$ lb.

Since rod is uniform weight acts at mid point "m" as rod.

Weights 1lb, 2 lb., 3lb, 4lb, 5lb. are attached to rod at A, C, M, D, B resp.

As shown in fig.

$$C.G = \frac{\sum m_i x_i}{\sum m_i} = \frac{1(0)+2(1)+(3+6)2+4(3)+5(4)}{1+2+9+4+5}$$

$$C.G = \frac{2+18+12+20}{21}$$

$$C.G = \frac{52}{21} = 2.48 = \boxed{2.5\ ft}\ \text{Ans}$$

Note:- Right circular cone is symmetric about its axis.

$\Rightarrow$ Its c.m lies on its Axis.

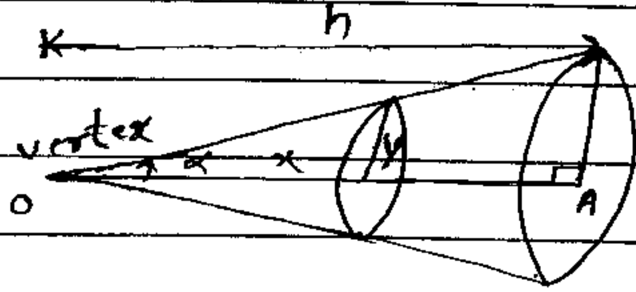
vip **Example # 11:-** page 82:-

Find the c.m of right circular solid cone?

Solution:-

$\Rightarrow$ Right circular cone is symmetric about its axis (A line through vertex and centre of base).

$\Rightarrow$ C.m lies on axis of cone.



Let vertex of cone is at origin and its axis OA (where A is centre of base) along the x-axis.

Note:-
$$\rho = \frac{\delta m}{\delta V}$$

$$\rho \, \delta V = \delta m$$

$$\bar{r} = \frac{\int r \, dm}{\int dm}$$

$$x = \frac{\int x \, dm}{\int dm}.$$

Solid cone is considered to made up of circular slices of thickness δx parallel to base.

Let α be the semi-vertical angle. and y be the radius of one circular slice at a distance x from O.

$$\frac{y}{x} = \tan\alpha$$

$$y = x\tan\alpha \longrightarrow (i)$$

$$\delta V = \text{(Area) (Thickness)}$$

$$\delta V = \pi y^2 \delta x$$

$$\delta V = \pi x^2 \tan^2\alpha \, \delta x \qquad \because \text{ using (i)}$$

If ρ is the density of material then,

$$\rho = \frac{\delta m}{\delta V}.$$

$$\delta m = \rho \, \delta V = \rho \pi x^2 \tan^2\alpha \, \delta x.$$

C.m,

$$\bar{x} = \frac{\int_0^h x \, dm}{\int_0^h dm}$$

$$\bar{x} = \frac{\int_0^h x \cdot \rho\pi x^2 \tan^2\alpha \, dx.}{\int_0^h \rho\pi x^2 \tan^2\alpha \, dx.}$$

$$\bar{x} = \frac{\cancel{\rho\pi\tan^2\alpha} \int_0^h x^3 dx.}{\cancel{\rho\pi\tan^2\alpha} \int_0^h x^2 dx.}$$

$$\bar{x} = \frac{\int_0^h x^3 dx.}{\int_0^h x^2 dx.}$$

$$\bar{x} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h}$$

$$\bar{x} = \frac{h^{\cancel{4}\,1}}{4} \cdot \frac{3}{\cancel{h^3}}$$

$$\bar{x} = \frac{3}{4} \cdot h \qquad \text{Ans}$$

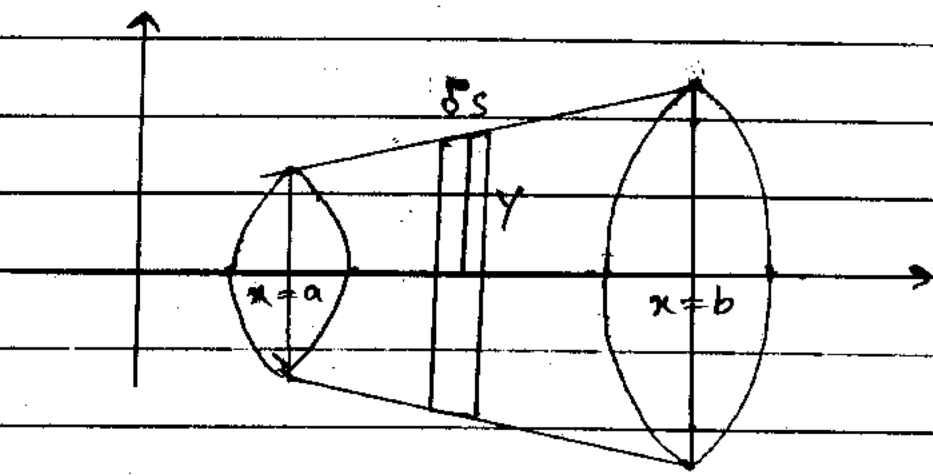
Define:-

Surface of revolution:-

Surface formed by the rotation of a plane curve about a line in the same plane:-

* Cm of surface of revolution.

Let $y = f(x)$ be a plane curve where x varies from $x=a$ to $x=b$ and let the axis of rotation be taken as x-axis.



The area of the element of surface of revolution $= 2\pi y \delta s$

$\bar{y} = 0$ $\because$ cm lies on axis of rotation i.e one axis.

$$\bar{x} = \frac{\int \rho x \, 2\pi y \, ds}{\int \rho \, 2\pi y \, ds.}$$

$$\because \rho = \frac{\delta m}{\delta(\text{surface area})}$$

$$\rho = \frac{\delta m}{2\pi y \delta s}$$

$$\rho\, 2\pi y \delta s = \delta m$$

$$\bar{x} = \frac{\int \rho\, 2\pi x y \, ds}{\int 2\pi \rho y \, ds.}$$

$$\bar{x} = \frac{\cancel{2\pi\rho} \int x y \, ds}{\cancel{2\pi\rho} \int y \, ds}$$

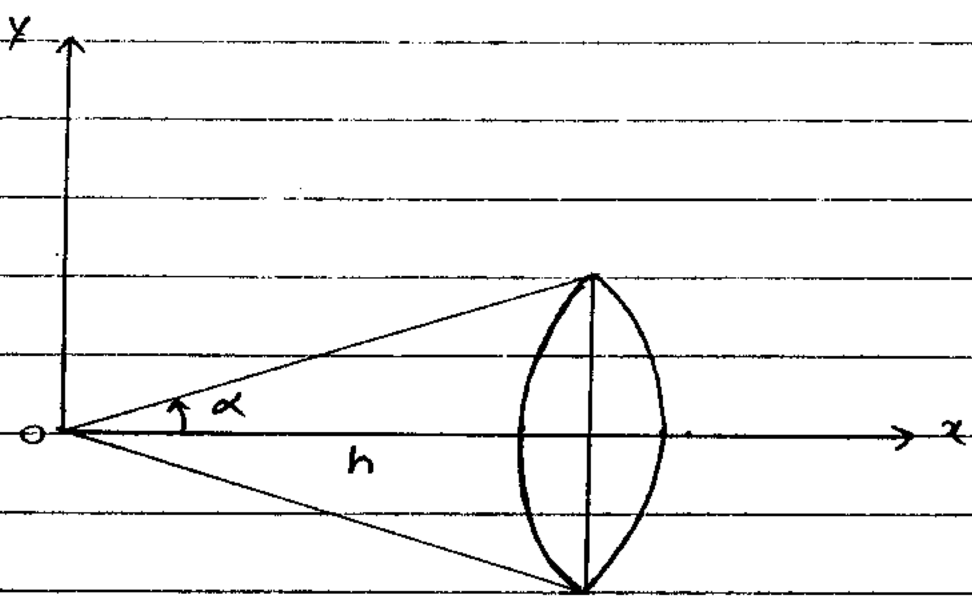
$$\bar{x} = \frac{\int_a x y \, ds}{\int_a^b y \, ds.}$$

Example #(1):- **Page :93 :-**

Find c.m of hollow right circular cone of semi-verticle angle α and height h.

Solution:-

A hollow circular cone can be regarded as a surface of revolution generated by a line (passes through origin and making an angle α).



$P(x_1, y_1) \Rightarrow P(0,0)$; $m = \tan\alpha$.

Equation of generator line.

$$y - y_1 = m(x - x_1).$$

$$y - 0 = \tan\alpha\,(x - 0).$$

$$y = x\tan\alpha$$

c.m of hollow cone lies on its axis of rotation i.e on x-axis.

$$\Rightarrow \bar{y} = 0$$

$$\bar{x} = \frac{\int xy\,ds}{\int y\,ds.}$$

$$\because ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

$$\because \frac{dy}{dx} = \tan\alpha.$$

$$ds = \sqrt{1 + \tan^2\alpha}\,dx.$$

$$= \sqrt{\sec^2\alpha}\,dx.$$

$$ds = \sec\alpha\,dx$$

$$\Rightarrow \bar{x} = \frac{\int_0^h x\tan\alpha \cdot x\sec\alpha\,dx}{\int_0^h \tan\alpha \cdot x\sec\alpha\,dx.}$$

$$\bar{x} = \frac{\cancel{\tan\alpha}\,\cancel{\sec\alpha}\int_0^h x^2\,dx}{\cancel{\tan\alpha}\,\cancel{\sec\alpha}\int_0^h x\,dx}$$

$$\bar{x} = \frac{\left[x^3/3\right]_0^h}{\left[x^2/2\right]_0^h}$$

$$\bar{x} = \frac{h^3/3}{h^2/2}$$

$$\bar{x} = \frac{2}{3}h. \quad \text{Ans}$$

N.B: In a numerical case the value of ds in terms of:-

Cartesian coordinates:-

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Polar coordinates:-

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Parameter:-

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

may be substituted as found convenient.

Solid of Revolution:-

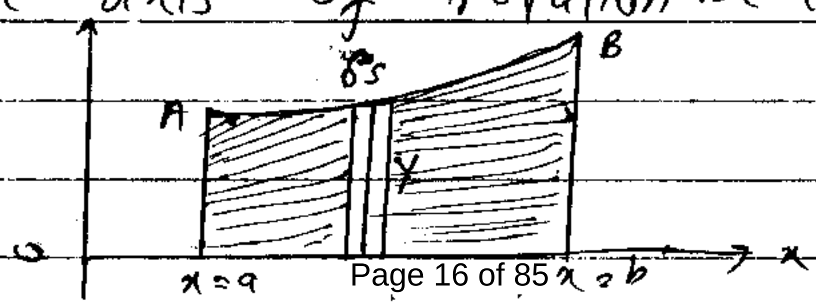
⇒ A solid whose boundary is obtained by revolving a given plane curve, about a line in its plane.

* C.m of a solid of revolution.

Let the given curve be

$$y = f(x).$$

where x varies from $x = a$ to $x = b$ and let the axis of rotation be taken as x-axis



The solid can be regarded as made up of thin circular slices $\parallel$ lar to x-axis. The mass of slice at a distance x from the origin is $\rho \pi y^2 \delta x$

$\bar{y} = 0 \quad \because$ c.m lies on axis of rotation i.e one axis.

$$\bar{X} = \frac{\int_a^b x \cdot \rho \pi y^2 dx}{\int_a^b \rho \pi y^2 dx}$$

$$= \frac{\cancel{\rho\pi} \int_a^b x y^2 dx}{\cancel{\rho\pi} \int_a^b y^2 dx}$$

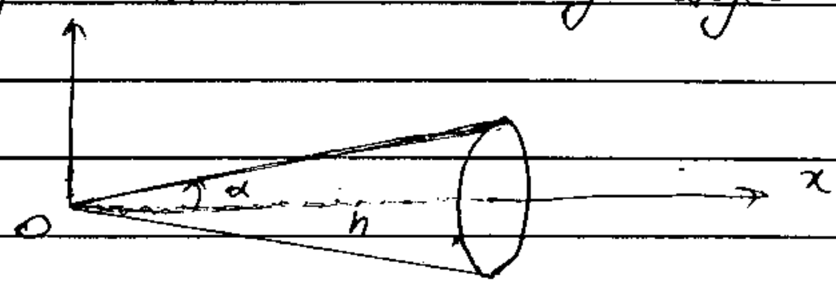
$$\bar{X} = \frac{\int_a^b x y^2 dx}{\int_a^b y^2 dx}$$

Example:- page 92:

A solid right circular cone?

Solution:-

A solid right circular cone can be regarded as a solid of revolution formed by the rotation of the line (passes through the origin and makeng angle α).



$P(x_1, y_1) \Rightarrow P(0,0)$; $m = \tan\alpha$

Equation of rotation line.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \tan\alpha\,(x - 0)$$

$$y = x\tan\alpha$$

$\therefore$ c.m lies on axis of rotation

i.e one axis.

$$\bar{x} = \frac{\int_0^h x y^2\,dx}{\int_0^h y^2\,dx}$$

$$\bar{x} = \frac{\int_0^h x \cdot x^2 \tan^2\alpha\,dx}{\int_0^h x^2 \tan^2\alpha\,dx}$$

$$= \frac{\cancel{\tan^2\alpha} \int_0^h x^3\,dx}{\cancel{\tan^2\alpha} \int_0^h x^2\,dx}$$

$$\bar{x} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h}$$

$$= \frac{\frac{h^4}{4}}{\frac{h^3}{3}}$$

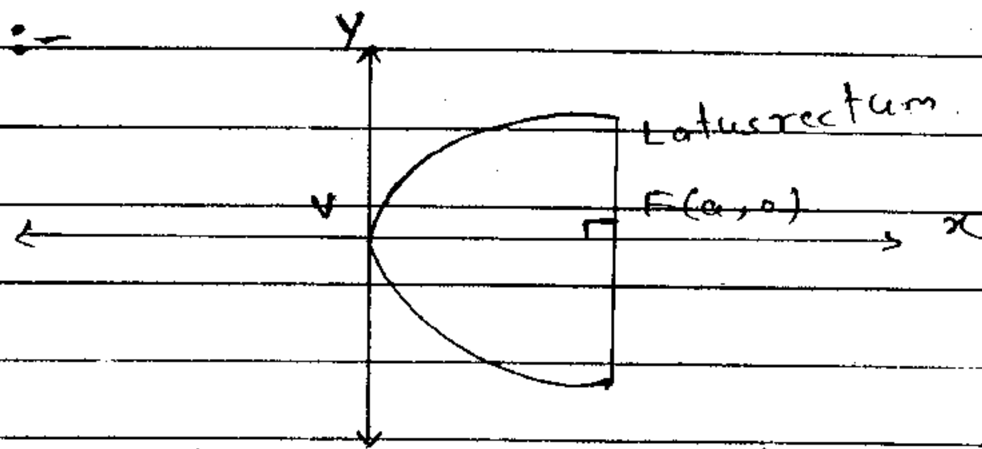
$$= \frac{\cancel{h^4}^{\,1}}{4} \times \frac{3}{\cancel{h^3}}$$

$$\bar{x} = \frac{3}{4}h \qquad \underline{\text{Ans}}$$

Example #2:- **Page 94:**

Find the c.m of the surface generated by the revolution of the arc of the parabola, lying between the vertex and the latus rectum, about the x-axis?

