Shahzad Ahmad Khan
M.Sc.(BZU),B.Ed.(AIOU)

Send by:
M. Tahir Aziz"
We are very thankful to him for sending these notes

Kinematics

The branch of mechanics which deals with the motion of a body without any reference to the forces acting on it is called kinematics.

Trajectory of a particle

The curve that is traced by a moving particle as the time progresses is called trajectory or path of the particle.
Considering a particle which is moving, and let at any instant it is at point 'P' whose position vector with respect to origion is $\vec{r}$ (i.e $\vec{r} = \vec{OP}$). Since the particle is moving, the vector $\vec{r}$ will change with time, so $\vec{r}$ is a function of time. We can therefore specify the path of particle by vector equation

$$\vec{r} = \vec{r}(t)$$

Velocity and Acceleration

Let $\vec{r}$ be the position of a particle P at any time 't'

then $\vec{r} = \vec{r}(t)$

we define the rate of displacement of the particle as velocity of the particle at any instant 't'.

So the vector $\frac{d\vec{r}}{dt}$ represents the velocity of the particle in this case.

The rate of change of velocity vector $\vec{v}$ is defined as acceleration of the particle placed at point P

i.e $\quad \vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right) = \dfrac{d^2\vec{r}}{dt^2}$

where $\vec{r} = \vec{OP}$ at any time 't'.

Example P-140

A particle is moving in such a way that its position at any time t is specified by $\vec{r} = (t^3+t^2)\hat{i} + (\cos t + \sin^2 t)\hat{j} + (e^t + \log t)\hat{k}$

Find its velocity and acceleration

Solution

Considering a particle which is moving in such a way that its position at any time 't' is given by

$$\vec{r} = (t^3+t^2)\hat{i} + (\cos t + \sin^2 t)\hat{j} + (e^t + \log t)\hat{k}$$

If $\vec{v}$ and $\vec{a}$ are respectively the velocity and acceleration of the particle then

$$\frac{d\vec{r}}{dt} = (3t^2+2t)\hat{i} + (-\sin t + 2\sin t\cos t)\hat{j} + \left(e^t + \frac{1}{t}\right)\hat{k}$$

$$\vec{v} = (3t^2+2t)\hat{i} + (\sin 2t - \sin t)\hat{j} + \left(e^t + \frac{1}{t}\right)\hat{k}$$

and $$\frac{d\vec{v}}{dt} = (6t+2)\hat{i} + (2\cos 2t - \cos t)\hat{j} + \left(e^t + \left(\frac{-1}{t^2}\right)\right)\hat{k}$$

$$\vec{a} = (6t+2)\hat{i} + (2\cos 2t - \cos t)\hat{j} + \left(e^t - \frac{1}{t^2}\right)\hat{k}$$

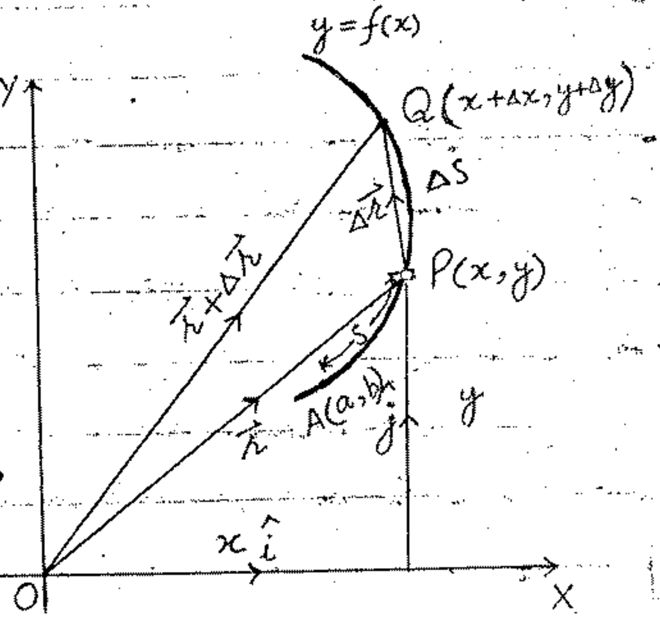
are velocity and acceleration expression for the moving particle at any time t.

Shahzad Ahmad Khan
M.Sc.(BZU),B.Ed.(AIOU)

Question

Find the cartesian components of velocity and acceleration

Answer



Considering a particle which is moving along the curve $y = f(x)$.
Let at any instant the particle is at the point $P(x,y)$ whose position vector w.r.t 'O' is $\vec{r}$ then,

$$\vec{r} = x\hat{i} + y\hat{j}$$

diff. both sides w.r.t. 't'

$$\frac{d\vec{r}}{dt} = \vec{V} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} \qquad (*)$$

So

The cartisian component of velocity along X-axis $= \frac{dx}{dt}$
& The cartisian component of velocity along Y-axis $= \frac{dy}{dt}$

The magnitude of velocity is given

$$V = |\vec{V}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$V = \frac{ds}{dt}$$

where s is the distance of the particle along the path from some fixed point $A(a,b)$ on the path.
Let $Q(x+\Delta x, y+\Delta y)$ be a point on the curve very close to $P(x,y)$.

Since $\vec{V} = \frac{d\vec{r}}{dt}$

so $\vec{V} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt}$

$= v \cdot \frac{d\vec{r}}{ds}$ ——— (1) $\left(\because \frac{ds}{dt} = v\right)$

where 'v' is the magnitude of the velocity vector and $\frac{d\vec{r}}{ds}$ shows the direction of the velocity.

