Exercise 3.3 (Solution)

Book: ***Algebra and Trigonometric*** **By:** ***Muhammad Fiaz Hussain***
Class: ***First Year***
fiaz.hussain24@yahoo.com

Q1. Evaluate the following determinants:

(i).

$$\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$$

$$=5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix}-(-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix}+(-4)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$=5(-2+3)+2(6-6)-4(3-2)$$

$$=5(1)+2(0)-4(1)$$

$$=1$$

(ii).

$$\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & 2 \end{vmatrix}$$

$$=5\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}-2\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix}+(-3)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$=5(2-1)-2(-6+2)-3(3-2)$$

$$=5(1)-2(-4)-3(1)$$

$$=10$$

(iii).

$$\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$$

$$=1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}-2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix}+(-3)\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$$

$$=1(18-20)-2(-6+8)-3(-5+6)$$

$$=1(-2)-2(2)-3(1)$$

$$=-9$$

(iv).

$$\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$$

$$=\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix} \quad C_1+C_2+C_3$$

$$=\begin{vmatrix} a+l+a-l+a & a-l & a \\ a+a+l+a-l & a+l & a-l \\ a-l+a+a+l & a & a+l \end{vmatrix}$$

$$=\begin{vmatrix} 3a & a-l & a \\ 3a & a+l & a-l \\ 3a & a & a+l \end{vmatrix}$$

$$=\begin{vmatrix} 3a & a-l & a \\ 3a & a+l & a-l \\ 3a & a & a+l \end{vmatrix}$$ By tacking common $3a$ from $C_{1.}$

$$=3a\begin{vmatrix} 1 & a-l & a \\ 1 & a+l & a-l \\ 1 & a & a+l \end{vmatrix}$$

$$=3a\left\{1\begin{vmatrix} a+l & a-l \\ a & a+l \end{vmatrix}-(a-l)\begin{vmatrix} 1 & a-l \\ 1 & a+l \end{vmatrix}+a\begin{vmatrix} 1 & a+l \\ 1 & a \end{vmatrix}\right\}$$

$$=3a\{1((a+l)^2-a(a-l))-(a-l)(a+l-a+l)+a(a-a-l)\}$$

$$=3a\{a^2+l^2+2al-a^2+al-(a-l)(2l)+a(-l)\}$$

$$=3a\{a^2+l^2+2al-a^2+al-2al+2l^2-al\}$$

$$=3a\{3l^2\}$$

$$=9al^2$$

$$=1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}-2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix}+(-3)\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$$

$$=1(18-20)-2(-6+8)-3(-5+6)$$

$$=1(-2)-2(2)-3(1)$$

$$=-9$$

(v).

$$\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$$

$$=1\begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix}-2\begin{vmatrix} -1 & -3 \\ 2 & -1 \end{vmatrix}+(-2)\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$=1(-1+12)-2(1+6)-2(-4-2)$$

$$=9$$

(vi).

$$\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$$

$$=abc\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$ Taking common a from R_1, b from

R_2 and c from R_3

$$= abc\left\{2\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix} - 1\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix} + 1\begin{vmatrix}1 & 2\\ 1 & 1\end{vmatrix}\right\}$$

$$= abc\{2(4-1)-1(2-1)+1(1-2)\}$$

$$= 4abc$$

Q2. Without expansion show that:

(i).

$$\begin{vmatrix}6 & 7 & 8\\ 3 & 4 & 5\\ 2 & 3 & 4\end{vmatrix} = 0$$

$$L.H.S = \begin{vmatrix}6 & 7 & 8\\ 3 & 4 & 5\\ 2 & 3 & 4\end{vmatrix}$$

By $c_2 - c_1$ and $c_3 - c_2$,

$$L.H.S = \begin{vmatrix}6 & 1 & 1\\ 3 & 1 & 1\\ 2 & 1 & 1\end{vmatrix} = 0 = R.H.S$$

(ii).

$$\begin{vmatrix}2 & 3 & -1\\ 1 & 1 & 0\\ 2 & -3 & 5\end{vmatrix} = 0$$

$$L.H.S = \begin{vmatrix}2 & 3 & -1\\ 1 & 1 & 0\\ 2 & -3 & 5\end{vmatrix}$$

$$L.H.S = \begin{vmatrix}-1 & 3 & -1\\ 0 & 1 & 0\\ 5 & -3 & 5\end{vmatrix} c_1 - c_2$$

$$L.H.S = 0 = R.H.S$$

(iii).