Solution:-



Let Equation of parabola.

$$y^2 = 4ax$$

$$y = \sqrt{4ax}$$

Curve of parabola. x-axis.

$$y = 2\sqrt{ax}$$

$$\frac{dy}{dx} = \cancel{2}\sqrt{a}\cdot\frac{1}{\cancel{2}\sqrt{x}}$$

$$\frac{dy}{dx} = \sqrt{\frac{a}{x}}$$

$$\therefore \; ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1+\frac{a}{x}}\, dx$$

$$ds = \frac{\sqrt{x+a}}{\sqrt{x}}\, dx.$$

$\because$ c.m lies on axis of revolution i.e on x-axis;

$$\bar{y} = 0$$

$$\bar{x} = \frac{\int_0^a x y \, ds}{\int_0^a y \, ds}$$

$$\bar{x} = \frac{\int_0^a x \cdot 2\sqrt{a}\,\cancel{\sqrt{x}} \cdot \frac{\sqrt{a+x}}{\cancel{\sqrt{x}}}\, dx}{\int_0^a 2\sqrt{a}\,\cancel{\sqrt{x}} \cdot \frac{\sqrt{a+x}}{\cancel{\sqrt{x}}}\, dx}$$

$$\bar{x} = \frac{\cancel{2\sqrt{a}} \int_0^a x\sqrt{a+x}\, dx}{\cancel{2\sqrt{a}} \int_0^a \sqrt{a+x}\, dx.}$$

$$\bar{x} = \frac{\int_0^a (x+a-a)\sqrt{x+a}\, dx}{\int_0^a \sqrt{x+a}\, dx}$$

$$= \frac{\int_0^a \left[(x+a)^{3/2} - a\sqrt{x+a}\right] dx}{\int_0^a \sqrt{x+a}\, dx}$$

$$= \frac{\left| \frac{2}{5}(x+a)^{5/2} - a\,\frac{2}{3}(x+a)^{3/2} \right|_0^a}{\left| \frac{2}{3}\cdot(x+a)^{3/2} \right|_0^a}$$

$$= \frac{\frac{2}{5}\cdot(2a)^{5/2} - a\cdot\frac{2}{3}(2a)^{3/2} - \frac{2}{5}a^{5/2} + a\cdot\frac{2}{3}a^{3/2}}{\frac{2}{3}\cdot(2a)^{3/2} - 2/3\,(a)^{3/2}}$$

$$= \frac{\frac{2}{5}\cdot 2^{5/2}\cdot a^{5/2} - \frac{2}{3}\cdot 2^{3/2}\cdot a^{5/2} - \frac{2}{5}a^{5/2} + \frac{2}{3}a^{5/2}}{\frac{2}{3}\cdot 2^{3/2}\, a^{3/2} - 2/3\, a^{3/2}}$$

$$= \frac{a^{5/2}\left[\frac{2}{5}\cdot 4\sqrt{2} - \frac{2}{3}\,2\sqrt{2} - \frac{2}{5} + \frac{2}{3}\right]}{a^{3/2}\left[\frac{2}{3}\cdot 2\sqrt{2} - 2/3\right]}$$

$$= \frac{a^{5/2-3/2}\left[\frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \frac{2}{5} + \frac{2}{3}\right]}{\left[\frac{4\sqrt{2}}{3} - \frac{2}{3}\right]}$$

$$= a \cdot \frac{\frac{24\sqrt{2} - 20\sqrt{2} - 6 + 10}{\cancel{15}\,5}}{\frac{4\sqrt{2} - 2}{\cancel{3}}}$$

$$= a \cdot \frac{24\sqrt{2} - 20\sqrt{2} - 6 + 10}{5(4\sqrt{2} - 2)}$$

$$= a \cdot \frac{4\sqrt{2} + 4}{20\sqrt{2} - 10}$$

$$= a \cdot \frac{\overset{2}{\cancel{4}}}{\underset{5}{\cancel{10}}}\left[\frac{\sqrt{2} + 1}{2\sqrt{2} - 1}\right]$$

$$= \frac{2a}{5}\left[\frac{\sqrt{2} + 1}{2\sqrt{2} - 1} \cdot \frac{2\sqrt{2} + 1}{2\sqrt{2} + 1}\right]$$

$$= \frac{2a}{5} \cdot \left[\frac{4 + \sqrt{2} + 2\sqrt{2} + 1}{8 - 1}\right]$$

$$= \frac{2a}{5} \cdot \left[\frac{5 + 3\sqrt{2}}{7}\right]$$

$$= \frac{10 + 6\sqrt{2}}{35} \cdot a$$

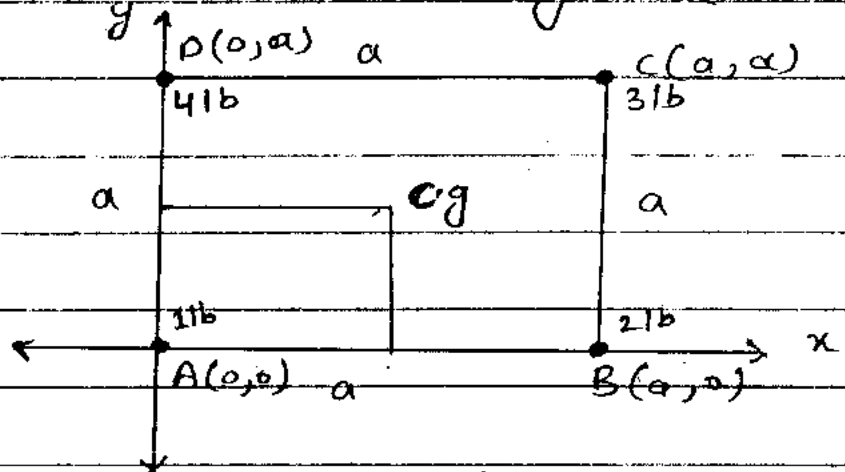
Ans

Q:No: 2:-

Weights of 1, 2, 3, 4 lb are placed at the corners A, B, C, D respectively of a square of side 8 inches. Find the distances of the c.g of the set of weights from AB and AD?

Solution:-

Consider a square ABCD of side length $a = 8''$ inches and origin at A with sides AB and AD along x and y-axis.



$$C.G = (\bar{x}, \bar{y})$$

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

$$\bar{x} = \frac{1(0) + 2(a) + 3(a) + 4(0)}{1+2+3+4}$$

$$\bar{x} = \frac{\cancel{5}a}{\cancel{10}_2}$$

$$\bar{x} = \frac{a}{2} \qquad \because a = 8$$

$$\bar{x} = {}^{4}\cancel{8}/\cancel{2}$$

$$\bar{x} = 4'' \text{ inches.}$$

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

$$\bar{y} = \frac{1(0) + 2(0) + 3(a) + 4(a)}{1+2+3+4}$$

$$\bar{y} = \frac{7a}{10} \qquad \because a = 8$$

$$\bar{y} = \frac{7 \cdot \cancel{8}^{4}}{\cancel{10}_{5}}$$

$$\bar{y} = \frac{28}{5} = 5.6'' \text{ inches}$$

$$C.G = (4'', 5.6'')$$

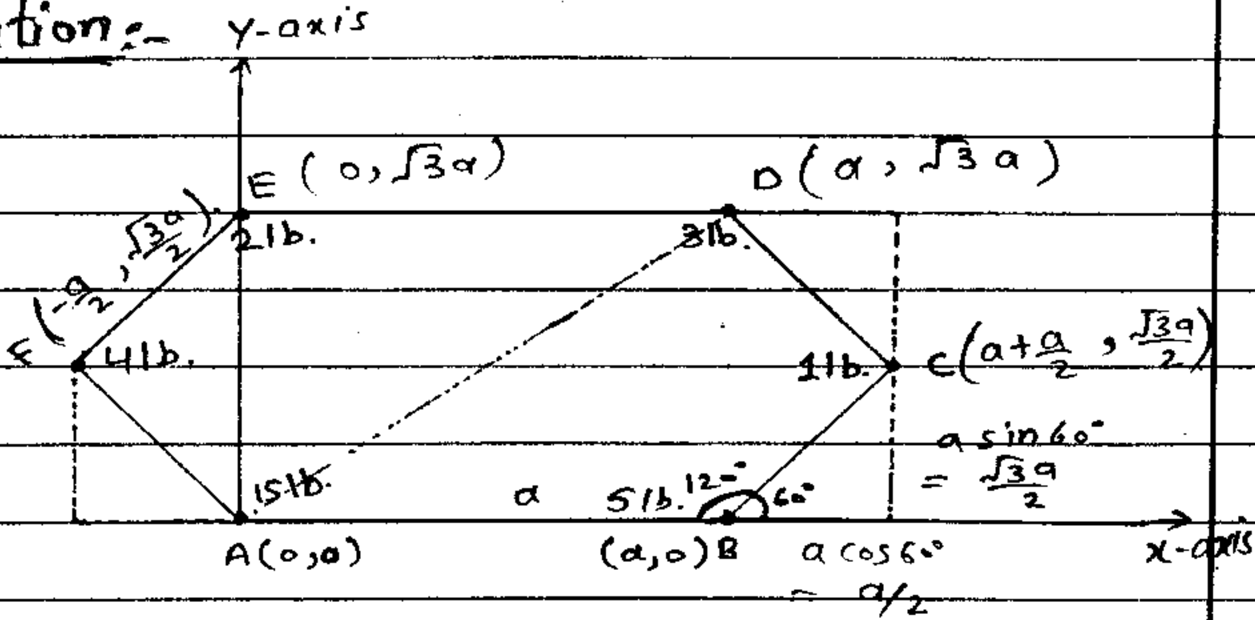
Distance of C.G from AB = 5.6 inches.
Distance of C.G from AD = 4 inches.

-: Q: No: 3 :-

Weights of 5, 1, 3, 2, 4 and 15 lb. are placed at the angular points of a regular hexagon taken in order. Find the distance of their c.g. from the 15 lb. weight?

Ans: Diagram:-

Solution:-



Consider regular hexagon ABCDEF with corner A at origin and side AB and direction AE along x and y-axis.

If "a" is the length of side then co-ordinates of $A(0,0)$, $B(a,0)$, $C\left(\frac{3a}{2}, \frac{\sqrt{3}a}{2}\right)$, $D(a, \sqrt{3}a)$, $E(0, \sqrt{3}a)$ and $F\left(-a/2, \frac{\sqrt{3}}{2}a\right)$.

$$C.G = (\bar{x}, \bar{y})$$

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

$$\bar{x} = \frac{15(0) + 5(a) + 1\cdot\frac{3}{2}a + 3a + 2(0) + \cancel{4}^{2}\left(-\frac{a}{\cancel{2}}\right)}{15+5+1+3+2+4}$$

$$\bar{x} = \frac{5a + \frac{3a}{2} + 3a - 2a}{30}$$

$$\bar{x} = \frac{10a + 3a + 6a - 4a}{60}$$

$$\bar{x} = \frac{15a}{60} = \frac{a}{4}$$

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

$$\bar{y} = \frac{15(0) + 5(0) + 1\cdot\frac{\sqrt{3}}{2}a + 3(\sqrt{3}a) + 2(\sqrt{3}a) + \cancel{4}^{2}\left(\frac{\sqrt{3}a}{\cancel{2}}\right)}{15+5+1+3+2+4}$$

$$= \frac{\frac{\sqrt{3}a}{2} + 3\sqrt{3}a + 2\sqrt{3}a + 2\sqrt{3}a}{30}$$

$$= \frac{\sqrt{3}a + 14\sqrt{3}a}{60}$$

$$= \frac{\cancel{15}\sqrt{3}a}{\cancel{60}_{4}} = \frac{\sqrt{3}a}{4}$$

$$C.G = \left(\frac{a}{4}, \frac{\sqrt{3}a}{4}\right)$$

Distance of C.G from A (i.e from 15 lb weights) $= \sqrt{x^2+y^2}$

$$= \sqrt{\frac{a^2}{16}+\frac{3a^2}{16}}$$

$$= \sqrt{\frac{4a^2}{16}}$$

$$= \sqrt{\frac{a^2}{4}}$$

$$= \frac{a}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{3}\cancel{a}}{\cancel{4}}}{\frac{\cancel{a}}{\cancel{4}}}$$

$$\tan\theta = \sqrt{3}$$

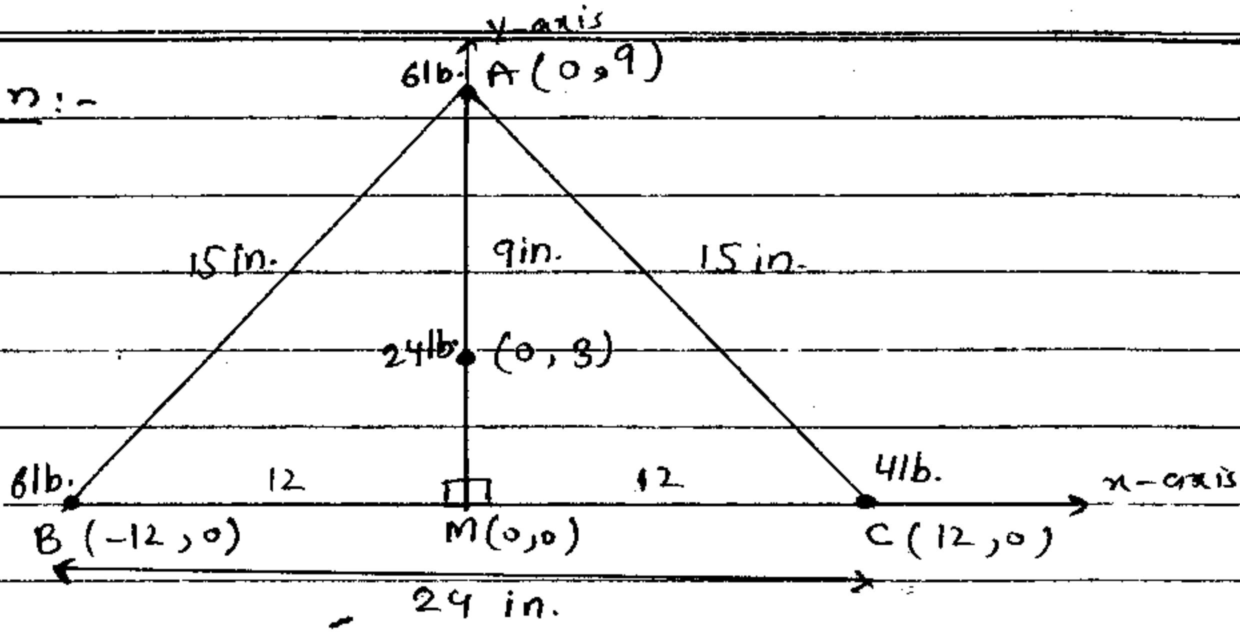
$$\theta = 60^\circ$$

$\Rightarrow$ C.G lies at a distance half of the length of side in the line joining 15 lb. and 3 lb. weights.

-: Q:No:4 :-

ABC is an isosceles triangular lamina in which AB = AC = 15 inches, BC = 24 inches. The weight of the lamina is 24 lb. and weights of 6, 6 and 4 lb. are placed at the corners A, B and C respectively. Find the distance of the c.g. of the system from BC?

Solution:-



Consider isosceles ΔABC $|AB| = |AC| = 15$ in.
Draw perpendicular $|AM|$ from A to side BC. $|BC| = 24$ in.
Take M as origin and BC and AM along x and y-axis respectively.
The co-ordinates of $B(-12,0)$, $C(12,0)$ $M(0,0)$, $A(0,9)$.

$\because$ In ΔAMC

$$|AM|^2 + |MC|^2 = |AC|^2$$

$$|AM|^2 = |AC|^2 - |MC|^2$$

$$= (15)^2 - (12)^2$$

$$= 225 - 144$$

$$\sqrt{|AM|^2} = \sqrt{81}$$

$$|AM| = 9 \text{ in.}$$

$\because$ triangle is symmetric about line $|AM|$ its c.g can lies on this line.

$$c.m \text{ of } \Delta ABC = \left(\frac{0-12+12}{3}, \frac{9+0+0}{3}\right)$$

$$= (0, 3).$$

Weights of 6, 6 and 4 lb. are attached at vertices A, B, C respectively.

Distance of c.g of the system from $\overline{BC} = \bar{y}$

$$(c.m)\ \bar{y} = \frac{6(9) + 6(0) + 4(0) + 24(3)}{6+6+4+24}$$

$$\because \quad c.m = \frac{\sum m_i y_i}{\sum m_i}$$

$$\bar{y} = \frac{54+72}{40}$$

$$\bar{y} = \frac{126}{40}$$

$$= 3.15 \text{ inches. Ans}$$

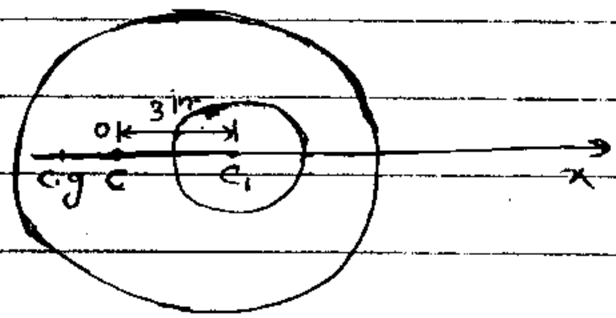
Example:- Page 76:-

In a uniform circular disc of 8" radius a circular hole of 2" radius is cut, the centre of the hole being 3" from the centre of the disc. Find the centre of the disc. Find the centre of mass of the Remainder of the disc?

Solution:-

Let M and M_1 be the masses

of large disc and hole removed resp. acting at c and c_1.