Since $\frac{d\vec{r}}{ds} = \underset{\Delta s \to 0}{Lt} \frac{\Delta \vec{r}}{\Delta s}$ is the vector parallel to the tangent at P and its magnitude is given by $\left|\frac{d\vec{r}}{ds}\right| = \left|\underset{\Delta s \to 0}{Lt} \frac{\Delta \vec{r}}{\Delta s}\right| = \underset{\Delta s \to 0}{Lt} \frac{|\Delta \vec{r}|}{\Delta s} = 1$

($\because$ as $\Delta s \to 0$, $Q \to P$ and Chord PQ $\approx$ Arc PQ

so $\frac{\text{Chord } PQ}{\text{Arc } PQ} \approx 1$)

Thus $\frac{d\vec{r}}{ds} = \hat{t}$

where $\hat{t}$ is the unit vector parallel to the tangent at point P.

So from (1) we have

$\vec{V} = v\hat{t}$

"This equation shows that at any instant the direction of motion of the particle is along the tangent to the path."

Now differentiating equation (*) w.r.t 't' we get the acceleration $\vec{a}$ of the particle

$$\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{V}}{dt} = \vec{a} = \left(\frac{d^2x}{dt^2}\right)\hat{i} + \left(\frac{d^2y}{dt^2}\right)\hat{j}$$

So, The cartisian Component of acceleration along X-axis $= \frac{d^2x}{dt^2}$

& The cartisian Component of acceleration along Y-axis $= \frac{d^2y}{dt^2}$

Shahzad Ahmad Khan
M.Sc.(BZU),B.Ed.(AIOU)

Example p#141

At any time 't' the position of a particle moving in a plane can be specified by $(a\cos wt, a\sin wt)$ where 'a' and 'w' are constants. Find the components of its velocity and acceleration along the coordinate axes.

Solution

Considering a particle which is moving in a plane and its position is specified by

$$(x, y) = (a\cos wt, a\sin wt)$$

then $x = a\cos wt$ and $y = a\sin wt$

diff. both sides w.r.t. 't'

$$\frac{dx}{dt} = a(-\sin wt)w = -aw\sin wt \quad \Big| \quad \frac{dy}{dt} = a(\cos wt)w = aw\cos wt$$

are cartesian components of velocity.

Again $\frac{dx}{dt} = -aw\sin wt \;\Big|\; \frac{dy}{dt} = aw\cos wt$

diff. both sides w.r.t. 't'

$$\frac{d^2x}{dt^2} = -aw(\cos wt)w = -aw^2\cos wt \quad \Big| \quad \frac{d^2y}{dt^2} = aw(-\sin wt)w = -aw^2\sin wt$$

are the Cartesian components of acceleration.

Question

Find the tangential and normal components of velocity and acceleration.

Solution

Considering a particle which is moving along a curve $y = f(x)$.

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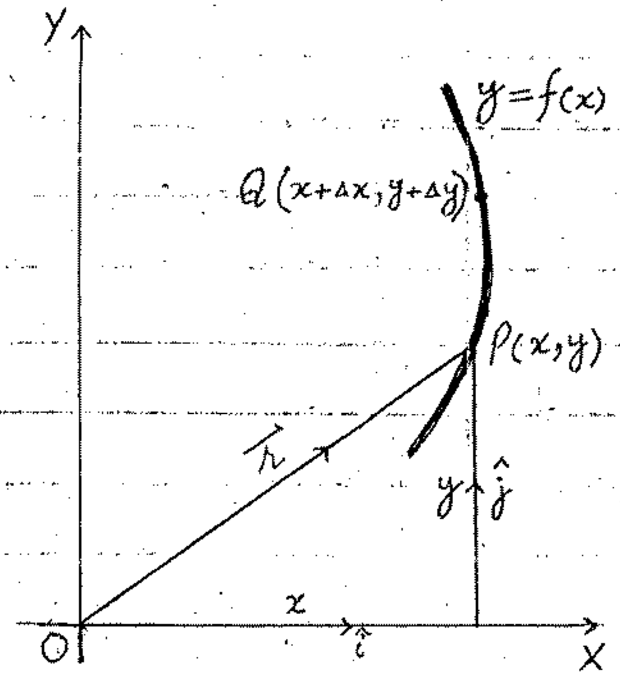
Let at any instant the particle is at point $P(x,y)$ whose position vector w.r.t. 'O' is $\vec{r}$ then,



$$\vec{r} = x\hat{i} + y\hat{j}$$

differentiating w.r.t. 't'

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j}$$

$$\vec{V} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j}$$

$$|\vec{V}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$V = \frac{ds}{dt}$$

Since $\vec{V} = \frac{d\vec{r}}{dt}$ is the velocity vector

$$= \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt}$$

$$= v\,\frac{d\vec{r}}{ds} \qquad \because \frac{ds}{dt} = v$$

where 'v' is the magnitude of the velocity vector and $\frac{d\vec{r}}{ds}$ represents the direction of the velocity. Since the velocity of the particle is all along the tangent line so,

$$\frac{d\vec{r}}{ds} = \hat{t} \text{ (unit vector along the tangent line)}$$

So

$$\vec{V} = v\hat{t}$$

$$\vec{V} = v\hat{t} + 0\hat{n}$$

So, tangential component of velocity $= v$

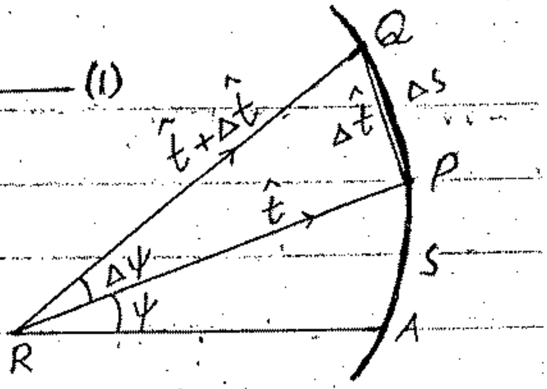
and normal component of velocity $= 0$

Again $\vec{v} = v\hat{t}$

diff. w.r.t. 't'

$$\frac{d\vec{v}}{dt} = v\frac{d\hat{t}}{dt} + \hat{t}\frac{dv}{dt}$$

$$\vec{a} = \frac{dv}{dt}\hat{t} + v\frac{d\hat{t}}{dt} \quad \text{——— (1)}$$



Since $\hat{t}\cdot\hat{t} = 1$

diff. w.r.t. 't'

$$\hat{t}\cdot\frac{d\hat{t}}{dt} + \hat{t}\cdot\frac{d\hat{t}}{dt} = 0$$

$$2\hat{t}\cdot\frac{d\hat{t}}{dt} = 0$$

$\div 2$

$$\hat{t}\cdot\frac{d\hat{t}}{dt} = 0$$

$\Rightarrow \frac{d\hat{t}}{dt}$ is a vector perpendicular to vector $\hat{t}$.

i.e $\frac{d\hat{t}}{dt}$ is a vector along the normal line.

