Chp (12) Proofs:-

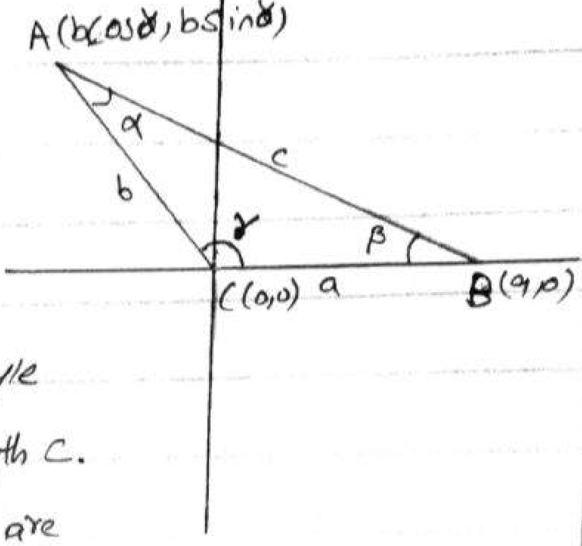
The law of cosine:-

(i) $a^2 = b^2 + c^2 - 2bc\cos\alpha$

(ii) $b^2 = a^2 + c^2 - 2ac\cos\alpha$

(iii) $c^2 = a^2 + b^2 - 2ab\cos\alpha$

Proof:-



Consider a triangle with sides of length a, b, c where γ is the measurement of the angle opposite the side of length c.

The tricoordinates of triangle are

$A = A(b\cos\gamma, b\sin\gamma)$ $\quad B = B(a,0)$ $\quad C = (0,0)$

Now by distance formula

$$|AB| = c = \sqrt{(a - b\cos\gamma)^2 + (0 - b\sin\gamma)^2}$$

take square b.s

$$c^2 = a^2 + b^2\cos^2\gamma - 2ab\cos\gamma + b^2\sin^2\gamma$$

$$= a^2 + b^2(\cos^2\gamma + \sin^2\gamma) - 2ab\cos\gamma$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\gamma}$$

Hence proved

Similarly other proofs.

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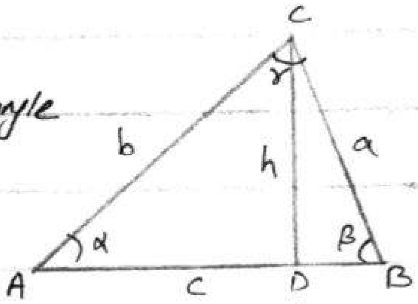
The law of Sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$



Proof: Let ABC be a triangle and let h be its altitude.
As we know the sine of angle in a triangle is the ratio of the length of its perpendicular to the length of its hypotenuse
From Fig there are two rt △ADC & rt △BCD
In triangle ACD

$$\sin\alpha = \frac{h}{b}$$

$$h = b\sin\alpha \quad -①$$

in triangle BCD

$$\sin\beta = \frac{h}{a}$$

$$h = a\sin\beta \quad -②$$

by compare ① & ②

$$b\sin\alpha = a\sin\beta$$

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \quad — Ⓐ$$

Similarly we can show

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \quad -Ⓑ$$

Now by Ⓐ & Ⓑ

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Hence proved.

The Law of Tangent

$$\frac{a-b}{a+b} = \frac{\tan\left(\alpha-\beta/2\right)}{\tan\left(\alpha+\beta/2\right)}$$

Proof As by law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\frac{a}{b} = \frac{\sin\alpha}{\sin\beta}$$

use componendo and dividendo

$$\frac{a+b}{a-b} = \frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}$$

$$\frac{a+b}{a-b} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)}$$

use invertendo

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

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Half Angle Formulas:-

(a) The sines of Half the angle in terms of the sides

(i) $\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Proof:- As we know that

$$2\sin^2\frac{\alpha}{2} = 1-\cos\alpha \quad -①$$

$$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc} \quad \text{put in ①}$$

$$2\sin^2\frac{\alpha}{2} = 1-\frac{b^2+c^2-a^2}{2bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{2bc-b^2-c^2+a^2}{4bc}$$

$$= \frac{a^2-(b^2+c^2-2bc)}{4bc} = \frac{a^2-(b-c)^2}{4bc}$$

$$= \frac{(a+b-c)(a-b+c)}{4bc} \quad - Ⓐ$$

$\because$ $a+b-c$
$a+b+c-c-c$
$2s-2c$
$2(s-c)$

$a-b+c$
$a+b+c-2b$
$2s-2b$
$2(s-b)$

Now Ⓐ becomes

$$\sin^2\frac{\alpha}{2} = \frac{\cancel{2}(s-c)\,\cancel{2}(s-b)}{\cancel{4}bc}$$

Take sq root.