Taking a line through cc_1 as x-axis (Axis of symmetry). So c.g of remaining portion lies on it.

M = mass of large disc.

= (Aera)(mass per unit area).

where m = mass per unit area.

$= \pi r^2 \cdot m$

$M = \pi (8)^2 m = 64\pi m$

M_1 = mass of circular hole.

$= \pi r^2 . m$

$= \pi (2)^2 \cdot m$

$M_1 = 4\pi m$

If taking c at origin. $oc = 0$, $oc_1 = 3''$. C.G of remaining portion is consider as a disc from which a circle of radius $2''$ is removed.

$$\bar{x} = \frac{M \cdot oc - M_1 \cdot oc_1}{M - M_1}$$

$$\bar{x} = \frac{64\pi m \cdot 0 - 4\pi m \cdot 3}{64\pi m - 4\pi m}$$

$$= -\frac{12\pi m}{60\pi m}$$

$$= -\frac{1}{5}$$

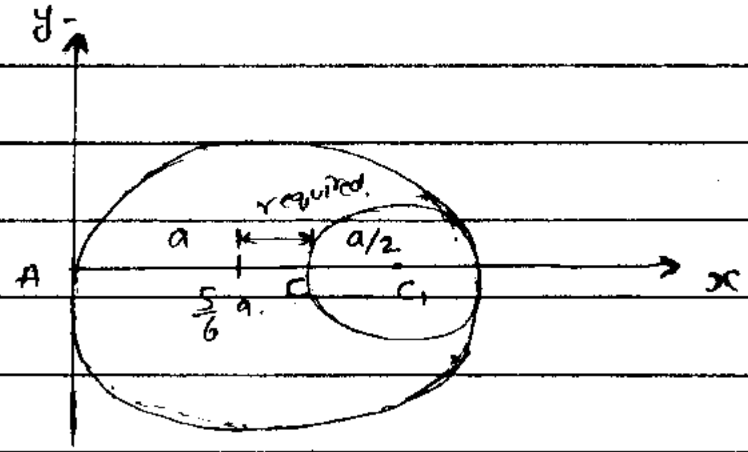
$\therefore$ -ive sign shows that c.g of remaining portion lies left to origin at distance $\frac{1}{5}$".

Q:No:10:-

From a uniform circular disc of radius a, a circular hole, having radius half that of the disc, is punched. Find the position of c.m of the remainder?

Solution:-

Let M and M_1 be the mass of circular disc of radius a and $\frac{a}{2}$ resp. Their c.g lies at their central pt c and c_1.



A line through c, c_1 is to bring as

x-axis (Axis of symmetry) so c.g lies on it.
If A is origin.

$AC = a.$

$AC_1 = AC + CC_1 = a + \frac{a}{2} = \frac{3}{2}a.$

M = mass of circular disc
$= $ Aera $\times$ mass/Aera where m = mass per unit Aera.
$= \pi a^2 . m$

M_1 = mass of circular hole.

$= \pi \left(\frac{a}{2}\right)^2 . m$

$= \frac{\pi a^2 m}{4}$

Remaining portion of disc regarded as a disc from which a circular hole is removed.

If c.G of remaining portion lies at G. Then

$$AG = \frac{M.AC - M_1.AC_1}{M - M_1}$$

$$= \frac{\pi a^2 m.a - \frac{\pi a^2 m}{4}.\frac{3}{2}a.}{\pi a^2 m - \frac{\pi a^2 m}{4}}$$

$$= \frac{\cancel{\pi a^2 m}\left(a - \frac{3a}{8}\right)}{\cancel{\pi a^2 m}\left(1 - \frac{1}{4}\right)}.$$

$$= \frac{8a - 3a}{8 - 2} = \frac{5a}{6}$$

c.g of remaining disc from C. $= a - \frac{5a}{6}$

$$= \frac{6a - 5a}{6}$$

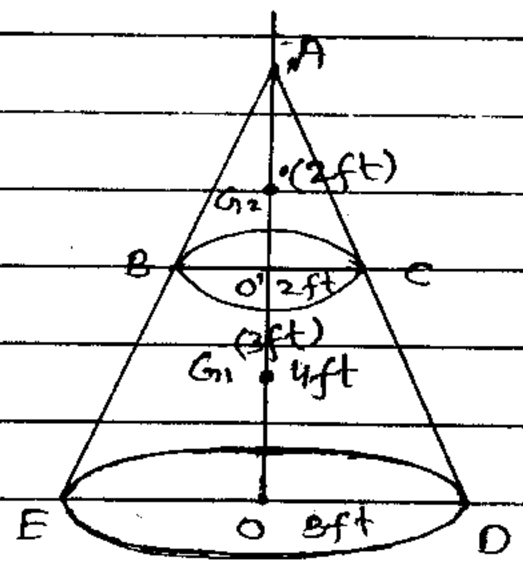
$$= \frac{a}{6}$$

At distance $\frac{1}{6}a$ from the centre of the disc.

-: Q. No: 5 :-

The radius of the faces of a frustum of a solid cone are 2ft. and 3ft. and the height of the frustum is 4ft. Find the distance of the c.g from the larger face?

Solution:-



Consider a frustrum BCDE. Extend sides EB and DC to complete the cone.

$OD = 3ft.$; $O'C = 2ft.$

$OO' = 4ft.$

If A is the verlex of cone and OA is the axis of symmetry c.g lies on this Axis.

$$\Delta AOD \cong \Delta AO'C$$

$$\frac{OA}{O'A} = \frac{OD}{O'C}$$

$$\frac{OA}{O'A} = \frac{3}{2}$$

$$\frac{OA}{O'A} - 1 = \frac{3}{2} - 1$$

$$\frac{OA - O'A}{O'A} = \frac{3-2}{2}$$

$$\frac{OO'}{O'A} = \frac{1}{2}$$

$$\frac{4}{O'A} = \frac{1}{2}$$

$O'A = 8ft.$

h = height of cone.

$h = OO' + O'A$

$= 4 + 8$

$h = 12 ft.$

Height of (Cone) big cone from vertex $= \frac{3}{4}h$.

Height of cone from base $= h - \frac{3}{4}h$

$= \frac{1}{4}h.$

$= \frac{1}{4}(12)$

$OG_1 = 3ft.$

height of small cone $= \frac{1}{4}h$

$O'G_2 = \frac{1}{4}(8) = 2ft$

$OG_2 = OO' + O'G_2$

$OG_2 = 4 + 2$

$OG_2 = 6ft.$

Let m be the mass per unit volume.

M = mass of big cone.

$= \text{volume} \cdot \frac{\text{mass}}{\text{volume}}$

$= \frac{1}{3}\pi r^2 h \cdot m.$

$= \frac{1}{\cancel{3}}\pi (3)^{\cancel{2}}(12) \cdot m$

$= 36\pi m$

M_1 = Mass of small cone.

$= \frac{1}{3}\pi r^2 h \cdot m$

$= \frac{1}{3}\pi (2)^2 \cdot 8 \cdot m$

$= \frac{32\pi m}{3}$

If G be the c.g of frustrum. Frustrum is consider as a cone from which a small cone is removed.

$$OG = \frac{M \cdot OG_1 - M_1 \cdot OG_2}{M - M_1}$$

$$= \frac{36\pi m \cdot 3 - \frac{32\pi m}{\cancel{3}} \cdot \cancel{6}^{2}}{36\pi m - \frac{32\pi m}{3}}$$

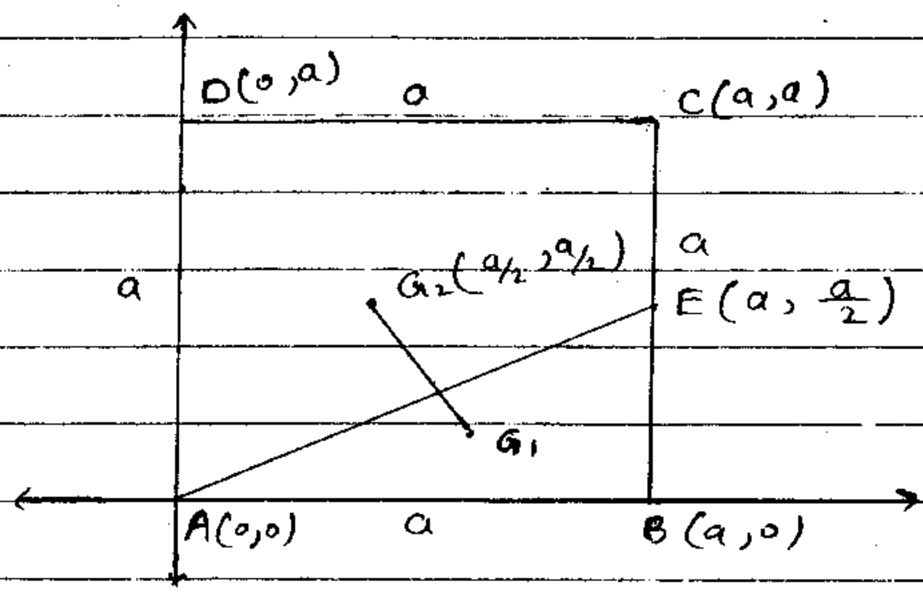
$$= \frac{\cancel{\pi m}(108 - 64)}{\cancel{\pi m}\left(\frac{108 - 32}{3}\right)}$$

$$= \frac{44 \cdot 3}{76} = \frac{132}{76}$$

$OG = \frac{33}{19}$ ft.

Solution:- :Q:No: 8:-

Let a be the length of side of square ABCD.



Consider vertex A at origin and side AB and AD on x and y-axis respectively.

Co-ordinates of $A(0,0)$, $B(a,0)$, $C(a,a)$ and $D(0,a)$.

$$E \text{ (mid pt BC)} = \left(\frac{a+a}{2}, \frac{a+0}{2}\right)$$

$$= \left(a, \frac{a}{2}\right).$$

$$G_1 = \text{c.m of } \Delta ABE$$

$$= \left(\frac{0+a+a}{3}, \frac{0+0+a/2}{3}\right)$$

$$= \left(\frac{2a}{3}, \frac{a}{6}\right).$$

$$G_2 = \text{c.m of square ABCD.}$$

$$= \left(\frac{a}{2}, \frac{a}{2}\right).$$

$$G = \text{c.m of quadrilateral AECD.}$$

If m = mass per unit area.

M = mass of square ABCD.

$M = (a)^2 m$

$M = a^2 m$

M_1 = mass of ΔABE

$= \frac{1}{2}|AB||BE|.m$

$= \frac{1}{2}a.\frac{a}{2}.m$

$M_1 = \frac{a^2}{4}m.$

Quad. AECD is regarded as a square ABCD from which ΔABE is removed.

So, $\bar{x} = \dfrac{M.\frac{a}{2} - M_1.\frac{2a}{3}}{M - M_1}$

$\bar{x} = \dfrac{a^2m.\frac{a}{2} - \frac{a^2m}{\cancel{4}_2}.\frac{\cancel{2}a}{3}}{a^2m - \frac{a^2m}{4}}$

$\bar{x} = \dfrac{\cancel{a^2m}\left(\frac{a}{2} - \frac{a}{6}\right)}{\cancel{a^2m}\left(1 - \frac{1}{4}\right)}$

$\bar{x} = \dfrac{6a - 2a}{12 - 3}$

$\bar{x} = \dfrac{4a}{9}.$

$\bar{y} = \dfrac{M.\frac{a}{2} - M_1.\frac{a}{6}}{M - M_1}$

$\bar{y} = \dfrac{a^2m.\frac{a}{2} - \frac{a^2m}{4}.\frac{a}{6}}{a^2m - \frac{a^2m}{4}}$

$\bar{y} = \dfrac{\cancel{a^2m}\left(\frac{a}{2} - \frac{a}{24}\right)}{\cancel{a^2m}\left(1 - \frac{1}{4}\right)}$

$\bar{y} = \dfrac{12a - a}{24 - 6}$

$$\bar{y} = \frac{11a}{18}$$

$$G = (\bar{x}, \bar{y}) = \left(\frac{4a}{9}; \frac{11a}{18}\right).$$

$$\text{Slope of } AE = \frac{\frac{a}{2} - 0}{a - 0} = \frac{a/2}{a} = \frac{1}{2}$$

$$\text{Slope of } G_1G = \frac{\frac{11a}{18} - \frac{a}{6}}{\frac{4a}{9} - \frac{2a}{3}}$$

$$= \frac{11a - 3a}{8a - 12a}$$

$$= \frac{8a}{-4a}$$

$$= -2$$

$$(\text{slope of } AE)(\text{slope of } G_1G) = \frac{1}{2}(-2)$$

$$= -1$$

$$\Rightarrow AE \perp G_1G$$

Q: A square lamina ABCD is divided into two parts by joining A to E, the middle point of BC. Prove that line joining the c.m of the triangular portion ABE to that of the quadrilateral portion AECD is perpendicular to AE?

VIP :- Q: No: 12 :-

Two uniform solid spheres, composed of the same material and whose diameters are 6 in. and 12 in. resp., are firmly united. Find the c.m of the combined body?

Solution:-

Volume of sphere $= \frac{4}{3}\pi r^3$

Diameter of big sphere $= 12$ in.

Radius of big sphere $= 6$ in.

Diameter of small sphere $= 6$ in.

Radius of small sphere $= 3$ in.

If $m =$ mass per unit volume

$M =$ mass of big sphere.

$= \frac{4}{3}\pi r^3 . m$

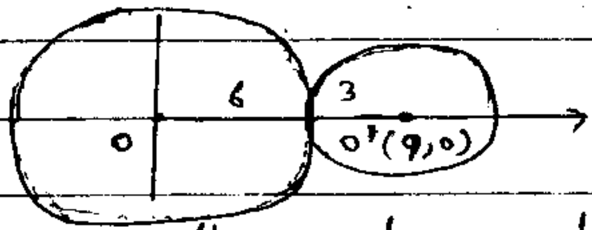
$= \frac{4}{3}\pi (6)^3 . m$

$M_1 =$ mass of small sphere

$= \frac{4}{3}\pi r^3 . m$

$M_1 = \frac{4}{3}\pi (3)^3 . m$

c.m of both sphere lies at their central points say O and O'.



line OO' through centre is taken as x-axis and O at origin.

If G is the c.m of combined body Then,

$$G = \frac{M(0) + M_1 \cdot OO'}{M + M_1}$$

$$= \frac{4/3 \pi 3^3 m \cdot 9}{4/3 \pi 6^3 m + \frac{4}{3} \pi 3^3 m}$$

$$= \frac{\cancel{\frac{4}{3} \pi m} (3^3 \cdot 9)}{\cancel{\frac{4}{3} \pi m} ((2 \times 3)^3 + 3^3)}$$

$$= \frac{3^3 \cdot 9}{2^3 \cdot 3^3 + 3^3}$$

$$= \frac{\cancel{3^3} \cdot 9}{\cancel{3^3}(2^3 + 1)}$$

$$= \frac{9}{8+1}$$

$$= \frac{9}{9}$$

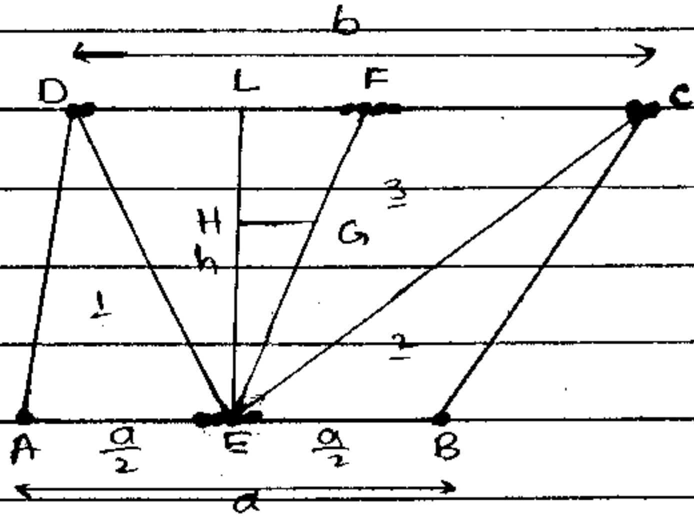
$$= 1''$$

$\Rightarrow$ G acts at 1 in. from centre of big sphere.

-: Q : No: 6 :-

ABCD is a trapezium which bounds a uniform lamina. AB, CD are parallel, and of lengths a, b respectively. Prove that the distance of the c.m of the lamina from AB is $\frac{1}{3} h \frac{a+2b}{a+b}$ where h is the distance between parallel sides?