So we can write $\frac{d\hat{t}}{dt} = \underset{\Delta t \to 0}{Lt}\frac{\Delta\hat{t}}{\Delta t}\hat{n}$

Hence from (1) we have

$$\vec{a} = \frac{dv}{dt}\hat{t} + v\left(\underset{\Delta t \to 0}{Lt}\frac{\Delta\hat{t}}{\Delta t}\hat{n}\right)$$

$$= \frac{dv}{dt}\hat{t} + v\left(\frac{\Delta\hat{t}}{\Delta\psi}\cdot\frac{\Delta\psi}{\Delta s}\cdot\frac{\Delta s}{\Delta t}\right)\hat{n} \qquad \left[\times\text{ing by } \frac{\Delta\psi}{\Delta\psi}\cdot\frac{\Delta s}{\Delta s}\right.$$

$$\underset{\substack{\Delta t \to 0\\ \Delta\psi \to 0\\ \Delta s \to 0}}{Lt}$$

$$\vec{a} = \frac{dv}{dt}\hat{t} + v(1\cdot K\cdot v)\hat{n}$$

$$= \frac{dv}{dt}\hat{t} + v^2\frac{1}{\rho}\hat{n} \qquad \because K = \frac{1}{\rho}$$

$$= \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$$

$\because \underset{\Delta\psi \to 0}{Lt}\frac{\Delta\hat{t}}{\Delta\psi} = 1$

$\underset{\Delta s \to 0}{Lt}\frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds} = K$ (curvature)

$\underset{\Delta t \to 0}{Lt}\frac{\Delta s}{\Delta t} = \frac{ds}{dt} = v$ (Velocity)

Tangential component of acc. $= \frac{dv}{dt}$

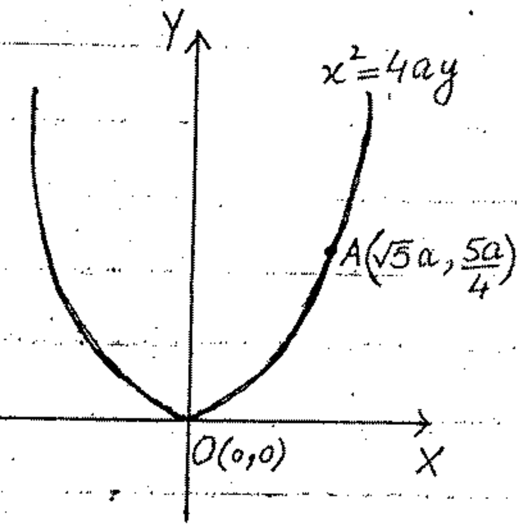
Normal component of acc. $= \frac{v^2}{\rho}$

Example 8-11.3

A particle is moving along the parabola $x^2 = 4ay$ with Constant speed v. Determine the tangential and normal components of its acc. when it reaches the point whose abscissa is $\sqrt{5}\,a$

Solution

Considering a particle Which is moving along a parabola $x^2 = 4ay$ with Constant velocity v. Let after a time 't' the particle reaches at point $A(\sqrt{5}a, \frac{5a}{4})$.



Since the particle is moving with Constant velocity v

so $\frac{dv}{dt} = 0$

i.e Tangential component of acc $= 0$

Now $4ay = x^2$

diff. w.r.t. x

$$4a\frac{dy}{dx} = 2x$$

$\div 4a$

$$\frac{dy}{dx} = \frac{x}{2a}$$

i.e $y_1 = \frac{x}{2a}$

Again diff. w.r.t. x

$$\frac{dy_1}{dx} = \frac{1}{2a}(1)$$

$$y_2 = \frac{1}{2a}$$

Since point A lies on parabola, so this point will satisfy the equation of parabola

$$4ay = x^2$$

Put $x = \sqrt{5}\,a$

$$4ay = (\sqrt{5}\,a)^2$$

$$4ay = 5a^2$$

$\div 4a$

$$y = \frac{5a}{4}$$

As $y_1 = \frac{x}{2a}$

Put $x = \sqrt{5}\,a$

$$y_1 = \frac{\sqrt{5}\,a}{2a}$$

$$= \frac{\sqrt{5}}{2}$$

Shahzad Ahmad Khan
M.Sc.(BZU).B.Ed.(AIOU)

Now $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+5/4)^{3/2}}{\frac{1}{2a}} = \frac{(9/4)^{3/2}}{\frac{1}{2a}} = \frac{(3/2)^{2\times 3/2}}{\frac{1}{2a}}$$

$$\rho = 2a(3/2)^3 = 2a.\frac{27}{8} = \frac{27a}{4}$$

Now The normal component of acceleration $= \dfrac{v^2}{\rho}$

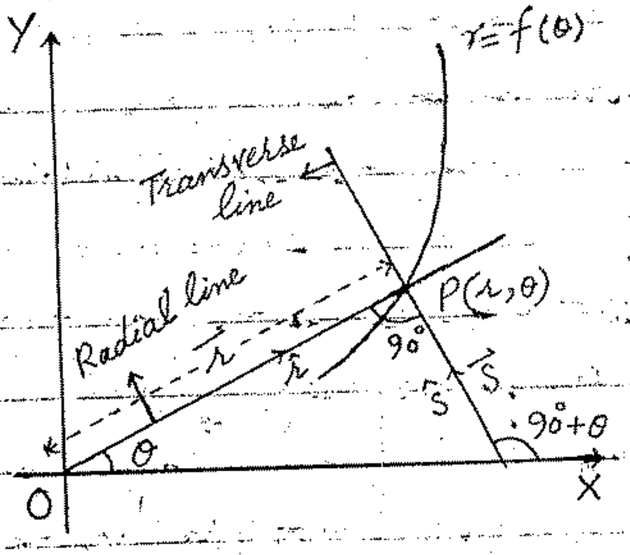
$$= \frac{v^2}{\frac{27a}{4}}$$

$$= \frac{4v^2}{27a}$$

Question

Find the radial and transverse components of velocity and acceleration.

