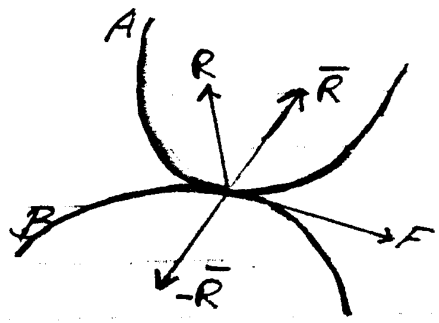
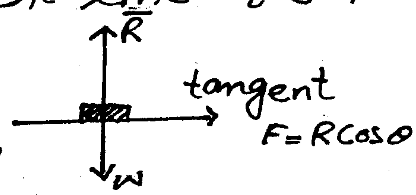
FRICTION



Let two bodies A & B are in contact at pt 'O' and are in equilibrium. Then by Newton's Third Law of motion "if body A exerts a force '$\bar{R}$' on body B. Then body B will also exert a force '$-\bar{R}$'. $\bar{R}$ & $-\bar{R}$ are called action & Reaction.

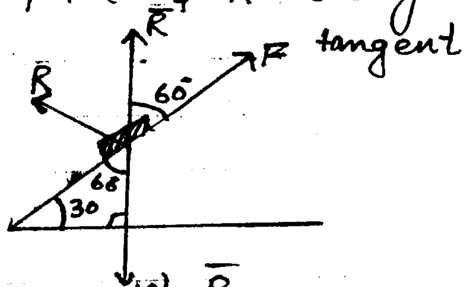
Now we resolve force $\bar{R}$ into two components F along the tangent and Second component R along the normal. R is called Normal reaction of B on A.

The component of $\bar{R}$ along tangent line i-e F is called force of friction.



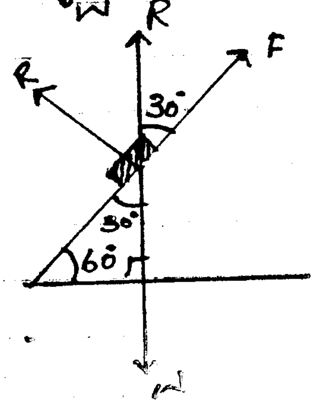
Examples:-

(i) Let a block of weight w is placed on a horizontal plane. $\bar{R}$ be the reaction of the plane. Then $\bar{R} = w$. Now resolved part of $\bar{R}$ along tangent i-e $F = R\cos 90 = 0$



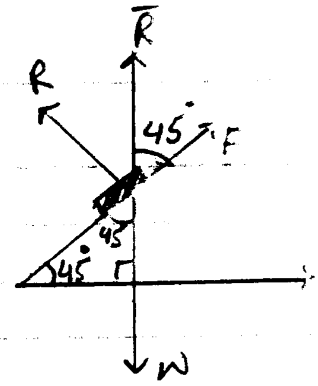
(ii) Let a block of weight w is placed on an inclined plane making an angle of 30° with x-axis

$$F = \bar{R}\cos 60^\circ = \bar{R}(1/2)$$



(iii) Let a block of weight w is placed on an Inclined plane making an angle of 60° with x-axis.

$$F = \bar{R}\cos 30^\circ = \bar{R}\cdot\frac{\sqrt{3}}{2}$$



(iv) Let a block of weight w is placed on an inclined plane making an angle of $45°$ with x-axis

$$F = \bar{R}\cos 45°$$

$$F = \bar{R}.\frac{1}{\sqrt{2}}$$

Laws of Friction:-

(1) Force of friction try to prevent the motion of the body.

Force of friction always acts opposite to the direction of motion-

The magnitude of the force of friction is equal to the force tending to produce motion-

The amount of friction (Limiting friction) is independent of of the areas and shape of the Surfaces in contact, provided the normal pressaure remains unchanged-

The magnitude of Limiting friction bears a constant ratio 'μ' to the normal component of the reaction i-e

$$\mu = \frac{F}{R} \Rightarrow F = \mu R$$

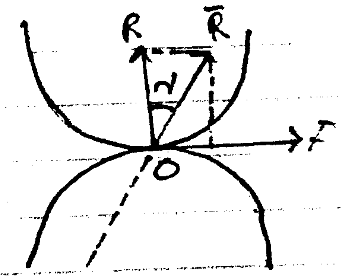
$\because$ μ is called co-officient of friction-

μ depends upon nature of the Surface and so has different Value for different Surfaces-

When a motion takes place, the friction Still opposes the motion

Friction is independent of velocity & is proportional to the normal reaction, but is slightly less Than the limiting friction-

Angle of Friction:-

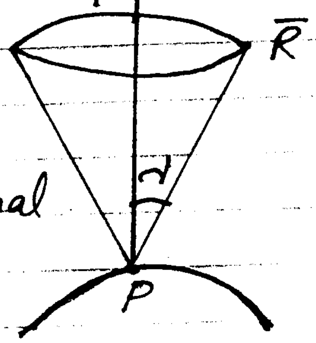


Let two bodies are in contact at a point 'o'. Then component of $\bar{R}$ (reaction) along tangent is F & along normal is R. The angle λ which the direction of reaction $\bar{R}$ makes with the normal R when the body is just on the point of motion is called the angle of friction.

$$\tan\lambda = \frac{F}{R} = \frac{\mu R}{R}$$

$$\tan\lambda = \mu \quad \because \lambda \text{ is angle of friction.}$$

Cone of Friction:-



A cone drawn with its vertex at the pt of contact P, its axis along the common normal at P, and whose semi-vertical angle is λ (The angle of friction) is called the cone of friction.

Types of Friction:-

(i) STATIC FRICTION → Ordinary Friction; Limiting Friction.

(ii) Dynamic Friction

STATIC Friction:-

When one body is in contact with one another and is not on the pt of sliding, the force of friction is still there and it is called Static Friction. i.e Friction exerted is just sufficient to maintain equilibrium. This force of static friction is also called ordinary friction.

Limiting Friction:-

When one body is in contact with some other body and the body is on the pt of moving i-e body is in limiting equilibrium that means force of friction F has attained its maximum value to prevent the motion. This maximum value of force of friction is called limiting friction.

Dynamic Friction:-

When one body is in contact with some other body and the body is moving upon another body In this case the force of friction is called dynamic friction.

Smooth Contact:-

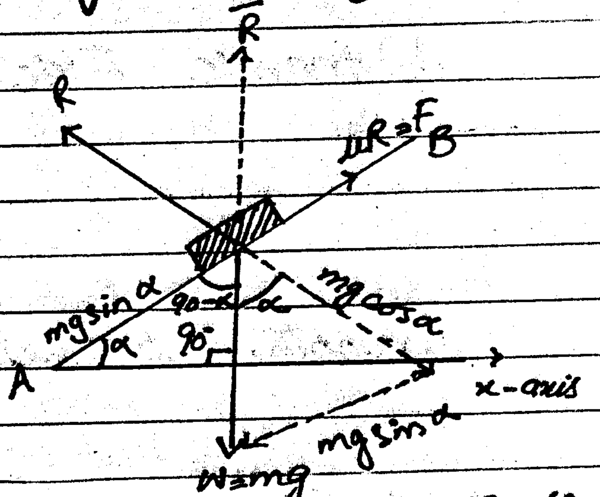
When the force of friction between two bodies in contact with each other is zero. the contact is smooth and if the force of friction is not zero then such a contact is called Rough contact.

Equilibrium OF a Particle ON a Rough Inclined Plane:-

Case - I

If a particle be in limiting eqbn on a on a rough inclined plane under its own weight, then the inclination 'α' the plane equals to the magnitude of angle of friction 'λ':

Proof:-



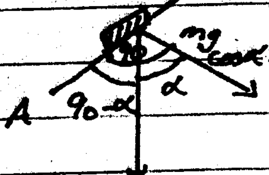
let a particle of mass 'm' is placed on a rough inclined plane AB. When the particle is just at the point of motion there exist limiting friction behind it.

Sum of angles in Δ = $180-(90+\alpha)$
$= 90-\alpha$

Then by resolving all the forces along and ⊥ar to plane AB.

$\mu R = mg\sin\alpha$ —①

$R = mg\cos\alpha$ —②

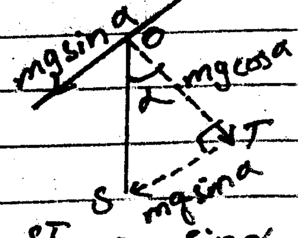


$90-(90-\alpha) = \alpha$

Dividing

$$\frac{\mu R}{R} = \frac{mg\sin\alpha}{mg\cos\alpha}$$

$\mu = \tan\alpha$



where μ is coefficient of friction.

But we know

$\mu = \tan\lambda$ for limiting friction

where λ is angle of friction

$\frac{ST}{OS} = \sin\alpha$

$ST = |OS|\sin\alpha$

$ST = mg\sin\alpha$

So

$\tan\alpha = \tan\lambda$

$\Rightarrow \alpha = \lambda$

Hence proved!!

$\dfrac{|OT|}{|OS|} = \cos\alpha$

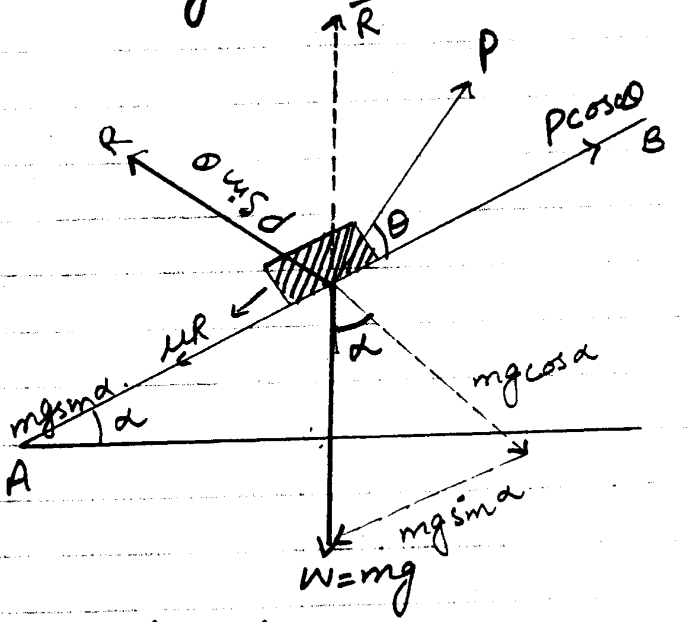
$|OT| = |OS|\cos\alpha$

$|OT| = mg\cos\alpha.$

Case - II

Find the least force to drag the particle up on rough inclined plane.

Proof:



Let a particle of mass 'm' is placed on a rough inclined plane AB. Let P be the least force acting on a particle making an angle θ with plane AB.

When the particle is just at the pt of moving up there exist limiting friction behind it.

Resolving the forces along & ⊥ar to the plane.

$$P\cos\theta = \mu R + mg\sin\alpha \quad \text{--- (1)}$$

$$P\sin\theta + R = mg\cos\alpha \quad \text{--- (2)}$$

$$R = mg\cos\alpha - P\sin\theta \quad \text{--- (3)}$$

Put (3) in (1)

$$P\cos\theta = \mu(mg\cos\alpha - P\sin\theta) + mg\sin\alpha$$

$$P\cos\theta + \mu P\sin\theta = \mu mg\cos\alpha + mg\sin\alpha.$$
$$P(\cos\theta + \mu\sin\theta) = mg(\mu\cos\alpha + \sin\alpha)$$
$$\because \mu = \tan\lambda = \frac{\sin\lambda}{\cos\lambda}$$
$$P\left[\cos\theta + \frac{\sin\lambda}{\cos\lambda}\cdot\sin\theta\right] = mg\left[\frac{\sin\lambda}{\cos\lambda}\cos\alpha + \sin\alpha\right]$$
$$P\left[\frac{\cos\theta\cos\lambda + \sin\theta\sin\lambda}{\cos\lambda}\right] = mg\left[\frac{\sin\lambda\cos\alpha + \cos\lambda\sin\alpha}{\cos\lambda}\right]$$
$$P[\cos(\lambda-\theta)] = mg\sin(\lambda+\alpha)$$
$$P = mg\frac{\sin(\lambda+\alpha)}{\cos(\lambda-\theta)}$$

P is least when denominator i-e $\cos(\lambda-\theta)$ is max.

i-e $\cos(\lambda-\theta) = 1$

$\because$ only $\cos(0) = 1$

$\because \lambda - \theta = 0$

$\lambda = \theta$

Therefore $P_{Least} = \dfrac{mg\sin(\lambda+\alpha)}{1} = mg\sin(\lambda+\alpha)$

For horizontal plane, $\alpha = 0$

$$P = mg\sin(\lambda+0)$$
$$P = mg\sin\lambda$$

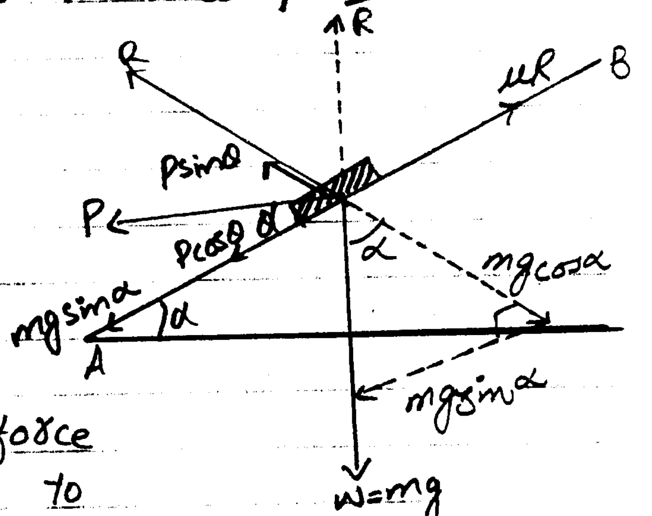
Ans.

Available at
www.mathcity.org

Case - III :-

Find the least force to drag the particle down the inclined plane.

Proof :-



Let a particle of mass 'm' is placed on a rough inclined plane AB making angle α with x-axis.
Let P be the least force acting on the particle to drag it down the inclined plane.
P makes angle θ with the plane AB. When the particle is just at the pt of moving down, there exist limiting friction behind it.

Resolving forces along & ⊥ar to plane AB.

$$\mu R = mg\sin\alpha + P\cos\theta \quad \text{--- (1)}$$

$$R + P\sin\theta = mg\cos\alpha \quad \text{--- (2)}$$

$$R = mg\cos\alpha - P\sin\theta \quad \text{--- (3)}$$

Putting value in (1)

$$\mu(mg\cos\alpha - P\sin\theta) = mg\sin\alpha + P\cos\theta$$

$$\mu mg\cos\alpha - \mu P\sin\theta = mg\sin\alpha + P\cos\theta$$

$$\because \mu = \tan\lambda$$

$$\left(\frac{\sin\lambda}{\cos\lambda}\right) mg\cos\alpha - mg\sin\alpha = \frac{\sin\lambda}{\cos\lambda} P\sin\theta + P\cos\theta$$

$$mg\left[\frac{\sin\lambda\cos\alpha - \sin\alpha\cos\lambda}{\cos\lambda}\right] = P\left[\frac{\sin\lambda\sin\theta + \cos\theta\cos\lambda}{\cos\lambda}\right]$$

$$mg\sin(\lambda - \alpha) = P\cos(\lambda - \theta)$$

$$P = mg\frac{\sin(\lambda-\alpha)}{\cos(\lambda-\theta)}$$

P is least if $\cos(\lambda-\theta) = Max.$

i-e $\cos(\lambda-\theta) = 1$

but only $\cos(0) = 1$

$\lambda - \theta = 0$

$\lambda = \theta$

Therefore least $P = \dfrac{mg\sin(\lambda-\alpha)}{1}$

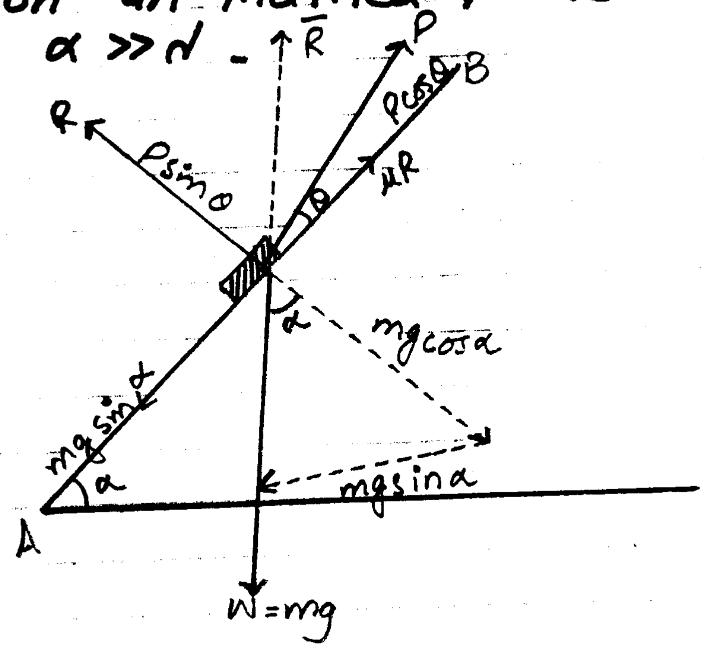
$$P_{Least} = mg\sin(\lambda-\alpha)$$

Ans

Case- IV

Find the necessary force just to support a heavy particle on an inclined plane of inclination $\alpha >> \lambda$.

Proof:



Let a particle of mass 'm' is placed on an inclined plane of inclination 'α'.
As $\alpha > \lambda$ the particle will itself slide down the planes.
So force P is necessary to support it.
Let P makes an angle θ with the plane.

Resolving forces along & ⊥or to plane.

$$P\cos\theta + \mu R = mg\sin\alpha \quad \text{—①}$$

$$R + P\sin\theta = mg\cos\alpha \quad \text{—②}$$

$$R = mg\cos\alpha - P\sin\theta \quad \text{—③ Put in ①}$$

$$P\cos\theta + \mu(mg\cos\alpha - P\sin\theta) = mg\sin\alpha$$

$$P\cos\theta + \mu mg\cos\alpha - \mu P\sin\theta = mg\sin\alpha$$

$$P(\cos\theta - \mu\sin\theta) = mg(\sin\alpha - \mu\cos\alpha)$$

$$\because \quad \mu = \tan\lambda$$

$$P\left[\cos\theta - \frac{\sin\lambda}{\cos\lambda}\sin\theta\right] = mg\left[\sin\alpha - \frac{\sin\lambda}{\cos\lambda}\cos\alpha\right]$$

$$P\left[\frac{\cos\theta\cos\lambda - \sin\theta\sin\lambda}{\cos\lambda}\right] = mg\left[\frac{\sin\alpha\cos\lambda - \sin\lambda\cos\alpha}{\cos\lambda}\right]$$

$$P\cdot\cos(\theta+\lambda) = mg\sin(\alpha-\lambda)$$

$$P = \frac{mg\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

P is least if denominator i-e $\cos(\theta+\lambda)$ is max.

i-e $\cos(\theta+\lambda) = 1$

but only $\cos(0) = 1$

$$\theta + \lambda = 0$$

$$\theta = -\lambda$$

It implies that P acts along CE & not along CD-α

$$P_{least} = \frac{mg\sin(\alpha-\lambda)}{1}$$

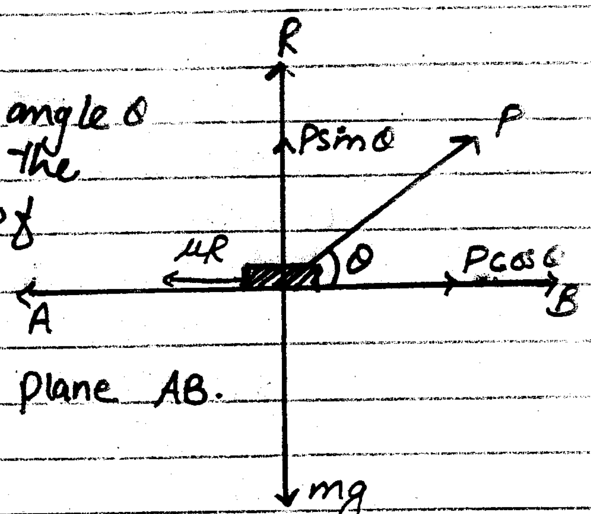
$$P_{least} = mg\sin(\alpha-\lambda)$$

Ans

Case V

Find the least force which will set into motion a particle at rest on rough horizontal plane.

Proof :-



Let P be the force making angle θ with the plane AB. When the particle is just at the pt of moving, the exist limiting friction behined it.