$$\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix} = 0$$

$$L.H.S = \begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}$$

$$L.H.S = \begin{vmatrix}1 & 1 & 1\\ 4 & 1 & 1\\ 7 & 1 & 1\end{vmatrix} \begin{matrix}C_2 - C_1\\ C_3 - C_2\end{matrix}$$

$$L.H.S = 0 = R.H.S$$

Q3. Show that:

(i)

$$\begin{vmatrix}a_{11} & a_{12} & a_{13}+\alpha_{13}\\ a_{21} & a_{22} & a_{23}+\alpha_{23}\\ a_{31} & a_{32} & a_{33}+\alpha_{33}\end{vmatrix} = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} + \begin{vmatrix}a_{11} & a_{12} & \alpha_{13}\\ a_{21} & a_{22} & \alpha_{23}\\ a_{31} & a_{32} & \alpha_{33}\end{vmatrix}$$

$$L.H.S = \begin{vmatrix}a_{11} & a_{12} & a_{13}+\alpha_{13}\\ a_{21} & a_{22} & a_{23}+\alpha_{23}\\ a_{31} & a_{32} & a_{33}+\alpha_{33}\end{vmatrix}$$

$$L.H.S = (a_{13}+\alpha_{13})\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix} - (a_{23}+\alpha_{23})\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}$$

$$+ (a_{33}+\alpha_{33})\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$$

$$L.H.S = a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix} + \alpha_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix} - a_{23}\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}$$

$$-\alpha_{23}\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} + a_{33}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} + \alpha_{33}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$$

$$L.H.S = \left\{a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix} - a_{23}\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} + a_{33}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\right\}$$

$$+\left\{\alpha_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix} - \alpha_{23}\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} + \alpha_{33}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\right\}$$

$$L.H.S = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} + \begin{vmatrix}a_{11} & a_{12} & \alpha_{13}\\ a_{21} & a_{22} & \alpha_{23}\\ a_{31} & a_{32} & \alpha_{33}\end{vmatrix} = R.H.S$$

(ii).

$$\begin{vmatrix}2 & 3 & 0\\ 3 & 9 & 6\\ 1 & 15 & 1\end{vmatrix} = 9\begin{vmatrix}2 & 1 & 0\\ 1 & 1 & 2\\ 2 & 5 & 1\end{vmatrix}$$

$$L.H.S = \begin{vmatrix}2 & 3 & 0\\ 3 & 9 & 6\\ 1 & 15 & 1\end{vmatrix}$$

Taking 3 common from R_2.

$$L.H.S = 3\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 2 \\ 1 & 15 & 1 \end{vmatrix}$$

Taking 3 common from R_2.

$$L.H.S = 3.3\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 5 & 1 \end{vmatrix}$$

$$L.H.S = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 5 & 1 \end{vmatrix} = R.H.S$$

(iii).

$$\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$$

$$L.H.S = \begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$$

$$L.H.S = \begin{vmatrix} 3a+l & a & a \\ 3a+l & a+l & a \\ 3a+l & a & a+l \end{vmatrix} C_1 + C_2$$

Taking common $3a+l$ from C_1.

$$L.H.S = (3a+l)\begin{vmatrix} 1 & a & a \\ 1 & a+l & a \\ 1 & a & a+l \end{vmatrix}$$

$$L.H.S = (3a+l)\begin{vmatrix} 1 & a & a \\ 0 & l & 0 \\ 0 & 0 & l \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$L.H.S = (3a+l)\begin{vmatrix} l & 0 \\ 0 & l \end{vmatrix}$$

$$L.H.S = (3a+l)(l^2 - 0)$$

$$L.H.S = l^2(3a+l) = R.H.S$$

(iv).

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$L.H.S = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & xy \end{vmatrix}$$

Multiplying C_1 by x, $C2$ by y and $C3$ by z.
Dividing det. By xyz.

$$L.H.S = \frac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

Taking common xyz from R_3.

$$L.H.S = \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$L.H.S = -1\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$L.H.S = (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$L.H.S = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = R.H.S$$

(v).