Answer



Considering a particle which is moving along the polar curve $r = f(\theta)$. Let at any instant 't' the particle is at the point $P(r,\theta)$ whose position vector w.r.t. 'O' is $\vec{r}$.

Let $\hat{r}$ be the unit vector along the vector $\vec{r}$ and $\hat{s}$ be the unit vector along the vector $\vec{s}$.

We know that relation connecting cartesian and the polar coordinates are $x = r\cos\theta$, $y = r\sin\theta$

Sol:

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \text{——— (1)}$$

diff. w.r.t. 't'

$$\frac{d\hat{r}}{dt} = -\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j}$$

$$= (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{d\theta}{dt}$$

$$\frac{d\hat{r}}{dt} = \hat{s}\frac{d\theta}{dt} \quad \text{——— (3)} \quad (\text{by }(2))$$

and

$$\hat{s} = \cos(90^\circ+\theta)\,\hat{i} + \sin(90^\circ+\theta)\,\hat{j}$$

$$\hat{s} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \quad \text{——— (2)}$$

diff. w.r.t. 't'

$$\frac{d\hat{s}}{dt} = -\cos\theta\frac{d\theta}{dt}\hat{i} + (-\sin\theta)\frac{d\theta}{dt}\hat{j}$$

$$= -(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\frac{d\theta}{dt}$$

$$\frac{d\hat{s}}{dt} = -\hat{r}\frac{d\theta}{dt} \quad \text{——— (4)} \quad (\text{by }(1))$$

we know that

$$\vec{r} = r\,\hat{r}$$

diff. w.r.t. 't'

$$\frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \hat{r}\frac{dr}{dt}$$

$$\vec{V} = r\left(\hat{s}\frac{d\theta}{dt}\right) + \left(\frac{dr}{dt}\right)\hat{r} \quad \text{by (3)}$$

$$\vec{V} = \left(\frac{dr}{dt}\right)\hat{r} + \left(r\frac{d\theta}{dt}\right)\hat{s}$$

So, The radial component of velocity $= \frac{dr}{dt}$

The transverse component of velocity $= r\frac{d\theta}{dt}$

Again $\vec{V} = \left(\frac{dr}{dt}\right)\hat{r} + \left(r\frac{d\theta}{dt}\right)\hat{s}$

diff. w.r.t. 't'

$$\frac{d\vec{V}}{dt} = \frac{dr}{dt}\left(\frac{d\hat{r}}{dt}\right) + \hat{r}\frac{d^2r}{dt^2} + r\frac{d\theta}{dt}\left(\frac{d\hat{s}}{dt}\right) + r\frac{d^2\theta}{dt^2}\hat{s} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{s}$$

Put $\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{s}$ and $\frac{d\hat{s}}{dt} = -\hat{r}\frac{d\theta}{dt}$ (by (3), (4))

$$\vec{a} = \frac{dr}{dt}\frac{d\theta}{dt}\hat{s} + \frac{d^2r}{dt^2}\hat{r} + r\frac{d\theta}{dt}\left(-\hat{r}\frac{d\theta}{dt}\right) + r\frac{d^2\theta}{dt^2}\hat{s} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{s}$$

$$= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{s}$$

so, The radial component of acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

The transverse component of acceleration $= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$

Shahzad Ahmad Khan
M.Sc.(BZU),B.Ed.(AIOU)

Example P-146

A particle P moves in a plane in such a way that at any time 't' its distance from a fixed point 'O' is $r = at + bt^2$ and the line connecting 'O' and 'P' makes an angle $\theta = ct^{3/2}$ with a fixed line OA. Find the radial and transverse components of the velocity and acceleration of the particle at $t = 1$

Solution

Considering a particle P which is moving in a plane whose distance from a fixed point 'O' is given by

$$r = at + bt^2$$

$r = at + bt^2$

$\frac{dr}{dt} = a + 2bt$

The radial component of velocity $= \frac{dr}{dt}$

$= a + 2bt$

Put $t = 1$

$= a + 2b(1)$

$= a + 2b$

Transverse Component of Velocity $= r\frac{d\theta}{dt}$

$= (at + bt^2)\frac{3c}{2}\sqrt{t}$

Put $t = 1$

$= (a(1) + b(1)^2)\frac{3c}{2}\sqrt{1}$

$= (a + b)\frac{3c}{2}$

Here

$\theta = ct^{3/2}$

$\frac{d\theta}{dt} = c \cdot \frac{3}{2} t^{1/2}$

$= \frac{3c}{2}\sqrt{t}$

$\frac{d^2\theta}{dt^2} = \frac{3c}{2}\frac{1}{2}t^{-1/2}$

$= \frac{3c}{4\sqrt{t}}$

Radial Component of acc $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$= 2b - (at + bt^2)\left(\frac{3c}{2}\sqrt{t}\right)^2$