Resolving forces along & lar to plane AB.

$$P\cos\theta = \mu R \quad \text{---} \ (1)$$

$$P\sin\theta + R = mg \quad \text{---} \ (2)$$

$$R = mg - P\sin\theta \quad \text{---} \ (3) \quad \text{Put in } (1)$$

$$P\cos\theta = \mu(mg - P\sin\theta)$$

$$P\cos\theta = \mu mg - \mu P\sin\theta$$

$$P(\cos\theta + \mu\sin\theta) = \mu mg.$$

$$P\left[\cos\theta + \frac{\sin\alpha}{\cos\alpha}\sin\theta\right] = \frac{\sin\alpha}{\cos\alpha} mg.$$

$$P\left[\frac{\cos\theta\cos\alpha + \sin\theta\sin\alpha}{\cos\alpha}\right] = \frac{\sin\alpha}{\cos\alpha} mg$$

$$P\cos(\theta - \alpha) = \sin\alpha \cdot mg$$

$$P = \frac{mg\sin\alpha}{\cos(\theta - \alpha)}$$

P is least if $\cos(\theta-\alpha)$ is max. i-e $\cos(\theta-\alpha) = 1$

$\cos(0) = 1$

$\theta - \alpha = 0 \Rightarrow \theta = \alpha$

$P = mg\sin\alpha / 1$

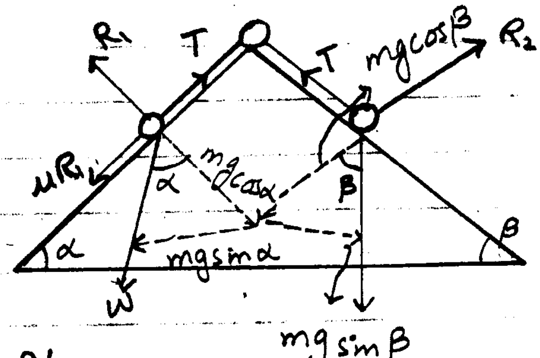
$P_{least} = mg\sin\alpha$

Example 3-

Two inclined planes have a common vertex, and a string, passing over a smooth pulley at the vertex, supports two equal weights. If one of the plane be rough and the other smooth, find the relation b/w the inclinations of the planes when the weight on smooth plane is on the pt of moving down.

Proof:-

Let W be the weights of both the particles α, β be the inclinations of the plane and μ the cofficient of friction of plane having angle α.



Resolving along ⊥ar to plane for Ist Particle.

$$T = \mu R_1 + mg\sin\alpha \quad \text{——} (1)$$

$$R_1 = mg\cos\alpha \quad \text{——} (2)$$

$$T = \mu\, mg\cos\alpha + mg\sin\alpha \quad \text{——} (3)$$

Similarly for 2nd Particle-

$$T = mg\sin\beta \quad \text{——} (4)$$

from (3) & (4)

$$mg\sin\beta = \mu mg\cos\alpha + mg\sin\alpha$$

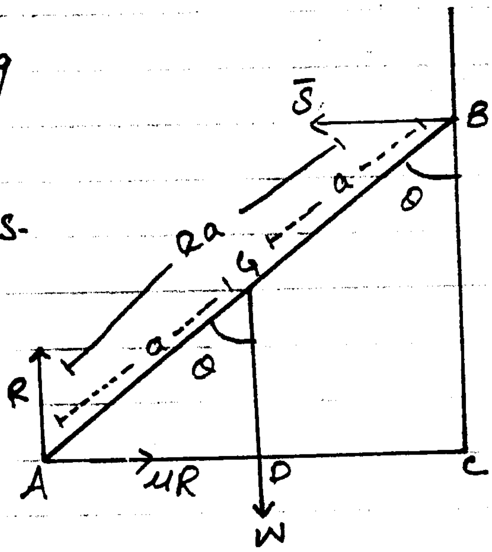
$$\sin\beta = \mu\cos\alpha + \sin\alpha$$

is the required relation b/w α & β.

Example 4:-

A uniform ladder rests in limiting eqbm with one end on a rough floor whose cofficient of friction is μ and with the other against a smooth wall. Show that its inclination to the vertical wall is $\tan^{-1}(2\mu)$.

Proof :-



The forces on ladder being all in one plane. So we apply conditions of eqbm of coplaner forces.

Resolving horizontally.

$$S = \mu R \quad \text{---} (1)$$

Resolving vertically

$\cancel{S = \mu}$

$$R = W \quad \text{---} (2)$$

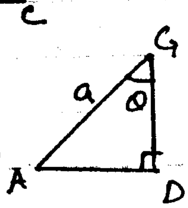
$$\Rightarrow \quad S = \mu W \quad \text{---} (3)$$

Taking moment about A :-

Let $2a$ is length of ladder and θ is its inclination with the vertical wall.

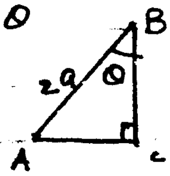
$$-W|AD| + S|BC| = 0$$

$$-W a \sin\theta + S \cdot 2a\cos\theta = 0$$



$$\frac{AD}{AG} = \sin\theta$$

$$AD = a\sin\theta$$



$$\frac{BC}{AB} = \cos\theta$$

$$BC = 2a\cos\theta.$$

$$\because S = \mu W$$

$$\mu W \cdot 2a\cos\theta = W a\sin\theta$$

$$2\mu = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = 2\mu$$

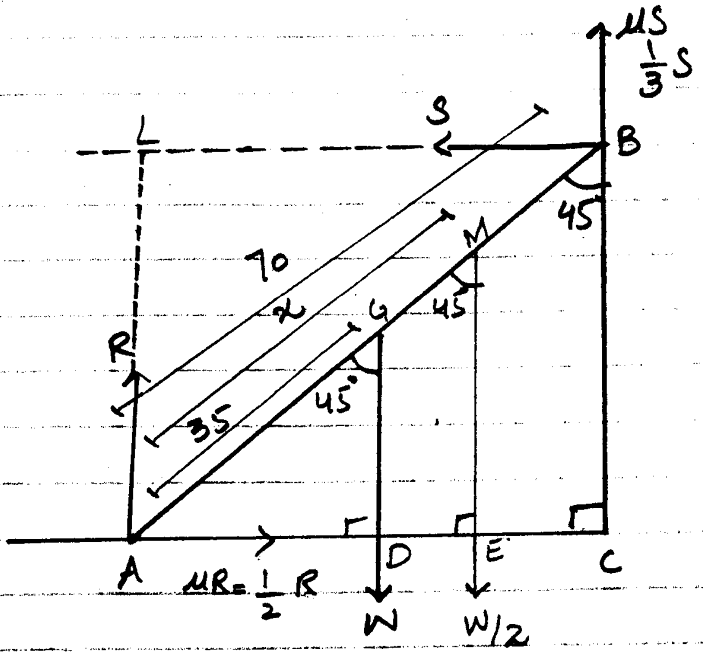
$$\theta = \tan^{-1}(2\mu)$$

Proved !!

Example 5:-

A Uniform, Ladder, of length 70 feet, rests against a vertical wall which it makes an angle 45° the cofficients of friction b/w the ladder and the wall and the ground respectively being $\frac{1}{3}$ & $\frac{1}{2}$. If a man, whose weight is one half that of the Ladder, ascends the ladder, where will he be when ladder slips-?

Proof :-



Let R, S, be the normal reactions of the ground and the wall respectively - W is the weight of ladder - W/2 is the weight of man -

|AB| the length of ladder = 70

|AM| the distance covered by man = x

$|AG| = \frac{70}{2} = 35$

{Limiting friction exists when the Ladder is just at the pt of sliding.

{ $\mu = 1/2$ for Ladder & ground.
$\mu = 1/3$ for Ladder & wall.

Resolving horizontally & vertically:-

$$\frac{1}{2}R = S \Rightarrow R = 2S \quad \text{--- (1)}$$

$$W+\frac{W}{2}=R+\frac{1}{3}S \Rightarrow \frac{3W}{2}=R+\frac{1}{3}S \quad \text{---(2)}$$

using (1) in (ii)

$$\frac{3W}{2}=2S+\frac{S}{3}$$

$$\Rightarrow \frac{3W}{2}=\frac{7S}{3}$$

$$\Rightarrow S=\frac{9W}{14} \quad \text{---(3)}$$

Taking moment about A.

$$S\,|BC| + \mu S\,|AC| - W|AD| - \frac{W}{2}|AE| = 0$$

$$S\cdot\frac{70}{\sqrt{2}}+\frac{1}{3}S\cdot\frac{70}{\sqrt{2}}-W\cdot\frac{35}{\sqrt{2}}-\frac{W}{2}\cdot\frac{x}{\sqrt{2}}=0$$

$$S\cdot\frac{70}{\sqrt{2}}+\frac{S}{3}\cdot\frac{70}{\sqrt{2}}=W\cdot\frac{35}{\sqrt{2}}+\frac{Wx}{2\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\left[70S+\frac{70}{3}S\right]=\frac{1}{\sqrt{2}}\left[35W+\frac{Wx}{2}\right]$$

$$\left(\frac{280}{3}\right)S=\frac{70W+Wx}{2}$$

$$\left(\frac{280}{3}\right)\left(\frac{9W}{14}\right)=W\left[\frac{70+x}{2}\right] \because \text{using (3)}$$

$$60W=\left(\frac{70}{2}+\frac{x}{2}\right)W$$

$$60=\frac{70}{2}+\frac{x}{2}$$

$$60-\frac{70}{2}=\frac{x}{2}$$

$$25=x/2$$

$$\Rightarrow x=50$$

Ans

Force $R, \mu R = 0$ ∵ Passing through A.

in ΔABC,

$$\frac{BC}{AB}=\cos 45^\circ$$

$$|AL| = BC = 70\cdot\frac{1}{\sqrt{2}}$$

ΔABC, $\frac{AC}{AB}=\sin 45^\circ$

$$AC=70\cdot\frac{1}{\sqrt{2}}$$

ΔAGD, $\frac{AD}{AG}=\sin 45^\circ$

$$AD=35\cdot\frac{1}{\sqrt{2}}$$

ΔANE, $\frac{AE}{AM}=\sin 45^\circ$

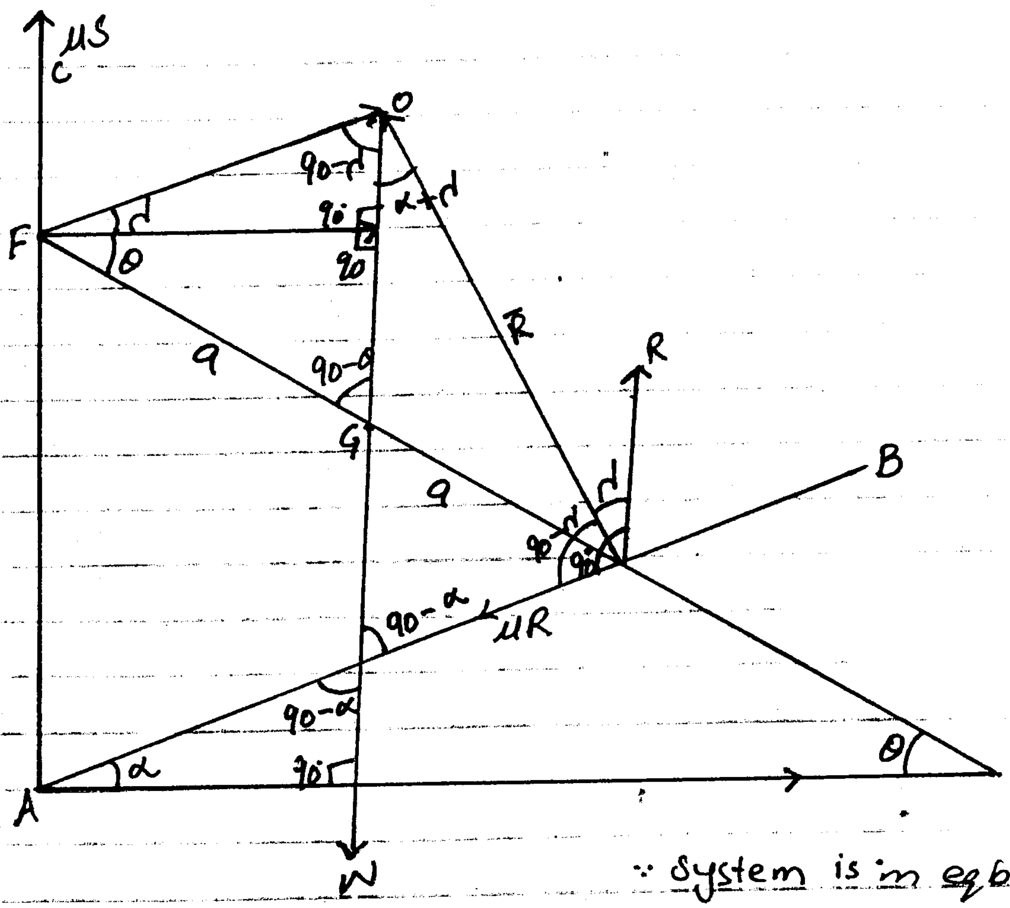
$$AE=x\cdot\frac{1}{\sqrt{2}}$$

Example 6:-

A Uniform rod Slides with its ends on two fixed equally rough rods, one being vertical and the other inclined at an angle α to the horizontal. Show that the angle θ to the horizontal of the moveable rod, when it is on the point of Sliding, is given by.

$$\tan\theta = \frac{1-2\mu\tan\mu-\mu^2}{2(\mu+\tan\alpha)}.$$

Proof:-



∵ System is in eqbm.

Let AB & Ac be two fixed rods and EF being a moveable rod of Length 2a.

W = weight of rod EF

$\bar{R}$ & $\bar{S}$ are reactions.

R & S are normal reactions. So resultant R & μR is $\bar{R}$, resultant of S & μS is $\bar{S}$.

There are the body by three forces on the reaction of the fixed rods $\bar{R}, \bar{S}$ and weight w of moveable rod.

Let the lines of reactions and vertical through G intersect in O i.e. concurrent.

$F\hat{O}G = 90 - \lambda$

$F\hat{G}O = 90 - \theta$

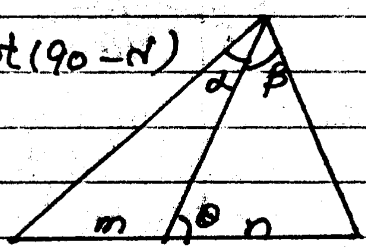
$E\hat{O}G = \alpha + \lambda$

$\because E\hat{O}G = 180 - [(90 - \lambda) + (90 - \alpha)]$

$= 180 - 90 - 90 + \alpha + \lambda$

$E\hat{O}G = \lambda + \alpha$ (m,n) Theorem.

Apply (m,n) Theorem on ΔFOE

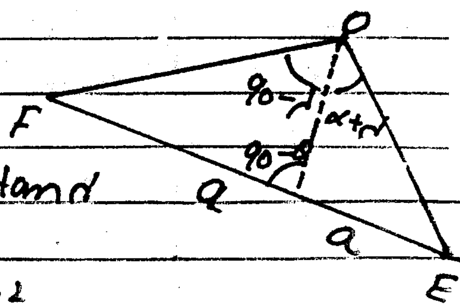


$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

$(a+a)\cot(90-\theta) = a\cot(\alpha+\lambda) - a\cot(90-\lambda)$

$2a\tan\theta = a\left[\frac{1}{\tan(\alpha+\lambda)} - \tan\lambda\right]$

$2\tan\theta = \left[\frac{1}{\frac{\tan\alpha + \tan\lambda}{1 - \tan\alpha\tan\lambda}} - \tan\lambda\right]$



$= \frac{1 - \tan\alpha\tan\lambda}{\tan\alpha + \tan\lambda} - \tan\lambda$

$= \frac{1 - \tan\alpha \cdot \mu}{\tan\alpha + \mu} - \mu \quad \because \mu = \tan\lambda$

$2\tan\theta = \frac{1 - \mu\tan\alpha - \mu\tan\alpha - \mu^2}{\mu + \tan\alpha}$

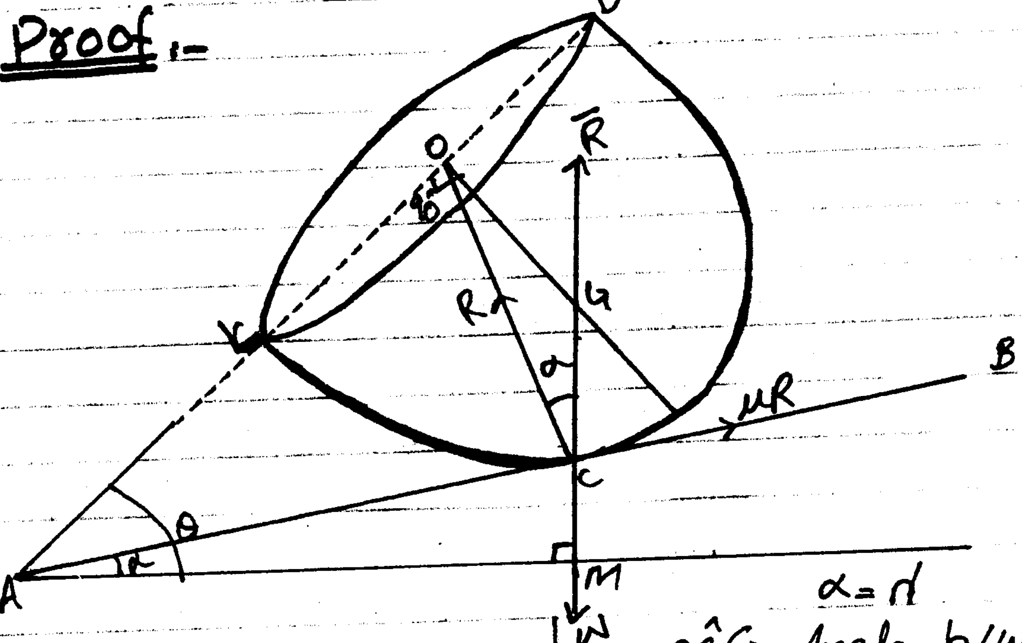
$\tan\theta = \frac{1 - 2\mu\tan\alpha - \mu^2}{2(\mu + \tan\alpha)}$

Proved !

Example 7:-

A hemispherical shell rests on a rough inclined plane whose angle of friction is λ. Show that the inclination of the plane base to the horizontal cannot be greater then $\sin^{-1}(2\sin\lambda)$.

Proof:-



$\alpha = \lambda$

Let the plane base LD makes an angle θ with horizontal.

$OG \perp LD$

Let 'a' be the radius of shell. W be it's weight.

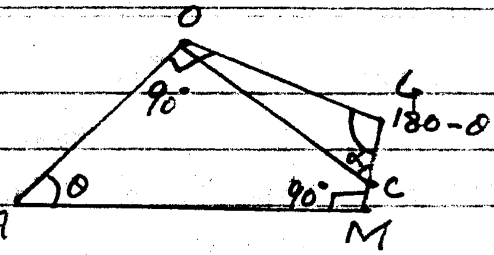
The forces acting on the shell are it's weight W, reaction $\bar{R}$ of plane & the friction μR when the hemisphere is at the point of slipping downward.

$< O\hat{C}G = \alpha$

$O\hat{C}G$ = Angle b/w R & $\bar{R}$ = λ

$\because$ angle b/w two lars on arms of an angle is equal to that angle

$\therefore \lambda = \alpha$



$\angle O\hat{G}C = 180-\theta$

Applying law of sine.
In $\triangle OCG$

$$\frac{OC}{\sin O\hat{G}C} = \frac{OG}{\sin O\hat{C}G}$$

$O\hat{G}C = 360-(90+90+\theta)$
$= 180-\theta$
$\because OC = a, \quad OG = \frac{a}{2}$

$$\frac{a}{\sin(180-\theta)} = \frac{a/2}{\sin\alpha}$$

$$\frac{a}{\sin\theta} = \frac{a}{2\sin\alpha}$$

$$\frac{a \cdot 2\sin\alpha}{a} = \sin\theta$$

$$2\sin\alpha = \sin\theta$$

$$\sin^{-1}(2\sin\alpha) = \theta$$

$$\theta = \sin^{-1}(2\sin\alpha)$$

$$\theta = \sin^{-1}(2\sin d) \quad \because \alpha = d$$

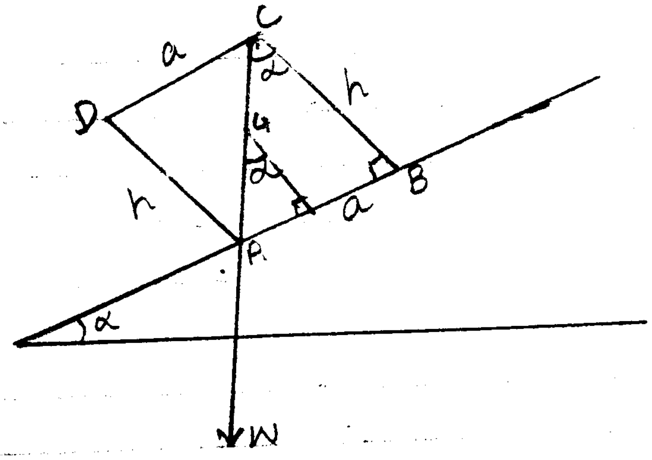
Hence proved !!

Example 8:-

A uniform rectangular block of hieght h whose base is a square of side a, rests on a rough horizontal plane. The plane is gradually tilted about a line about a line parallel to two edges of the base. Show that the block will slide or topple over according as $a \gtrless \mu h$, where μ is coffficient of Friction.