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-c)(s-b)}{bc}}$$

Similarly we can proof all other

(b) The Cosine of Half the angle of the sides:-

(i) $\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Proof: As we know that

$$2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$$

$$2\cos^2\frac{\alpha}{2} = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$2\cos^2\frac{\alpha}{2} = \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{(b+c)^2-a^2}{4bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{(b+c+a)(b+c-a)}{4bc}$$

$\because$ $s = \frac{a+b+c}{2}$

$2s = a+b+c$

$b+c-a$

$= b+c+a-2a$

$2s-2a$

$2(s-a)$

$$\cos^2\frac{\alpha}{2} = \frac{(\cancel{2}s)(\cancel{2}(s-a))}{\cancel{4}bc}$$

take sq root

$$\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

(c) The Tangent of Half the angle in terms of the sides:-

(i) $\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Proof:- As we know

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad -(1) \qquad \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}} \quad -(2)$$

divid (1) by (2)

$$\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)/\cancel{bc}}{s(s-a)/\cancel{bc}}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Area of Triangle:-

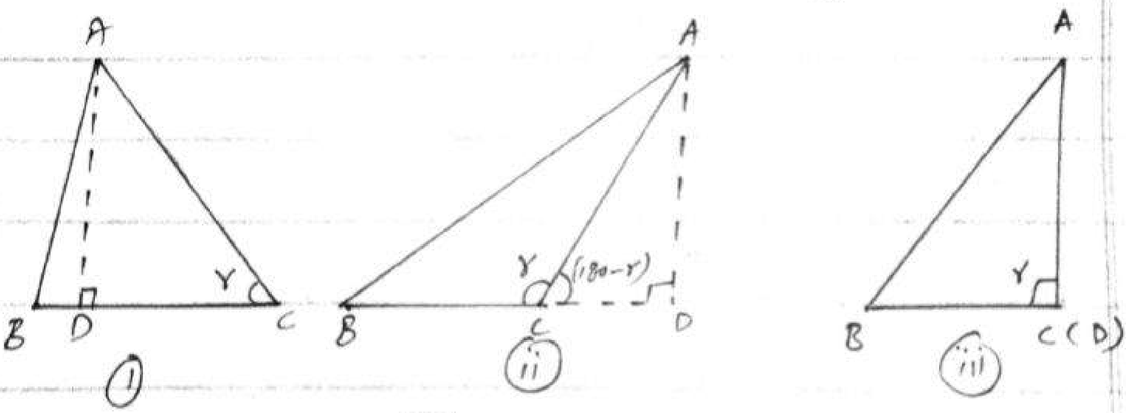
by Mirxxa Ahmad Younas

Case I

Area of Triangle in the Measures of Two sides and Their included angle

Proof:- Consider three different kind of triangle ABC with $m\angle C = \gamma$ as

(i) Acute (ii) obtuse (iii) right



in Fig (1) $\frac{\overline{AD}}{\overline{AC}} = \sin\gamma$

in Fig (ii) $\frac{\overline{AD}}{\overline{AC}} = \sin(180^\circ - \gamma) = \sin\gamma$

in Fig (iii) $\frac{AD}{AC} = 1 = \sin 90^\circ = \sin\gamma$

in all three cases we have

$$\overline{AD} = \overrightarrow{AC}\sin\gamma$$

$$= b\sin\gamma$$

Now Let Δ denote the area of triangle ABC

Now as by def of tringle area

$$\Delta = \frac{1}{2}(\text{base})(\text{altitude})$$

$$\Delta = \frac{1}{2}(\overline{BC})(\overline{AD}) = \frac{1}{2}ab\sin\gamma$$

Similarly we can prove

$$\Delta = \frac{1}{2}\sin\alpha\,(bc) = \frac{1}{2}ca\sin\beta$$

Case II

Area of Triangle in terms of the measures of the one side and Two angle.