Put $t = 1$

$= 2b - (a + b)\frac{9c^2}{4}$

$= \frac{1}{4}[8b - 9c^2(a + b)]$

$\frac{dr}{dt} = a + 2bt$

$\frac{d^2r}{dt^2} = 2b$

Transverse component of acc. $= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$

$$= 2(a+2bt)\frac{3c}{2}\sqrt{t} + (at+bt^2)\frac{3c}{4\sqrt{t}}$$

Put $t=1$

$$= 2(a+2b)\frac{3c}{2} + (a+b)\frac{3c}{4}$$

$$= \frac{3c}{4}[4a+8b+a+b]$$

$$= \frac{3c}{4}(5a+9b)$$

Exercise Set 7

Q-No.1

A particle starts from O at $t=0$. Find its velocity and accelration at any time 't' if its position at that time is given by

(i) $\vec{r} = (t^3+2t)\underline{i} + (5t^2-7)\underline{j}$

(ii) $\vec{r} = at^2\underline{i} + 4at\underline{j}$

(iii) $\vec{r} = a\cos t\,\underline{i} + b\sin t\,\underline{j}$

(iv) $\vec{r} = a(t-\cos t)\underline{i} + a(1+\sin t)\underline{j}$

Solution

Considering a particle which starts moving from point O at $t=0$. Let at any instant the particle is at point P whose position vector w.r.t. origion is given by

(i) $$\vec{r} = (t^3+2t)\underline{i} + (5t^2-7)\underline{j}$$

diff. w.r.t. 't'

$$\frac{d\vec{r}}{dt} = (3t^2+2(1))\underline{i} + (10t-0)\underline{j}$$

Shahzad Ahmad Khan
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$\vec{V} = (3t^2+2)\underline{i} + 10t\underline{j}$ is the velocity of the particle at any instant 't'.

Again diff. w.r.t. 't'

$$\frac{d\vec{V}}{dt} = (6t+0)\underline{i} + 10(1)\underline{j}$$

$\vec{a} = 6t\underline{i} + 10\underline{j}$ is the acceleration of the particle at any instant 't'.

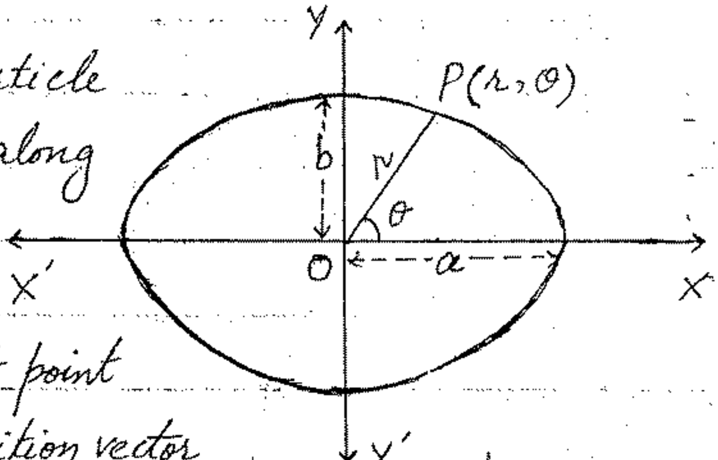
(ii), (iii) and (iv) can be solved similarly.

Q No. 2

The position of a particle moving along an ellipse is given by $\vec{r} = a\cos t\,\underline{i} + b\sin t\,\underline{j}$. If $a > b$, find the position of the particle where its velocity has a maximum or minimum magnitude.

Solution

Considering a particle which is moving along an ellipse. Let at any time 't' the particle is at point $P(r,\theta)$ whose position vector w.r.t. 'O' is $\vec{r}$



$a > b$

then $\vec{r} = a\cos t\,\underline{i} + b\sin t\,\underline{j}$

diff. w.r.t. 't'

$$\frac{d\vec{r}}{dt} = a(-\sin t)\underline{i} + b\cos t\,\underline{j}$$

$$\vec{V} = -a\sin t\,\underline{i} + b\cos t\,\underline{j}$$

$$|\vec{V}| = \sqrt{(-a\sin t)^2 + (b\cos t)^2}$$

$$V = \sqrt{a^2\sin^2 t + b^2\cos^2 t} \quad \text{———— (1)}$$

$$V^2 = a^2\sin^2 t + b^2\cos^2 t$$

Since for the velocity 'V' to be maximum

(i) $\frac{dv}{dt} = 0$ (ii) $\frac{d^2v}{dt^2} < 0$

and for the velocity 'V' to be minimum

(i) $\frac{dv}{dt} = 0$ (ii) $\frac{d^2v}{dt^2} > 0$

Again

$$V^2 = a^2\sin^2 t + b^2\cos^2 t$$

diff. w.r.t. 't'

$$2(v)'\frac{dv}{dt} = a^2(2\sin t\cos t) + b^2(2\cos t(-\sin t))$$

$$2v\frac{dv}{dt} = a^2\sin 2t - b^2\sin 2t$$

$$2v\frac{dv}{dt} = (a^2 - b^2)\sin 2t$$

Put $\frac{dv}{dt} = 0$

$$2V(0) = (a^2 - b^2)\sin 2t$$

$$(a^2 - b^2)\sin 2t = 0$$

$$\sin 2t = 0 \qquad \because a^2 - b^2 \neq 0 \text{ as } a > b$$

$$\Rightarrow 2t = 0, \pi, 2\pi, 3\pi, \ldots\ldots$$

$\div 2$

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots\ldots$$

Again

$$2v\frac{dv}{dt} = (a^2 - b^2)\sin 2t$$

diff. w.r.t. 't'

$$2\left(v\frac{d^2v}{dt^2} + \frac{dv}{dt}\frac{dv}{dt}\right) = (a^2 - b^2)\cos 2t \cdot 2$$

$\div 2$

$$v\frac{d^2v}{dt^2} + \left(\frac{dv}{dt}\right)^2 = (a^2 - b^2)\cos 2t$$

$V = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$

Put $t = 0$

then $V = b$

Put $t = 0$, $\frac{dv}{dt} = 0$, $v = b$

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$$b\frac{d^2v}{dt^2}+0=(a^2-b^2)\cos 0$$

$$\frac{d^2v}{dt^2}=\frac{a^2-b^2}{b}$$

$$\Rightarrow \frac{d^2v}{dt^2}>0 \qquad \text{as } a>b$$

$$\boxed{\text{Clearly for } t=0;\ \text{(i) } \frac{dv}{dt}=0 \quad \text{(ii) } \frac{d^2v}{dt^2}>0} \quad \text{—— (i)}$$