Proof :-

Let α be the inclination of the plane to the horizontal when the block is on the point of topple over.



The vertical line through, the centre of gravity of the block, must fall with in the base.
(Till the time the vertical line is with in the base the block will not fall, when it comes out of the base the block will topple over).

$$\tan\alpha = \frac{AB}{BC} = \frac{a}{h} \quad \text{--- (1)}$$

Also the inclination 'θ' of the plane to the horizontal when the block is about to slide is given by:

$$\theta = \alpha$$
$$\tan\theta = \tan\alpha$$
$$\tan\theta = \mu \quad \text{--- (ii)}$$

The block will slide or topple over according as.

$$\theta \lesseqgtr \alpha$$

{ i-e block will slide for $\theta > \alpha$
{ block will topple ove for $\theta < \alpha$

$$\tan\theta \lesseqgtr \tan\alpha$$
$$\mu \lesseqgtr \frac{a}{h} \quad \text{using (1) \& (ii)}$$

$$\mu h \lesseqgtr a$$

{ i-e block will slide for $\mu h < a$
{ block will topple over for $\mu h > a$.

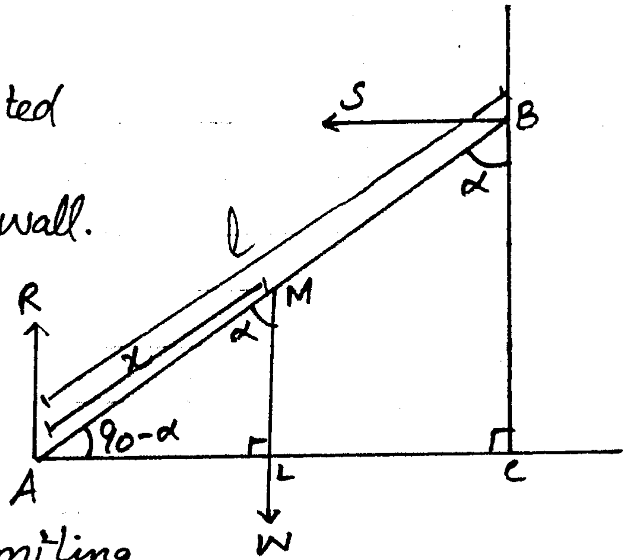
Hence proved !!

Exercise Set 5

Q No 1-

A light ladder is supported on a rough floor and leans against a smooth wall. How far up the ladder can a man climb without slipping take place-?

Proof:-



Let 'l' be the length of the ladder represented by AB. Ladder makes an angle 'α' with the wall. Let a man of weight 'W' climb a distance 'x' up the ladder. When the ladder is just at the pt of slipping there exist limiting friction at pt A. R & S are the reactions of the rough floor and smooth wall.

Resolving the force horizontally & vertically.

$\because$ ladder is light so it is considered without wt.

$\therefore \; S = \mu R$ — ①

$\therefore \; R = W$ — ②

From ① & ②

$S = \mu W$ — ③

Now taking moment about 'A'.

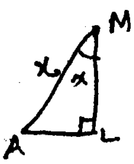
$-W\,|AL| + S\,|BC| = 0$

$-Wx\sin\alpha + Sl\cos\alpha = 0$

$Wx\sin\alpha = \mu W \cdot l\cos\alpha$

$x = \mu l \cot\alpha.$

in ΔAML



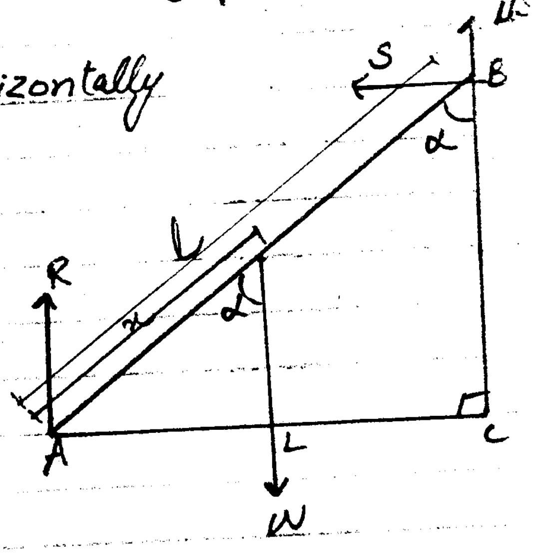
$AL = x\sin\alpha$

In ΔABC

$\frac{BC}{AB} = \cos\alpha$

$BC = l\cos\alpha$

Q No. 2.
Repeat Q. 1 when both wall & floor are rough having the same coefficient of friction μ.



Proof:
Resolving the forces horizontally and vertically.

$\mu R = S$ — (1)

$R + \mu S = W$ — (2)

from (1)

$R = \dfrac{S}{\mu}$ put in (2)

$\dfrac{S}{\mu} + \mu S = W$

$\Rightarrow S + \mu^2 S = \mu W$

$S(1+\mu^2) = \mu W \Rightarrow S = \dfrac{\mu W}{1+\mu^2}$

Now taking moment about A.

$-W|AL| + S|BC| + \mu S|AC| = 0$

$-Wx \sin\alpha + Sl\cos\alpha + \mu S l \sin\alpha = 0$

$Wx\sin\alpha = Sl(\cos\alpha + \mu\sin\alpha)$

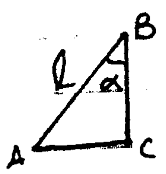
$Wx\sin\alpha = \left(\dfrac{\mu W}{1+\mu^2}\right)l(\cos\alpha + \mu\sin\alpha)$

$x = \left(\dfrac{\mu W}{1+\mu^2}\right)\dfrac{l}{W}\left[\dfrac{\cos\alpha}{\sin\alpha} + \dfrac{\mu\sin\alpha}{\sin\alpha}\right]$

$= \left(\dfrac{\mu l}{1+\mu^2}\right)(\cot\alpha + \mu)$

$x = \left(\dfrac{\mu l}{1+\mu^2}\right)(\mu + \cot\alpha)$

Ans



$\dfrac{AC}{BC} = \sin\alpha$

$AC = l\sin\alpha$.

Q No: 3.

A rod 4ft long rests on a rough floor against the smooth edge of a table of height 3ft. If the rod is on the point of slipping when inclined at an angle of 60° to the horizontal, find the coefficient of friction.

Proof:-

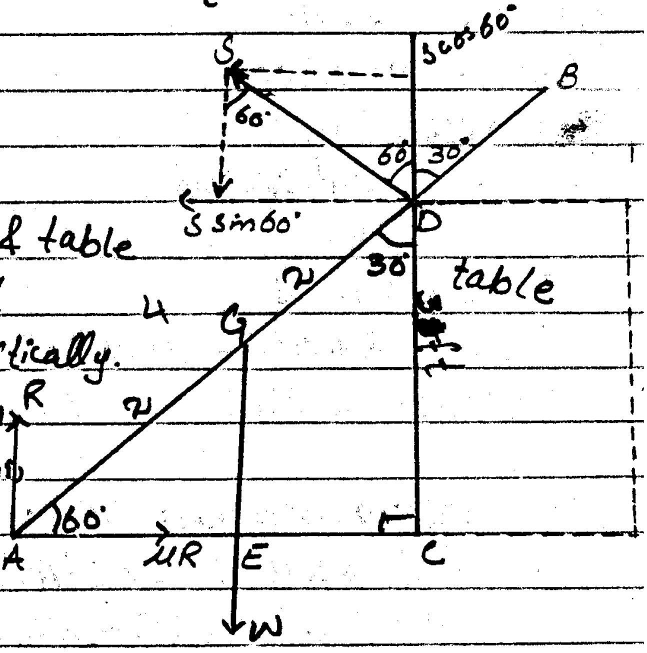
AB = length of rod = 4ft

AG = 2ft.

CD = height of table = 3ft.

'D' is pt of contact of rod & table

G = centre of gravity of rod.



Resolving forces horizontally & vertically.

$S\sin 60° = \mu R \Rightarrow S\cdot\frac{\sqrt{3}}{2} = \mu R$ —(1)

$R + S\cdot\cos 60 = W \Rightarrow R + S\cdot\frac{1}{2} = W$ —(2)

From (2) $W - \frac{S}{2} = R$

Put in (1)

$$S\cdot\frac{\sqrt{3}}{2} = \mu\left(W - \frac{S}{2}\right)$$

$$\frac{S\cdot\sqrt{3}}{2} = \frac{\mu(2W - S)}{2} \Rightarrow S\cdot\sqrt{3} = 2\mu W - \mu S$$

$$S\cdot\sqrt{3} + \mu S = 2\mu W.$$

$$\Rightarrow W = \frac{S(\sqrt{3}+\mu)}{2\mu} \quad \text{—(3)}$$

Now taking moment about A.

$$-W|AE| + S\cdot|AD| = 0$$

$$-W\cdot 1 + S\cdot 2\sqrt{3} = 0$$

$$S\cdot 2\sqrt{3} = W$$

$$\cancel{S}\cdot 2\sqrt{3} = \cancel{S}\cdot\frac{(\sqrt{3}+\mu)}{2\mu}$$

$$4\sqrt{3}\,\mu = \sqrt{3} + \mu.$$

$$\mu(4\sqrt{3} - 1) = \sqrt{3}$$

in ΔAGE

$$\frac{AE}{AG} = \cos 60°$$

$$AE = 2\cdot\frac{1}{2}$$

$$AE = 1$$

in ΔADC

$$\frac{DC}{AD} = \sin 60$$

$$\frac{DC}{\sin 60} = AD$$

$$\mu = \frac{\sqrt{3}}{4\sqrt{3}-1}$$

$$AD = \frac{3.2}{\sqrt{3}} = 2\sqrt{3}$$

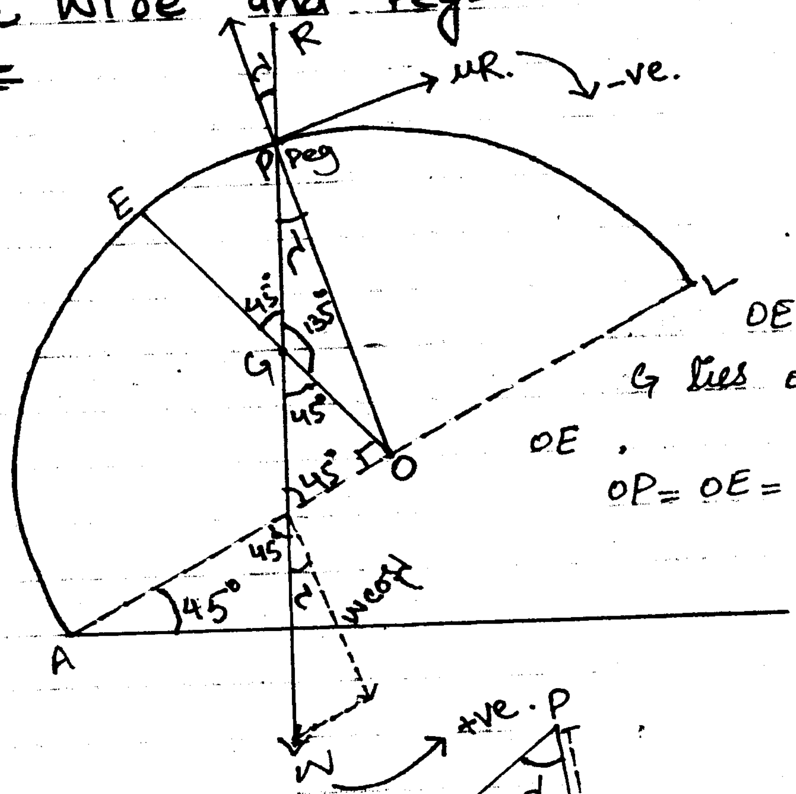
$$\mu = 0.292$$

Ans:

Q NO: 4.

A Uniform semi-circular wire hangs on a rough Peg. the line Joining its extremities making an angle of 45° with the horizontal. If it is just on the point of slipping, find the co-fficient of friction b/w the wire and Peg.

Proof:-

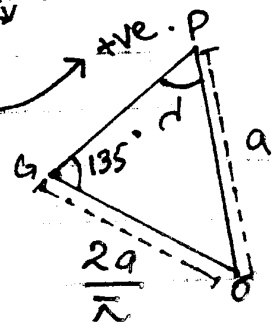


OE ⊥ AL.

G lies on the line OE.

OP = OE = Radius = a.

From Fig



Apply Sine Law.

$$\frac{a}{\sin 135^\circ} = \frac{2a/\pi}{\sin\alpha}$$

ہمیشہ centre سے c.g تک کا
فاصلہ 2a/π distance ہوتا ہے۔ جہاں
a = Radius ہے۔

$$\sin\alpha = \frac{2a/\pi \,\sin 135^\circ}{a}$$

$$\sin\alpha = \frac{2}{\pi}\left(\frac{1}{\sqrt{2}}\right), \quad \sin\alpha = \frac{\sqrt{2}}{\pi} \text{ --- (1)}$$

Since. $\mu = \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

$$= \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$$

$$= \frac{\sqrt{2}/\pi}{\sqrt{1-2/\pi^2}} = \frac{\sqrt{2}}{\sqrt{\pi^2-2}} = 0.504$$

$$\mu = 0.504$$

Ans.

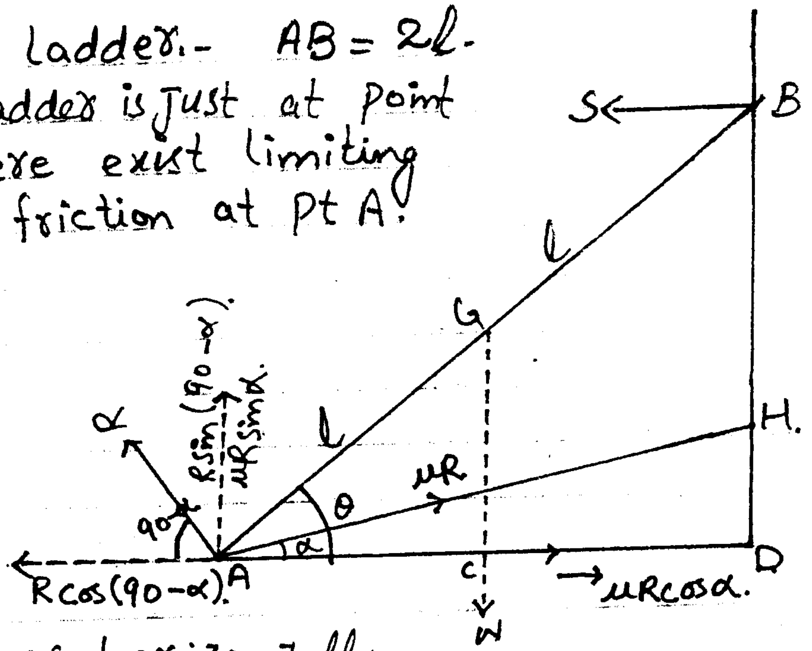
Q No: 5.-

One end of a uniform Ladder, of weight W, rests against a smooth wall, and the other end on rough ground, which slopes down from the wall at angle α to the horizon. Find the inclination of the ladder to the horizontal when it is at the point of slipping and show that reaction of the wall is then $W\tan(\lambda-\alpha)$, where λ is angle of friction.

Proof:-

Let AH be the rough ground which slopes at down from the wall at an angle α to AD (horizon).

Length of ladder:- $AB = 2l$.
When the ladder is just at point of slipping there exist limiting (equilibrium) friction at pt A.



Resolving forces horizontally & vertically.

$\mu R\cos\alpha = R\cos(90-\alpha) + S \Rightarrow \mu R\cos\alpha = R\sin\alpha + S.$

$R(\mu\cos\alpha - \sin\alpha) = S$ --- (1)

$\mu R\sin\alpha + R\sin(90-\alpha) = W$

$\Rightarrow \mu R\sin\alpha + R\cos\alpha = W.$

$\Rightarrow R(\mu\sin\alpha + \cos\alpha) = W$ --- (ii) $\Rightarrow R = \dfrac{W}{\mu\sin\alpha + \cos\alpha}$

from Eq (1)

$$\left[\frac{W}{\mu\sin\alpha + \cos\alpha}\right](\mu\cos\alpha - \sin\alpha) = S$$

$$W\left[\frac{(\sin\lambda/\cos\lambda)\cos\alpha - \sin\alpha}{(\sin\lambda/\cos\lambda)\sin\alpha + \cos\alpha}\right] = S.$$

$$\Rightarrow W\left[\frac{\sin\lambda\cos\alpha - \sin\alpha\cos\lambda}{\sin\lambda\sin\alpha + \cos\lambda\cos\alpha}\right] = S.$$

$$\Rightarrow S = W\frac{\sin(\lambda-\alpha)}{\cos(\lambda-\alpha)}$$

$$S = W\tan(\lambda-\alpha) \quad \text{-----(iii)}$$

Taking moment about 'A':

$\because R = 0 = \mu R$ $\because$ line of action of R & μR passing through A.

$S|BD| - W|AC| = 0$

$S \cdot 2l\sin(\theta+\alpha) - W \cdot l\cos(\theta+\alpha) = 0$

$\because$

$S = W\tan(\lambda-\alpha)$

$\Rightarrow [W\tan(\lambda-\alpha)]\,2l\sin(\theta+\alpha) - Wl\cos(\theta+\alpha) = 0$

$W\tan(\lambda-\alpha)\,2l\sin(\theta+\alpha) = Wl\cos(\theta+\alpha)$

$\tan(\lambda-\alpha) = \dfrac{Wl\cos(\theta+\alpha)}{W\cdot 2l\sin(\theta+\alpha)}$

$\tan(\lambda-\alpha) = \dfrac{1}{2}\cot(\theta+\alpha)$

Taking reciprocal: $\cot(\lambda-\alpha) = 2\tan(\theta+\alpha)$

$\dfrac{1}{2}\cot(\lambda-\alpha) = \tan(\theta+\alpha)$

$\Rightarrow \theta+\alpha = \tan^{-1}\left[\dfrac{1}{2}\cot(\lambda-\alpha)\right]$

Ans.

in ΔABD

$\dfrac{BD}{AB} = \sin(\theta+\alpha)$

$BD = 2l\sin(\theta+\alpha)$

ΔAGC.

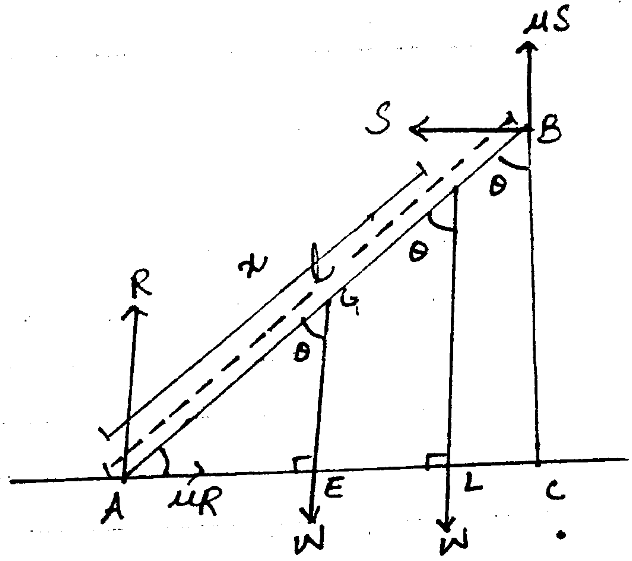
$\dfrac{AC}{AG} = \cos(\theta+\alpha)$

$AC = \dfrac{2l}{2}\cos(\theta+\alpha)$

$AC = l\cos(\theta+\alpha)$

Q No: 6

The upper end of a uniform ladder rests against a rough wall and the other end on a rough horizontal plane, the coefficient of friction in both cases being $1/3$ - prove that if the inclination of the ladder to the vertical is $\tan^{-1} 1/2$, a weight equal to that of ladder cannot be attached to it at a point, more than $9/10$ of the distance from the foot of it without destroying the equilibrium:-



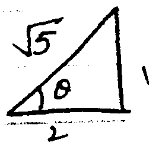
Proof.-

Let 'l' be the length of ladder. Let 'W' be the weight of ladder. Let weight equal to that of ladder is attached at D at the distance x from A.

Let θ be the angle of ladder with vertical.

$\theta = \tan^{-1}(1/2)$ given.

$\tan\theta = \frac{1}{2}$



$\sin\theta = \frac{1}{\sqrt{5}}$

$\cos\theta = 2/\sqrt{5}$

Resolving horizontally and vertically.