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

$$L.H.S = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$L.H.S = \begin{vmatrix} b+c-b-c & a-c-a-c & a-b-a-b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} R_1 - R_2 - R_3$$

$$L.H.S = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$L.H.S = 2\begin{vmatrix} 0 & -c & -b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$L.H.S=2\begin{vmatrix}0 & -c & -b\\ b & a & 0\\ c & 0 & a\end{vmatrix}\begin{matrix}R_2+R_1\\ R_3+R_1\end{matrix}$$

$$L.H.S=2\left\{0\begin{vmatrix}a & 0\\ & a\end{vmatrix}-(-c)\begin{vmatrix}b & 0\\ c & a\end{vmatrix}+(-b)\begin{vmatrix}b & a\\ c & 0\end{vmatrix}\right\}$$

$$L.H.S=2\{c(ba-0)-b(0-ac)\}$$

$$L.H.S=2(abc+abc)=4abc=R.H.S$$

(vi).

$$\begin{vmatrix}b & -1 & a\\ a & b & 0\\ 1 & a & b\end{vmatrix}=a^3+b^3$$

$$L.H.S=\begin{vmatrix}b & -1 & a\\ a & b & 0\\ 1 & a & b\end{vmatrix}$$

$$L.H.S=b\begin{vmatrix}b & 0\\ a & b\end{vmatrix}-(-1)\begin{vmatrix}a & 0\\ 1 & b\end{vmatrix}+a\begin{vmatrix}a & b\\ 1 & a\end{vmatrix}$$

$$L.H.S=b(b^2-0)+1(ab-0)+a(a^2-b)$$

$$L.H.S=b^3+ab+a^3-ab$$

$$L.H.S=a^3+b^3=R.H.S$$

(vii).

$$\begin{vmatrix}r\cos\phi & 1 & -\sin\phi\\ 0 & 1 & 0\\ r\sin\phi & 0 & \cos\phi\end{vmatrix}=r$$

$$L.H.S=\begin{vmatrix}r\cos\phi & 1 & -\sin\phi\\ 0 & 1 & 0\\ r\sin\phi & 0 & \cos\phi\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}r\cos\phi & 1 & -\sin\phi\\ -r\cos\phi & 0 & \sin\phi\\ r\sin\phi & 0 & \cos\phi\end{vmatrix}R_2-R_1$$

Expanding by C_2,

$$L.H.S=-\begin{vmatrix}-r\cos\phi & \sin\phi\\ r\sin\phi & \cos\phi\end{vmatrix}$$

$$L.H.S=-(-r\cos^2\phi-r\sin^2\phi)$$

$$L.H.S=-(-r)(\cos^2\phi+\sin^2\phi)=r=R.H.S$$

(viii).

$$\begin{vmatrix}a & b+c & a+b\\ b & c+a & b+c\\ c & a+b & c+a\end{vmatrix}=a^3+b^3+c^3-3abc$$

$$L.H.S=\begin{vmatrix}a & b+c & a+b\\ b & c+a & b+c\\ c & a+b & c+a\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}a+b+c & b+c & a+b\\ a+b+c & c+a & b+c\\ a+b+c & a+b & c+a\end{vmatrix}C_1+C_2$$

taking common $a+b+c$ from C_1,

$$L.H.S=(a+b+c)\begin{vmatrix}1 & b+c & a+b\\ 1 & c+a & b+c\\ 1 & a+b & c+a\end{vmatrix}$$

$$L.H.S=(a+b+c)\begin{vmatrix}1 & b+c & a+b\\ 0 & a-b & c-a\\ 0 & a-c & c-b\end{vmatrix}\begin{matrix}R_2-R_1\\ R_3-R_1\end{matrix}$$

expanding from C_1,

$$L.H.S=(a+b+c)\begin{vmatrix}a-b & c-a\\ a-c & c-b\end{vmatrix}$$

$$L.H.S=(a+b+c)\{(a-b)(c-b)-(c-a)(a-c)\}$$

$$L.H.S=(a+b+c)\{ac-ab-bc+b^2-(ac-c^2-a^2+ac)\}$$

$$L.H.S=(a+b+c)\{ac-ab-bc+b^2-ac+c^2+a^2-ac\}$$

$$L.H.S=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

$$L.H.S=a^3+b^3+c^3-3abc=R.H.S$$

(ix).