Again

$$v\frac{d^2v}{dt^2}+\left(\frac{dv}{dt}\right)^2=(a^2-b^2)\cos 2t$$

Put $t=\frac{\pi}{2}$; $\frac{dv}{dt}=0$, $v=a$

$$a\frac{d^2v}{dt^2}+0=(a^2-b^2)\cos \cancel{2}\frac{\pi}{\cancel{2}}$$

$$\frac{d^2v}{dt^2}=\frac{-(a^2-b^2)}{a}$$

$$\Rightarrow \frac{d^2v}{dt^2}<0 \qquad \text{as } a>b$$

$v=\sqrt{a^2\sin^2t+b^2\cos^2t}$
Put $t=\frac{\pi}{2}$
then $v=a$

$$\boxed{\text{Clearly for } t=\frac{\pi}{2};\ \text{(i) } \frac{dv}{dt}=0 \quad \text{(ii) } \frac{d^2v}{dt^2}<0} \quad \text{—— (ii)}$$

Again

$$v\frac{d^2v}{dt^2}+\left(\frac{dv}{dt}\right)^2=(a^2-b^2)\cos 2t$$

Put $t=\pi$, $\frac{dv}{dt}=0$, $v=b$

$$b\frac{d^2v}{dt^2}+0=(a^2-b^2)\cos 2\pi$$

$$\frac{d^2v}{dt^2}=\frac{a^2-b^2}{b}$$

$v=\sqrt{a^2\sin^2t+b^2\cos^2t}$
put $t=\pi$
$v=b$

$$\Longrightarrow \frac{d^2v}{dt^2}>0 \qquad \text{as } a>b$$

$$\boxed{\text{Clearly for } t=\pi;\ \text{(i) } \frac{dv}{dt}=0 \quad \text{(ii) } \frac{d^2v}{dt^2}>0} \quad \text{—— (iii)}$$

Again $v\dfrac{d^2v}{dt^2}+\left(\dfrac{dv}{dt}\right)^2=(a^2-b^2)\cos 2t$

Put $t=\dfrac{3\pi}{2}$; $\dfrac{dv}{dt}=0$, $v=a$

$v=\sqrt{a^2\sin^2t+b^2\cos^2t}$

Put $t=\dfrac{3\pi}{2}$

then $v=a$

$a\dfrac{d^2v}{dt^2}+0=(a^2-b^2)\cos\cancel{2}\dfrac{3\pi}{\cancel{2}}$

$\dfrac{d^2v}{dt^2}=\dfrac{-(a^2-b^2)}{a}$

$\Rightarrow \dfrac{d^2v}{dt^2}<0$ as $a>b$

$\boxed{\text{Clearly for } t=\dfrac{3\pi}{2} \text{ (i) } \dfrac{dv}{dt}=0 \text{ (ii) } \dfrac{d^2v}{dt^2}<0}$ —— (iv

Clearly the velocity will be minimum for $t=0, \pi, \ldots$ (from (i) and (iii))

Now $\vec{r}=a\cos t\,\underline{i}+b\sin t\,\underline{j}$

Put $t=0, \pi$

we get $\vec{r}=a\underline{i}\,,\,-a\underline{i}$

$\vec{r}=\pm a\underline{i}$ are the positions of the particle where it attains its minimum velocity.

Clearly the velocity will be maximum for $t=\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$ (from (ii) and (iv))

Now $\vec{r}=a\cos t\,\underline{i}+b\sin t\,\underline{j}$

Put $t=\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

we get

$\vec{r}=b\underline{j}\,,-b\underline{j}$

$\vec{r}=\pm b\underline{j}$ are the positions of the particle where it attains its maximum velocity.

Shahzad Ahmad Khan
M.Sc.(BZU),B.Ed.(AIOU)

Q. No. 3

A particle is moving with uniform (or constant) speed 'v' along the curve

$$x^2 y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)$$

Show that its acceleration has the maximum value $\frac{10v^2}{9a}$

Solution

Considering a particle which is moving along the curve

$$x^2 y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)$$

with constant speed 'v'.

then $\frac{dv}{dt} = 0$

so, The tangential component of acc. $= 0$

and the normal component of acceleration $= \frac{v^2}{\rho}$

Now $x^2 y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)$

$\div x^2$

$$y = a\left(1 + \frac{a^2}{\sqrt{5}} x^{-2}\right)$$

diff. w.r.t x

$$\frac{dy}{dx} = a\left(0 + \frac{a^2}{\sqrt{5}}(-2)(x)^{-3}\right)$$

$$y_1 = -\frac{2a^3}{\sqrt{5}} x^{-3}$$

Again diff. w.r.t. x

$$\frac{dy_1}{dx} = \frac{-2a^3}{\sqrt{5}}(-3)\, x^{-4}$$

$$y_2 = \frac{6a^3}{\sqrt{5}\, x^4}$$

Now

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left[1 + \left(\frac{-2a^3}{\sqrt{5}\, x^3}\right)^2\right]^{3/2}}{\frac{6a^3}{\sqrt{5}\, x^4}}$$

$$\rho = \frac{\left(1+\frac{4a^6}{5x^6}\right)^{3/2}}{\frac{6a^3}{\sqrt{5}\,x^4}}$$

$$= \left(\frac{5x^6+4a^6}{5x^6}\right)^{3/2} \cdot \frac{\sqrt{5}\,x^4}{6a^3}$$

$$= \frac{(5x^6+4a^6)^{3/2}}{5\cancel{\sqrt{5}}\,\cancel{x^9}\,x^5} \cdot \frac{\cancel{\sqrt{5}}\,\cancel{x^4}}{6a^3}$$

$$= \frac{(5x^6+4a^6)^{3/2}}{30a^3x^5}$$

Now the normal Component of acceleration $= \frac{v^2}{\rho}$

$$A = \frac{v^2}{\frac{(5x^6+4a^6)^{3/2}}{30a^3x^5}}$$

$$= \frac{v^2 \cdot 30a^3x^5}{(5x^6+4a^6)^{3/2}}$$

$$A = 30a^3v^2\, x^5(5x^6+4a^6)^{-3/2}$$

Since 'A' depends on $B = x^5(5x^6+4a^6)^{-3/2}$

$\therefore$ 'A' will be maximum if 'B' is maximum.