$\mu R = S \quad --- (1)$

$R + \mu S = 2W \Rightarrow R = -\mu S + 2W$

$\Rightarrow \quad \mu(-\mu S + 2W) = S$

$\Rightarrow \quad -\mu^2 S + 2\mu W = S$

$2\mu W = S + \mu^2 S. = S(1+\mu^2)$

$2\left(\frac{1}{3}\right) W = S\left(1 + \frac{1}{9}\right) \quad \because \mu^2 = \frac{1}{9} \ \& \ \mu = \frac{1}{3}$

$\frac{2}{3} W = \frac{10}{9} S.$

$S = \frac{3W}{5}$

Taking moment about A. $\quad R = 0 = \mu R.$

$-W|AE| - W|AL| + \mu S|AC| + S|BC| = 0$

$-W.\frac{l}{2}\sin\theta - Wx\sin\theta + \mu S\, l\sin\theta + Sl\cos\theta = 0$

In $\triangle AEG$

$\frac{AE}{AG} = \sin\theta$

$AE = \frac{l}{2}\sin\theta.$

$-W.\frac{l}{2}\left(\frac{1}{\sqrt{5}}\right) - Wx\left(\frac{1}{\sqrt{5}}\right) + \mu S l\left(\frac{1}{\sqrt{2}}\right) + Sl(2/\sqrt{5}) = 0$

$$\mu S\left(\frac{l}{\sqrt{5}}\right)+Sl\left(\frac{2}{\sqrt{5}}\right)=\frac{Wl}{2\sqrt{5}}+\frac{Wx}{\sqrt{5}}$$

$$Sl\left[\frac{\mu+2}{\sqrt{5}}\right]=W\left[\frac{l+2x}{2\sqrt{5}}\right]$$

$$\because \quad S=\frac{3W}{5}$$

$$\frac{3W}{5}l\left[\frac{1}{3}+2\right]=W\left(\frac{l+2x}{2}\right)$$

$$\left(\frac{3}{5}\right)\left(\frac{7l}{3}\right)=\left(\frac{l+2x}{2}\right)$$

$$\frac{7l}{5}=\frac{l}{2}+x$$

$$\frac{7l}{5}-\frac{l}{2}=x \quad \therefore \quad x=\frac{14l-5l}{10}$$

$$x=\frac{9}{10}l$$

Ans.

in ΔADL. $\frac{AL}{AD}=\sin\theta$

$AL=x\sin\theta$

ΔABC, $\frac{AC}{AB}=\sin\theta$

$AC=l\sin\theta$

in ΔABC, $\frac{BC}{AB}=\cos\theta$

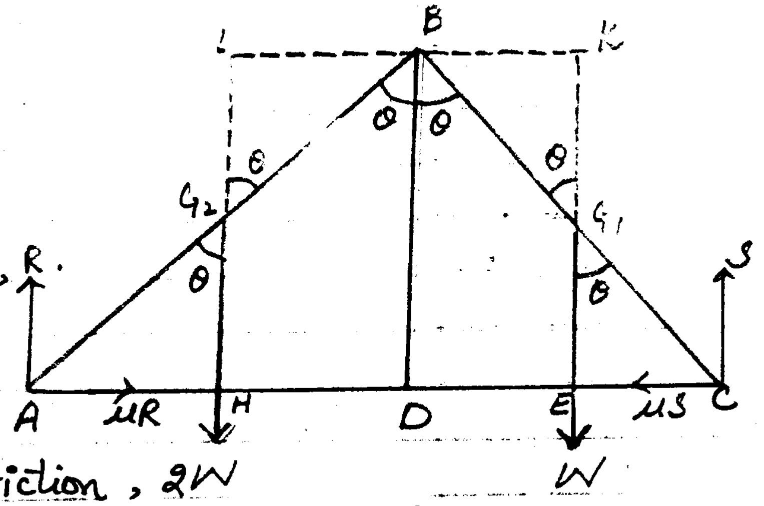
$BC=l\cos\theta$.

Q NO: 7

Two uniform beams AB, BC of equal length, are freely jointed at B, and rest in equilibrium in a vertical plane with the ends A & C on a rough horizontal plane. If the weight of AB is twice that of BC, show that there cannot be limiting friction both at A & C, and that if there is limiting friction at either of these points, it is at C. Find also the co-fficient of friction if the greatest angle that the rods can make with each other is a right angle.

Proof.–

Let $2a$ = length of each beam, i,e AB & BC.



Suppose the friction is Limiting at both ends A & C.

$BG_1 = BG_2 = a$

$AB = BC = 2a.$

Resolving the forces.

$$R + S = W + 2W$$

$$R + S = 3W \quad --- \quad (i)$$

$$\mu R = \mu S \quad \text{(When eqbm).}$$

$$R = S \quad --- \quad (ii)$$

using (ii) in (i)

$$R + R = 3W$$

$$R = \frac{3W}{2} \quad --- \quad (iii)$$

Now consider Limiting eqbm at A.

$R = 2W$ but this contradiction with (iii)

Hence friction is not Limiting at A, i-e Limiting friction is not at both ends.

$\because$ $\mu R > \mu S$. So the Stage of limiting friction at The end C reduces earlier Than at A.

Taking moment about B for the eqbm of rod BC.

$$-\mu S\,|BD| - W\,|DE| + S\,|DC| = 0$$

$\Delta BDC \Rightarrow \frac{BD}{BC} = \cos\theta$

$S|DC| = \mu S|BD| + W|DE|$

$S(2a\sin\theta) = \mu S(2a\cos\theta) + W(a\sin\theta)$.

$S\,a(2\sin\theta - \mu\cdot 2\cos\theta) = Wa\sin\theta$.

$\because$ put S becomes

$$S = \frac{W\sin\theta}{2(\sin\theta - \mu\cos\theta)} \quad \text{---(iv)}$$

Put in (i)

$R + S = 3W$

$$R + \frac{W\sin\theta}{2(\sin\theta - \mu\cos\theta)} = 3W.$$

$$R = 3W - \frac{W\sin\theta}{2(\sin\theta - \mu\cos\theta)}$$

$$= \frac{3W.2(\sin\theta - \mu\cos\theta) - W\sin\theta}{2(\sin\theta - \mu\cos\theta)}.$$

$$= \frac{6W\sin\theta - 6W\mu\cos\theta - W\sin\theta}{2(\sin\theta - \mu\cos\theta)}.$$

$$R. = \frac{5W\sin\theta - 6\mu W\cos\theta}{2(\sin\theta - \mu\cos\theta)} \quad \text{----(V)}$$

$BD = 2a\cos\theta$.

$DE = BK$.

$\triangle BG_1K$; $\frac{BK}{BG_1} = \sin\theta$

$\therefore DE = BK = a\sin\theta$

$\triangle BDC$, $\frac{DC}{BC} = \sin\theta$

$DC = 2a\sin\theta$

Now moment of forces on the rod AB about B.

$-R|AD| + 2W|HD| + \mu R|BD| = 0$

$-R(2a\sin\theta) + 2W(a\sin\theta) + \mu S(2a\cos\theta) = 0$

$2a(W\sin\theta) + \mu S(2a\cos\theta) = R(2a\sin\theta)$

$2a[W\sin\theta + \mu S.\cos\theta] = R(2a\sin\theta)$

$W\sin\theta + \mu S\cos\theta = R.\sin\theta$

when we have limiting friction at C then static friction at A. $\mu R = \mu S$

$\frac{AD}{AB} = \sin\theta$, $AD = 2a\sin\theta$

$\therefore LB = HD$

$\frac{LB}{BG_2} = \sin\theta$

$LB = HD = a\sin\theta$

$\because BD = 2a\cos\theta$.

using values of R & S from (iv) & (V).

$$W\sin\theta + \mu\left[\frac{W\sin\theta}{2(\sin\theta - \mu\cos\theta)}\right]\cos\theta = \left[\frac{5W\sin\theta - 6\mu W\cos\theta}{2(\sin\theta - \mu\cos\theta)}\right]\cdot\sin\theta.$$

$$\frac{W\sin\theta\cdot 2(\sin\theta - \mu\cos\theta) + \mu W\sin\theta\cos\theta}{\cancel{2(\sin\theta - \mu\cos\theta)}} = \frac{5W\sin^2\theta - 6\mu W\cos\theta\sin\theta}{\cancel{2(\sin\theta - \mu\cos\theta)}}$$

$$\cancel{W}(2\sin^2\theta - 2\mu\sin\theta\cos\theta + \mu\sin\theta\cos\theta) = 5\cancel{W}\sin^2\theta - 6\mu\cos\theta\sin\theta\cdot\cancel{W}$$

$$6\mu\sin\theta\cos\theta - 2\mu\sin\theta\cos\theta + \mu\sin\theta\cos\theta = 5\sin^2\theta - 2\sin^2\theta.$$

$$5\mu\sin\theta\cos\theta = 3\sin^2\theta.$$

$$\mu = \frac{3\sin^2\theta}{5\sin\theta\cos\theta} = \frac{3}{5}\tan\theta.$$

When rods are right angle. i-e $2\theta = 90^\circ$

$\theta = 45^\circ$

$$\mu = \frac{3}{5}\tan 45^\circ$$

$$\mu = \frac{3}{5}(1)$$

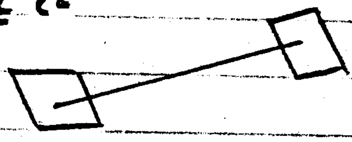
$$\mu = 3/5$$

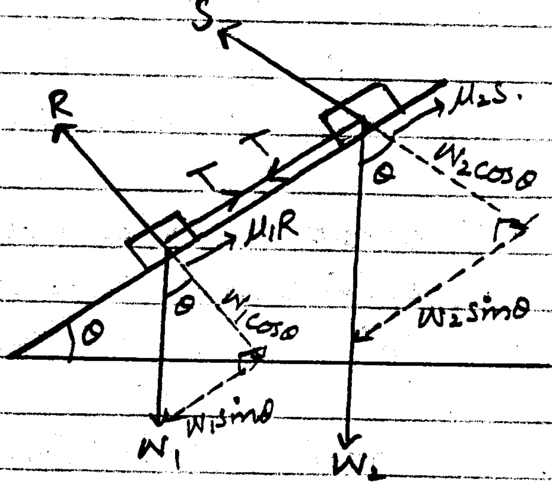
Ans.

Q NO:- 08:

Two bodies, weights W_1, W_2 are placed on an inclined plane and are connected by a light string which concides with a line of greatest slope of the plane. If the cofficients of friction b/w the bodies and plane be respectively μ_1 & μ_2, find the inclination of the plane to the horizon when both bodies are on the point of motion; it being assumed that the smoother body is below the other.

Proof:-





The line Joining mid pts is called line of greatest slope.

Let R & S be the normal reactions of the bodies.
T is the tension in the string.
$\mu_1 < \mu_2$ ∴ Smooth body is below other one.
The System is in limiting eqbm.
consider the eqbm of body of weight W_1

$$R = W_1\cos\theta \quad \text{— (1)}$$
$$\mu_1 R = W_1\sin\theta \quad \text{— (2)}$$

Similarly the eqbm of body 'wt' W_2

$$S = W_2\cos\theta \quad \text{— (3)}$$
$$\mu_2 S = W_2\sin\theta \quad \text{— (4)}$$

Adding (2) & (4)

$$\mu_1 R + \mu_2 S = W_1\sin\theta + W_2\sin\theta$$
$$\mu_1 R + \mu_2 S = (W_1 + W_2)\sin\theta \quad \text{— (5)}$$

using (1), (3) in (5)

$$\mu_1(W_1\cos\theta) + \mu_2(W_2\cos\theta) = (W_1 + W_2)\sin\theta.$$
$$(\mu_1 W_1 + \mu_2 W_2)\cos\theta = (W_1 + W_2)\sin\theta.$$
$$\frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

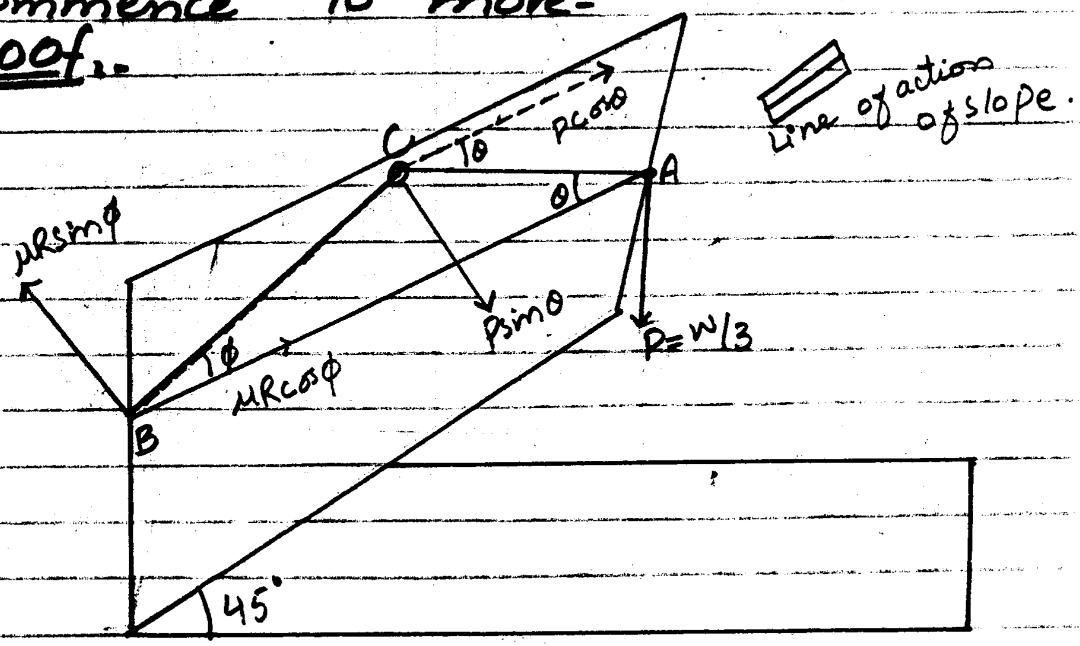
$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}\right) \quad \text{Ans.}$$

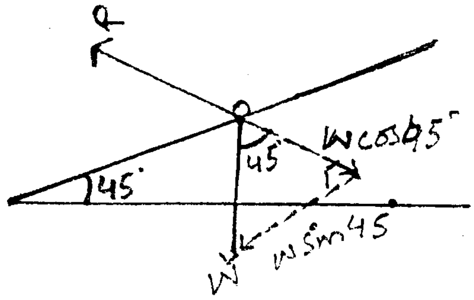
Q No:- 9.

A weight w, is laid upon a rough plane $\left(\mu = \frac{1}{\sqrt{3}}\right)$ inclined at $45°$ to the horizon, and is connected by a string passing through a smooth ring A, at the top of the plane, with a weight P, hanging vertically. if $W = 3P$, show that if θ be the greatest possible inclination of the string AW to the incline of greatest slope in the plane then $\cos\theta = \frac{2\sqrt{2}}{3}$

Find also the direction in which w would commence to move.

Proof:-





Let AB be the line of greatest slope. So $\angle CAB = \theta$.
As weight W tends to advance along CB Then μR will act along BC.
It makes an angle ϕ with BA.
Resolving the forces along & ⊥ar to AB.

$$P\cos\theta + \mu R\cos\phi = W\sin 45° \quad \text{---(i)}$$
$$R = W\cos 45°$$
$$R = \frac{W}{\sqrt{2}} \quad \text{---(ii) put in (i)}$$

$$P\cos\theta + \mu\cos\phi \cdot \frac{W}{\sqrt{2}} = W \cdot \frac{1}{\sqrt{2}}$$
$$\frac{W}{3}\cos\theta + \frac{1}{\sqrt{3}}\cos\phi\frac{W}{\sqrt{2}} = \frac{W}{\sqrt{2}}.$$
$$\because P = \frac{W}{3} \text{ \& } \mu = \frac{1}{\sqrt{3}}.$$
$$\left(\frac{\cos\theta}{3} + \frac{\cos\phi}{\sqrt{3}\cdot\sqrt{2}}\right)W = \frac{W}{\sqrt{2}}$$
$$\frac{\cos\phi}{\sqrt{3}\cdot\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{\cos\theta}{3}$$
$$\cos\phi = \frac{\sqrt{3}\cdot\sqrt{2}}{\sqrt{2}} - \frac{\cos\theta\cdot\sqrt{2}\cdot\sqrt{3}}{3}$$
$$\cos\phi = \sqrt{3} - \frac{\sqrt{2}}{\sqrt{3}}\cos\theta \quad \text{---(v)}$$

Resolving forces.
$$\mu R\sin\phi = P\sin\theta \quad \text{---(iii)}$$
$$R = W\cos 45°$$
$$R = \frac{W}{\sqrt{2}} \quad \text{---(iv)}$$
$$\frac{1}{\sqrt{3}}\cdot\frac{W}{\sqrt{2}}\sin\phi = \frac{W}{3}\sin\theta$$
$$\sin\phi = \frac{\sqrt{3}\cdot\sqrt{2}}{3}\sin\theta$$
$$\sin\phi = \frac{\sqrt{2}}{\sqrt{3}}\sin\theta \quad \text{---(vi)}$$

Squaring and adding (v) & (vi)
$$\cos^2\phi + \sin^2\phi = \left(\sqrt{3} - \frac{\sqrt{2}}{\sqrt{3}}\cos\theta\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{3}}\sin\theta\right)^2$$

$$1 = 3 + \frac{2}{3}\cos^2\theta - 2\sqrt{3}\cdot\frac{\sqrt{2}}{\sqrt{3}}\cos\theta + \frac{2}{3}\sin^2\theta$$

$$1 = 3 + \frac{2}{3}\left(\cos^2\theta + \sin^2\theta\right) - 2\sqrt{2}\cos\theta.$$

$$1 = 3 + \frac{2}{3} - 2\sqrt{2}\cos\theta$$

$$2\sqrt{2}\cos\theta = 3 + 2/3 - 1$$

$$2\sqrt{2}\cos\theta = \frac{8}{3} \Rightarrow \cos\theta = \frac{8}{3\cdot 2\sqrt{2}} = \frac{4}{3\sqrt{2}}.$$

$$\cos\theta = \frac{2\sqrt{2}}{3}$$

Now. from Eq (V)

$$\cos\phi = \sqrt{3} - \frac{\sqrt{2}}{\sqrt{3}}\cos\theta.$$

$\because \cos\theta = 2\sqrt{2}/3$ put in (V)

$$\cos\phi = \sqrt{3} - \frac{\sqrt{2}}{\sqrt{3}} \times \frac{2\sqrt{2}}{3}$$

$$\cos\phi = \sqrt{3} - \frac{4}{3\sqrt{3}} = \frac{9-4}{3\sqrt{3}} = \frac{5}{3\sqrt{3}}$$

$$\phi = \cos^{-1}\left(\frac{5}{3\sqrt{3}}\right) \quad \text{Ans.}$$

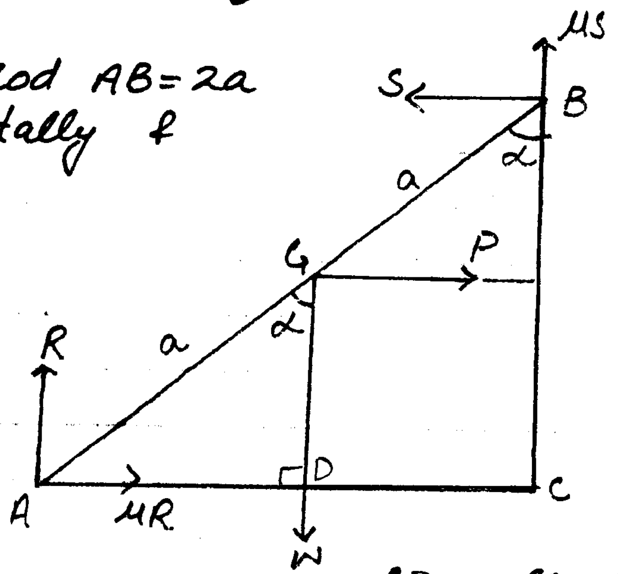
Q No: 10-

A uniform rod of weight W is placed with its lower end on a rough horizontal floor and its upper end against an equally rough vertical wall. The rod makes an angle α with the wall and is Just prevented from slipping down by a horizontal force P applied at its middle point. Prove That

$$P = W\tan(\alpha - 2\lambda).$$

Where λ is angle of friction and $\lambda < \frac{1}{2}\alpha$.

Proof:-



Let length of rod $AB = 2a$

Resolving horizontally & vertically.

$\mu R + P = S$ — (i)

$R + \mu S = W$

$R = W - \mu S$ — (ii)

Put in (i)

$S = \mu(W - \mu S) + P$

$S = \mu W - \mu^2 S + P$

$S(1+\mu^2) = \mu W + P$

$\Rightarrow \quad S = \dfrac{\mu W + P}{1+\mu^2}$

$\dfrac{AD}{AG} = \sin\alpha \Rightarrow AD = a\sin\alpha.$

$\dfrac{GD}{AG} = \cos\alpha \quad GD = a\cos\alpha$

$\dfrac{BC}{AC} = \sin\alpha$

$BC = 2a\sin\alpha$

$\dfrac{AC}{AB} = \sin\alpha$

$AC = 2a\sin\alpha$

Taking moment about A.