Solution:-



Consider a trapezium ABCD in which $|AB| \parallel |CD|$ and $|AB| = a$ and $|CD| = b$. Take E and F mid point of side AB and CD respectively.

Join E to C and D. Then trapezium is equalent to three triangle AED, EBC and ECD.

$h = \perp$lar distance b/w parallel sides.

= Altitude of each triangle

m = mass per unit area.

mass of ΔEAD = Area . mass/Area.

$= \frac{1}{2}\left(\frac{a}{2}\right)h \cdot m$ = mass of ΔEBC.

$= \frac{ahm}{4}$ = mass of ΔEBC.

mass of $\Delta ECD = \frac{1}{2} b \cdot h \cdot m = \frac{bhm}{2}$.

Whole mass of triangle is equalent to 3 particle of mass $\frac{1}{3}$ of the mass of triangle. place at the vertices.

Then mass at $A = \frac{ahm}{12}$ = mass at B.

mass at $C = \frac{ahm}{12} + \frac{bhm}{6}$ = mass at D

mass at $C = \frac{hm}{12}(a+2b)$ = mass at D

mass at $E = \frac{ahm}{12} + \frac{ahm}{12} + \frac{bhm}{6}$

$= \frac{hm}{12}(a+a+2b)$

$= \frac{hm}{6}(a+b)$.

mass at A and B are equalent to mass

$= \frac{ahm}{12} + \frac{ahm}{12}$ at E.

$= \frac{ahm}{6}$

Then mass at $E = \frac{hm}{6}(a+b) + \frac{ahm}{6}$

$= \frac{hm}{6}(2a+b)$.

mass at C and D are Eqvalent to mass at F.

$$= \frac{hm}{12}(a+2b) + \frac{hm}{12}(a+2b).$$

$$= \frac{2hm}{12}(a+2b).$$

$$= \frac{hm}{6}(a+2b)$$

If G is the c.m of whole system acting at E and F.

$$\frac{EG}{GF} = \frac{\text{mass at } F}{\text{mass at } E} = \frac{\cancel{\frac{hm}{6}}(a+2b)}{\cancel{\frac{hm}{6}}(2a+b)}$$

Draw ⊥lar from G to line EL which is ⊥lar from E to CD.

ΔEGH is similar with ΔELF.

$$\frac{EH}{HL} = \frac{EG}{GF} = \frac{a+2b}{2a+b}$$

$$\frac{EH}{HL} = \frac{a+2b}{2a+b}$$

$$\frac{HL}{EH} = \frac{2a+b}{a+2b}$$

$$\frac{HL}{EH} + 1 = \frac{2a+b}{a+2b} + 1$$

$$\frac{HL+EH}{EH} = \frac{2a+b+a+2b}{a+2b}$$

$$\frac{EL}{EH} = \frac{3a+3b}{a+2b}$$

$$\frac{h}{EH} = \frac{3(a+b)}{a+2b}$$

$$\frac{h(a+2b)}{3(a+b)} = EH$$

$$EH = \frac{1}{3} h \frac{(a+2b)}{a+b}.$$

Distance of c.m from AB.

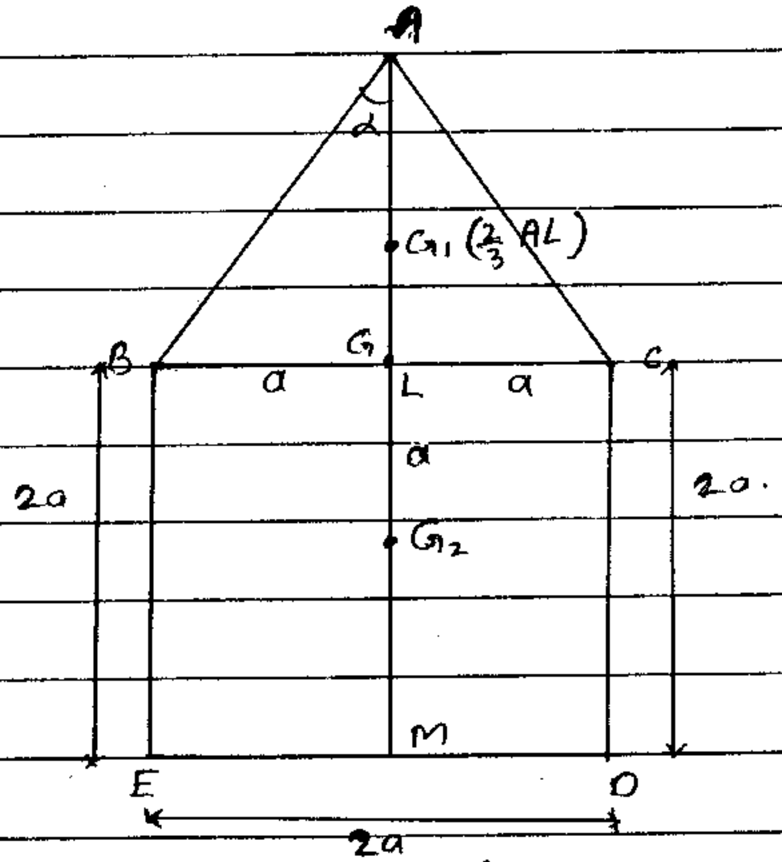
Note:- Centroid is the point of concurrency of medians.

Centroid divide medians 2 : 1.

-: Q: No: 9 :-

A lamina is in the shape of a square described on the base of an isosceles triangle. Find the tangent of the semi-vertical angle of the triangle if the c.m of the whole lamina is at the middle point of the base?

Solution:- Diagram:-



AM axis of symmetry so, c.m lies on it.

BCDE is a square of side length 2a. describe at the base of isosceles triangle ABC.

$\angle\alpha$ = semi-verticle angle

In ΔALB.

$$\frac{BL}{AL} = \tan\alpha \Rightarrow \frac{a}{AL} = \tan\alpha$$

$$a = AL\tan\alpha$$

$$a\cot\alpha = AL$$

G_1 is c.m of ΔABC.

$$AG_1 = \frac{2}{3}AL$$

$$= \frac{2}{3}a\cot\alpha$$

G_2 is c.m of square BCDE.

$$AG_2 = AL + LG_2$$

$$AG_2 = a\cot\alpha + a.$$

If m = mass per unit area.

M = mass of square.

$$= (2a)^2 \cdot m = 4a^2 m.$$

M_1 = mass of ΔABC.

$$= \frac{1}{2}|BC|.|AL|\cdot m$$

$$= \frac{1}{\cancel{2}}(\cancel{2}a)\,a\cot\alpha\, m$$

$$= a^2\cot\alpha\, m$$

If G is c.m of combined body.

$$AG = \frac{M\cdot AG_2 + M_1\cdot AG_1}{M + M_1}$$

$$= \frac{4a^2m(a + a\cot\alpha) + a^2\cot\alpha\, m \cdot \frac{2}{3}a\cot\alpha}{4a^2m + a^2\cot\alpha \cdot m}$$

$\because$ c.m of combined body lies at mid pt L of base so, $AL = AG$

$\because AL = a\cot\alpha$.

$$a\cot\alpha = \frac{\cancel{a^2m}\left(4a + 4a\cot\alpha + \frac{2}{3}a\cot^2\alpha\right)}{\cancel{a^2m}\,(4 + \cot\alpha)}$$

$$a\cot\alpha(4 + \cot\alpha) = 4a + 4a\cot\alpha + \frac{2}{3}a\cot^2\alpha$$

$$\cancel{12a\cot\alpha} + 3a\cot^2\alpha = 12a + \cancel{12a\cot\alpha} + 2a\cot^2\alpha$$

$$\cancel{a}\cot^2\alpha = 12\cancel{a}$$

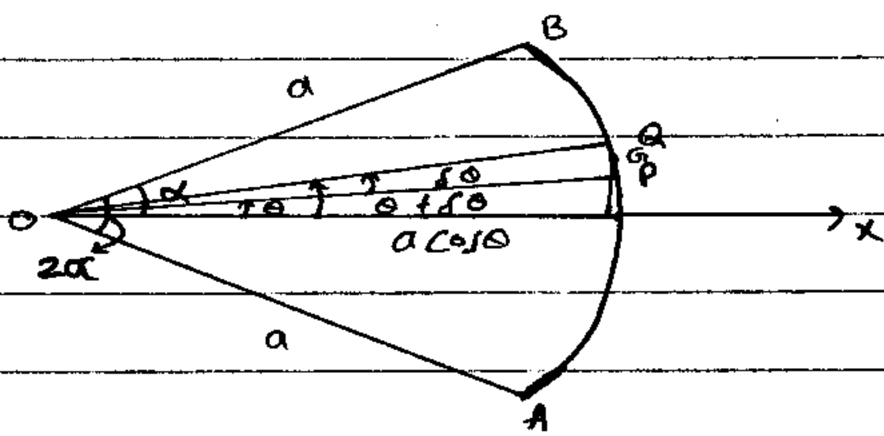
$$\cot\alpha = 2\sqrt{3}$$

$$\tan\alpha = \frac{1}{2\sqrt{3}} \text{ Ans}$$

Example # 14:- **Page : 14:-**

Find the c.m of a uniform circular cone..

Solution:-



Consider a circular arc AB of circle of radius a, making an angle 2α at centre, α is semi-verticle angle.

Consider a part PQ of arc s.t

$\angle POX = \theta$

$\angle QOX = \theta + \delta\theta$

$\angle POQ = \delta\theta$

$\widehat{PQ} = a\,\delta\theta \qquad \because \ l = r\theta$

, m = mass per unit length.

mass of $\widehat{PQ} = a\,\delta\theta \cdot m$

$X = a\cos\theta$

$$\bar{x} = \frac{\int x\,dm}{\int dm}$$

$$= \frac{\int_{-\alpha}^{\alpha} a\cos\theta \cdot am\,d\theta}{\int_{-\alpha}^{\alpha} am\,d\theta}$$

$$= \frac{\cancel{a^2 m}\int_{-\alpha}^{\alpha} \cos\theta\,d\theta}{\cancel{am}\int_{-\alpha}^{\alpha} d\theta}$$

$$= \frac{a\left[\sin\theta\right]_{-\alpha}^{\alpha}}{\left[\theta\right]_{-\alpha}^{\alpha}}$$

$$= \frac{a\left[\sin\alpha - \sin(-\alpha)\right]}{\left[\alpha - (-\alpha)\right]}$$

$$= \frac{a(\sin\alpha + \sin\alpha)}{\alpha + \alpha}$$

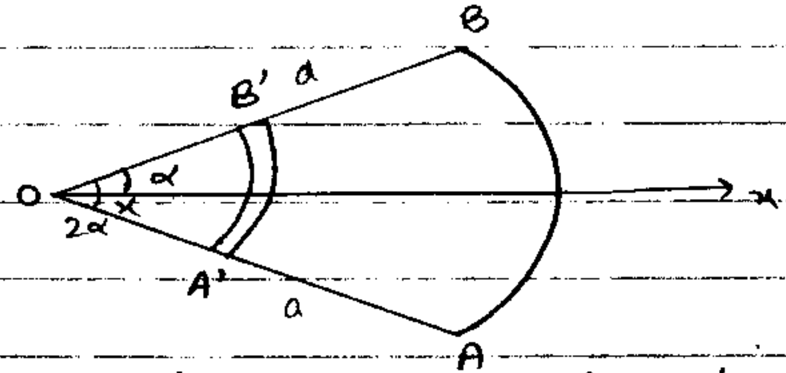
$$= \frac{\cancel{2}a\sin\alpha}{\cancel{2}\alpha}$$

$$\bar{x} = a \cdot \frac{\sin\alpha}{\alpha}. \quad \text{Ans.}$$

Example #15:- **Page: 84:-**

Find the c.m of a uniform sector of a circular lamina?

Solution:-



Let AOB be a sector of a circular lamina of radius a. Let the measure of $\angle AOB$ be 2α. By Symmetry the c.m lies on Ox, the bisector of $\angle AOB$. The sector may be regarded as consisting of strips concentric with the given circle as in Fig.

The length of a strip $A'B'$ of radius x.

$= 2\alpha . x.$

$\therefore$ mass of this strip

$= \rho 2\alpha \cdot x \cdot \delta x$

where ρ is the density of the sector and δx is the breadth of strip.

The c.m of the strip is at a distance $\frac{x \sin\alpha}{\alpha}$ from O.

$\therefore$ the position of the c.m of the sector is given by,

$$\bar{x} = \frac{\int_0^a \rho 2\alpha x\,dx \cdot \frac{x\sin\alpha}{\alpha}}{\int_0^a \rho 2\alpha x\,dx}$$

$$= \frac{\sin\alpha}{\alpha} \cdot \frac{\int_0^a x^2\,dx}{\int_0^a x\,dx}$$

$$= \frac{\sin\alpha}{\alpha} \cdot \frac{\left[\frac{x^3}{3}\right]_0^a}{\left[\frac{x^2}{2}\right]_0^a}$$

$$= \frac{\sin\alpha}{\alpha} \cdot \frac{\left[\frac{a^3}{3}\right]}{\left[\frac{a^2}{2}\right]}$$

$$= \frac{\sin\alpha}{\alpha} \cdot \frac{a^3}{3} \cdot \frac{2}{a^2}$$

$$\bar{x} = \frac{2a}{3} \cdot \frac{\sin\alpha}{\alpha} \quad \text{Ans}$$

Note:-

(i):- C.m of uniform circular arc $= a \cdot \frac{\sin\alpha}{\alpha}$

a = radius of circle, α = semi-vertical angle.

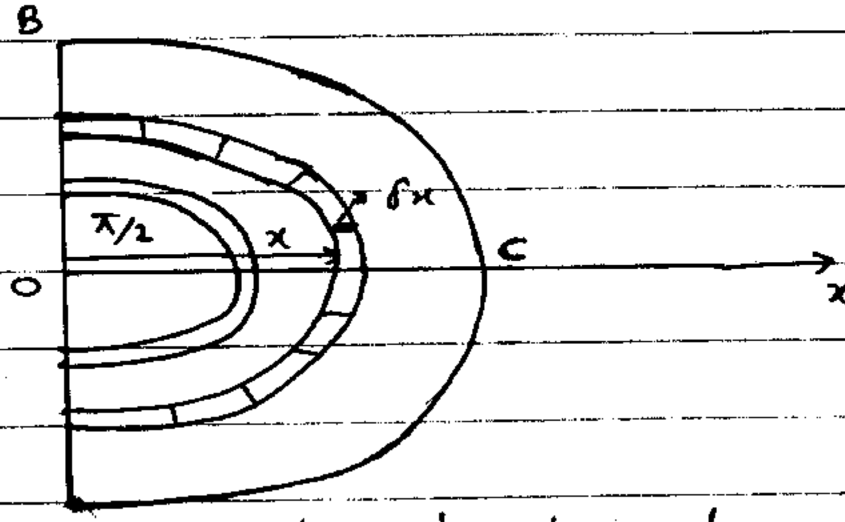
(ii):- C.m of a sector of circular lamina

$$= \frac{2}{3} a \cdot \frac{\sin\alpha}{\alpha}.$$

:Q:No:14:-

Find the c.g. of a semi-circular lamina of radius r when the density varies as the cube of the distance from the centre?

Solution:-



Consider OC radius ⊥lar to diameter AB of semi-circular lamina of radius r. OC is along x-axis (Axis of symmetry so c.g lies on it.)