Now for B to be maximum

(i) $\frac{dB}{dx} = 0$ (ii) $\frac{d^2B}{dx^2} < 0$

Now

$$B = x^5(5x^6+4a^6)^{-3/2}$$

diff. both sides w.r.t. 'x'

$$\frac{dB}{dx} = x^5(-3/2)(5x^6+4a^6)^{-5/2}(30x^5) + (5x^6+4a^6)^{-3/2}\cdot 5x^4$$

$$\frac{dB}{dx} = -45x^{10}(5x^6+4a^6)^{-5/2} + 5x^4(5x^6+4a^6)^{-3/2}$$

$$= -5x^4(5x^6+4a^6)^{-5/2}\left[9x^6-(5x^6+4a^6)\right]$$

$$= -5x^4(5x^6+4a^6)^{-5/2}(4x^6-4a^6)$$

Put $\frac{dB}{dx} = 0$

$$-5x^4(5x^6+4a^6)^{-5/2}(4x^6-4a^6) = 0$$

$-5x^4 = 0$ | $(5x^6+4a^6)^{-5/2} \neq 0$ | $4x^6-4a^6 = 0$

$\Rightarrow x = 0$ | | $4x^6 = 4a^6$

$\div 4$ $x^6 = a^6$

$\Rightarrow x = a$

Again

$$\frac{dB}{dx} = -5\left[x^4(5x^6+4a^6)^{-5/2}(4x^6-4a^6)\right]$$

diff. w.r.t. x

$$\frac{d^2B}{dx^2} = -5\left[x^4(5x^6+4a^6)^{-5/2}\cdot 24x^5 + x^4(4x^6-4a^6)\left(-\frac{5}{2}\right)(5x^6+4a^6)^{-7/2}\cdot 30x^5 + (5x^6+4a^6)^{-5/2}(4x^6-4a^6)\cdot 4x^3\right]$$

when $x=0$, $\frac{d^2B}{dx^2} = 0$.

when $x = a$

$$\frac{d^2B}{dx^2} = -5\left[a^4(9a^6)^{-5/2}\,24a^5\right]$$

i.e $\frac{d^2B}{dx^2} < 0$

Clearly for $x = a$

(i) $\frac{dB}{dx} = 0$ (ii) $\frac{d^2B}{dx^2} < 0$

Clearly for $x = a$, $B = x^5(5x^6+4a^6)^{-3/2}$ will be maximum. Hence $A = 30a^3v^2x^5(5x^6+4a^6)^{-3/2}$ will be maximum.

Now $A = \frac{30a^3v^2x^5}{(5x^6+4a^6)^{3/2}}$ Put $x = a$

$$= \frac{30a^3v^2a^5}{(5a^6+4a^6)^{3/2}}$$

$$= \frac{30a^8v^2}{(9a^6)^{3/2}} = \frac{30a^8v^2}{27a^9} = \frac{10v^2}{9a}$$

as required.

Q. No: 4

Find the tangential and normal Components of acceleration of a point describing the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with uniform speed 'V' when the particle is at (0,b)

constant

Solution

Considering a point which is moving along the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with uniform (Constant) speed V then

$$\frac{dv}{dt}=0$$

i.e The tangential Component of acceleration $=0$
(No tangential Component of acceleration)
The normal Component of acceleration $=\frac{v^2}{\rho}$

Now $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\times a^2b^2$

$$b^2x^2+a^2y^2=a^2b^2$$

diff. w.r.t. x

$$b^2(2x)+a^2(2y)\frac{dy}{dx}=0$$

$\div 2$

$$b^2x+a^2y\frac{dy}{dx}=0$$

$$a^2y\frac{dy}{dx}=-b^2x$$

$\div a^2y$

$$\frac{dy}{dx}=\frac{-b^2}{a^2}\frac{x}{y}$$

$(0,b)$
x, y

Put $x=0,\ y=b$

$$y_1=\frac{-b^2}{a^2}\cdot\frac{0}{b}$$

$$y_1=0$$

Since $y_1 = -\frac{b^2}{a^2}\frac{x}{y}$

diff w.r.t. x

$$\frac{dy_1}{dx} = -\frac{b^2}{a^2}\left[\frac{y(1) - x\frac{dy}{dx}}{y^2}\right] \quad ; \quad \underset{x,\,y}{(0,b)}$$

Put $\frac{dy}{dx} = 0$, $x = 0$, $y = b$

$$y_2 = -\frac{\cancel{b^2}}{a^2}\left[\frac{b - 0(0)}{\cancel{b^2}}\right]$$

$$y_2 = -\frac{b}{a^2}$$

Now
$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+0^2)^{3/2}}{-\frac{b}{a^2}}$$

$$\rho = -\frac{a^2}{b}$$

Normal Component of acceleration $= \frac{v^2}{\rho}$

$$= \frac{v^2}{-\frac{a^2}{b}}$$

$$= -\frac{bv^2}{a^2}$$

Q No

Find the radial and transverse components of the velocity of a particle moving along the curve $ax^2 + by^2 = 1$ at any time 't' if the polar angle $\theta = ct^2$

Solution

Considering a particle which is moving along the curve $ax^2 + by^2 = 1$ ——— (1)