$-W|AD| - P|GD| + S|BC| + \mu S|AC| = 0$

$-Wa\sin\alpha - Pa\cos\alpha + S2a\cos\alpha + \mu S 2a\sin\alpha = 0$

$a(S2\cos\alpha + \mu S\cdot 2\sin\alpha) = Wa\sin\alpha + Pa\cos\alpha.$

$S2\cos\alpha + \mu S\cdot 2\sin\alpha = \dfrac{\cancel{a}(W\sin\alpha + P\cos\alpha)}{\cancel{a}}$

$\left(\dfrac{\mu W + P}{1+\mu^2}\right)2\cos\alpha + \mu\left(\dfrac{\mu W + P}{1+\mu^2}\right)2\sin\alpha = W\sin\alpha + P\cos\alpha$

$\dfrac{\mu W\cdot 2\cos\alpha + 2P\cos\alpha + \mu^2 W 2\sin\alpha + \mu P 2\sin\alpha}{1+\mu^2} = W\sin\alpha + P\cos\alpha$

$$2\mu W\cos\alpha + 2P\cos\alpha + \mu^2 W 2\sin\alpha + 2P\mu\sin\alpha = (1+\mu^2)(W\sin\alpha + P\cos\alpha).$$

$$2\mu W\cos\alpha + 2P\mu\sin\alpha + \mu^2 W 2\sin\alpha + 2P\cos\alpha = W\sin\alpha + P\cos\alpha + \mu^2 W\sin\alpha + \mu^2 P\cos\alpha$$

$$2P\cos\alpha + 2P\mu\sin\alpha - P\cos\alpha - \mu^2 P\cos\alpha = -2\mu W\cos\alpha - \mu^2 W 2\sin\alpha + W\sin\alpha + \mu^2 W\sin\alpha.$$

$$P\cos\alpha + 2P\mu\sin\alpha - \mu^2 P\cos\alpha = W\sin\alpha - 2\mu W\cos\alpha - \mu^2 W\sin\alpha.$$

$$P\left[\cos\alpha + 2\mu\sin\alpha - \mu^2\cos\alpha\right] = W\left[-2\mu\cos\alpha - \mu^2\sin\alpha + \sin\alpha\right]$$

$$P\left[\cos\alpha + 2\frac{\sin\lambda}{\cos\lambda}\sin\alpha - \frac{\sin^2\lambda}{\cos^2\lambda}\cos\alpha\right] = W\left[-\frac{2\sin\lambda\cos\alpha}{\cos\lambda} - \frac{\sin^2\lambda}{\cos^2\lambda}\sin\alpha + \sin\alpha\right]$$

$$P\left[\frac{\cos\alpha\cos^2\lambda + 2\sin\lambda\cos\lambda\sin\alpha - \sin^2\lambda\cos\alpha}{\cancel{\cos^2\lambda}}\right] = W\left[\frac{-2\sin\lambda\cos\lambda\cos\alpha - \sin^2\lambda\sin\alpha + \sin\alpha\cos^2\lambda}{\cancel{\cos^2\lambda}}\right]$$

$$P\left[\cos\alpha(\cos^2\lambda - \sin^2\lambda) + \sin 2\lambda\sin\alpha\right] = W\left[-\sin 2\lambda\cos\alpha + \sin\alpha(\cos^2\lambda - \sin^2\lambda)\right]$$

$$P\left[\cos\alpha\cos 2\lambda + \sin 2\lambda\sin\alpha\right] = W\left[\sin\alpha\cos 2\lambda - \cos\alpha\sin 2\lambda\right]$$

$$P\cos(\alpha - 2\lambda) = W\sin(\alpha - 2\lambda).$$

$$P = W\frac{\sin(\alpha - 2\lambda)}{\cos(\alpha - 2\lambda)}$$

$$P = W\tan(\alpha - 2\lambda).$$

For 'P' to be +ve we must have $\alpha > 2\lambda$

$\Rightarrow \tan(\alpha - 2\lambda)$ is +ve.

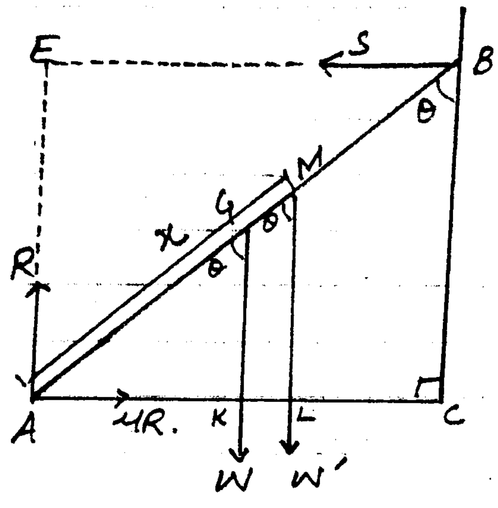
Hence proved !!

Q NO: 11.

A uniform ladder rests in limiting equilibrium with one end on a rough horizontal plane, and the other against a smooth vertical wall. A man ascends the ladder. Show that he cannot go more than half way up

Proof:-

Length of ladder $= 2a = AB$



Let man ascends the ladder and its weight is w' at a distance x from A. w is weight of ladder.

$AG = a$

$AM = x$.

Resolving horizontally & vertically.

$S = \mu R$ —— (i)

$R = W + W'$ —— (ii)

$S = \mu(W + W')$ —— (iii) using (ii) in (i)

Taking moments about A.

$S|AE| - W|AK| - w'|AL| = 0$

ΔABC, $\dfrac{BC}{AB} = \cos\theta$

$S \cdot 2a\cos\theta - w\,a\sin\theta - w' \cdot x\sin\theta = 0$

$AE = BC = 2a\cos\theta$

$2aS\cos\theta = (wa + w'x)\sin\theta$

ΔAGK, $\dfrac{AK}{AG} = \sin\theta$ $\quad AK = a\sin\theta$

$\dfrac{2aS}{wa + w'x} = \dfrac{\sin\theta}{\cos\theta}$

$\dfrac{2a\mu(W + w')}{wa + w'x} = \tan\theta$

ΔAML, $\dfrac{AL}{AM} = \cos\theta$

$AL = x\sin\theta$

$$\frac{2a\mu(w+w')}{wa+w'x} = 2\mu$$

$\because$ As the ladder is in limiting eqbm with one end on rough floor & other end on smooth wall then its inclination to the vertical is

$$\tan^{-1}(2\mu) = \theta$$
$$2\mu = \tan\theta.$$

$$a(w+w') = wa + w'x$$
$$aw + w'a = wa + w'x$$
$$w'a = w'x$$
$$a = x$$

Hence Proved !!

Q NO: 12.

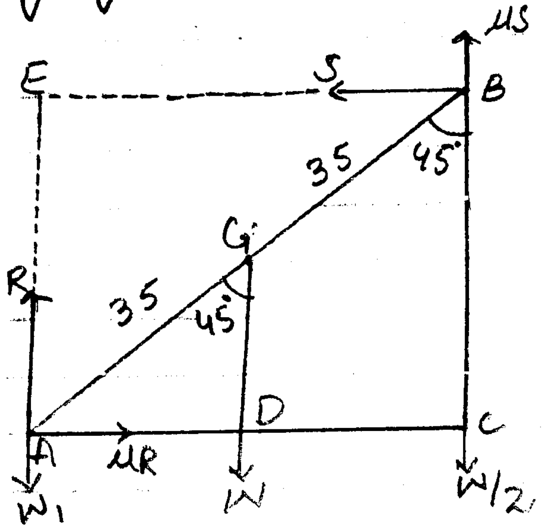
A uniform ladder of length 70 feet, rests against a vertical wall with which it makes an angle of 45°, the co-fficients of friction b/w the ladder and wall and ground respectively being 1/3 & 1/2, If a man whose weight is one half of that of the laddee, ascends the laddee, how high will be he when the ladder slips-?

If a boy now stands on the bottom rung of the ladder what must be his least weight so that the man may go to the top of ladder?

Solution:-

First solve example 5 complete then.

Now let a boy of weight w_1 stand on the bottom rung so that the man



AB=70

can go up - System is in equilibrium.
Resolving horizontally & vertically -

$S = \mu R = \frac{1}{2} R \Rightarrow R = 2S$ —— (i)

$R + \mu S = W_1 + W + W/2$

$R + \frac{1}{3} S = W_1 + \frac{3}{2} W \quad \because \mu = 1/3.$

$2S + \frac{1}{3} S = W_1 + 3/2\, W \quad \because$ using (i)

$\frac{7S}{3} = W_1 + 3W/2$

$S = \frac{3}{7} \left[\frac{3W}{2} + W_1 \right]$ —— (ii)

Now taking moment about A.

$W_1 = R = \mu R = 0$

$-W|AD| - W/2\, |AC| + S|AE| + \mu S|AC| = 0 \qquad \because \Delta ADG,\ \frac{AD}{AG} = \sin 45$

$S|AE| + \mu S|AC| = W|AD| + W/2\, |AC|$

$S \cdot \frac{70}{\sqrt{2}} + \mu S \cdot \frac{70}{\sqrt{2}} = W \cdot \frac{35}{\sqrt{2}} + \frac{W}{2} \cdot \frac{70}{\sqrt{2}} \qquad AD = 35 \sin 45^\circ$

$\qquad AD = 35/\sqrt{2}$

$\frac{1}{\cancel{\sqrt{2}}} \left[70S + 70 \cdot \mu S \right] = \left[W \cdot 35 + \frac{70}{2} W \right] \cdot \frac{1}{\cancel{\sqrt{2}}} \qquad \Delta ABC,\ \frac{AC}{AB} = \sin 45^\circ$

$70 \left(S + \frac{1}{3} S \right) = 35W + 35W \qquad AC = 70 \cdot 1/\sqrt{2}$

$\qquad \Delta ABC,\ \frac{BC}{AB} = \cos 45^\circ$

$70 \left(\frac{4S}{3} \right) = 70W$

$\qquad AE = BC = 70/\sqrt{2}$

$\frac{4}{\cancel{3}} \left[\frac{\cancel{3}}{7} \left(\frac{3W}{2} + W_1 \right) \right] = W$

$4 \left[\frac{3W}{2} + W_1 \right] = 7W \Rightarrow \frac{\overset{2}{\cancel{4}} \cdot 3W}{\cancel{2}} + 4W_1 = 7W$

$6W + 4W_1 = 7W$

$4W_1 = W$

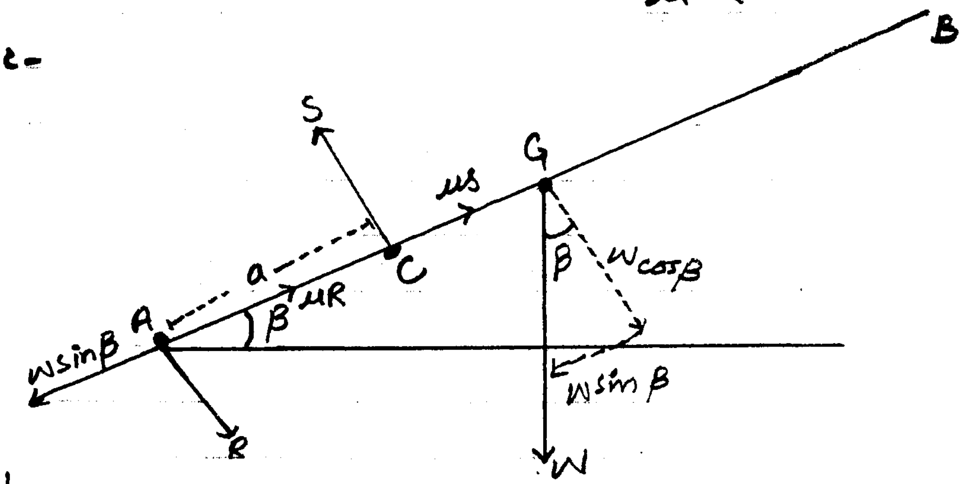
$W_1 = \frac{W}{4}$

Ans.

Q No: 13.

A thin uniform rod passes over one peg and under another. The cofficient of friction between each peg and the rod being μ. The distance between the pegs is a, and the straight line joining them makes an angle β with the horizontal. Show that equilibrium is not possible unless the length of the rod is greater than $\frac{a}{\mu}(\mu + \tan\beta)$.

Proof:-



Let $|AB| = x$

be the rod resting over the peg C and under the peg 'A'.

$|AG| = x/2$

$|AC| = a$

Resolving forces along and $\perp$ar to AB.

$\mu R + \mu S = W\sin\beta$ ___ (i)

$R + W\cos\beta = S$

$R = S - W\cos\beta$ ___ (ii) put in (i)

$\mu(S - W\cos\beta) + \mu S = W\sin\beta$

$\mu S - \mu W\cos\beta + \mu S = W\sin\beta.$

$$2\mu S = \mu W\cos\beta + W\sin\beta$$
$$= W(\mu\cos\beta + \sin\beta)$$
$$S = \frac{W}{2\mu}\left[\mu\cos\beta + \sin\beta\right] \quad \text{——(iii)}$$

Now taking moment of the forces about A.

$$S|AC| - W\cos\beta|AG| = 0 \qquad \because R = \mu R = \mu S = 0$$
$$W\sin\beta = 0$$
$$\Rightarrow S|AC| = W\cos\beta\,|AG|$$
$$S\cdot a = W\cos\beta\cdot\frac{x}{2}$$

So we are left with $W\cos\beta$, S.

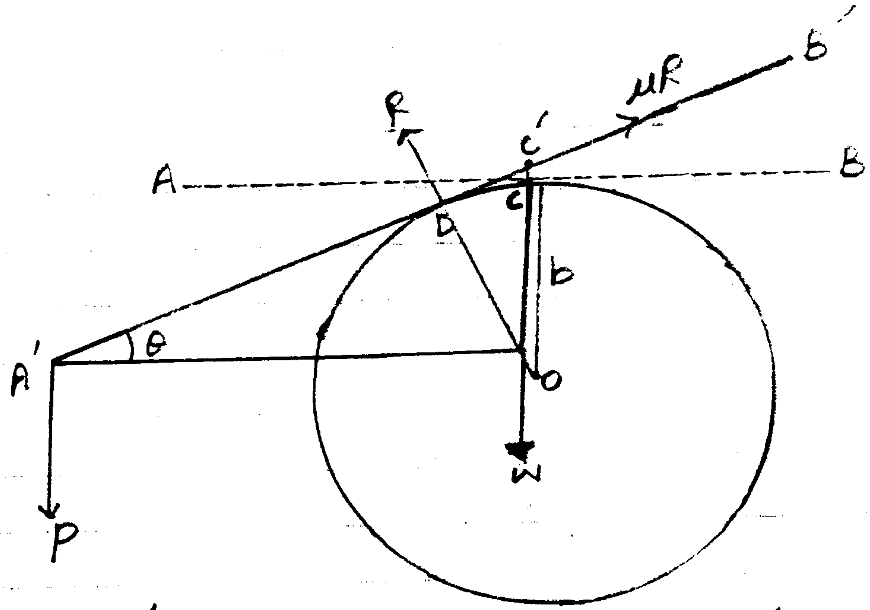
$$\frac{W}{2\mu}\left[\mu\cos\beta + \sin\beta\right]\cdot a = W\cos\beta\cdot\frac{x}{2}$$
$$\frac{a}{\mu}\left[\mu\cos\beta + \sin\beta\right] = x\cdot\cos\beta$$
$$\Rightarrow x = \frac{a}{\mu}\left[\frac{\mu\cos\beta}{\cos\beta} + \frac{\sin\beta}{\cos\beta}\right]$$
$$x = \frac{a}{\mu}\left[\mu + \tan\beta\right]$$

proved !!

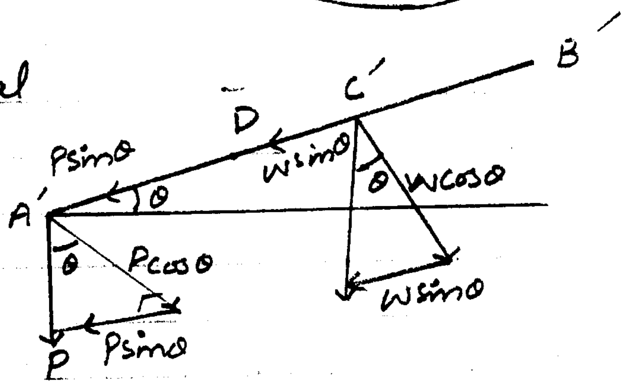
Q No- 14.

A uniform (ladder)^x rod of Length $2a$ and weight W, rests with its middle point upon a rough horizontal cylinder whose axis is perpendicular to the rod. Show that the greatest weight that can be attached to one end of the rod, without sliding it off the cylinder is $\frac{b\lambda}{a-b\lambda}W$, where b is the radius of the cylinder and λ is the angle of friction.

Proof :-



Let AB be the rod resting in horizontal position over the cylinder with it's middle point (C.G) in contact with cylinder. on



suspending a weight 'P' at the end 'A' the rod turns over the cylinder in the position A'B'. D is the new point of contact and C takes the position C'.

$$AB = A'B' = 2a$$

$$OD = OC = b$$

$$|DC'| = \overset{\frown}{DC}$$

$$= b(\angle COD) \text{ --- (i)} \qquad \text{using } S = r\theta = b(\angle COD)$$

Suppose the rod A'B' makes an angle θ with the horizontal. Resolving along and ⊥ar to rod.

$$\mu R = W\sin\theta + P\sin\theta \text{ --- (I)}$$

$$R = W\cos\theta + P\cos\theta \text{ --- (II)}$$

Put R in (i)

$$\mu(W\cos\theta + P\cos\theta) = W\sin\theta + P\sin\theta$$

$$\mu\,(W+P)\cos\theta = (W+P)\sin\theta$$

$$\mu = \frac{(W+P)\sin\theta}{(W+P)(\cos\theta)}$$

$$\mu = \tan\theta.$$

$$\because \mu = \tan\alpha$$

$$\Rightarrow \tan\alpha = \tan\theta.$$

$$\alpha = \theta \Rightarrow \angle COD = \theta = \alpha.$$

Taking moment about D.

$$|A'D|P\cos\theta - |DC'|W\cos\theta = 0$$

$$|A'D|P\cos\theta = |DC'|W\cos\theta$$

$$(a - b\alpha)\,P = b\alpha\,W$$

$$P = \frac{b\alpha}{a - b\alpha}W.$$

from (i)

$DC' = b(\angle COD)$

$= b\alpha$

$|A'D| = |A'C'| - |DC'|$

$|A'D| = a - b\alpha$

$\because A'C' = a$

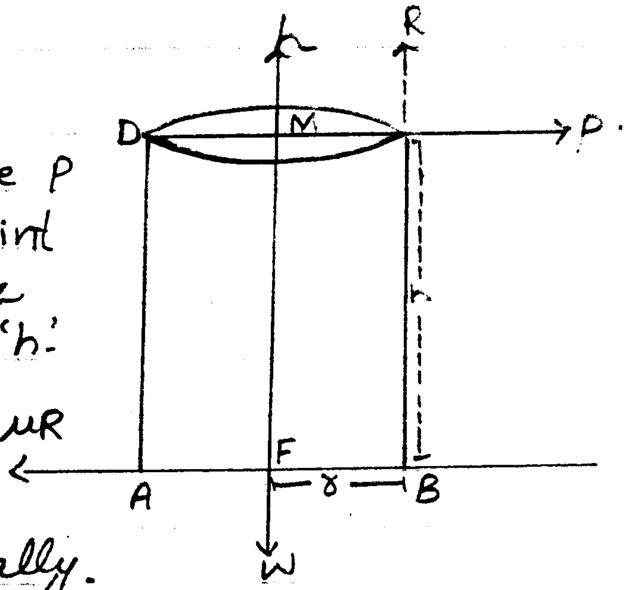
$C'B' = a$

proved !!

Q NO: 15-

A solid cylinder rests on a rough horizontal plane with one of it's flat ends on the plane, and is acted on by a horizontal force through the centre of it's upper end. If this force be just sufficient to move the solid, show that it will slide and not topple over, if the coefficient of friction be less than the ratio of radius of the base of cylinder to it's height.

Proof:-



Let the horizontal force P is applied at the midpoint of upper Base M. The hieght of cylinder is 'h'. The radius of base = r

Resolving forces horizontally & vertically.

$P = \mu R$ —(i)

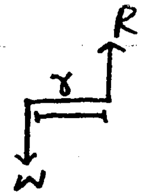
$R = W$ —(ii)

$P = \mu W$ —(iii) using (ii) in (i)

Moment of First couple-

$G_1 = Wr$

(i)

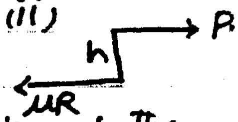


Moment of Second couple:-

$G_2 = P \cdot h$

$= \mu Wh$ using (iii)

(ii)



The cylinder will remain upright & slide if

$G_1 > G_2$

$Wr > \mu Wh$

$r > \mu h$

$\frac{r}{h} > \mu$

$\mu < \frac{\text{radius of the base}}{\text{hieght of cylinder}}$

Hence proved!!

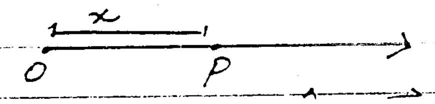
(i) [It keeps the cylinder slightly without toppling over].