$$\begin{vmatrix}a+\lambda & b & c\\ a & b+\lambda & c\\ a & b & c+\lambda\end{vmatrix}=\lambda^2(a+b+c+\lambda)$$

$$L.H.S=\begin{vmatrix}a+\lambda & b & c\\ a & b+\lambda & c\\ a & b & c+\lambda\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}a+b+c+\lambda & b & c\\ a+b+c+\lambda & b+\lambda & c\\ a+b+c+\lambda & b & c+\lambda\end{vmatrix}C_1+C_2+C_3$$

taking common $a+b+c+\lambda$ from C_1,

$$L.H.S=(a+b+c+\lambda)\begin{vmatrix}1 & b & c\\ 1 & b+\lambda & c\\ 1 & b & c+\lambda\end{vmatrix}$$

$$L.H.S=(a+b+c+\lambda)\begin{vmatrix}1 & b & c\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{vmatrix}\begin{matrix}R_2-R_1\\ R_3-R_1\end{matrix}$$

Expanding by C_1

$$L.H.S=(a+b+c+\lambda)\begin{vmatrix}\lambda & 0\\ 0 & \lambda\end{vmatrix}$$

$$L.H.S=(a+b+c+\lambda)(\lambda^2-0)$$

$$L.H.S=\lambda^2(a+b+c+\lambda)=R.H.S$$

(x).

$$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(a-b)(b-c)(c-a)$$

$$L.H.S=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}0 & 0 & 1\\ a-b & b-c & c\\ a^2-b^2 & b^2-c^2 & c^2\end{vmatrix}\begin{matrix}C_1-C_2\\ C_2-C_3\end{matrix}$$

Expanding by R_1,

$$L.H.S=\begin{vmatrix}a-b & b-c\\ a^2-b^2 & b^2-c^2\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}a-b & b-c\\ (a-b)(a+b) & (b-c)(b+c)\end{vmatrix}$$

$$L.H.S=(a-b)(b-c)\begin{vmatrix}1 & 1\\ a+b & b+c\end{vmatrix}$$

$$L.H.S=(a-b)(b-c)(b+c-a-b)$$

$$L.H.S=(a-b)(b-c)(c-a)=R.H.S$$

(xi).

$$\begin{vmatrix}b+c & a & a^2\\ c+a & b & b^2\\ a+b & c & c^2\end{vmatrix}=(a+b+c)(a-b)(b-c)(c-a)$$

$$L.H.S=\begin{vmatrix}b+c & a & a^2\\ c+a & b & b^2\\ a+b & c & c^2\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}a+b+c & a & a^2\\ a+b+c & b & b^2\\ a+b+c & c & c^2\end{vmatrix}C_1+C_2$$

taking common $a+b+c$ from C_1,

$$L.H.S=(a+b+c)\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

$$L.H.S=(a+b+c)\begin{vmatrix}0 & a-b & a^2-b^2\\ 0 & b-c & b^2-c^2\\ 1 & c & c^2\end{vmatrix}\begin{matrix}R_1-R_2\\ R_2-R_3\end{matrix}$$

Expanding by C_1

$$L.H.S=(a+b+c)\begin{vmatrix}a-b & a^2-b^2\\ b-c & b^2-c^2\end{vmatrix}$$

$$L.H.S=(a+b+c)\begin{vmatrix}a-b & (a-b)(a+b)\\ b-c & (b-c)(b+c)\end{vmatrix}$$

$$L.H.S=(a+b+c)(a-b)(b-c)\begin{vmatrix}1 & a+b\\ 1 & b+c\end{vmatrix}$$

$$L.H.S=(a+b+c)(a-b)(b-c)(b+c-a-b)$$

$$L.H.S=(a+b+c)(a-b)(b-c)(c-a)=R.H.S$$

Q4. If $A=\begin{bmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & -2 & 1\end{bmatrix}$; $B=\begin{bmatrix}5 & -2 & 5\\ 3 & -1 & 4\\ -2 & 1 & -2\end{bmatrix}$ then find,

$$A=\begin{bmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & -2 & 1\end{bmatrix};\ B=\begin{bmatrix}5 & -2 & 5\\ 3 & -1 & 4\\ -2 & 1 & -2\end{bmatrix}$$