Semi-circular lamina is considered as a made up of infinity many semi circular strips. Consider one such strip at a distance x from O. of width δx.

$$\text{c.g of this strip} = a \cdot \frac{\sin\alpha}{\alpha}$$

$$= \frac{x \sin \pi/2}{\pi/2}$$

$$= \frac{2x}{\pi}$$

M = mass of strip.

$$= \frac{1}{2}(2\pi r)\,\delta x \cdot \rho \qquad \because \rho \propto x^3$$

$$= \pi x\,\delta x \cdot \lambda x^3 \qquad \rho = \lambda x^3$$

$$= \lambda \pi x^4 \delta x$$

If G is the c.g of whole lamina.

$$OG = \frac{\int x\,dm}{\int dm}.$$

$$= \frac{\int_0^r \frac{2x}{\cancel{\pi}} \cdot \lambda x^4 \cancel{\pi}\, dx}{\int_0^r \lambda \pi x^4\, dx}$$

$$= \frac{2\cancel{\lambda}\int_0^r x^5\, dx}{\cancel{\lambda}\pi \int_0^r x^4\, dx}$$

$$= \frac{2}{\lambda} \cdot \frac{\left[\frac{x^6}{6}\right]_0^r}{\left[\frac{x^5}{5}\right]_0^r}$$

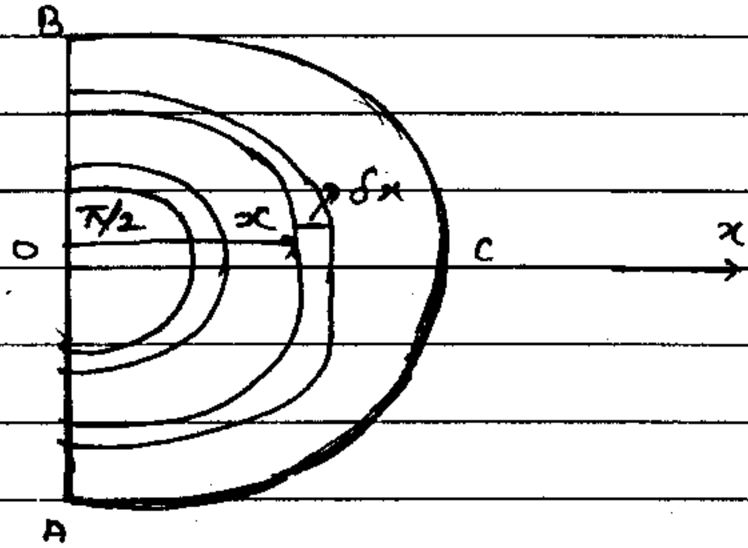
$$= \frac{\cancel{2}}{\lambda} \cdot \frac{r^{\cancel{6}}}{\cancel{6}_3} \cdot \frac{5}{\cancel{r^5}}$$

$$= \frac{5}{3} \cdot \frac{r}{\lambda} \quad \text{Ans}$$

Example # 16:- **page :85:**

Find the c.m of a semi-circular lamina of radius a whose density varies as the square of the distance from the centre?

Solution:-



Consider OC radius ⊥lar to diameter

AB of semi circular lamina of radius a. Oc is along x-axis (Axis of symmetry so c.m lies on it).

Semi circular lamina is considered as a made up of infinity many semi-circular strips. Consider one such strip at a distance x from O of width δx.

c.m of this strip $= a \frac{\sin\alpha}{\alpha}$

$$= \frac{x \cdot \sin \pi/2}{\pi/2}$$

$$= \frac{2x}{\pi}$$

M = mass of strip.

$$= \frac{1}{\cancel{2}}(\cancel{2}\pi x) \cdot \delta x \cdot \rho$$

$$= \pi x \cdot \delta x \cdot \rho \qquad \because \rho \propto x^2$$

$$\qquad \rho = \lambda x^2$$

$$= \pi x \cdot \delta x \cdot \lambda x^2$$

$$= \lambda \pi x^3 \delta x$$

If the G position of c.m of the whole lamina is given by $OG = \frac{\int x\, dm}{\int dm}$

$$OG = \frac{\int_0^a \frac{2x}{\pi} \cdot \lambda \cancel{\pi} x^3\, dx}{\int_0^a \lambda \pi x^3 dx.}$$

$$= \frac{2\cancel{\lambda} \int_0^a x^4 dx}{\cancel{\lambda}\pi \int_0^a x^3\, dx}$$

$$= \frac{2}{\pi} \cdot \frac{\left[\frac{x^5}{5}\right]_0^a}{\left[\frac{x^4}{4}\right]_0^a}$$

$$= \frac{2}{\pi} \cdot \frac{a^5}{5} \cdot \frac{4}{a^4}$$

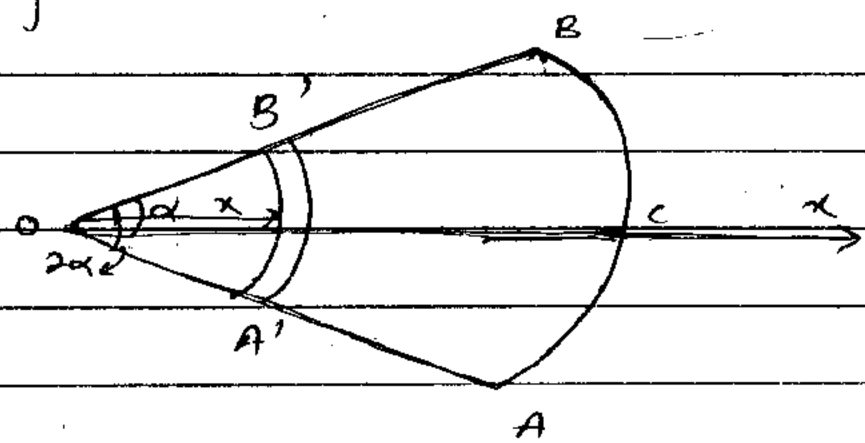
$$= \frac{8a}{5\pi} \quad \text{Ans}$$

-: Q.No: 15. -

The density at any point in a sector of a circular lamina varies as the distance from the centre. Find c.g of the sector?

Solution:

Let AOB be a sector of a circular lamina of radius 'a'.



Let $\angle AOB = 2\alpha$

OC (line bisecting angle is the axis of symmetry taken along x-axis)

c.g lies on OC.

Consider sector of circular lamina is made up by infinitly many strips. A'B' is one such strip at a distance x from O. δx is the width of strip.

Length of strip $A'B' = 2\alpha x$ $\quad \because l = r\theta$, $l = x\delta x$.

Mass = Area × density.

$= 2\alpha x \delta x \cdot \rho$ $\quad \because \rho \propto x$, $\rho = \lambda x$

$= 2\alpha x \delta x \cdot \lambda x$

$M = 2\alpha x^2 \lambda \delta x.$

The centre of gravity (c.g) of arc $A'B'$ lies on OC at $OG_1 = \dfrac{r\sin\alpha}{\alpha}$

$= \dfrac{x \sin\alpha}{\alpha}.$

c.g of sector of lamina $= \dfrac{\int x\,dm}{\int dm}$

$$= \frac{\int_0^a x \frac{\sin\alpha}{\cancel{\alpha}} \cdot 2\cancel{\alpha} x^2 \lambda\, dx}{\int_0^a 2\alpha \lambda x^2\, dx}$$

$$= \frac{\cancel{2\lambda}\sin\alpha \int_0^a x^3\, dx}{\cancel{2\lambda}\alpha \int_0^a x^2\, dx}$$

$$= \frac{\sin\alpha}{\alpha} \cdot \frac{\left[\frac{x^4}{4}\right]_0^a}{\left[\frac{x^3}{3}\right]_0^a}$$

$$= \frac{\sin\alpha}{\alpha} \cdot \frac{a^{\cancel{4}}}{4} \cdot \frac{3}{\cancel{a^3}}$$

$$= \frac{3}{4} a \frac{\sin\alpha}{\alpha}$$ Ans

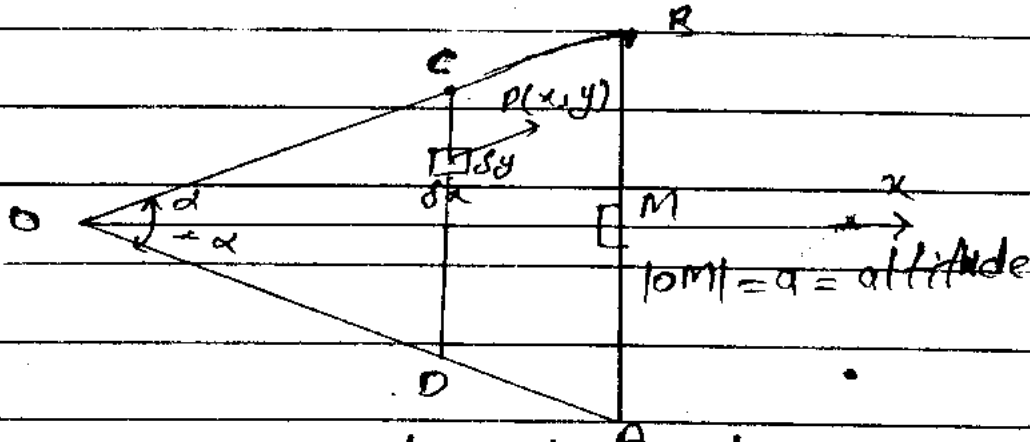
vip

-: Q:No: 16: -

An isosceles triangular lamina is such

that its mass per unit area at every point is proportional to the sum of the distances of the point from the equal sides of the triangle. Prove that the distance of the c.m. from the vertex is three-fourths of the altitude?

Ans :-



Consider an isosceles triangle OAB
OM ⊥lar AB, |OM| = a = altitude.
OM is the axis of symmetry taken along x-axis c.m lies on it.

Equation of line OB, $Y = x\tan\alpha$.

$\because \quad y - y_1 = m(x - x_1)$

$y - 0 = \tan\alpha\,(x - 0)$

$y = x\tan\alpha$

Equation of line OA, $y = -x\tan\alpha$

$\because \quad y - y_1 = m(x - x_1)$

$y - 0 = \tan(-\alpha)(x - 0)$

$y = -x\tan\alpha$

Taking an element of length δx and width δy at a point $P(x,y)$ of triangle.

Distance of $P(x,y)$ from line OB.

$(x\tan\alpha - y = 0)$.

$$|PC| = \frac{|x\tan\alpha - y|}{\sqrt{\tan^2\alpha + 1}}$$

$$= \frac{\left|x\cdot\frac{\sin\alpha}{\cos\alpha} - y\right|}{\sqrt{\sec^2\alpha}}$$

$$|PC| = \frac{x\sin\alpha - y\cos\alpha}{\cos\alpha\cdot\sec\alpha}.$$

$$|PC| = x\sin\alpha - y\cos\alpha$$

Similarly,

Distance of $P(x,y)$ from OA.

$(x\tan\alpha + y = 0)$.

$$|PD| = x\sin\alpha + y\cos\alpha.$$

$$\rho \propto |PC| + |PD| = 2x\sin\alpha.$$

$$\rho = 2\lambda x\sin\alpha.$$

Mass = Area . density.

$$= \delta x\cdot\delta y\cdot 2\lambda x\sin\alpha$$

$$c.m = \frac{\int x\,dm}{\int dm}$$

$$= \frac{\int_0^a\int_{-x\tan\alpha}^{x\tan\alpha} x\cdot 2\lambda x\sin\alpha\,dx\,dy}{\int_0^a\int_{-x\tan\alpha}^{x\tan\alpha} 2\lambda x\sin\alpha\,dx\,dy}.$$

$$= \frac{\int_0^a 2x^2 \lambda \sin\alpha \left(\int_{-x\tan\alpha}^{x\tan\alpha} 1\, dy \right) dx}{\int_0^a 2x\lambda \sin\alpha \left(\int_{-x\tan\alpha}^{x\tan\alpha} 1 \cdot dy \right) dx}$$

$\because$

$$\int_{-x\tan\alpha}^{x\tan\alpha} 1 . dy = \Big[y \Big]_{-x\tan\alpha}^{x\tan\alpha}$$

$$= x\tan\alpha + x\tan\alpha = 2x\tan\alpha$$

$$c.m = \frac{\int_0^a 2x^2 \lambda \sin\alpha \cdot 2x\tan\alpha\, dx}{\int_0^a 2\lambda x \sin\alpha \cdot 2x\tan\alpha\, dx}$$

$$= \frac{\int_0^a x^3\, dx}{\int_0^a x^2\, dx}$$

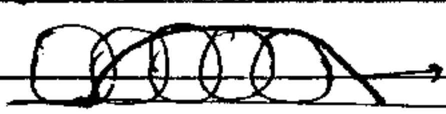
$$= \frac{\left[\frac{x^4}{4}\right]_0^a}{\left[\frac{x^3}{3}\right]_0^a}$$

$$= \frac{\cancel{a^4}}{4} \cdot \frac{3}{\cancel{a^3}}$$

$$= \frac{3}{4}a. \quad \text{Ans}$$

Note:- cycloid:-

A curve that is generated by a point on the circumference of the circle and it roled on a straight line.



cycloid:

$$\bar{x} = \frac{\int x\, ds}{\int ds} \quad ; \quad \bar{y} = \frac{\int y\, ds}{\int ds}$$

vip -: Q: No: 17 :-

Find the centroid of the arc of the cycloid $x = a(\theta + \sin\theta)$
$y = a(1 - \cos\theta)$
which lies in the first Quadrant?

Solution:-

$x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$

$\frac{dx}{d\theta} = a(1 + \cos\theta)$; $\frac{dy}{d\theta} = a\sin\theta$

squaring and adding.

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2(1 + \cos^2\theta + 2\cos\theta + \sin^2\theta)$$

$$= a^2(2 + 2\cos\theta)$$

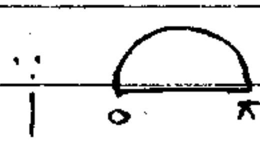
$$= 2a^2(1 + \cos\theta)$$

$$= 2a^2 \cdot 2\cos^2\theta/2$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \sqrt{4a^2\cos^2\theta/2}\, d\theta$$

$$ds = 2a\cos\theta/2\, d\theta$$

$$\bar{x} = \frac{\int x\, ds}{\int ds}$$



$$\bar{x} = \frac{\int_0^\pi a(\theta + \sin\theta) \cdot 2a\cos\theta/2\, d\theta}{\int_0^\pi 2a\cos\theta/2\, d\theta}$$

$$\bar{x} = \frac{\cancel{2}a^{\cancel{2}}\int_0^\pi (\theta\cos\theta/2 + \sin\theta\cos\theta/2)\, d\theta}{\cancel{2a}\int_0^\pi \cos\theta/2\, d\theta}$$

$$= a \cdot \frac{\int_0^{\pi} \frac{\theta}{\pi} \cos\frac{\theta}{2}\, d\theta + \int_0^{\pi} 2\sin\frac{\theta}{2} \cos^2\theta/_2\, d\theta}{\int_0^{\pi} \cos\theta/_2\, d\theta}$$

$$= a \cdot \frac{\int_0^{\pi} \frac{\theta}{\pi} \cos\theta/_2\, d\theta + 2\int_0^{\pi} \sin\theta/_2 \cos^2\theta/_2\, d\theta.}{\int_0^{\pi} \cos\theta/_2\, d\theta}$$

$$= a. \frac{\left[2\theta \cdot \sin\frac{\theta}{2} - \int 2 \cdot \sin\frac{\theta}{2} \cdot 1\, d\theta\right]_0^{\pi} - 4\int_0^{\pi} \cos^2\frac{\theta}{2}\left(-\frac{1}{2}\sin\frac{\theta}{2}\right) d\theta}{\left[2\sin\theta/_2\right]_0^{\pi}}$$

$$= \frac{a \cdot \left[2\theta \cdot \sin\theta/_2 + 4\cos\frac{\theta}{2}\right]_0^{\pi} - 4\left[\frac{\cos^3\pi/_2}{3}\right]_0^{\pi}}{2\sin\frac{\pi}{2} - 2\sin 0}$$

$$= a \frac{\left[\left\{2\pi \sin\frac{\pi}{2} + 4\cos\frac{\pi}{2}\right\} - \left\{0 + 4\cos 0\right\}\right] - \frac{4}{3}\left[\left(\cos\frac{\pi}{2}\right)^3 - (\cos 0)^3\right]}{2(1) - 0}$$

$$= a. \frac{\left[2\pi + 0 - 0 - 4 + \frac{4}{3}\right]}{2}$$

$$= a. \frac{\left[\frac{6\pi - 12 + 4}{3}\right]}{2}$$

$$= \frac{a. (6\pi - 8)}{6} = a. \frac{\cancel{2}(3\pi - 4)}{6/_3}$$

$$\bar{x} = \frac{1}{3} a (3\pi - 4).$$

$$\bar{y} = \frac{\int y\, ds}{\int ds.}$$

$$\bar{y} = \frac{\int_0^{\pi} a(1 - \cos\theta) \cdot 2a\cos\theta/_2\, d\theta}{\int_0^{\pi} 2a\cos\theta/_2\, d\theta}$$

$$= \frac{\cancel{2}a^{\cancel{2}}\int_0^{\pi}(1-\cos\theta)\cdot\cos\theta/_2\,d\theta}{\cancel{2a}\int_0^{\pi}\cos\theta/_2\,d\theta}$$

$$= a\cdot\frac{\int_0^{\pi}2\sin^2\theta/_2\cdot\cos\frac{\theta}{2}\,d\theta}{\int_0^{\pi}\cos\theta/_2\,d\theta}$$

$$= a\cdot 2\cdot 2\cdot\frac{\int_0^{\pi}\sin^2\frac{\theta}{2}\left(\frac{1}{2}\cos\frac{\theta}{2}\right)d\theta}{\int_0^{\pi}\cos\theta/_2\,d\theta}$$

$$= \frac{4a\cdot\left[\frac{\sin^3\theta/_2}{3}\right]_0^{\pi}}{\left[2\sin\frac{\theta}{2}\right]_0^{\pi}}.$$

$$= \frac{4a\cdot\left[\frac{(\sin\pi/_2)^3}{3}-\frac{(\sin 0)^3}{3}\right]}{2\sin\frac{\pi}{2}-2\sin 0^\circ}$$

$$= \frac{4a\cdot\left[\frac{1}{3}-0\right]}{2(1)-0}$$

$$= \frac{\frac{4a}{3}}{2} = \frac{\overset{2}{\cancel{4}}a}{3\times\cancel{2}}$$

$$\bar{y} = \frac{2a}{3}$$

$$\text{Centroid} = (\bar{x},\bar{y}).$$

$$= \left(\frac{a(3\pi-4)}{3},\frac{2a}{3}\right)$$

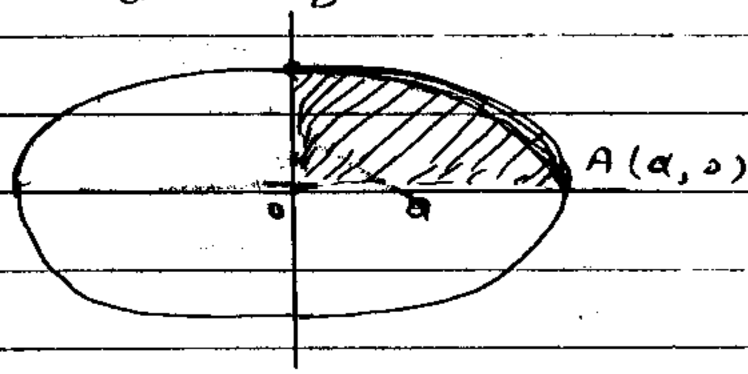
Ans

VIP **-: Q: No: 18 :-**

Find the position of the centroid of a quadrant of an elliptic lamina?