$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ $\qquad$ $\frac{dr}{dt}$, $\frac{rd\theta}{dt}$

$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$

we know that relation connecting cartesian and Polar coordinates are

$$x = r\cos\theta \;;\; y = r\sin\theta$$

so from (1) we have

$$a(r\cos\theta)^2 + b(r\sin\theta)^2 = 1$$

$$a r^2\cos^2\theta + b r^2\sin^2\theta = 1$$

$$r^2(a\cos^2\theta + b\sin^2\theta) = 1$$

$$r^2 = \frac{1}{a\cos^2\theta + b\sin^2\theta}$$

$$r = \frac{1}{\sqrt{a\cos^2\theta + b\sin^2\theta}}$$

$$r = (a\cos^2\theta + b\sin^2\theta)^{-\frac{1}{2}}$$

The radial component of velocity $= \frac{dr}{dt}$

The transverse component of velocity $= r\frac{d\theta}{dt}$

Now $r = (a\cos^2\theta + b\sin^2\theta)^{-1/2}$

diff. w.r.t. 't'

$$\frac{dr}{dt} = -\frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-3/2} \cdot \frac{d}{dt}(a\cos^2\theta + b\sin^2\theta)$$

$$\frac{dr}{dt} = \frac{-1}{2(a\cos^2\theta + b\sin^2\theta)^{3/2}}\left[a\left(2\cos\theta(-\sin\theta)\frac{d\theta}{dt}\right) + b\left(2\sin\theta\cos\theta\frac{d\theta}{dt}\right)\right]$$

$$= \frac{-1}{2(a\cos^2\theta + b\sin^2\theta)^{3/2}}\left[-a\sin 2\theta\frac{d\theta}{dt} + b\sin 2\theta\frac{d\theta}{dt}\right]$$

$$= \frac{-1\left(-\sin 2\theta\frac{d\theta}{dt}\right)(a-b)}{2(a\cos^2\theta + b\sin^2\theta)^{3/2}}$$

$$\theta = ct^2$$

$$\frac{d\theta}{dt} = 2ct$$

$$= \frac{(a-b)\sin 2\theta}{2(a\cos^2\theta + b\sin^2\theta)^{3/2}}\frac{d\theta}{dt}$$

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$$\frac{dr}{dt} = \frac{(a-b)\sin 2\theta}{\cancel{2}(a\cos^2\theta + b\sin^2\theta)^{3/2}} \cdot \cancel{2}ct$$

$\therefore$ Radial Component of velocity $= \dfrac{(a-b)ct\sin 2\theta}{(a\cos^2\theta + b\sin^2\theta)^{3/2}}$

Transverse Component of velocity $= r\dfrac{d\theta}{dt}$

$$= (a\cos^2\theta + b\sin^2\theta)^{-1/2}\, 2ct$$

$$= \frac{2ct}{\sqrt{a\cos^2\theta + b\sin^2\theta}}$$

Q.No.6

Find the radial and transverse and transverse Components of acceleration of a particle moving along a circle $x^2+y^2=a^2$ with Constant angular velocity c.

Solution

Considering a particle which is moving along a circle

$$x^2+y^2=a^2 \quad \text{--- (1)}$$

we know that the relation connecting Cartesian and polar Coordinates are

$$x = r\cos\theta \; ; \; y = r\sin\theta$$

So from (1) we have

$$(r\cos\theta)^2 + (r\sin\theta)^2 = a^2$$

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2$$

$$r^2(\cos^2\theta + \sin^2\theta) = a^2$$

$$r^2(1) = a^2$$

$$\Rightarrow \quad r = a$$

The radial component of acceleration

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= 0 - a(c)^2$$

$$= -ac^2$$

The transverse component of acceleration $= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$

$$= 2(0)(c) + a(0)$$

$$= 0 + 0$$

$$= 0$$

$r = a$

$\frac{dr}{dt} = 0$

$\frac{d^2r}{dt^2} = 0$

Angular velocity $= c$

i.e. $\frac{d\theta}{dt} = c$

$\frac{d^2\theta}{dt^2} = 0$

consider a partical along ellipse. The position of particle

$r = a\cos t\,\underline{i} + b\sin t\,\underline{j}$

diff w.r.t t

$\frac{dr}{dt} = a(-\sin t)\underline{i} + b\cos t\,\underline{j}$

$v = a(-\sin t)\underline{i} + b\cos t\,\underline{j}$

$|v| = \sqrt{(-a\sin t)^2 + (b\cos t)^2}$

$= \sqrt{a^2\sin^2 t + b^2\cos^2 t}$

$= a^2(1-\cos^2 t) + b^2\cos^2 t$

$= a^2 - a^2\cos^2 t + b^2\cos^2 t$

$= a^2 - (a^2-b^2)\cos^2 t$

Case I: magnitude of velocity will be max if $\cos^2 t$ is minimum

minimum value of $\cos^2 t = 0$

$\cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

at $t = \frac{\pi}{2}, \frac{3\pi}{2}$ velocity max

Put $t = \pi/2$

$r = a\cos\frac{\pi}{2}\,i + b\sin\frac{\pi}{2}\,j$

Put $t = 3\pi/2$

$r = a\cos\frac{3\pi}{2}\,i + b\sin\frac{3\pi}{2}\,j$

$= 0 + b(-1) = -bj$

$r = \pm bj$ are the positions of particle it attains its max velocity

Case II: magnitude of velocity will be mini if $\cos^2 t$ is max

max value of $\cos^2 t = 1$

~~$\cos t = 1$~~

$t = 0, \pi$

So at $t = 0, \pi$ velocity mini

Put $t = 0$

Now $r = a\cos 0\,\underline{i} + b\sin 0\,\underline{j}$

$= a\underline{i}$

Put $t = \pi$

Now $r = a\cos\pi\,i + b\sin\pi\,j$

$= a(-1)i = -ai$

$r = \pm ai$ are the positions where its attains minimum velocity.