(i) A_{12}, A_{32}, A_{23} and $|A|$

As we know that,

$$A_{ij}=(-1)^{i+j}M_{ij}$$

where, M_{ij} are the minor det.

$$A_{12}=(-1)^{1+2}\begin{vmatrix}0 & 0\\ -2 & 1\end{vmatrix}=0\quad A_{22}=(-1)^{2+2}\begin{vmatrix}1 & -3\\ -2 & 1\end{vmatrix}=-5$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1 & -3\\ 0 & 0\end{vmatrix}=0\quad |A|=\begin{vmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & -2 & 1\end{vmatrix}$$

$$|A|=1(-2)-2(0)-3(-4)=10$$

(ii) B_{21}, B_{22}, B_{23} and $|B|$

As we know that,

$$B_{ij}=(-1)^{i+j}M_{ij}$$

where, M_{ij} are the minor det.

$$B=\begin{bmatrix}5 & -2 & 5\\ 3 & -1 & 4\\ -2 & 1 & -2\end{bmatrix}\quad B_{21}=(-1)^{2+1}\begin{vmatrix}-2 & 5\\ 1 & -2\end{vmatrix}=1$$

$$B_{22}=(-1)^{2+2}\begin{vmatrix}5 & 5\\ -2 & -2\end{vmatrix}=0$$

$$B_{23}=(-1)^{2+3}\begin{bmatrix}5 & -2\\ -2 & 1\end{bmatrix}=-1$$

$$|B|=\begin{vmatrix}5 & -2 & 5\\ 3 & -1 & 4\\ -2 & 1 & -2\end{vmatrix}$$

$$|B|=5\begin{vmatrix}-1 & 4\\ 1 & -2\end{vmatrix}+2\begin{vmatrix}3 & 4\\ -2 & -2\end{vmatrix}+5\begin{vmatrix}3 & -1\\ -2 & 1\end{vmatrix}$$

$$|B|=5(-2)+2(2)+5(1)=-1$$

Q5. Without expansion verify that:

(i).

$$\begin{vmatrix}\alpha & \beta+\gamma & 1\\ \beta & \gamma+\alpha & 1\\ \gamma & \alpha+\beta & 1\end{vmatrix}=0$$

$$L.H.S=\begin{vmatrix}\alpha & \beta+\gamma & 1\\ \beta & \gamma+\alpha & 1\\ \gamma & \alpha+\beta & 1\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}\alpha+\beta+\gamma & \beta+\gamma & 1\\ \alpha+\beta+\gamma & \gamma+\alpha & 1\\ \alpha+\beta+\gamma & \alpha+\beta & 1\end{vmatrix}C_1+C_2$$

taking common $\alpha+\beta+\gamma$ from C_1,

$$L.H.S=(\alpha+\beta+\gamma)\begin{vmatrix}1 & \beta+\gamma & 1\\ 1 & \gamma+\alpha & 1\\ 1 & \alpha+\beta & 1\end{vmatrix}$$

$$L.H.S=(\alpha+\beta+\gamma)(0)=0=R.H.S$$

(ii).

$$\begin{vmatrix}1 & 2 & 3x\\ 2 & 3 & 6x\\ 3 & 5 & 9x\end{vmatrix}=0$$

$$L.H.S=\begin{vmatrix}1 & 2 & 3x\\ 2 & 3 & 6x\\ 3 & 5 & 9x\end{vmatrix}$$

taking common $3x$ from C_3,

$$L.H.S=3x\begin{vmatrix}1 & 2 & 1\\ 2 & 3 & 2\\ 3 & 5 & 3\end{vmatrix}$$

$$L.H.S=3x(0)=0=R.H.S$$

(iii).