$$\Delta = \frac{a^2\sin\beta\sin\gamma}{\sin\alpha}$$

Proof By law of Sines we know that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$a = c\frac{\sin\alpha}{\sin\gamma} \quad \& \quad b = c\frac{\sin\beta}{\sin\gamma}$$

Now As we know Area of trigle ABC is

$$\Delta = \frac{1}{2}ab\sin\gamma$$

$$= \frac{1}{2}\left(c\frac{\sin\alpha}{\sin\gamma}\right)\left(c\frac{\sin\beta}{\cancel{\sin\gamma}}\right)\cancel{\sin\gamma}$$

$$\Delta = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

Similarly we can proof other also.

Case III

Area of triangle in terms of the Measures of its sides.

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Proof As we know area of triangle ABC is

$$\Delta = \frac{1}{2} bc \sin\alpha$$

$$= \frac{1}{\cancel{2}} bc\, \cancel{2}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$\sin\alpha$
$= \sin(2(\frac{\alpha}{2}))$
$= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

which is called Hero's Formula.

Ahmad Younas

Acute Angle

An Acute angle (acute meaning small is an angle smaller then a right angle. the range of acute angle is between 0° & 90°

Obtuse Angle:-

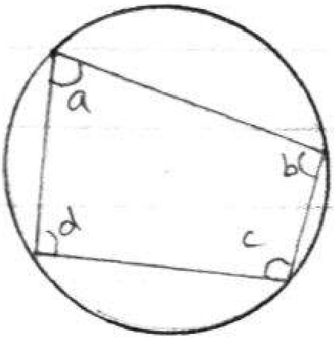
The obtuse angle is the smaller angle. It is more then 90° and less then 180°. The smaller angle is an obtuse angle but the larger angle is Reflex Angle

Cyclic quadrilateral.

A cyclic quadrilateral or inscribed quadrilateral is a quadrilateral whose vertices all lie on a single circle. This circle is called the circumcirle and the vertices are said to by concyclic.

or

A cyclic quadrilateral has all its vertices on the circumference of the circle.



opposite angle add up to 180°

$$\angle a + \angle c = 180^\circ$$

$$\angle b + \angle d = 180^\circ$$

Circum-circle:-

The circle passing through the three vertices of a traingle is called a circum circle. Its centre is called the circum centre.

(9) Prove that

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

with usual notation?

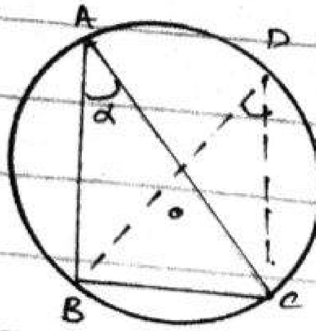


Fig I
acute

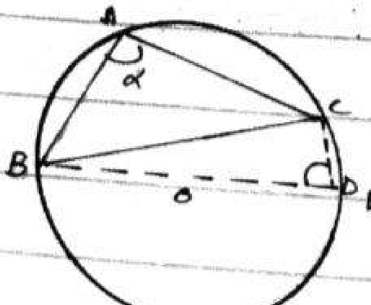


Fig II
obtuse

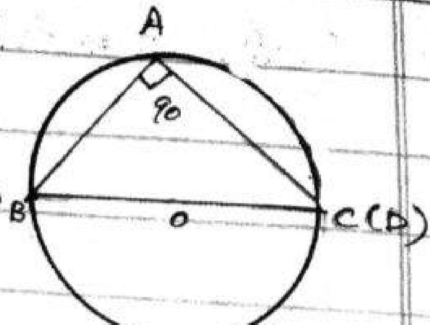


Fig III
right

Proof consider three different kinds of triangle ABC with $m\angle A = \alpha$ (i) acute (ii) obtuse (iii) right

Let O be the circum centre of $\triangle ABC$. Join B to O and produce $\overline{BO}$ to meet the circle again at D. Join C to D. Thus we have the measure of diameter $m\overline{BD} = 2R$ & $m\overline{BC} = a$