Solution:-

Equation of elliptical lamina in standard form is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$
$$= \frac{a^2 - x^2}{a^2}$$
$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$
$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

Consider portion of elliptical lamina in Ist Quad. Then,

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\bar{x} = \frac{\int_0^a x \cdot y \, dx}{\int_0^a y \, dx}$$

$\because$ limits 0 and a are due to portion in I Quad.

$$\bar{x} = \frac{\int_0^a x \cdot \frac{b}{a}\sqrt{a^2 - x^2}\, dx}{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx}$$

$$\bar{x} = \frac{\frac{b}{a}\cdot\frac{1}{(-2)}\int_0^a (a^2-x^2)^{1/2}(-2x)\,dx}{\frac{b}{a}\cdot\int_0^a \sqrt{a^2-x^2}\,dx.}$$

$$\bar{x} = -\frac{1}{2}\cdot\frac{\left[\frac{(a^2-x^2)^{3/2}}{3/2}\right]_0^a}{\left[\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}\right]_0^a}$$

$$\bar{x} = -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot\frac{\left[(a^2-a^2)^{3/2}-(a^2-0)^{3/2}\right]}{\left[\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)+\frac{a}{2}\sqrt{a^2-a^2}\right]-\left[\frac{a^2}{2}\sin^{-1}(0)+0\right]}$$

$$= -\frac{1}{3}\cdot\frac{\left(0-a^{\cancel{2}\cdot\frac{3}{\cancel{2}}}\right)}{\frac{a^2}{2}\sin^{-1}(1)+0-0-0}$$

$$= -\frac{1}{3}\cdot\frac{-a^3}{\frac{a^2}{2}\cdot\frac{\pi}{2}}$$

$$\bar{x} = \frac{4\,a^{\cancel{3}}}{3\,\cancel{a^2}\,\pi} = \frac{4a}{3\pi}$$

$$\bar{y} = \frac{\int_0^a \frac{y^2}{2}\,dx}{\int_0^a y\,dx} = \frac{\frac{1}{2}\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx}{\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx}$$

$$\bar{y} = \frac{\frac{1}{2}\cdot\frac{b^{\cancel{2}}}{a^{\cancel{2}}}\int_0^a (a^2-x^2)\,dx}{\frac{\cancel{b}}{\cancel{a}}\int_0^a \sqrt{a^2-x^2}\,dx.}$$

$$= \frac{1}{2}\cdot\frac{b}{a}\cdot\frac{\left[a^2x-\frac{x^3}{3}\right]_0^a}{\frac{a^2\pi}{4}}$$

$$\because \int_0^a \sqrt{a^2-x^2}\,dx = \frac{a^2\pi}{4}.$$

$$= \frac{b}{\cancel{2}a}\cdot\frac{\overset{2}{\cancel{4}}}{a^2\pi}\cdot\left[a^3-\frac{a^3}{3}\right]-\left[0-0\right]$$

$$\bar{y} = \frac{2b}{a^3\pi}\left(\frac{3a^3 - a^3}{3}\right).$$

$$= \frac{2b}{\cancel{a^3}\pi} \cdot \frac{2\cancel{a^3}}{3}$$

$$= \frac{4b}{3\pi}$$

$$\text{Centroid} = (\bar{x}, \bar{y}).$$

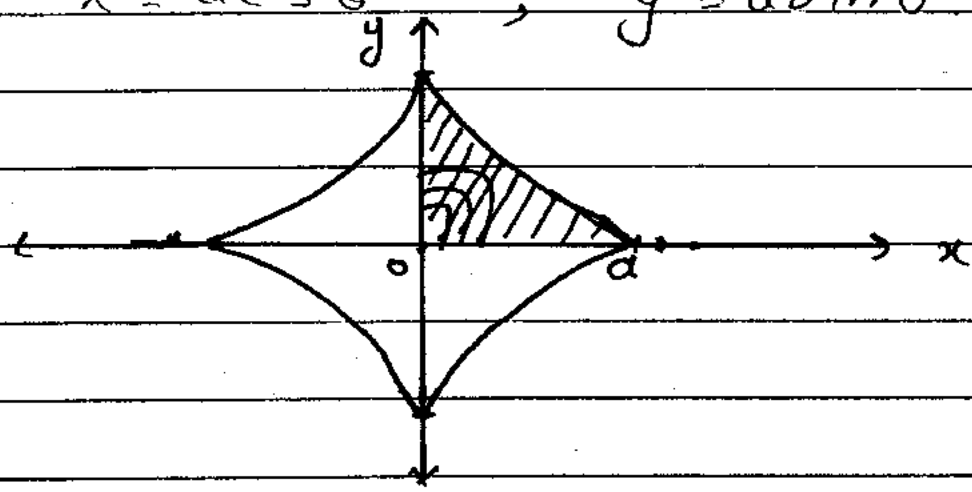
$$= \left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right). \text{ Ans}$$

VIP **Q:No:19:**

A lamina is bounded by the astroid
$x = a\cos^3\theta$, $y = a\sin^3\theta$.
Find the centroid of its portion that lies in the first Quadrant?

Solution:

$$x = a\cos^3\theta \quad ; \quad y = a\sin^3\theta$$



$$dx = 3a\cos^2\theta(-\sin\theta)\,d\theta$$

$$dx = -3a\cos^2\theta\cdot\sin\theta\,d\theta.$$

(62).

Centroid $= (\bar{x}, \bar{y})$.

$$\bar{x} = \frac{\int_0^a xy\,dx}{\int_0^a y\,dx}$$

$$= \frac{\int_{\pi/2}^{0} a\cos^3\theta \cdot a\sin^3\theta\,(-3a\cos^2\theta\sin\theta)\,d\theta}{\int_{\pi/2}^{0} a\sin^3\theta\,(-3a\cos^2\theta\sin\theta)\,d\theta}$$

$\because$ when $x = 0$ $\cancel{\cos\theta = 0} \Rightarrow \frac{\pi}{2} = \theta$

when $x = a$ $\cos\theta = 1 \Rightarrow \theta = 0$

$$= \frac{(\cancel{-3}a^{\cancel{3}\,1})(-1)\int_0^{\pi/2} \sin^4\theta\cos^5\theta\,d\theta}{(\cancel{-3}a^2)(-1)\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta}$$

$$= a \cdot \frac{\frac{\cancel{(4-1)(4-3)(5-1)(5-3)}}{9\cdot7\cdot5\cdot3\cdot1}}{\frac{(4-1)(4-3)(2-1)}{6\cdot4\cdot2}\cdot\frac{\pi}{2}}$$

<u>Note:-</u>

(i) If p is even and q is odd.

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\cdot\cos^q\theta\,d\theta = \frac{(p-1)(p-3)\cdots(q-1)(q-3)\cdots}{(p+q)(p+q-2)\cdots\cdots}$$

(ii) If p is even and q is even.

$$\int_0^{\pi/2} \sin^p\theta\cdot\cos^q\theta\,d\theta = \frac{(p-1)(p-3)\cdots(q-1)(q-3)\cdots}{(p+q)(p+q-2)\cdots\cdots}\cdot\frac{\pi}{2}$$

$$\bar{x} = \frac{\frac{\cancel{3}\cdot1\cdot4\cdot2}{9\cdot7\cdot5\cdot\cancel{3}}}{\frac{\cancel{3}\cdot1\cdot1\cdot\pi}{\cancel{6}_2\cdot4\cdot2\cdot2}} = a\cdot\frac{\frac{8}{315}}{\frac{\pi}{32}}$$

$$\bar{x} = a \cdot \frac{8}{315} \cdot \frac{32}{\pi}$$

$$\bar{x} = a \cdot \frac{256}{315\pi}$$

$$\bar{x} = \bar{y} = \frac{256a}{315\pi}$$

$$\text{Centroid} = (\bar{x}, \bar{y}).$$

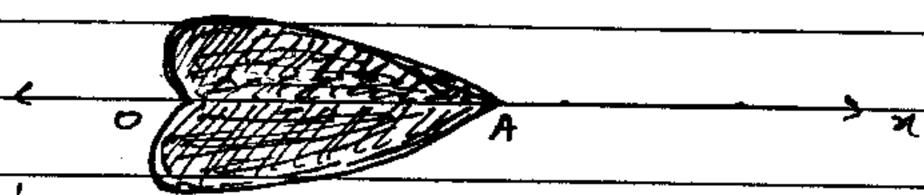
$$= \left(\frac{256a}{315\pi}, \frac{256a}{315\pi}\right).$$

:Q: No: 20: -

Find the centroid of the surface formed by the revolation of the cardioide $r = a(1+\cos\theta)$ about the initial line?

Solution:- Equation of cardioide:-

$$r = a(1+\cos\theta).$$



In polar form,

$$x = r\cos\theta \quad ; \quad y = r\sin\theta$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$ds = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$$

$$= \sqrt{a^2(1+\cos^2\theta + 2\cos\theta + \sin^2\theta)}\ d\theta.$$

$$ds = \sqrt{a^2(2+2\cos\theta)}\cdot d\theta$$

$$ds = \sqrt{2a^2(1+\cos\theta)}\cdot d\theta.$$

$$ds = \sqrt{2a^2 . 2\cos^2\theta/2}$$

$$ds = \sqrt{4a^2\cos^2\theta/2}\, d\theta$$

$$ds = 2a\cos\theta/2\, d\theta.$$

$\because$ Cardioide is revolving about a line (x-axis). So centroid lies on it.

$$xy\,ds = r\cos\theta . r\sin\theta . 2a . \cos\theta/2\, d\theta$$

$$= r^2\cos\theta\sin\theta . 2a\cos\theta/2\, d\theta$$

$$= a^2(1+\cos\theta)^2 (2\cos^2\theta/2 - 1)\, 2\cdot\sin\tfrac{\theta}{2}\cos\theta/2 . 2a\cos\tfrac{\theta}{2}\, d\theta$$

$$= 4a^3(2\cos^2\theta/2)^2(2\cos^2\theta/2 - 1)\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2}\, d\theta$$

$$= 16a^3\cos^6\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\left(2\cos^2\tfrac{\theta}{2} - 1\right) d\theta$$

$$= 16a^3\left[2\cos^8\tfrac{\theta}{2}\sin\tfrac{\theta}{2} - \cos^6\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\right] d\theta.$$

$$\int_0^\pi xy\,ds = 16a^3\left[-2\cdot 2\int\cos^8\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\cdot\tfrac{1}{2}\right)d\theta + 2\int\cos^6\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}\cdot\tfrac{1}{2}\right)d\theta\right]_0^\pi$$

$$= 16a^3\left[-4\cdot\frac{\cos^9\theta/2}{9} + 2\cdot\frac{\cos^7\theta/2}{7}\right]_0^\pi$$

$$= 16a^3\left[-\frac{4}{9}(0-1) + \frac{2}{7}(0-1)\right]$$

$$= 16a^3\left[\frac{4}{9} - \frac{2}{7}\right]$$

$$= 16a^3\cdot\left[\frac{28-18}{63}\right] = 16a^3\cdot\left[\frac{10}{63}\right]$$

$$\int_0^\pi xy\,ds = \frac{160a^3}{63}$$

$$y\,ds = r\sin\theta \cdot 2a\cos\frac{\theta}{2}\,d\theta$$

$$= a(1+\cos\theta)\sin\theta \cdot 2a\cos\theta/_2\,d\theta$$

$$= a \cdot 2\cos^2\frac{\theta}{2} \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,2a\cos\frac{\theta}{2}\,d\theta$$

$$= 8a^2\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta$$

$$\int_0^{\pi} y\,ds = 8a^2\int_0^{\pi}\cos^4\frac{\theta}{2} \cdot \sin\frac{\theta}{2}\,d\theta.$$

$$= -16a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\cdot\frac{1}{2}\right)d\theta$$

$$= -16a^2\left[\frac{\cos^5\frac{\theta}{2}}{5}\right]_0^{\pi}$$

$$= -16a^2\left[0 - \frac{1}{5}\right]$$

$$= \frac{16a^2}{5}$$

$$\bar{x} = \frac{\int_0^{\pi} xy\,ds}{\int_0^{\pi} y\,ds} = \frac{\frac{160a^3}{63}}{\frac{16a^2}{5}}$$

$$\bar{x} = \frac{\cancel{160}^{10}\,a^{\cancel{3}}}{63} \cdot \frac{5}{\cancel{16}_{1}\cancel{a^2}}$$

$$= \frac{10a}{63} \cdot \frac{5}{1}$$

$$= \frac{50a}{63}$$

Centroid $= (\bar{x}, \bar{y})$

Centroid $= \left(\frac{50a}{63}, 0\right).$ Ans

Formula's:-

Surface of solid of revolution in xy-plane

$$\bar{x} = \frac{\int x y^2 dx}{\int y^2 dx}$$

Revolution of curve in hollow form.

$$\bar{x} = \frac{\int x y \, dS}{\int y \, dS}$$

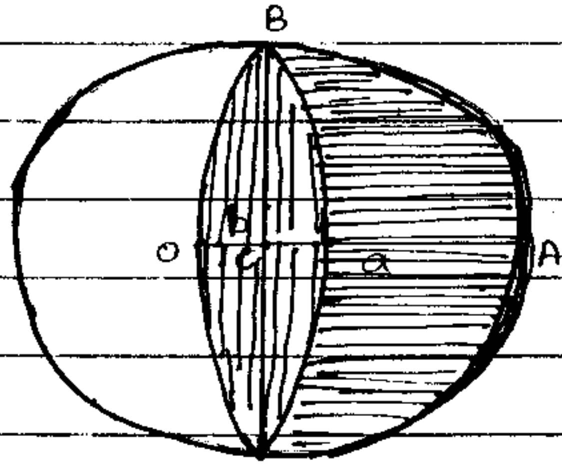
In plane xy-

$$\bar{x} = \frac{\int x y \, dx}{\int y \, dx}$$

-: Q: No: 21 :-

Show that the c.m. of a segment of a solid sphere of radius a, at a distance b from the centre of the sphere is at a distane $\frac{3}{4}\frac{(a+b)^2}{(2a+b)}$ from the centre?