$$\begin{vmatrix}1 & a^2 & \frac{a}{bc}\\ 1 & b^2 & \frac{b}{ca}\\ 1 & c^2 & \frac{c}{ab}\end{vmatrix}=0$$

$$L.H.S=\begin{vmatrix}1 & a^2 & \frac{a}{bc}\\ 1 & b^2 & \frac{b}{ca}\\ 1 & c^2 & \frac{c}{ab}\end{vmatrix}$$

multiplying C_3 by abc, and dividing det. by abc,

$$L.H.S=\frac{1}{abc}\begin{vmatrix}1 & a^2 & abc.\frac{a}{bc}\\ 1 & b^2 & abc.\frac{b}{ca}\\ 1 & c^2 & abc.\frac{c}{ab}\end{vmatrix}$$

$$L.H.S=\frac{1}{abc}\begin{vmatrix}1 & a^2 & a^2\\ 1 & b^2 & b^2\\ 1 & c^2 & c^2\end{vmatrix}$$

$$L.H.S=\frac{1}{abc}\begin{vmatrix}1 & a^2 & a^2\\ 1 & b^2 & b^2\\ 1 & c^2 & c^2\end{vmatrix}=\frac{1}{abc}(0)$$

$$L.H.S=0=R.H.S$$

(iv).

$$\begin{vmatrix}a-b & b-c & c-a\\ b-c & c-a & a-b\\ c-a & a-b & b-c\end{vmatrix}=0$$

$$L.H.S=\begin{vmatrix}a-b & b-c & c-a\\ b-c & c-a & a-b\\ c-a & a-b & b-c\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}a-b+b-c+c-a & b-c & c-a\\ b-c+c-a+a-b & c-a & a-b\\ c-a+a-b+b-c & a-b & b-c\end{vmatrix}C_1+C_2+C_3$$

$$L.H.S=\begin{vmatrix}0 & b-c & c-a\\ 0 & c-a & a-b\\ 0 & a-b & b-c\end{vmatrix}$$

$$L.H.S=0=R.H.S$$

(v).

$$\begin{vmatrix}bc & ca & ab\\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}\\ a & b & c\end{vmatrix}=0$$

$$L.H.S=\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$$

Multiplying R_2 by $abc,$ and dividing det. by $abc,$

$$L.H.S=\frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ abc.\frac{1}{a} & abc.\frac{1}{b} & abc.\frac{1}{c} \\ a & b & c \end{vmatrix}$$

$$L.H.S=\frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a & b & c \end{vmatrix}$$

$$L.H.S=\frac{1}{abc}(0)=0=R.H.S$$

(vi).

$$\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}=\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

$$L.H.S=\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}$$

multiply R_1 by $l,$ R_2 by m and R_3 by $n,$ also dividing det. by $lmn,$

$$L.H.S=\frac{1}{mnl}\begin{vmatrix} mnl & l^2 & l^3 \\ mnl & m^2 & m^3 \\ mnl & n^2 & n^3 \end{vmatrix}$$

Taking common mnl from C_1,

$$L.H.S=\frac{mnl}{mnl}\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

$$L.H.S=\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}=R.H.S$$

(vii).

$$\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}=0$$

$$L.H.S=\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

taking common 2 from R_1,

$$L.H.S=2\begin{vmatrix} a & b & c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$L.H.S=2\begin{vmatrix} a & b & c \\ b & b & b \\ c & c & c \end{vmatrix}\begin{matrix} R_2-R_1 \\ R_3-R_1 \end{matrix}$$

taking common b from R_2 and c from R_3,

$$L.H.S=2bc\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$L.H.S=2bc(0)=0=R.H.S$$

(viii).

$$\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix}=\begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix}+\begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

$$L.H.S=\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix}=\begin{vmatrix} 7 & 2 & 7-1 \\ 6 & 3 & 5-3 \\ -3 & 5 & -3+4 \end{vmatrix}$$

$$L.H.S=(7-1)\begin{vmatrix} 6 & 3 \\ -3 & 5 \end{vmatrix}-(5-3)\begin{vmatrix} 7 & 2 \\ -3 & 5 \end{vmatrix}+(-3+4)\begin{vmatrix} 7 & 2 \\ 6 & 3 \end{vmatrix}$$

$$L.H.S=\left\{7\begin{vmatrix} 6 & 3 \\ -3 & 5 \end{vmatrix}-5\begin{vmatrix} 7 & 2 \\ -3 & 5 \end{vmatrix}+(-3)\begin{vmatrix} 7 & 2 \\ 6 & 3 \end{vmatrix}\right\}$$

$$+\left\{-1\begin{vmatrix} 6 & 3 \\ -3 & 5 \end{vmatrix}-(-3)\begin{vmatrix} 7 & 2 \\ -3 & 5 \end{vmatrix}+4\begin{vmatrix} 7 & 2 \\ 6 & 3 \end{vmatrix}\right\}$$

$$L.H.S=\begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix}+\begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