(ii) [It cause the cylinder to topple over].

Rectilinear Motion

"The motion of a particle along a straight line is called Rectilinear motion".

Let a particle moves along a straight line taken as x-axis i,e from 'O' as fixed point and x be the distance of particle from 'O' at any time 't'. then magnitude of velocity and accelration is



$r = x$

$V = \dfrac{dx}{dt}$ —— (1) $\because \vec{r} = x\hat{i} = \vec{OP}$

$\because \vec{V} = \dfrac{dx}{dt}\hat{i}$

$a = \dfrac{d^2x}{dt^2}$ —— (2) $\because \vec{a} = \dfrac{d^2x}{dt^2}\hat{i}$

Also $a = \dfrac{dv}{dt} = \dfrac{dv}{dx} \cdot \dfrac{dx}{dt}$ Also $\vec{a} = \dfrac{d\vec{V}}{dt}$

$= V \cdot \dfrac{dv}{dx}$ —— (III)

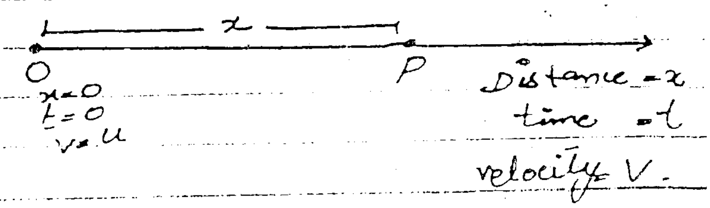
These eq's (1), (II) & (III) are differential eq's of a particle describing the rectilinear motion.

Motion With Constant Accelration:-

Suppose a particle moves along a straight line with constant (uniform) accelration 'a'. At time t = 0 particle is at 'O' as where its velocity be 'u' After time 't'. Let the particle be at a distance 'x' from 'O' where it's velocity be 'V'. Then

$\dfrac{dV}{dt} = a$

$\Rightarrow dV = a\,dt.$



$\Rightarrow \int dV = a\int dt$

$V = at + A$

Now at 'O', $t=0$, $V=u$

$\therefore u = 0 + A \Rightarrow u = A$

$\Rightarrow V = at + u$ ——— (I)

$\Rightarrow \dfrac{dx}{dt} = at + u$

$\Rightarrow \int dx = \int (at+u)\,dt$

$x = \dfrac{at^2}{2} + ut + B$

Now at $t=0$, $x=0$

$\therefore 0 = 0 + 0 + B$

$\Rightarrow B = 0$

$\Rightarrow x = ut + \dfrac{1}{2}at^2$ ——— (II)

from ① $V = u + at$

$x = u\left(\dfrac{V-u}{a}\right) + \dfrac{1}{2}a\left(\dfrac{V-u}{a}\right)^2$

$2ax = 2u(V-u) + (V-u)^2$

$2ax = \cancel{2uV} - 2u^2 + V^2 + u^2 - \cancel{2uV}$

$2ax = V^2 - u^2$ ——— (III)

2nd Method

$V\dfrac{dV}{dx} = a$

$\int V\,dV = \int a\,dx$

$\Rightarrow \dfrac{V^2}{2} = ax + c$

Now at 'O'

$V = u, x = 0$

$\therefore \dfrac{u^2}{2} = 0 + c$

$\dfrac{V^2}{2} = \dfrac{2}{2}ax + \dfrac{u^2}{2}$

$V^2 = 2ax + u^2$

$V^2 - u^2 = 2ax$

Eq's (I), (II), (III) are eq's of motion of particle moving with constant accelration.

Motion of the particle with variable acc:-

When the particle moves with variable accelration, the accelration may be expressed as a function of time, velocity or distance.

Time Dependent Accelration:-

When the accelration is function of time only, then

$$a = f(t)$$

$$\frac{dv}{dt} = f(t)$$

$$\int dv = \int f(t)\,dt$$

$$v = \phi(t) + A \quad \text{, where } \phi(t) = \int f(t)\,dt$$

$$\frac{dx}{dt} = \phi(t) + A$$

$$\int dx = \int \phi(t)\,dt + A\int dt$$

$$x = \int \phi(t)\,dt + At + B. \; ; \; A \,\&\, B$$ are constants of integration can be found by initial condition.

Velocity dependent Accelration:-

$$a = f(v)$$

$$v\frac{dv}{dx} = f(v)$$

$$\frac{v\,dv}{f(v)} = dx$$

$$\int \frac{v\,dv}{f(v)} = \int dx$$

$$\Rightarrow \int \frac{v\,dv}{f(v)} + A = x$$

Also, $a = f(v)$

$$\frac{dv}{dt} = f(v)$$

$$\int \frac{dv}{f(v)} = \int dt$$

$$\int \frac{dv}{f(v)} + B = t.$$

Accelration depend upon distance:-

$$a = f(x)$$

$$v\frac{dv}{dx} = f(x)$$

$$\int v\,dv = \int f(x)\,dx$$

$$\frac{v^2}{2} = \psi(x) + \frac{A}{2}$$

$$v^2 = 2\psi(x) + A$$

$$v = \pm\sqrt{2\psi(x) + A}$$

$$\int \frac{dx}{\pm\sqrt{2\psi(x)+A}} = \int dt \qquad \because v = \frac{dx}{dt}$$

$$\pm\int \frac{dx}{\sqrt{2\psi(x)+A}} = t.$$

Example:-

Distance covered in nth unit of time:-

Let x_1 & x_2 be the distances travelled by the particle in the first n & $n-1$ seconds respectively. let

we know, $x = ut + \frac{1}{2}at^2$

when $x = x_1$, $t = n$ $\qquad x_1 = un + \frac{1}{2}an^2$ ——(1)

When, $x = x_2$, $t = n-1$, $\qquad x_2 = u(n-1) + \frac{1}{2}a(n-1)^2$ ——(ii)

The distance travelled in nth unit of time as $x_1 - x_2$

$$\therefore\ x_1 - x_2 = \left(un + \frac{1}{2}an^2\right) - \left\{u(n-1) + \frac{1}{2}a(n-1)^2\right\}$$

$$= \cancel{un} + \frac{1}{2}an^2 - \cancel{un} + u - \frac{1}{2}a(n^2 - 2n + 1)$$

$$= \cancel{\frac{1}{2}an^2} + u - \cancel{\frac{1}{2}an^2} + \frac{a.2n}{2} - \frac{a}{2}$$

$$x_1 - x_2 = u + \frac{1}{2}a(2n-1)$$

Example:-

Find the distance travelled by a particle moving in a straight line and the velocity ; at any time 't' if it starts from rest at 't' and subject to an accelration $t^2+\sin t+e^t$.

Solution:-

$$\frac{d^2x}{dt^2} = t^2+\sin t+e^t \quad \text{(Time dependent Acc).}$$

Integrating w.r.t 't'

$$\frac{dx}{dt} = \frac{t^3}{3} - \cos t + e^t + A$$

when $t=0,\ x=0,\ v=\frac{dx}{dt}=0$

$$\Rightarrow \quad 0 = 0-1+1+A \quad \Rightarrow \quad A=0$$

$$\therefore \quad v=\frac{dx}{dt} = \frac{t^3}{3} - \cos t + e^t$$

Required velocity.

Integrating w.r.t 't'.

$$x = \frac{t^4}{3.4} - \sin t + e^t + B$$

At $t=0,\ x=0,\ \Rightarrow\ 0=0-0+1+B \Rightarrow B=-1$

$$\therefore \quad x = \frac{t^4}{12} - \sin t + e^t - 1$$

Ans.

Example:-

A particle moves in a straight line with an accelration KV^3. If it's initial velocity is u, find the velocity and the time spent when the particle has travelled a distance x.

Solution:-

$$V\frac{dV}{dx} = KV^3, \quad \text{Velocity dependent Acc.}$$

$$\frac{v\,dv}{v^3} = K\,dx$$

Integrating w.r.t 'v'
$$\int \frac{dv}{v^2} = \int K\,dx$$

$$-\frac{1}{v} = Kx + A$$

when $x=0$, vel. $v=u$

$$-\frac{1}{u} = 0 + A$$

$$\therefore -\frac{1}{v} = Kx - \frac{1}{u}.$$

$$-\frac{1}{v} = \frac{uKx-1}{u}.$$

$$\frac{u}{1-uKx} = v$$ Required velocity at distance 'x'.

Now
$$a = \frac{dv}{dt} = Kv^3$$

$$\frac{dv}{v^3} = K\,dt$$ ∵ we want to find time so $dv/dt = a$

$$\int \frac{dv}{v^3} = K\int dt$$

$$\frac{-1}{2v^2} = Kt + B$$ Initially when $t=0$, vel $v=u$.

$$\Rightarrow -\frac{1}{2}u^2 = B$$

$$\therefore -\frac{1}{2v^2} = Kt - \frac{1}{2u^2}$$

$$Kt = \frac{1}{2u^2} - \frac{1}{2v^2}$$

$$Kt = \frac{1}{2}\left[\frac{1}{u^2} - \left(\frac{1-uKx}{u}\right)^2\right]$$

$$t = \frac{1}{2Ku^2}\left[\cancel{1} - \cancel{1} - u^2K^2x^2 + 2uKx\right]$$

$$= \frac{uKx}{2Ku^2}\left(2 - uKx\right)$$

$$t = \frac{x}{2u}\left(2 - uKx\right)$$

Ans.

Example:- Discuss the motion of a particle moving in a straight line with an acc. x^3 where x is the distance of the particle from a fixed point 'O' on the line, if it starts at $t=0$, from a point $x=c$, with velocity $c^2/\sqrt{2}$.

Solution:-

$$V\frac{dV}{dx} = x^3$$

Distance dependent acc.

$$\int V\,dV = \int x^3\,dx$$

$$\frac{V^2}{2} = \frac{x^4}{4} + A$$

when $x=c,\ t=0,\ V=c^2/\sqrt{2}$

$$\Rightarrow \frac{c^4}{4} = \frac{c^4}{4} + A \Rightarrow A = 0$$

$$\therefore \frac{V^2}{2} = \frac{x^4}{4}$$

$$\Rightarrow V = \frac{x^2}{\sqrt{2}}$$

$$\frac{dx}{dt} = \frac{x^2}{\sqrt{2}} \Rightarrow \int \sqrt{2}\,\frac{dx}{x^2} = \int dt$$

$$-\frac{\sqrt{2}}{x} + B = t$$

; when $t=0,\ x=c,\ \therefore B = +\frac{\sqrt{2}}{c}$

$$\therefore -\frac{\sqrt{2}}{x} + \frac{\sqrt{2}}{c} = t \Rightarrow t = \frac{\sqrt{2}}{c} - \frac{\sqrt{2}}{x}$$

$$t = \sqrt{2}\left(\frac{1}{c} - \frac{1}{x}\right)$$

Required time.

Example 4:-

Find the distance travelled by a particle moving along a straight line with uniform acceleration 'a' in the nth units of time. Also if x, y, z be the distance travelled in the pth, qth, rth seconds respectively show that

$$(q-r)x + (r-p)y + (p-q)z = 0$$

Solution:-

Let x_1, & x_2 be the distances travelled by the particle in first n & $n-1$ seconds resp.

let initially velocity $v = u$

$$x_1 = un + \frac{1}{2}an^2 \qquad \because \text{put } x = x_1 \text{ in } x = ut + \frac{1}{2}at^2,\ t = n, \text{in}$$

$$\because \text{Put } x = x_2, \text{ in } x = ut + \frac{1}{2}at^2,\ t = n-1$$

$$x_2 = u(n-1)\ \frac{1}{2}a(n-1)^2$$

$$x_1 - x_2 = un + \frac{1}{2}an^2 - un + u - \frac{1}{2}a(n^2 + 1 - 2n)$$

$$= \frac{1}{2}an^2 + u - \frac{1}{2}an^2 - \frac{a}{2} + \frac{2na}{2}$$

$$x_1 - x_2 = u + \frac{a}{2}(2n-1) \quad \text{—— (A)}$$

$$x = u + \frac{a}{2}(2p-1) \quad \text{—— (1)}$$

$$y = u + \frac{a}{2}(2q-1) \quad \text{—— (2)}$$

$$z = u + \frac{a}{2}(2r-1) \quad \text{—— (3)}$$

(2) − (3) we get.

$$③ - ⓘ \quad y - z = a(q - r) \quad — (iv)$$

$$\cancel{y}\, z - x = a(r - p) \quad — (v)$$

$$① - ② \Rightarrow \quad x - y = a(p - q) \quad — (vi)$$

Multiplying (iv) by (x) (v) by y & (vi) by (z) and adding

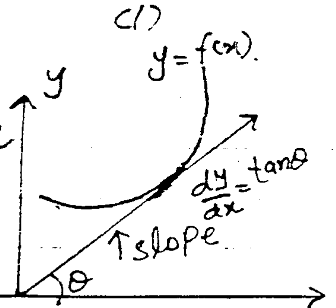
$$(q-r)x + (r-p)y + (p-q)z = \frac{1}{a}[yx - zx + zy - xy + xz - yz]$$

$$\Rightarrow (q-r)x + (r-p)y + (p-q)z = 0$$

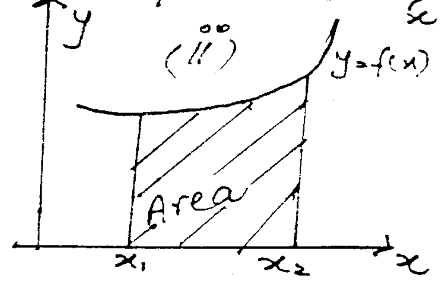
Proved!!

Graphical Method:-

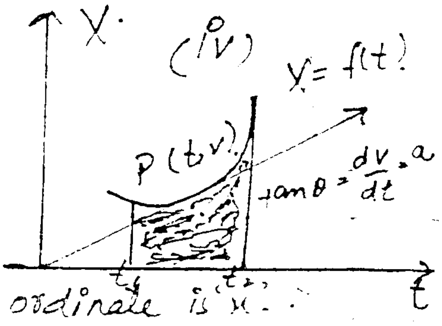
(i) If we draw a graph of the curve $y = f(x)$ then $\frac{dy}{dx}$ at pt of curve gives slope of tangent to the curve at that point.

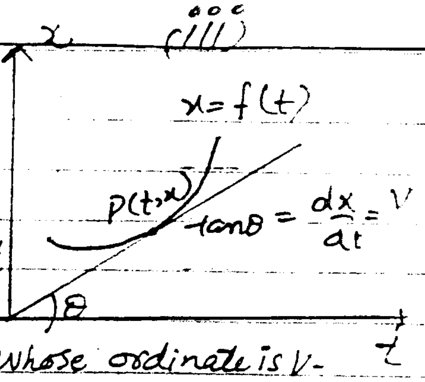


(ii) The area bounded by the curve $y = f(x)$ the x-axis and ordinates $x = x_1$, $x = x_2$ is given by $\int_{x_1}^{x_2} y\,dx$.



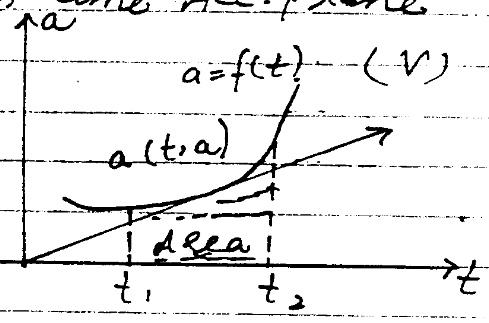
(iii) If we consider $x = f(t)$ in time space plane i.e (t, x) plane Then
$\frac{dx}{dt}$ = velocity of particle at a, = slope of tangent to the space distance x. time curve at a point whose ordinate is x.





(iv) If we consider $v=f(t)$ in velocity time plane i.e (t,v) Plane then
$\frac{dv}{dt}$ = Accelration of Particle moving in straight line = Slope of tangent to the velocity time curve at a point whose ordinate is v.
If the Acc. is constant then velocity time curve will be a straight line whose slope gives acc. The area under the curve gives the distance travelled by the particle.

(v) If we consider $a=f(t)$ in time Acc. Plane i.e (t,a) Plane then
The area under the curve gives the velocity of Particle.



Exercise

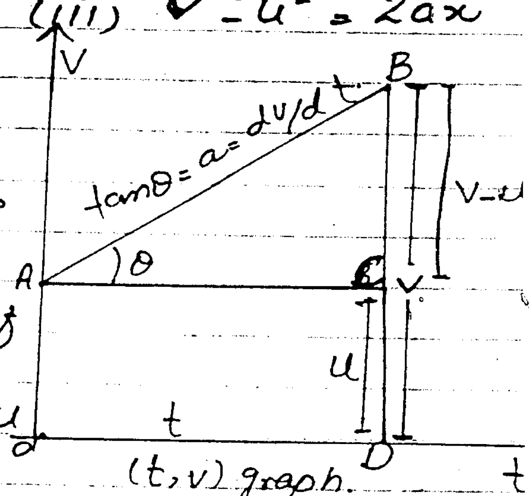
Q NO: 1

Obtain the eq's of motion by Graphical Method-

(i) $V = u + at$

(ii) $x = ut + \frac{1}{2}at^2$

(iii) $V^2 - u^2 = 2ax$



(t, v) graph.

Solution :-

Let a particle moves with constant accelration 'a' in (t, v) plane

$\therefore a = \frac{dv}{dt}$ = Slope of tangent of straight line $\overline{AB}$

Let initial velocity is $u \therefore OA = u$

Let final velocity is $V \therefore OB = V$

Let time taken is $t \therefore OD = t = AC$

Let distance covered is $x \therefore x$ = Area of ABDO.

Now in $\triangle ABC$

$BC = BD - CD = V - u$ ——— (i)

$\frac{BC}{AC} = \tan\theta = a$

$BC = a\,AC$ ——— (ii)

Equating (i) & (ii)

$V - u = a\,AC$

$V - u = at \Rightarrow V = u + at$ ——— (iii)

Now distance covered by particle from 'O' to 'B' is the total area under the graph

x = Area of trapizum ABDO

= Area of ODCA + Area of $\triangle ABC$

$= |OA||OD| + \frac{1}{2}|AC||BC|$

$= ut + \frac{1}{2}t(V - u)$

$$x = ut + \frac{1}{2}t(at)$$

$$x = ut + \frac{1}{2}at^2 \quad \text{——— (iv)}$$

Also

$x =$ Area of trapizum ABDC

$= \frac{1}{2}$ (Sum of ∥ sides) (distance b/w ∥ sides).

$= \frac{1}{2}\left(|OA| + |DB|\right)\left(|OD|\right)$

$= \frac{1}{2}(u+v)(t)$

$= \frac{1}{2}(u+v)\left(\frac{v-u}{a}\right)$ using (iii)

$$x = \frac{v^2-u^2}{2a} \Rightarrow 2ax = v^2-u^2 \quad \text{——— (v)}$$

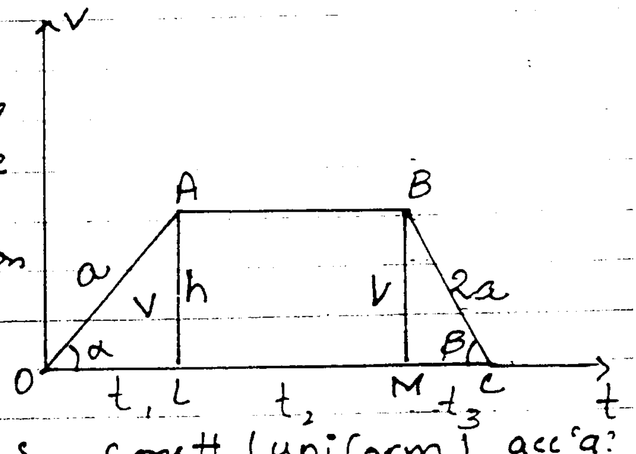
Hence proved !!

Example:-

If a particle starts from rest with constant acc 'a', when it's velocity acquires a certain value V, it moves uniformly it's velocity starts decreasing with a constt retardation 2a till it comes to rest. Find the distance travelled by the particle if the time taken from rest to rest is t.

Solution:-

Let the time spent during the accelrated motion be the $t_1 = OL$, and the time spent in uniform motion be $t_2 = LM$, & the time spent in retarted motion is $t_3 = MC$



∴ The particle describes constt (uniform) acc 'a' so the velocity time curve will be a straight

Line whose slope gives acc.

$a =$ slope of $OA = \tan\alpha = V/t_1$

$t_1 = \frac{V}{a}$ —— (i) $\qquad \because t = t_1 + t_2 + t_3$

Similarly $2a = \tan\beta = V/t_3$

$t_3 = \frac{V}{2a}$

$t_2 = t - (t_1 + t_3)$

$t_2 = t - \left(\frac{V}{a} + \frac{V}{2a}\right)$

$t_2 = t - \frac{3V}{2a}$ —— (iii)

Now distance covered in time 't' is

$X =$ Area of $\Delta OAL +$ Area of $\square ABML +$ Area of ΔBMC

$= \frac{1}{2}|OL||AL| + |LM||AL| + \frac{1}{2}|MC||MB|$

$= \frac{1}{2}t_1 V + t_2 V + \frac{1}{2}t_3 V = \frac{1}{2}V(t_1 + 2t_2 + t_3)$

$= \frac{1}{2}V\left(\frac{V}{a} + 2\left(t - \frac{3V}{2a}\right) + \frac{V}{2a}\right)$

$= \frac{V}{2}\left(\frac{3V}{2a} + 2t - \frac{6V}{2a}\right)$

$X = \frac{V}{2}\left(2t - \frac{3V}{2a}\right)$ Ans.