Solution:- Segment of solid sphere is the solid of revolution of a portion ABC at a distance b from centre O of sphere of radius a about OA (x-axis). Axis of symmetry. So, c.m lies on it.



$\bar{x} = ?$; $\bar{y} = 0$

Equation of circular lamina.

$$x^2+y^2=a^2$$

$$y^2 = a^2-x^2 \qquad \text{limits } \; b \le x \le a$$

$$\bar{x} = \frac{\int_b^a x y^2\,dx}{\int_b^a y^2\,dx}$$

$$= \frac{\int_b^a x\cdot(a^2-x^2)\,dx}{\int_b^a (a^2-x^2)\,dx}$$

$$= \frac{\int_b^a (a^2x-x^3)\,dx}{\int_b^a (a^2-x^2)\,dx}$$

$$= \frac{\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_b^a}{\left[a^2x-\frac{x^3}{3}\right]_b^a}$$

$$\bar{x} = \frac{\left[\frac{a^4}{2}-\frac{a^4}{4}\right]-\left[\frac{a^2b^2}{2}-\frac{b^4}{4}\right]}{\left[a^3-\frac{a^3}{3}\right]-\left[a^2b-\frac{b^3}{3}\right]}$$

$$= \frac{\frac{a^4}{2}-\frac{a^4}{4}-\frac{a^2b^2}{2}+\frac{b^4}{4}}{a^3-\frac{a^3}{3}-a^2b+\frac{b^3}{3}}$$

$$= \frac{\frac{1}{4}\left[2a^4-a^4-2a^2b^2+b^4\right]}{\frac{1}{3}\cdot\left[3a^3-a^3-3a^2b+b^3\right]}$$

$$= \frac{3}{4} \cdot \frac{a^4 + b^4 - 2a^2b^2}{2a^3 - 3a^2b + b^3}$$

$$= \frac{3}{4} \cdot \frac{(a^2 - b^2)^2}{2a^3 - 2a^2b - a^2b + b^3}$$

$$= \frac{3}{4} \cdot \frac{(a-b)^2(a+b)^2}{2a^2(a-b) - b(a^2-b^2)}$$

$$= \frac{3}{4} \cdot \frac{(a-b)^{\cancel{2}}(a+b)^2}{\cancel{(a-b)}[2a^2 - ab - b^2]}$$

$$= \frac{3}{4} \cdot \frac{(a-b)(a+b)^2}{2a^2 - 2ab + ab - b^2}$$

$$= \frac{3}{4} \cdot \frac{(a-b)(a+b)^2}{2a(a-b) + b(a-b)}$$

$$= \frac{3}{4} \cdot \frac{\cancel{(a-b)}(a+b)^2}{\cancel{(a-b)}(2a+b)}$$

$$\bar{x} = \frac{3}{4} \cdot \frac{(a+b)^2}{2a+b} \quad \text{Ans}$$

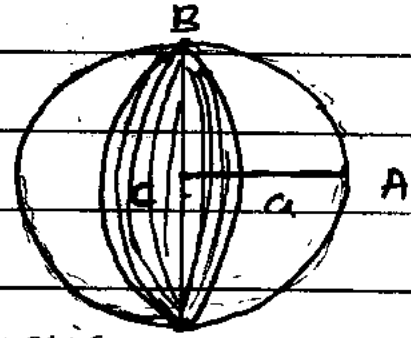
vip -: Q: No: 22: -

Prove that the c.m of a hemispherical shell of radius a is at a distance $\frac{a}{2}$ from the centre?

Solution:-

A hemispherical shell is a hollow half sphere generated by a curve $\overset{\frown}{AB}$ about radius CA.

(along x-axis).



Axis of symmetry.
So, c.m lies on it.
Radius of shell = a
∵ c.m lies on x-axis.

So, $\bar{x} = ?$, $\bar{y} = 0$

Equation of circule whose Arc AB is consider.

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

$$y = \sqrt{a^2 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{a^2 - x^2}}(-\cancel{2}x)$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx$$

$$= \sqrt{\frac{a^2 - \cancel{x^2} + \cancel{x^2}}{a^2 - x^2}}\, dx$$

$$ds = \frac{a}{\sqrt{a^2 - x^2}}\, dx$$

$$\bar{x} = \frac{\int_0^a xy\,ds}{\int_0^a y\,ds}$$

$$\bar{X} = \frac{\int_0^a x\cdot\sqrt{a^2-x^2}\cdot\frac{a}{\sqrt{a^2-x^2}}\,dx}{\int_0^a \sqrt{a^2-x^2}\cdot\frac{a}{\sqrt{a^2-x^2}}\,dx}$$

$$= \frac{a}{a}\cdot\frac{\int_0^a x\,dx}{\int_0^a 1\cdot dx}$$

$$= \frac{\left[\frac{x^2}{2}\right]_0^a}{[x]_0^a} = \frac{a^2}{2}\cdot\frac{1}{a}$$

$$\bar{x} = \frac{a}{2}$$

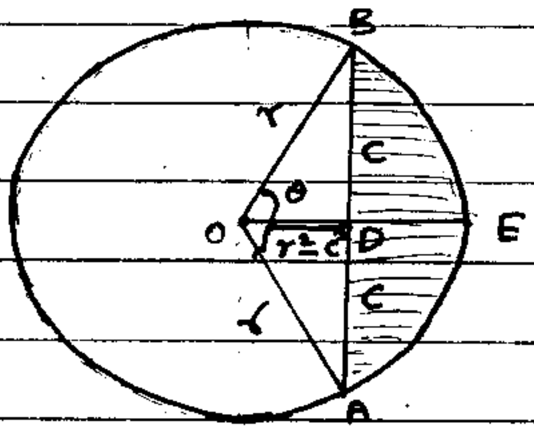
$\Rightarrow$ c.m is at a distance $\frac{a}{2}$ from centre.

-: Q:No:7: -

A portion of a circular disc of radius r is cut off by a straight cut of length $2c$. Find the position of c.m of the larger portion. If $r = 1$ ft., $c = 6$ in., calculate the distance of the mass-centre from the centre of the circle?

Note. disc is a plan fig.

Solution:-



if $\overline{AB} = 2c$ straight cut

Consider radius OE $\perp$lar to AB segment ABE is removed from the disc.

Aera of segment ABE = 2 (Aera of OEB)

$= 2\,[\text{Aera of sector OBE} - \text{Aera of ODB}]$

$= 2\left[\frac{1}{2}r^2\theta - \frac{1}{2}|OD|.|BD|\right]$

$= r^2\theta - \sqrt{r^2-c^2}\,.\,c$

$= r^2\theta - c.\sqrt{r^2-c^2} \qquad \because \sin\theta = \frac{c}{r}$

$= r^2\sin^{-1}\left(\frac{c}{r}\right) - c\sqrt{r^2-c^2} \quad \Big| \; \theta = \sin^{-1}\left(\frac{c}{r}\right)$

$= \lambda$

If m is the mass per unit area of disc.

M_1 = Mass of segment ABE.

$M_1 = \lambda m$.

M = mass of disc

$= \pi r^2.\,m$.

c.m of disc = $(0, 0)$.

Equation of circular disc.

$x^2+y^2 = r^2$

$y^2 = r^2 - x^2$

$y = \sqrt{r^2-x^2}$ Equation of Arc $\overset{\frown}{AB}$

$\because \sqrt{r^2-c^2} \le x \le r$

$$\bar{x} = \frac{\int_{\sqrt{r^2-c^2}}^{r} xy\,dx}{\int_{\sqrt{r^2-c^2}}^{r} y\,dx}$$

$$\bar{x} = \frac{\int_{\sqrt{r^2-c^2}}^{r} x\cdot\sqrt{r^2-x^2}\,dx}{\int_{\sqrt{r^2-c^2}}^{r} \sqrt{r^2-x^2}\,dx}$$

$$= \frac{-\frac{1}{2}\int_{\sqrt{r^2-c^2}}^{r} (r^2-x^2)^{1/2}(-2x)\,dx}{\int_{\sqrt{r^2-c^2}}^{r} \sqrt{r^2-x^2}\,dx}$$

$$= \frac{-\frac{1}{2}\left[\frac{(r^2-x^2)^{3/2}}{3/2}\right]_{\sqrt{r^2-c^2}}^{r}}{\left[\frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)+\frac{x}{2}\sqrt{r^2-x^2}\right]_{\sqrt{r^2-c^2}}^{r}}$$

$$= \frac{-\frac{1}{3}\cdot\left[(r^2-r^2)^{3/2}-(r^2-r^2+c^2)^{3/2}\right]}{\left[\frac{r^2}{2}\sin^{-1}(1)+\frac{r}{2}\sqrt{r^2-r^2}\right]-\left[\frac{r^2}{2}\sin^{-1}\left(\frac{\sqrt{r^2-c^2}}{r}\right)+\frac{\sqrt{r^2-c^2}}{2}\sqrt{r^2-r^2+c^2}\right]}$$

$$= \frac{-\frac{1}{3}\left[-c^3\right]}{\frac{r^2}{2}\left[\sin^{-1}(1)-\sin^{-1}\frac{\sqrt{r^2-c^2}}{r}\right]-\frac{c\sqrt{r^2-c^2}}{2}}$$

$$\because\ \sin^{-1}A-\sin^{-1}B=\sin^{-1}\left(A\sqrt{1-B^2}-B\sqrt{1-A^2}\right)$$

$$\sin^{-1}(1)-\sin^{-1}\frac{\sqrt{r^2-c^2}}{r}=\sin^{-1}\left(1\sqrt{1-\frac{r^2-c^2}{r^2}}-0\right)$$

$$=\sin^{-1}\sqrt{\frac{r^2-r^2+c^2}{r^2}}$$

$$=\sin^{-1}\left(\frac{c}{r}\right)$$

$$\bar{x} = \frac{\frac{c^3}{3}}{\frac{r^2}{2}\sin^{-1}\left(\frac{c}{r}\right)-\frac{c\sqrt{r^2-c^2}}{2}}$$

$$\because\ r^2\sin^{-1}\left(\frac{c}{r}\right)-c\sqrt{r^2-c^2}=\lambda$$

$$\bar{x} = \frac{\frac{c^3}{3}}{\frac{1}{2}\lambda} = \frac{2c^3}{3\lambda}$$

c.m of Big portion of disc.

$$cm = \frac{M \cdot 0 - M_1 \bar{x}}{M - M_1}$$

$$= \frac{-\cancel{\lambda} m \left(\frac{2c^3}{3\cancel{\lambda}}\right)}{\pi r^2 m - \lambda m}$$

$$= \frac{-\cancel{m}\left(\frac{2c^3}{3}\right)}{\cancel{m}(\pi r^2 - \lambda)}$$

$$= \frac{-\frac{2c^3}{3}}{(\pi r^2 - \lambda)}.$$

$\because$ $\lambda = r^2 \sin^{-1}\left(\frac{c}{r}\right) - c \cdot \sqrt{r^2 - c^2}$.

$$\lambda = (1)^2 \sin^{-1}\left(\frac{1}{2} \cdot \frac{1}{1}\right) - \frac{1}{2}\sqrt{1 - \frac{1}{4}}.$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \frac{1}{2} \cdot \sqrt{\frac{4-1}{4}}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2 \cdot 2}$$

$$\lambda = 0.091$$

$$c.m = \frac{-2\left(\frac{1}{2}\right)^3}{3(3.14(1)^2 - 0.091)}$$

$$= \frac{-\cancel{2} \cdot \frac{1}{\cancel{2} \cdot 4}}{9.15}$$

$$= -\frac{1}{4 \times 9.15} = -0.0273 \text{ ft}$$

$$= -0.0273 \times 12 \text{ inches}$$

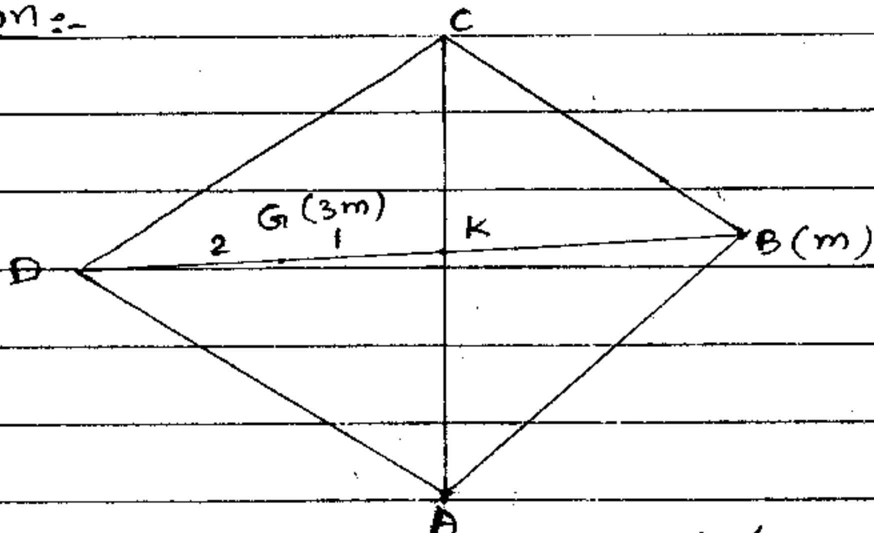
$$= 0.328 \text{ in.}$$

$\therefore$ c.m is shift 0.328 inches left to origional centre of disc.

-: Q: No: 25 :-

If the c.g of a quadrilateral lamina is the same as that of four equal particles placed at its angular points, show that the bounding quadrilateral must be a parallelogram?

Solution:-



Let ABCD be the quadrilateral lamina

If m be the mass of each particle placed at the angular points. A, B, C and D.

Join A to C. if G is the centroid of ΔACD.

$\Rightarrow$ A mass of 3m acts at G.

We are left with two particles of mass 3m acts at G and mass m acts at B.

If K is the mid point of AC.

G lies on median DK.

If K is the c.g of masses at G and B.

$$\frac{GK}{KB} = \frac{\cancel{m}}{3\cancel{m}}$$

$$3GK = KB \longrightarrow (i)$$

$\because$ G divides DK in 2 : 1.

$\Rightarrow GK = \frac{1}{3} DK$ using in (i)

$\cancel{3}.\frac{1}{\cancel{3}}.DK = KB$

$DK = KB$

$\Rightarrow$ K is the mid point of BD.

$\because$ K is also mid point of AC

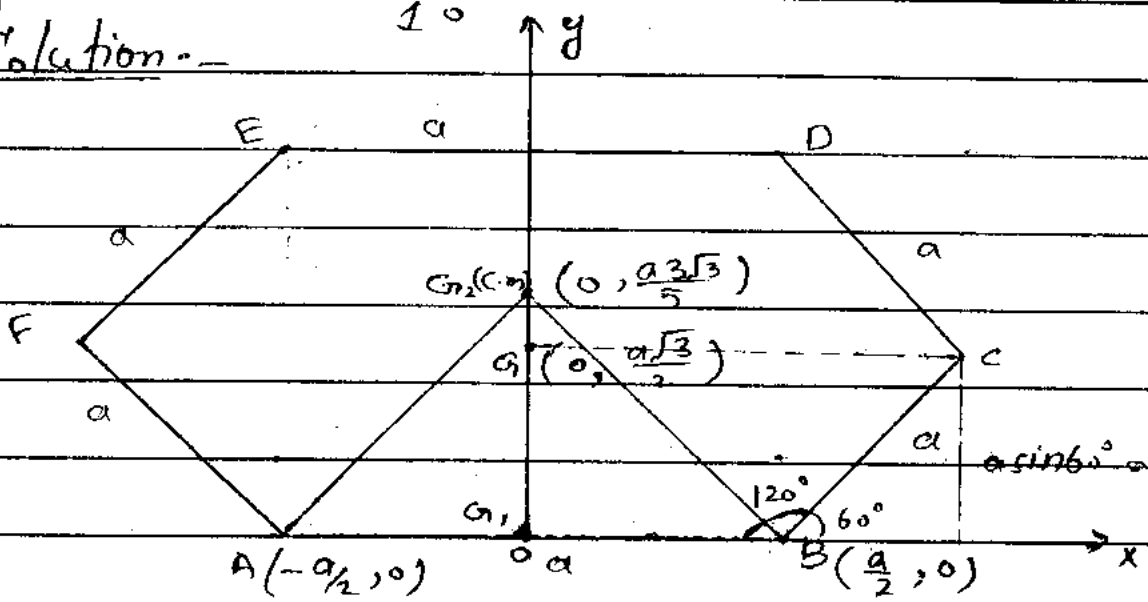
$\Rightarrow$ Diagonal bisect each other

$\Rightarrow$ ABCD is a parallelogram.