Fig I $\quad m\angle BDC = m\angle A = \alpha$

in rt $\triangle$ BCD $\quad \dfrac{m\overline{BC}}{m\overline{BD}} = \sin\alpha \quad$ -(1)

Fig II $\quad m\angle BDC + m\angle A = 180°$ (sum of opposite angle of a cyclic quadrilatral = 180°)

$m\angle BDC = 180° - \alpha$

in rt $\triangle$ BCD $\quad \dfrac{m\overline{BC}}{m\overline{BD}} = \sin m\angle(BDC) = \sin(180 - \alpha) = \sin\alpha \quad$ -(2)

Fig III $\quad m\angle A = \alpha = 90$

$$\frac{m\overline{BC}}{m\overline{BD}} = 1 = \sin 90° = \sin\alpha \quad -(3)$$

by (1) (2) & (3)

$$\frac{m\overline{BC}}{m\overline{BD}} = \sin\alpha$$

$$\frac{a}{2R} = \sin\alpha$$

$$\boxed{\frac{a}{2\sin\alpha} = R}.$$

Similarly we can prove others.

(b) Prove that $R = \frac{abc}{4R}$

Prove As we know

$$R = \frac{a}{2\sin\alpha} \qquad \left(\sin\alpha = \sin 2\left(\tfrac{\alpha}{2}\right) = 2\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\right)$$

$$= \frac{a}{2\cdot 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{a}{4\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}}$$

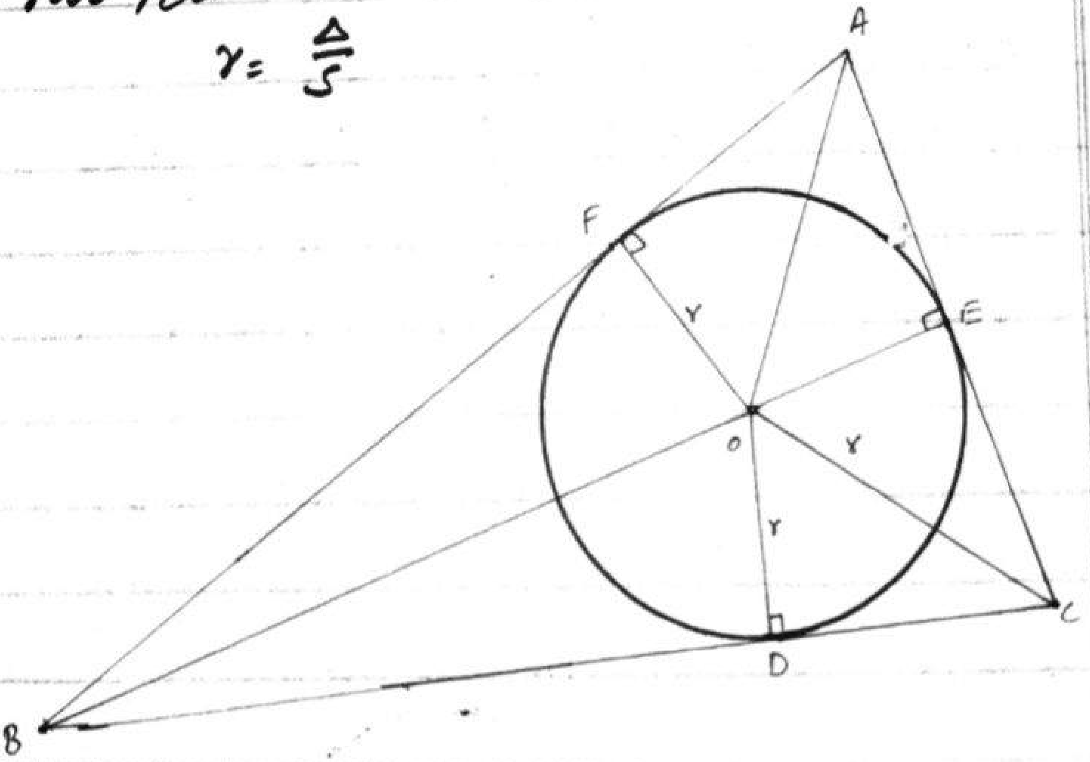
$$= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4\Delta}.$$

In circle:-

The circle make/drawn inside a triangle touching its three sides is called its inscribed circle or in circle. Its centre known as the in centre.