Vertical motion of a free Particle:-

A particle which is let fall freely from a hieght is subject to earth's gravitational pull. The acc produced by this pull is denoted by 'g'. 'g' is +ve while particle falls downward & 'g' is −ve when particle is projected upward.

$g = 32\ ft/sec^2$ in FPS System.

$g = 9.81\ m/s^2$ in MKS System.

$g = 981\ cm/sec^2$ in CGS system.
If air resistance is neglected then eq's of downward motion of particle is

$V = gt$ instead of $V = u + at$ ∴ u = initial velocity
$u = 0$, $a = g$.

$x = \frac{1}{2}gt^2$ instead of $x = ut + \frac{1}{2}at^2$

$V^2 = 2gx$ instead of $V^2 - u^2 = 2ax$.

If air resistance is neglected the eq's of upward motion of particle is

$V = u - gt$ $\quad a = -g$
$x = ut - \frac{1}{2}gt^2$
$V^2 - u^2 = -2gx$.

Example:-
A Stone is let fall freely from a hieght of 100 ft. Find the time that it takes and velocity that it acquires on reaching the ground.

Solution:-
For velocity, $V^2 = 2gx$ $\quad$ ∵ $u = 0$, $x = 100$, $g = 32\ ft/sec^2$
$V^2 = 2(32)(100)$
$V = 80\ ft/sec$

For time, $V = gt$
$80 = 32 \times t \Rightarrow t = \frac{80}{32} = \frac{5}{2}$ sec. Ans

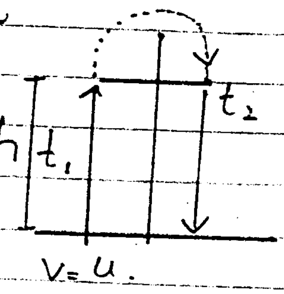
Example:-
A particle projected vertically upward is $t = 0$, with a velocity 'u', passes a point at a hieght 'h' at $t = t_1$ & $t = t_2$. Show that

$t_1 + t_2 = 2u/g$, & $t_1 t_2 = 2h/g$.

Solution:-

The distance x travelled by a particle in time 't' is

$x = ut - \frac{1}{2} g t^2$



t_1, t_2 are roots of eq.

$$h = ut - \frac{1}{2} g t^2$$
$$2h = 2ut - g t^2$$
$$g t^2 - 2ut + 2h = 0 \quad \text{quadratic in 't'}$$

Sum of roots $= t_1 + t_2 = \frac{-b}{a} = \frac{2u}{g}$

Product of root $= t_1 t_2 = \frac{c}{a} = \frac{2h}{g}$

Hence proved!!

Q NO:2.- A particle moving in a straight line starts from rest and is accelrated uniformly to attain a velocity of 60miles/hour in 4sec. Find the acc of motion and the distance travelled by the particle in the Last three seconds.

Solution:-

$u = 0$

$V = u + at$

$t = 4\,sec$

$\Rightarrow 88 = 0 + a(4)$

$\Rightarrow a = 22\ ft/sec^2$

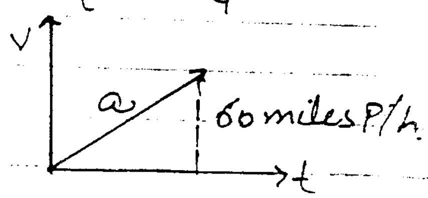
$\because V = 60\,miles/hour$

$$V = \frac{60 \times 1760 \times 3}{60 \times 60}$$

$V = 88\ ft/sec.$

2nd Method

$a = \tan\theta = \frac{V}{t} = \frac{88}{4} = 22\ ft/sec^2$



Distance covered in 4 seconds: $x = ut + \frac{1}{2}at^2$

$x = (0\times4) + \frac{1}{2}(22)(4)^2 = 176 ft$

Distance covered in first sec. $x_1 = (0)(1) + \frac{1}{2}(22)(1)^2$

$x_1 = 11 ft$

Distance covered in last '3' sec is, $x - x_1 = 176 - 11$

$x - x_1 = 165 ft$

Ans.

Q NO:3 Two particles start simultaneously from pt 'O' and move in straight line - one with velocity 45 miles/hour and $a = 2 ft/sec^2$ & other with velocity 90mph & retardation $8 ft/sec^2$. Find the time after which the velocities of both will be same and the distance of 'O', from the point where they meet again -

Solution :-

For Ist particle

$a = 2 ft/sec^2$

$u = 45 mph$

$= \frac{45 \times 1760 \times 3}{60 \times 60}$

$u = 66 ft/sec$

$\because V_1 = u + at$

$V_1 = 66 + 2t$ —(1)

For 2nd Particle.

$a = -8 ft/sec^2$

$u = 90 mph.$

$= \frac{90 \times 1760 \times 3}{60 \times 60}$

$u = 132 ft/sec.$

$\because V_2 = u + at$

$V_2 = 132 - 8t$ —(11)

According to Question $V_1 = V_2$

$66 + 2t = 132 - 8t$

$10t = 66$

$t = \frac{66}{10} = 6.6 sec$

let Both Particle meet at a distance x after time 't' then

For Ist particle.

$x = ut' + \frac{1}{2}at'^2$

$x_1 = 66t' + \frac{1}{2} \times 2t'^2$

$x_1 = 66t' + t'^2$ ——(iii)

For 2nd particle.

$x = ut' + \frac{1}{2}at'^2$

$x_2 = 132t' + \frac{1}{2}(-8)t'^2$

$x_2 = 132t' - 4t'^2$ ——(iv)

$\therefore x_1 = x_2$ $\because$ distance covered is same by both.

$66t' + t'^2 = 132t' - 4t'^2$

$5t'^2 = 66t'$

$5t'^2 - 66t' = 0$

$t'(5t' - 66) = 0 \Rightarrow t' = 0,\ t' = 66/5$ sec

Now when $t' = 0$ refers to initial instant and therefore $t' = 66/5$ is the required time after which particles meet.

so from (iii)

$x_1 = 66t' + t'^2$

$= 66 \times \frac{66}{5} + \left(\frac{66}{5}\right)^2$

$x = 1045.44\,ft.$

OR

$x_2 = 132t' - 4t'^2$

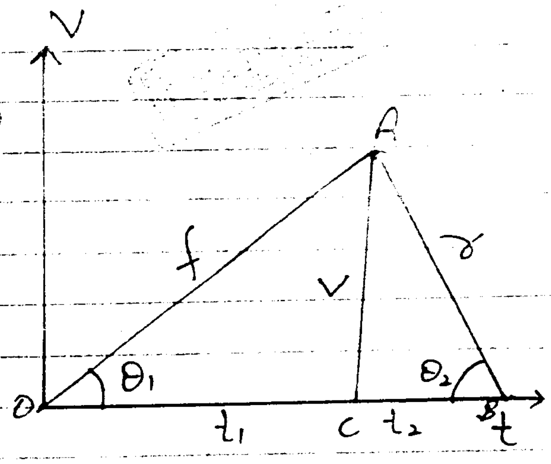
$= 132 \times \frac{66}{5} - 4\left(\frac{66}{5}\right)^2$

$x = 1045.44\,ft$

Q No. 4

A particle moving along a straight line starts from rest & is accelrated uniformly till it attains a velocity 'V'. The motion is then retarted and the particle comes to rest after traversing a total distance x. If the accelration is f, find the retardation and the total time taken by the particle from rest to rest.

Solution:-



Let t_1 & t_2 be times for accelration & retardational motion. The distance covered by particle from O to B is x then

x = Area of Δ OAB
= Area of Δ OAC + Area of Δ ABC

$$x = \frac{1}{2} t_1 V + \frac{1}{2} t_2 V$$

$$x = \frac{1}{2} V(t_1 + t_2)$$

$$\frac{2x}{V} = t_1 + t_2 \text{ ——— (i) Total time.}$$

Now Slope of OA = $f = \tan\theta_1$

$$f = \frac{V}{t_1} \Rightarrow t_1 = \frac{V}{f}$$

Again Slope of AB = $r = \tan\theta_2$

$$r = \frac{V}{t_2} \Rightarrow t_2 = \frac{V}{r}$$

From (i)

$$\frac{2x}{V} = \frac{V}{f} + \frac{V}{r}$$

$$\frac{2x}{V} - \frac{V}{f} = \frac{V}{r} \Rightarrow \frac{2fx - V^2}{Vf} = \frac{V}{r}$$

$$\Rightarrow r = \frac{V^2 f}{2fx - V^2} \quad \text{Ans}$$

Q NO: 5

Two particles travel along a straight line and are accelrated uniformly at different rates. The motion is such that when a particle attains the max. velocity v, its motion is retarded uniformly. The two particles come to rest simultaneously at a distance x from the starting point. If the acc. of the first is a and that of second is a/2, find the distance b/w the points where the two particles attain their max. hieght (velocities).

Solution:-

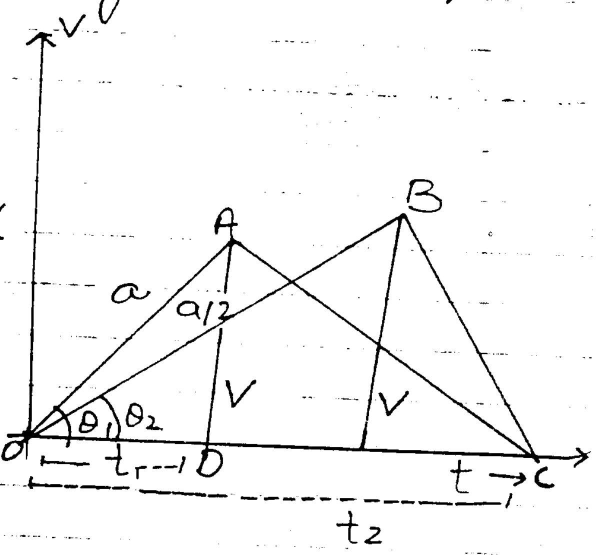
Let both particles attain maximum velocity 'V' at time t_1 & t_2 resp, then

Slope of OA $= a = \tan\theta_1 = \dfrac{V}{t_1}$

$\Rightarrow t_1 = \dfrac{V}{a}$ —(i)

Slope of OB $= \dfrac{a}{2} = \tan\theta_2 = \dfrac{V}{t_2}$

$\Rightarrow t_2 = \dfrac{2V}{a}$ —(ii)



Let x_1 & x_2 be the distances covered by the Ist & 2nd particles to attain vel 'V' in time t_1 & t_2 resp. Then

$x_1 =$ Area of OAD $= \dfrac{1}{2} V t_1$

$= \dfrac{1}{2} V\left(\dfrac{V}{a}\right) = \dfrac{V^2}{2a}$ using (i)

$x_2 =$ Area of OBE $= \dfrac{1}{2} V\left(\dfrac{2V}{a}\right) = \dfrac{V^2}{a}$ using (ii)

Required Distance, $x_2 - x_1 = \dfrac{V^2}{a} - \dfrac{V^2}{2a} = \dfrac{V^2}{2a}$

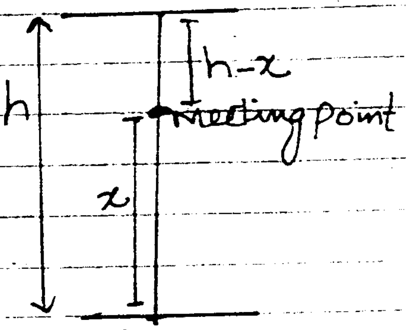
So $x_2 - x_1 = \dfrac{V^2}{}$

Q NO.6

A particle is projected vertically upwarc with a velocity $\sqrt{2gh}$ and another is let fall from a hieght 'h' at the same time - Find the hieght of the point where they meet each other-

Solution:-

Let the two particles meet each other at a hieght x after time 't'-



Projected Particle

$u = \sqrt{2gh}$, $x = x$, $t = t$

$x = ut + \frac{1}{2}gt^2$

$x = \sqrt{2gh}\, t + \frac{1}{2}(-g)t^2$ = upward. —(i)

Falling Particle:- $u = 0$, Distance $= h - x$, $t = t$

(Distance) $x = ut + \frac{1}{2}gt^2$

$h - x = 0 + \frac{1}{2}gt^2$

$h - x = \frac{1}{2}gt^2$

$x = h - \frac{1}{2}gt^2$ —(ii)

Comparing (i) & (ii)

$\sqrt{2gh}\, t - \cancel{\frac{1}{2}gt^2} = h - \cancel{\frac{1}{2}gt^2}$

$\sqrt{2gh}\, t = h \Rightarrow t = \frac{h}{\sqrt{2gh}}$

$t = \sqrt{\frac{h}{2g}}$ Put in (ii) or (i)

$x = h - \frac{1}{2}gt^2 = h - \frac{1}{2}g\left(\frac{h}{2g}\right) = h - \frac{h}{4} = \frac{3h}{4}$

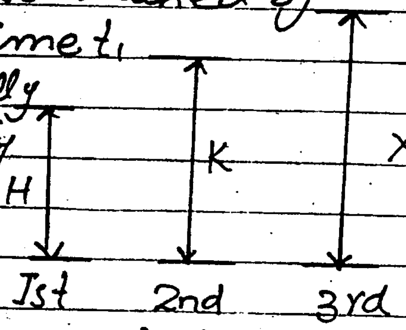
Ans.

Q NO: 7

Two particles are projected simultaneously in the vertically upward direction with velocities $\sqrt{2gh}$ and $\sqrt{2gK}$ $(K>h)$. After a time 't' when the two particles are still in flight, another particle is projected upward with a velocity u. Find the condition so that the third particles may meet the first two during their upward flight.

Solution:-

Let H, K be the max. hieghts reached by the particles P_1 & P_2 in time t_1 & t_2 respectively. In vertically upward motion final velocity at Max. hieght is '0' i.e $V = 0$ & g is $-ve$



For First Particle

$V = 0,\quad u = \sqrt{2gh},\quad t = t_1$

$V = u + at$

$0 = \sqrt{2gh} + (-g)t_1$

$t_1 = \dfrac{\sqrt{2gh}}{g} = \sqrt{\dfrac{2h}{g}}$

$V^2 - u^2 = 2ax \qquad \because x = H$

$0 - 2gh = 2(-g)H$

$h = H$

Similarly for 2nd Particle

$V = u + at$

$0 = \sqrt{2gK} + (-g)t_2$

$t_2 = \dfrac{\sqrt{2gK}}{g} = \sqrt{\dfrac{2K}{g}}$

$V^2 - u^2 = 2ax$

$-2gk = 2(-g)K \quad \because x = k$

$k = K$

$\Rightarrow \quad t_1 < t_2 \quad \because h < K$

After time 't' the third particle P_3

with velocity 'u' Since the P_3 should meet particles P_1 & P_2 during their upward motion, so t must be less than the shorter time of P_1 & P_2.

So $t < t_1$

$\Rightarrow t < \sqrt{\frac{2h}{g}}$

t is wasted time of P_3. P_1 takes shorter time t_1 to reach max. hieght H. then P_2 which takes t_2 time to reach max. hieght K as H < K. If t is not less than t_1 then P_1 will attain it's max. hieght H before the projection of P_3 - so the condn that the P_3 should meet P_1 & P_2 during their upward flight is not fulfilled ∴ t is less than t_1.

The time in which P_3 must cover the distance X > K. is $T = \sqrt{\frac{2h}{g}} - t$

$$\therefore X = uT - \frac{1}{2}gT^2$$

$$K < uT - \frac{1}{2}gT^2 \quad \because K < X$$

$$K + \frac{1}{2}gT^2 < uT$$

$$\frac{K + \frac{1}{2}gT^2}{T} < u$$

$$\frac{K}{T} + \frac{1}{2}gT < u$$

$$\Rightarrow \frac{K}{\sqrt{\frac{2h}{g}} - t} + \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - t\right) < u.$$

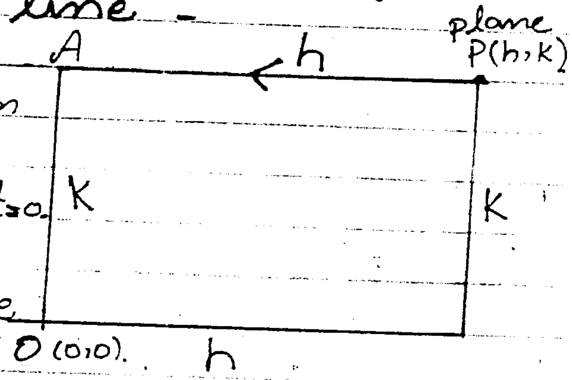
Ans

Q NO. 8.

A gunner detects a plane at t=0 approaching him with a velocity V. The horizontal and verticle distances of the plane being H & K respectively. His gun can fire a shell vertically upward with an initial velocity u. Find the time when he should fire the gun and the condition on" so that he may be able to hit the plane if it continues it's flight in the same horizontal line.

<u>Solution:-</u>

Let O(0,0) be the position of gunner. Let $P(h,K)$ be the position of plane at $t=0$. Let t_1 & t_2 be the time taken by shell and plane to reach A respectively.



<u>Motion of Shell</u>

$t_1 =$ time to reach hieght 'K' vertically upwards.

$u =$ initial velocity

$a = -g$

$OA = x = K$

$$\therefore \quad x = ut + \frac{1}{2}at^2$$

$$K = ut_1 - \frac{1}{2}gt_1^2$$

$$gt_1^2 - 2ut_1 + 2K = 0$$

$$t_1 = \frac{2u \pm \sqrt{4u^2 - 8gK}}{2g}$$

$$t_1 = \frac{u \pm \sqrt{u^2 - 2gK}}{g}$$

<u>Motion of Plane</u>

$V =$ horizontal velocity of plane.

$t_2 =$ time to cover distance h (horizontal)

$$\therefore \quad S = Vt \qquad AP = h = S$$

$$h = Vt_2 \quad \Rightarrow \quad t_2 = \frac{h}{V}$$

Now the shell will hit the plane if it is fired after time $t_2 - t_1$.

$$t_2 - t_1 = \frac{h}{V} - \left(\frac{u \pm \sqrt{u^2 - 2gK}}{g} \right)$$

Now condition for 'u':

The shell will hit the plane at A if time has real value i.e.

$$\sqrt{u^2 - 2gK} \geq 0 \Rightarrow u^2 \geq 2gK.$$

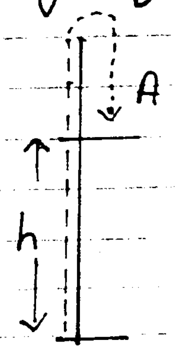
Ans.

QNO:9

A particle is projected vertically upward. After a time 't' another particle is sent up from the same point with the same velocity and meets the first at hieght h during the downward flight of the first. Find the velocity of projection.

Solution:-

Let 'T' be the time when the two particles meet at hieght 'h.'



$$x = ut - \frac{1}{2}gt^2$$

$$h = uT - \frac{1}{2}gT^2$$

$$2h = 2uT - gT^2$$

$$gT^2 - 2uT + 2h = 0$$

$$T = \frac{2u \pm \sqrt{4u^2 - 8gh}}{2g}$$

$$T = \frac{u \pm \sqrt{u^2 - 2gh}}{g}$$

$t_1 = \dfrac{u + \sqrt{u^2 - 2gh}}{g}$ i.e time taken by 1st particle to reach A during downward flight.

$t_2 = \dfrac{u - \sqrt{u^2 - 2gh}}{g}$ i.e time taken by 2nd particle to reach A during upward flight.

Now

$$t = t_1 - t_2 \qquad \because\ t_1 = t + t_2$$

$$t = \frac{u + \sqrt{u^2-2gh}}{g} - \left(\frac{u - \sqrt{u^2-2gh}}{g}\right)$$

$$t = \frac{2\sqrt{u^2-2gh}}{g}$$

$$\frac{gt}{2} = \sqrt{u^2-2gh}$$

$$\frac{g^2t^2}{4} = u^2 - 2gh$$

$$u^2 = \frac{g^2t^2}{4} + 2gh$$

$$u^2 = \frac{g^2t^2 + 8gh}{4} \qquad u = \frac{\sqrt{g^2t^2+8gh}}{2} \quad \text{Ans.}$$

Q NO: 10.

Discuss the motion of a particle moving in a straight line if it starts from rest at $t=0$ and it's acc is equal to t^n.