Q: No: 13 :

A rod of length 5a is bent so as to form 5 sides of a regular hexagon. Show that the distance of its c.m. from either end of rod is $\frac{\sqrt{133}}{10}a$?

Solution.–



Consider a regular hexagon ABCDEF of side length a. Taking side AB along x-axis and line joining the mid point of AB and DE as y-axis. (Axis of symmetry) c.m of hexagon lies on it. If G is c.m of complete hexagon.

$$G = \left(0, \frac{a\sqrt{3}}{2}\right).$$

If m is mass per unit length.

M = mass of hexagon.

$= 6a.m.$

if AB is the missing rod. Then, its c.m. acts at O (mid point of AB).

$$G_1 = (0, 0).$$

M_1 = mass of rod AB.

$= a.m.$

c.m of 5 sides is consider as a hexagon whose one side is missing.

$$c.m = \frac{M.OG - M_1.OG_1}{M - M_1}$$

$$= \frac{\cancel{6}^{3}am.\frac{a\sqrt{3}}{\cancel{2}} - am.(0)}{6am - am}$$

$$= \frac{3a^2\cancel{m}\sqrt{3}}{5\cancel{am}}$$

$$= \frac{a.3\sqrt{3}}{5}$$

$\Rightarrow G_2$ = c.m of 5 sides $= \left(0, a\frac{3\sqrt{3}}{5}\right)$.

Co-ordinates of $A\left(-\frac{a}{2}, 0\right)$ and $B\left(\frac{a}{2}, 0\right)$

~~Distance of c.m G_2 from A~~ $= \sqrt{\left(0+\frac{a}{2}\right)^2 + \left(\frac{a3\sqrt{3}}{5} - 0\right)^2}$

$$= \sqrt{\frac{a^2}{4} + a^2\frac{27}{25}}$$

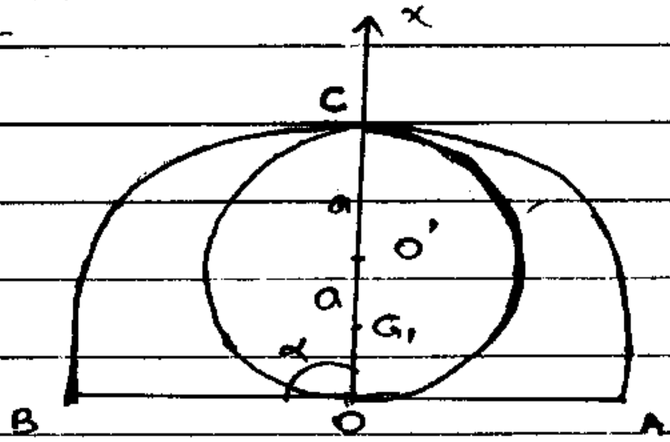
$$= \sqrt{a^2 \cdot \frac{25+108}{100}}$$

$$= a \cdot \frac{\sqrt{133}}{10} \quad \text{Ans.}$$

-: Q:No.11. -

From a semi-circular lamina of radius 2a a circular lamina of radius a is removed. Prove that the c.m. of the remainder is at a distance $\frac{16a}{3\pi} - a$ from the diameter?

Solution:-



Consider a semi circular lamina of radius 2a. AB be its diameter and OC its radius ⊥lar to AB. Consider as x-axis (Axis of symmetry) c.m lies on it.

If G_1 be c.m of semi circular lamina

$$OG_1 = \frac{2}{3} r \frac{\sin\alpha}{\alpha}.$$

$$= \frac{2}{3} \cdot 2a \frac{\sin\frac{\pi}{2}}{\pi/2}$$

$\because \alpha =$ semi vertical angle.
Angle in semi circle is π
So, semi angle is $\frac{\pi}{2}$.

$$= \frac{4a}{3} \cdot \frac{1}{\frac{\pi}{2}}$$

$$= \frac{8a}{3\pi}$$

Radius of disc removed $= a$

its c.m acts at o' centre of disc $oo' = a$

if m is mass per unit area.

$M =$ Mass of semi circular disc.

$$= \frac{1}{2}(\pi r^2).m \qquad \because r = 2a$$

$$= \frac{1}{\cancel{2}} \pi \cancel{4}^2 a^2.m$$

$$= 2\pi a^2 m$$

$M_1 =$ mass of removed disc.

$$= \pi r^2 \cdot m \qquad \because r = (a)$$

$$= \pi a^2.m$$

c.m of remaining portion of semi circular lamina $= \dfrac{M\, OG_1 - M_1 \cdot OO'}{M - M_1}$

$$= \frac{2\pi a^2 m \cdot \frac{8a}{3\pi} - \pi a^2 m \cdot a}{2\pi a^2 m - \pi a^2 m}$$

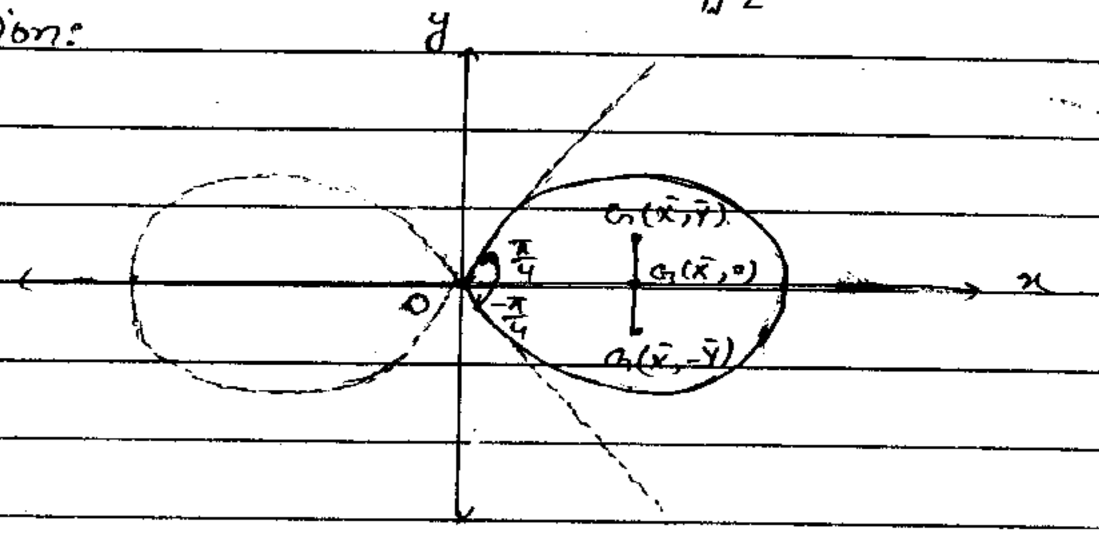
$$= \frac{\cancel{\pi a^2 m}\left(\frac{16a}{3\pi} - a\right)}{\cancel{\pi a^2 m}(2 - 1)}$$

$$= \frac{16a}{3\pi} - a. \quad \text{Ans}$$

-: Q: NO: 23 :-

Show that the c.g. of the lamina bounded by a loop of the lemniscate $r^2 = a^2\cos 2\theta$ is on the initial line at a distance $\frac{\pi a}{4\sqrt{2}}$ from the pole?

Solution:



$$r^2 = a^2\cos 2\theta$$

For limits of integration, put $r=0$

$$0 = a^2\cos 2\theta$$

$$0 = \cos 2\theta$$

$$\cos^{-1}(0) = 2\theta$$

$$-\frac{\pi}{2}, \frac{\pi}{2} = 2\theta$$

$$-\frac{\pi}{4}, \frac{\pi}{4} = \theta$$

c.g of whole loop is same as the c.g. of upper half loop.

$$\bar{X} = \frac{\frac{2}{3}\int_0^{\frac{\pi}{4}} r^3\cos\theta\, d\theta}{\int_0^{\frac{\pi}{4}} r^2\, d\theta}$$

$\because \quad r^2 = a^2\cos 2\theta$

$r = a\cos^{1/2} 2\theta$

$r^3 = a^3\cos^{3/2} 2\theta$

$$\bar{x} = \frac{\frac{2}{3}\int_0^{\frac{\pi}{4}} a^3\cos^{3/2} 2\theta \cdot \cos\theta\, d\theta}{\int_0^{\pi/4} a^2\cos 2\theta\, d\theta}$$

$$\bar{x} = \frac{\frac{2}{3}\cancel{a^3}\int_0^{\frac{\pi}{4}} (1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta}{\cancel{a^2}\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta}$$

$$= \frac{2}{3}a\,\frac{\int_0^{\frac{\pi}{4}}\left[1-(\sqrt{2}\sin\theta)^2\right]^{3/2}\cos\theta\, d\theta}{\int_0^{\frac{\pi}{4}}\cos 2\theta\, d\theta} \longrightarrow (i).$$

Neminator

$$\int_0^{\frac{\pi}{4}}\left[1-(\sqrt{2}\sin\theta)^2\right]^{3/2}\cos\theta\, d\theta$$

put $\sqrt{2}\sin\theta = \sin t$

$\sqrt{2}\cos\theta\, d\theta = \cos t\, dt$

$\cos\theta\, d\theta = \dfrac{\cos t\, dt}{\sqrt{2}}$

When $\theta = 0$; $\sqrt{2}\sin 0^\circ = \sin t$

$\Rightarrow 0 = \sin t$

$\Rightarrow 0 = t$

when $\theta = \frac{\pi}{4}$; $\sqrt{2}\sin\frac{\pi}{4} = \sin t$

$\Rightarrow 1 = \sin t$

$\Rightarrow \frac{\pi}{2} = t$

$$\int_0^{\frac{\pi}{4}}\left(1-(\sqrt{2}\sin\theta)^2\right)^{3/2}\cos\theta\, d\theta = \int_0^{\pi/2}\left[1-\sin^2 t\right]^{3/2}\frac{\cos t}{\sqrt{2}}\, dt.$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2}(\cos^2 t)^{3/2}\cos t\, dt$$

(80%)

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \cos^3 t . \cos t\, dt.$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \cos^4 t\, dt.$$

Using when p is even.

$$\int_0^{\pi/2} \cos^p\theta\, d\theta = \frac{(p-1)(p-3)\cdots}{p(p-2)(p-4)\cdots} . \pi/2$$

$$\frac{1}{\sqrt{2}}\int_0^{\pi/2} \cos^4 t\, dt = \frac{1}{\sqrt{2}} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$$

$$= \frac{3\pi}{16\sqrt{2}}.$$

denominator $\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta = \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$

$$= \frac{1}{2}\left[\sin \cancel{2}\left(\frac{\pi}{\cancel{4}}\right) - \sin 0\right]$$

$$= \frac{1}{2}\left[1 - 0\right]$$

$$= \frac{1}{2}$$

using b. results in Equation (i).

$$\bar{x} = \frac{2a}{3} . \frac{\frac{3\pi}{16\sqrt{2}}}{1/2}$$

$$= \frac{\cancel{2}a}{\cancel{3}} . \frac{\cancel{3}\pi}{\cancel{16}_4\sqrt{2}} * \frac{\cancel{2}}{1}$$

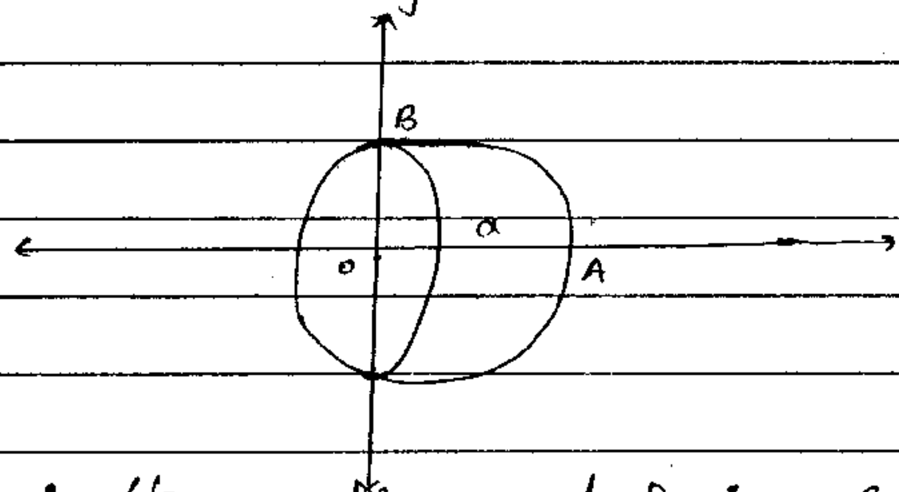
$$= \frac{\pi a}{4\sqrt{2}}. \text{ Ans}$$

-: Q:No:24 :-

Find the position of the c.g of an octant of a uniform solid sphere?

Solution:-

First we find c.g of solid hemisphere. It is solid of revolution of a portion OAB of circular lamina of radius a.



if OA is the radius and Axis of symmetry then c.g lies on it.

Take OA as x-axis, then

Equation of circular lamina.

$$x^2+y^2=a^2$$

$$y^2=a^2-x^2 \qquad \because 0\leq x\leq a.$$

$$\bar{x}=\frac{\int_0^a x y^2 dx}{\int_0^a y^2 dx}$$

$$\bar{x}=\frac{\int_0^a x(a^2-x^2)\,dx}{\int_0^a (a^2-x^2)\,dx}$$

$$\bar{x} = \frac{\left[a\cdot\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a}{\left[a^2x - \frac{x^3}{3}\right]_0^a}$$

$$= \frac{\left[\frac{a^4}{2} - \frac{a^4}{4}\right] - [0-0]}{\left[a^3 - a^3/3\right]}$$

$$= \frac{\frac{2a^4 - a^4}{4}}{\frac{3a^3 - a^3}{3}}$$

$$= \frac{a^{\cancel{4}}}{4}\cdot\frac{3}{2\cancel{a^3}}$$

$$= \frac{3}{8}a. \quad \text{Ans.}$$

$\Rightarrow$ c.g of first octant of solids sphere is $\bar{x} = \bar{y} = \frac{3a}{8}$.

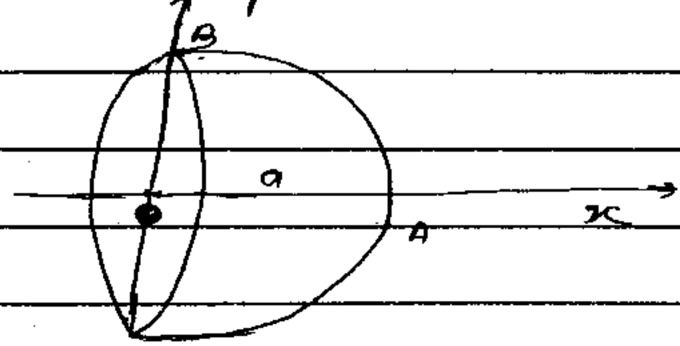
$\because$ Octant of sphere is symmetric from each plane face

$\Rightarrow$ c.g of octant $= \left(\frac{3a}{8}, \frac{3a}{8}\right)$.

Example # 13:- Page 83:-

A uniform solid hemisphere. Find c.m.

Solution:-



First we find c.g of solid hemisphere It is solid of revolution of a portion

(x3)

OAB of circular lamina of radius a.
if OA is the radius and Axis of symmetry the c.g lies on it.
Take OA as x-axis, Then
∴ The position of the c.m of the solid is given by,
Equation of circular lamina.

$$x^2+y^2=a^2$$

$$y^2=a^2-x^2 \qquad \because 0\le x\le a.$$

c.m.

$$\bar{x}=\frac{\int_0^a xy^2\,dx}{\int_0^a y^2\,dx}$$

$$=\frac{\int_0^a x(a^2-x^2)\,dx}{\int_0^a (a^2-x^2)\,dx}$$

$$=\frac{\left[a^2\cdot\frac{x^2}{2}-\frac{x^4}{4}\right]_0^a}{\left[a^2x-\frac{x^3}{3}\right]_0^a}$$

$$=\frac{\left[\frac{a^4}{2}-\frac{a^4}{4}\right]-\left[0-0\right]}{\left[a^3-\frac{a^3}{3}\right]}$$

$$=\frac{\frac{2a^4-a^4}{4}}{\frac{3a^3-a^3}{3}}$$

$$=\frac{\frac{a^4}{4}}{\frac{2a^3}{3}}=\frac{a^4}{4}\cdot\frac{3}{2a^3}$$

$$=\frac{3}{8}a. \text{ Ans}$$