Prove that

$$r = \frac{\Delta}{s}$$



Proof Let the internal bisectors of angle of triangle ABC meet at o, the in centre.

$\overline{OD} \perp \overline{BC}$, $\overline{OE} \perp \overline{AC}$, & $\overline{OF} \perp \overline{AB}$

Let $m\overline{OD} = m\overline{OE} = m\overline{OF} = r$

Area of ΔABC = Area of ΔOBC + Area of ΔOCA + Area of ΔOAB

$$\Delta = \frac{1}{2}(BC \times OD) + \frac{1}{2}(CA \times OE) + \frac{1}{2}(AB \times OF)$$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$= \frac{1}{2}r(a+b+c) = \frac{2sr}{2}$$

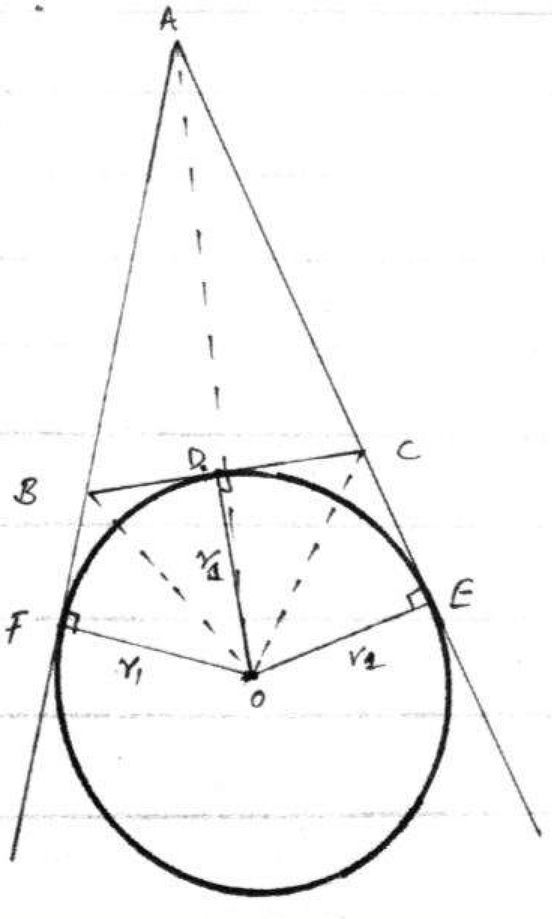
$$\boxed{\frac{\Delta}{s} = r}$$

Escribed Circles:-

A circle, which touches one side of the triangle externally and the other two produced sides is called an escribed circle or ex-circle or e-circle.

Prove that

$$r_1 = \frac{\Delta}{s-a} \; ; \; r_2 = \frac{\Delta}{s-b} \; ; \; r_3 = \frac{\Delta}{s-c}$$



Proof Let O be the centre of the escribed circle opposite to the vertex A of $\triangle ABC$

From O draw $\overline{OD} \perp \overline{BC}$, $\overline{OE} \perp \overline{AC}$ and $\overline{OF} \perp \overline{AB}$, then Join A, B & C

Let $m\overline{OD} = m\overline{OE} = m\overline{OF} = r_1$

$$\triangle ABC = \triangle OAB + \triangle OAC - \triangle OBC$$
$$= \frac{1}{2} AB \times OF + \frac{1}{2} AC \times OE - \frac{1}{2} BC \times OD$$
$$= \frac{1}{2} cr_1 + \frac{1}{2} br_1 - \frac{1}{2} ar_1$$
$$= \frac{1}{2} r_1 (c+b-a)$$
$$= \frac{1}{2} r_1 \, 2(s-a) \qquad \because \left(\begin{array}{c} a+b+c-2a \\ 2s-2a \\ 2(s-a) \end{array} \right)$$
$$\Delta = r_1 (s-a)$$
$$\frac{\Delta}{s-a} = r_1$$

Proved

Similarly we prove

$$r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}$$

Dadicated to my Teacher

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(MSc Mathematics.)