(i) $\text{Acc} = \frac{dv}{dt} = t^n$. (time dependent).

$$\int dv = \int t^n dt$$

$$v = \frac{t^{n+1}}{n+1} + A \qquad \text{When } t=0,\ v=0 \Rightarrow A=0$$

$$\therefore\ v = \frac{t^{n+1}}{n+1}$$

$$\frac{dx}{dt} = \frac{t^{n+1}}{n+1}$$

$$\int dx = \frac{1}{n+1}\int t^{n+1}\cdot dt$$

$$x = \frac{1}{n+1}\cdot\frac{t^{n+2}}{n+2} + B$$

when $x = 0,\ t = 0, \Rightarrow B = 0$

$\therefore\ x = \dfrac{t^{n+2}}{(n+1)(n+2)}.$

(ii) $Acc = \dfrac{dv}{dt} = a\cos t + b\sin t.$ (time dependent).

$\int dv = \int (a\cos t + b\sin t)\, dt.$

$V = a\sin t - b\cos t + A$

when $t = 0,\ V = 0, \Rightarrow A = b.$

$\therefore\ V = a\sin t - b\cos t + b.$

$\dfrac{dx}{dt} = a\sin t - b\cos t + b$

$\int dx = \int (a\sin t - b\cos t + b)\, dt$

$x = -a\cos t - b\sin t + bt + B$

when $t = 0,\ x = 0, \Rightarrow B = a.$

$x = -a\cos t - b\sin t + bt + a$

$x = a(1 - \cos t) + b(t - \sin t)$

Ans.

(iii) $Acc = V\dfrac{dv}{dx} = n^2 x.$ (distance dependent).

$\int v\, dv = \int n^2 x\, dx$ OR. $V\dfrac{dv}{dx} = -n^2 x$

$\dfrac{V^2}{2} = \dfrac{n^2 x^2}{2} + \dfrac{A}{2}$ $\int v\, dv = -\int n^2 x\, dx$

$V^2 = n^2 x^2 + A$ $\dfrac{V^2}{2} = -\dfrac{n^2 x^2}{2} + \dfrac{A}{2}$

when $x = 0,\ V = 0 \Rightarrow A = 0$ $V^2 = A - n^2 x^2$

$V^2 = n^2 x^2.$ $V = \sqrt{A - n^2 x^2}$

$V = nx$ $\dfrac{dx}{dt} = \sqrt{A - n^2 x^2}$

$\dfrac{dx}{dt} = nx.$

$\int \dfrac{dx}{x} = \int n\, dt$ $\Rightarrow \int \dfrac{dx}{\sqrt{A - n^2 x^2}} = \int dt.$

$$\ln x = nt + B$$
Ans.

$$\Rightarrow \int \frac{dx}{n\sqrt{\left(\frac{\sqrt{A}}{n}\right)^2 - x^2}} = \int dt$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{\sqrt{A}/n}\right) = nt + B$$

$$\frac{xn}{\sqrt{A}} = \sin(nt+B)$$
Ans

Q NO: 11

A particle starts with a velocity u & moves in a st line. If it suffers a retardation equal to the square of the velocity, find the distance travelled by the particle in a time 't'.

Solution:-

$$\frac{dV}{dt} = -V^2 \ (\because \text{retardation}).$$

$$\int \frac{dv}{v} = -\int dt$$

$$-\frac{1}{V} = -t + A \qquad \text{when } t = 0, \ V = u.$$

$$\Rightarrow A = -1/u.$$

$$-\frac{1}{V} = -t - \frac{1}{u}$$

$$\frac{1}{V} = \frac{ut+1}{u}$$

$$\Rightarrow V = \frac{u}{ut+1}$$

$$\frac{dx}{dt} = \frac{u}{ut+1}$$

$$\int dx = \int \frac{u}{ut+1}\,dt$$

$$x = \ln(ut+1) + B.$$

when $t=0, x=0$

$\ln(0+1) + B = 0$

$B = 0$

$$\Rightarrow x = \ln(ut+1)$$
Ans

Q NO: 12-

Discuss the motion of a particle moving in a straight line if it starts from rest at a distance 'a' from a point 'O' and move with an accelration equal to μ times its distance from 'O'.

Solution :-

Let x be the distance of the particle from the fixed point 'O' & V is the velocity

$O \longrightarrow a$

$$Acc = \mu x.$$

$$\Rightarrow V\frac{dV}{dx} = \mu x.$$

$$\Rightarrow \int V dV = \int \mu x dx$$

$$\frac{V^2}{2} = \frac{\mu x^2}{2} + \frac{A}{2}$$ when $x = a$, $V = 0$, $A = -\mu a^2$

$$\therefore V^2 = \mu x^2 - \mu a^2$$

$$V = \sqrt{\mu(x^2 - a^2)}$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\mu}\sqrt{x^2 - a^2}$$

$$\Rightarrow \int \frac{dx}{\sqrt{x^2 - a^2}} = \sqrt{\mu}\int dt$$

$$\cosh^{-1}(x/a) = \sqrt{\mu}\, t + B$$ when $x = a$, $t = 0$

$\cosh^{-1}(1) = B \Rightarrow B = 0$

$$x = a\cosh\sqrt{\mu}\, t$$

Ans

Q NO: 13-

A particle moving in a straight line starts with a velocity u and has acc V^3, where V is the velocity of particle at time t. Find the velocity and the time as functions of distance travelled by particle.

Solution:-

$$V\frac{dV}{dx} = V^3 \quad \text{(distance dependent)}$$

$$\Rightarrow \int \frac{V\,dV}{V^3} = \int dx$$

$$\int \frac{dV}{V^2} = x + A$$

$$-\frac{1}{V} = x + A \quad \text{when } x=0,\ V=u,\ \Rightarrow A = -\frac{1}{u}.$$

$$\Rightarrow -\frac{1}{V} = x - \frac{1}{u} \Rightarrow -\frac{1}{V} = \frac{ux-1}{u}.$$

$$\frac{-1}{dx/dt} = \frac{ux-1}{u}$$

$$\Rightarrow \frac{dx}{dt} = \frac{u}{1-ux}$$

$$\int (1-ux)\,dx = \int u\,dt.$$

$$x - \frac{ux^2}{2} = ut + B \quad \text{when } t=0,\ x=0 \rightarrow B=0$$

$$\Rightarrow x - \frac{ux^2}{2} = ut \Rightarrow t = \frac{x}{u}\left[\frac{2-ux}{2}\right] \text{ Ans.}$$

Simple Harmonic Motion:-

It is the motion of a particle moving in a straight line with an accelration which is always directed towards a fixed point in the line and is proportional to the distance of the particle from that point.

Let the fixed point be origin 'O'. At any time 't' suppose the particle is at 'P' which is at a distance 'x' from 'O' towards it's right side. then

$\leftarrow O \xleftrightarrow{\;x\;} P \rightarrow$

$$\frac{d^2x}{dt^2} \propto -x$$

-ve sign only indicates direction not magnitude. -ve sign indicates that acc. is directed towards 'O'. i.e against the direction in which 'x' increases α is constt of Per Portionality

$$\frac{d^2x}{dt^2} = -\alpha x$$

$$V\frac{dV}{dx} = -\alpha x$$

$$\int VdV = -\alpha\int x\,dx$$

$$\frac{V^2}{2} = -\frac{\alpha x^2}{2} + A$$

$\because$ Initially suppose particle is at $x = a$ then $t=0$, $V=0$ $x=a$

$\because \alpha a^2/2 = A$

$$\therefore \frac{V^2}{2} = -\frac{\alpha x^2}{2} + \frac{\alpha a^2}{2}$$

$$V^2 = \alpha(a^2 - x^2)$$

$$V = \pm\sqrt{\alpha(a^2-x^2)}$$

$$\int\frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\alpha}\int dt$$

$\because V = \frac{dx}{dt}$ for +ve value of 'V'.

$$\therefore \sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\alpha}\,t + B$$

at $t=0$, $x=a$

$\Rightarrow B = \sin^{-1}(1) = \pi/2$

$$\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\alpha}\,t + \pi/2$$

$$\frac{x}{a} = \sin\left(\sqrt{\lambda}\,t + \frac{\pi}{2}\right)$$

$$\Rightarrow x = a\cos\sqrt{\lambda}\,t$$

Now consider

$$\frac{dx}{dt} = -\sqrt{\lambda}\sqrt{a^2-x^2} \quad \text{for -ve value of } v$$

$$-\int \frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\lambda}\int dt$$

$$\cos^{-1}(x/a) = \sqrt{\lambda}\,t + C$$

At $t=0$, $x=a$. $C = \cos^{-1}(1) = 0$

$$\Rightarrow \quad x = a\cos\sqrt{\lambda}\,t$$

Another case.

Suppose at $t=0$, $x=0$, when the particle is at 'O'

then $\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{\lambda}\,t + B$ — so at $t=0$, $x=0$

$\because \sin^{-1}(0) = B \Rightarrow B = 0$

$$\therefore \sin^{-1}(x/a) = \sqrt{\lambda}\,t$$

$$x = a\sin(\sqrt{\lambda}\,t) \quad \text{for +ve 'V'}$$

Also at $t=0$ $x=0$

then $\cos^{-1}(x/a) = \sqrt{\lambda}\,t + C$ $\quad C = \pi/2$

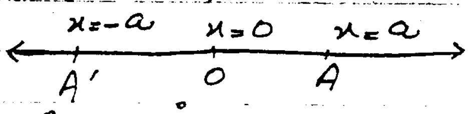
$$\Rightarrow x/a = \cos(\sqrt{\lambda}\,t + \pi/2)$$

$$\Rightarrow x = -a\sin\sqrt{\lambda}\,t$$

Nature of Simple Harmonic Motion:-

consider the eq $x = a\cos\sqrt{\lambda}\,t$ —(i)

$\because \quad -1 \le \cos\sqrt{\lambda}\,t \le 1$



$\therefore \quad -a \le a\cos\sqrt{\lambda}\,t \le a$

$-a \le x \le a$ using (i)

At $x=a$, $V = \sqrt{\lambda}\sqrt{a^2-x^2} = \sqrt{\lambda}\sqrt{a^2-a^2} = 0$

At $x=a$, minimum velocity.

$Acc = -\lambda x = -\lambda a$ Max. acc.

At $x=0$, $V = \sqrt{\lambda}\sqrt{a^2-0} = \sqrt{\lambda}\,a$ Max. velocity

$Acc = -\lambda x = -\lambda(0) = 0$ Min acc.

Now let

$$x = a\cos\sqrt{\alpha}\,t \quad \text{——— (I)}$$
$$x = a\cos(\sqrt{\alpha}\,t + 2\pi)$$
$$x = a\cos\sqrt{\alpha}\left(t + \frac{2\pi}{\sqrt{\alpha}}\right) \quad \text{——— (II)}$$

from (I) & (II) the distance at time 't' and $t + \frac{2\pi}{\sqrt{\alpha}}$ is same.

$$V = \frac{dx}{dt} = -a\sqrt{\alpha}\sin\sqrt{\alpha}\,t \quad \text{——— (III)}$$
$$\Rightarrow V = -a\sqrt{\alpha}\sin(\sqrt{\alpha}\,t + 2\pi)$$
$$V = -a\sqrt{\alpha}\sin\sqrt{\alpha}\left(t + \frac{2\pi}{\sqrt{\alpha}}\right) \quad \text{——— (IV)}$$

from (III) & (IV) the velocity at time 't' & $t + 2\pi/\sqrt{\alpha}$. Therefore the motion is repeated after $t = 2\pi/\sqrt{\alpha}$ and the particle oscillates between $x = -a$ and $x = a$. i,e between A' & A.

⇒ The max. (amplitude)/(displacement) from centre 'O' is called "Amplitude".

⇒ The fixed point 'O' is "centre of Motion"

⇒ As the particle moves towards the point 'A' the velocity of the particle decreases as 'x' increases till it reaches at pt 'A'. But At A acc. has max. magnitude "$-\alpha a$" and acc is directed towards 'O'. Due to acc the particle moves towards 'O'.

At pt 'O'. $x = 0$ so its acc. is zero but its velocity is max. "$\sqrt{\alpha}\,a$". Due to velocity the particle moves towards A'.

At Pot A', $x = -a$ it's velocity is zero and acc. is max and due to acc. the particle moves towards 'O' and finally comes to rest at 'A' - so the particle completes one "oscillation or vibration".

⇒ The motion is then repeated after time $\frac{2\pi}{\sqrt{\lambda}}$

⇒ The number of oscillations completed in unit time is called frequency

$$\nu = \frac{1}{T} = \frac{\sqrt{\lambda}}{2\pi}$$

⇒ The time to complete one oscillation is called time period $T = \frac{2\pi}{\sqrt{\lambda}}$.

NOTE:-

Time period & frequency depends on λ and are independent of Amplitude.

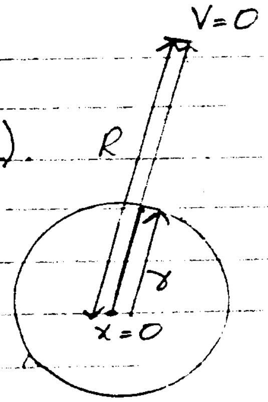
Q NO: 14.

The accelration of a particle falling freely under the gravitational pull is equal to K/x^2, where x is the distance of the particle from the centre of earth. Find the velocity of the particle if it is let fall from an altitude 'R' on striking the surface of earth if the radius of earth is 'r' and the air offers no resistance to motion.

Solution:-

$a = -\frac{K}{x^2}$ (-ve sign indicates that the direction of motion is opposite to the direction in which x increases i,e acc and distance

increases in opposite directions
i.e as distance of particle
from centre decreases, Acc increases).



$$V\frac{dv}{dx} = -\frac{K}{x^2}$$

$$\int V dV = -K\int \frac{1}{x^2}dx$$

$$\frac{V^2}{2} = \frac{K}{x} + A$$

when At t=0 $\Rightarrow A = -\frac{K}{R}$
$x = R,\ V = 0$

$$\therefore \frac{V^2}{2} = \frac{K}{x} - \frac{K}{R}$$

$$V^2 = 2K\left(\frac{1}{x} - \frac{1}{R}\right)$$

$$V^2 = 2K\left(\frac{1}{r} - \frac{1}{R}\right)$$ at earth's Surface $x = r$

$$V = \sqrt{2K\left(\frac{1}{r} - \frac{1}{R}\right)}$$ Ans

Q NO: 15-

A particle describes SHM with frequency N. If greatest velocity is V find the amplitude and max. value of acc of the particle. Also show that the velocity 'v' at a distance 'x' from centre of motion is $V = 2\pi N\sqrt{a^2 - x^2}$ where a is amplitude-

Solution:-

Frequency $= 'N' = \frac{\sqrt{\alpha}}{2\pi}$

$$2\pi N = \sqrt{\alpha} \quad \text{---} (1)$$

Now Greatest velocity $= V = \sqrt{\alpha}\, a$

$$V = 2\pi N a \quad \text{using (1)}$$

$$\frac{V}{2\pi N} = a$$

Max. Accelration $= \lambda a$

$$= (4\pi^2 N^2)\, a \quad \text{using (1)}$$

$$= (4\pi^2 N^2)\left(\frac{V}{2\pi N}\right)$$

$$= 2\pi N V$$

required velocity at a distance 'x' is

$$V = \sqrt{\lambda}\,\sqrt{a^2 - x^2}$$

$$V = 2\pi N \sqrt{a^2 - x^2}$$

Ans.

Q NO: 16.

A Particle describing SHM has Velocities 5m/sec and 4m/sec. when it's distance from centre are 12m & 13m resp. Find time Period of motion.-

Solution :-

$$T = \frac{2\pi}{\sqrt{\lambda}} \quad \text{—(1)} \qquad \sqrt{\lambda} = ?$$

$$V = \sqrt{\lambda(a^2 - x^2)} \Rightarrow V^2 = \lambda(a^2 - x^2)$$

when $v = 5,\ x = 12$

$$(5)^2 = \lambda(a^2 - (12)^2)$$

$$25 = \lambda(a^2 - 144) \quad \text{—(II)}$$

when $V = 4,\ x = 13$

$$4^2 = \lambda(a^2 - 13^2) \Rightarrow 16 = \lambda(a^2 - 169) \quad \text{—(III)}$$

Solving (II) & (III)

$$25 = \lambda a^2 - 144\lambda$$

$$\pm 16 = \pm \lambda a^2 \mp 169\lambda$$

$$9 = 25\lambda$$

$\alpha = \frac{9}{25} \Rightarrow \sqrt{\alpha} = \frac{3}{5}$

from (1) $T = \frac{2\pi}{\sqrt{\alpha}} = \frac{2\pi}{3/5} = \frac{10\pi}{3}$ Ans.

Q NO: 17

The max. velocity of a particle describing SHM of amplitude 'a' attains is 'V'. If it is disturbed in such a way that its max. velocity becomes nV, find change in amplitude and time period of motion-

Solution:-

$V_{max} = V = \sqrt{\alpha}\, a$ —(1) when Amp is a. when

$V_{max} = nV = \sqrt{\alpha}\, A$. disturbed Amp is A.

using (1) $n\sqrt{\alpha}\, a = \sqrt{\alpha}\, A$

$na = A$ —(11)

change in Amplitude $= A - a$

$= na - a$

$\therefore T = \frac{2\pi}{\sqrt{\alpha}}$ is unchanged, $= (n-1)\, a$

since π & $\sqrt{\alpha}$ remains unchanged.

So there is no change in time period.

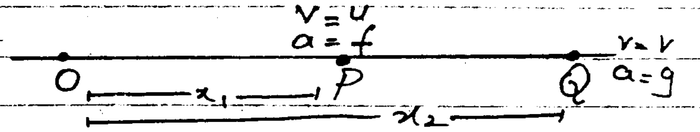
Q NO: 18:-

A point describes SHM that it's velocity and Acc. at a point P are u & f resp. and the corresponding quantities at another point Q are V & g. Find the distance PQ-

Solution:-

If $OP = x_1$

$OQ = x_2$



then $u^2 = \alpha(a^2 - x_1^2)$ —(1)

& $V^2 = \alpha(a^2 - x_2^2)$ —(2) $\quad \because V = \sqrt{\alpha(a^2 - x^2)}$

At point P Max. acc $f = x_1 \lambda \Rightarrow x_1 = \frac{f}{\lambda}$ $\quad \because a = \lambda x$

At Q Max. acc $g = x_2 \lambda \Rightarrow x_2 = \frac{g}{\lambda}$

$$u^2 - v^2 = -\lambda x_1^2 + \lambda x_2^2$$
$$= \lambda(x_2^2 - x_1^2)$$
$$= \lambda(x_2 - x_1)(x_2 + x_1)$$
$$\frac{u^2 - v^2}{\lambda(x_2 + x_1)} = (x_2 - x_1)$$
$$\frac{u^2 - v^2}{\lambda\left[\frac{g}{\lambda} + \frac{f}{\lambda}\right]} = x_2 - x_1$$
$$\frac{u^2 - v^2}{\cancel{\lambda}\left(\frac{g+f}{\cancel{\lambda}}\right)} = x_2 - x_1 \Rightarrow PQ = \frac{u^2 - v^2}{g+f}$$

Ans

Q 19:- A point moves with a velocity v is given by $v^2 = n^2(ax^2 + 2bx + c)$. Show that P executes a SHM. Find the centre, the amplitude and the time period of the motion.

Solution:-

$$v^2 = n^2(ax^2 + 2bx + c)$$
$$= n^2 a\left(x^2 + \frac{2b}{a}x + \frac{c}{a}\right)$$
$$= n^2 a\left(x^2 + \frac{2b}{a}x + \frac{b^2}{a^2} - \frac{b^2}{a^2} + \frac{c}{a}\right)$$
$$= n^2 a\left(\left(x + \frac{b}{a}\right)^2 + \frac{c}{a} - \frac{b^2}{a^2}\right)$$
$$= n^2 a\left\{\left(x + \frac{b}{a}\right)^2 + \left(\frac{ac - b^2}{a^2}\right)\right\}$$

$$V^2 = n^2a\left\{\left(x+\frac{b}{a}\right)^2 - \left(\frac{b^2-ac}{a^2}\right)\right\}$$

$$V^2 = -n^2a\left\{-\left(x+\frac{b}{a}\right)^2 + \left(\frac{b^2-ac}{a^2}\right)\right\}$$

$$V^2 = -n^2a\left\{-\left(x+\frac{b}{a}\right)^2 + \sqrt{\left(\frac{b^2-ac}{a^2}\right)^2}\right\}$$

$$V^2 = -n^2a\left\{\left(\sqrt{\frac{b^2-ac}{a^2}}\right)^2 - \left(x+\frac{b}{a}\right)^2\right\}$$

Compare with $V^2 = d(a^2 - x^2)$

$\therefore$ the centre is given by

$x + \frac{b}{a} = 0$ $\quad\because$ At centre $x=0$

$\Rightarrow x = -\frac{b}{a}$

Amplitude $= \sqrt{\frac{b^2-ac}{a^2}}$ ($\because$ Amplitude is 'a')

$= \frac{\sqrt{b^2-ac}}{a}$

Time period $= \frac{2\pi}{\sqrt{d}}$

$= \frac{2\pi}{\sqrt{n^2a}}$ $\quad\because d = n^2a$

$= \frac{2\pi}{n\sqrt{a}}$

Ans