(ix).

$$\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}=0$$

$$L.H.S=\begin{vmatrix}-a & 0 & c\\ 0 & a & -b\\ b & -c & 0\end{vmatrix}$$

Multiply R_1 by b, R_2 by c and R_3 by a, also dividing det. By abc,

$$L.H.S=\frac{1}{abc}\begin{vmatrix}-ab & 0 & bc\\ 0 & ac & -bc\\ ab & -ac & 0\end{vmatrix}$$

$$L.H.S=\frac{1}{abc}\begin{vmatrix}-ab+0+ab & 0+ac-ac & bc-bc+0\\ 0 & ac & -bc\\ ab & -ac & 0\end{vmatrix} R_1+R_2+R_3$$

$$L.H.S=\frac{1}{abc}\begin{vmatrix}0 & 0 & 0\\ 0 & ac & -bc\\ ab & -ac & 0\end{vmatrix}$$

$$L.H.S=\frac{1}{abc}(0)=0=R.H.S$$

Q6. Find the values of x if,

(i).

$$\begin{vmatrix}3 & 1 & x\\ -1 & 3 & 4\\ x & 1 & 0\end{vmatrix}=-30$$

Expanding by R_1,

$$3\begin{vmatrix}3 & 4\\ 1 & 0\end{vmatrix}-1\begin{vmatrix}-1 & 4\\ x & 0\end{vmatrix}+x\begin{vmatrix}-1 & 3\\ x & 1\end{vmatrix}=-30$$

$$3(0-4)-1(0-4x)+x(-1-3x)=-30$$

$$-12+4x-x-3x^2=-30$$

$$-3x^2+3x+18=0$$

$$x^2-x-6=0$$

$$x^2-3x+2x-6=0$$

$$x(x-3)+2(x-3)=0$$

$$(x-3)(x+2)=0$$

$$x-3=0;\ x+2=0$$

$$x=3;\ x=-2$$

$$x=\{(-2,3)\}$$

(ii).

$$\begin{vmatrix}1 & x-1 & 3\\ -1 & x+1 & 2\\ 2 & -2 & x\end{vmatrix}=0$$

Expanding by R_1,

$$1\begin{vmatrix}x+1 & 2\\ -2 & x\end{vmatrix}-(x-1)\begin{vmatrix}-1 & 2\\ 2 & x\end{vmatrix}+3\begin{vmatrix}-1 & x+1\\ 2 & -2\end{vmatrix}=0$$

$$1(x(x+1)-2(-2))-(x-1)(-x-4)+3(2-2(x+1))=0$$

$$1(x^2+x+4)-(-x^2-4x+x+4)+3(2-2x-2)=0$$

$$x^2+x+4+x^2+4x-x-4+6-6x-6=0$$

$$2x^2-2x=0$$

$$2x(x-1)=0$$

$$2x=0;\ x-1=0$$

$$x=0;\ x=1$$

$$x=\{(0,1)\}$$

(iii).

$$\begin{vmatrix}1 & 2 & 1\\ 2 & x & 2\\ 3 & 6 & x\end{vmatrix}=0$$

Expanding by R_1,

$$1\begin{vmatrix}x & 2\\ 6 & x\end{vmatrix}-2\begin{vmatrix}2 & 2\\ 3 & x\end{vmatrix}+1\begin{vmatrix}2 & x\\ 3 & 6\end{vmatrix}=0$$

$$1(x^2-12)-2(2x-6)+1(12-3x)=0$$

$$x^2-12-4x+12+12-3x=0$$

$$x^2-7x+12=0$$

$$x^2-4x-3x+12=0$$

$$x(x-4)-3(x-4)=0$$

$$(x-4)(x-3)=0$$

$$x-4=0\ :\ x-3=0$$

$$x=4\ :\ x=3$$

$$x=\{(3,4)\}$$

Q6. Evaluate the followings:

(i).

$$\begin{vmatrix}3 & 4 & 2 & 7\\ 2 & 5 & 0 & 3\\ 1 & 2 & -3 & 5\\ 4 & 1 & -2 & 6\end{vmatrix}$$

$$=\begin{vmatrix}0 & -2 & 11 & -8\\ 0 & 1 & 6 & -7\\ 1 & 2 & -3 & 5\\ 0 & -7 & 10 & -14\end{vmatrix}\begin{matrix}R_1-3R_3\\ R_2-2R_3\\ R_4-4R_3\end{matrix}$$

$$=\begin{vmatrix}-2 & 11 & -8\\ 1 & 6 & -7\\ -7 & 10 & -14\end{vmatrix}$$

$$=-2\begin{vmatrix}6 & -7\\ 10 & -14\end{vmatrix}-11\begin{vmatrix}1 & -7\\ -7 & -14\end{vmatrix}+(-8)\begin{vmatrix}1 & 6\\ -7 & 10\end{vmatrix}$$

$$=-2(-84+70)-11(-14-49)-8(10+42)$$

$$=-2(-14)-11(-63)-8(52)$$

$$=28+693-416=305$$

(ii).

$$\begin{vmatrix}2&3&1&-1\\4&0&2&1\\5&2&-1&6\\3&-7&2&-2\end{vmatrix}$$

$$=\begin{vmatrix}2&3&1&-1\\0&-6&0&3\\7&5&0&5\\-1&-13&0&0\end{vmatrix}\begin{matrix}R_2-2R_1\\R_3+R_1\\R_4-2R_1\end{matrix}$$

$$=\begin{vmatrix}0&-6&3\\7&5&5\\-1&-13&0\end{vmatrix}$$

$$=0\begin{vmatrix}5&5\\-13&0\end{vmatrix}-(-6)\begin{vmatrix}7&5\\-1&0\end{vmatrix}+3\begin{vmatrix}7&5\\-1&-13\end{vmatrix}$$

$$=0+6(0+5)+3(-91+5)=6(5)+3(-86)=-228$$

(iii).

$$\begin{vmatrix}-3&9&1&1\\0&3&-1&2\\9&7&-1&1\\-2&0&1&-1\end{vmatrix}$$

$$=\begin{vmatrix}-3&9&1&1\\-3&12&0&3\\6&16&0&2\\1&-9&0&-2\end{vmatrix}\begin{matrix}R_2+R_1\\R_3+R_1\\R_4-R_1\end{matrix}$$

$$=\begin{vmatrix}-3&12&3\\6&16&2\\1&-9&-2\end{vmatrix}$$

$$=-3\begin{vmatrix}16&2\\-9&-2\end{vmatrix}-12\begin{vmatrix}6&2\\1&-2\end{vmatrix}+3\begin{vmatrix}6&16\\1&-9\end{vmatrix}$$

$$=-3(-14)-12(-14)+3(70)$$

$$=-3(-14)-12(-14)+3(-70)=0$$

Q8. Show that,

$$\begin{vmatrix}x&1&1&1\\1&x&1&1\\1&1&x&1\\1&1&1&x\end{vmatrix}=(x+3)(x-1)^3$$

$$L.H.S=\begin{vmatrix}x&1&1&1\\1&x&1&1\\1&1&x&1\\1&1&1&x\end{vmatrix}$$

$$L.H.S=\begin{vmatrix}x+3&1&1&1\\x+3&x&1&1\\x+3&1&x&1\\x+3&1&1&x\end{vmatrix}C_1+C_2+C_3+C_4$$

$$L.H.S=(x+3)\begin{vmatrix}1&1&1&1\\1&x&1&1\\1&1&x&1\\1&1&1&x\end{vmatrix}$$

$$L.H.S=(x+3)\begin{vmatrix}1&1&1&1\\0&x-1&0&0\\0&0&x-1&0\\0&0&0&x-1\end{vmatrix}\begin{matrix}R_2-R_1\\R_3-R_1\\R_4-R_1\end{matrix}$$

$$L.H.S=(x+3)\begin{vmatrix}x-1&0&0\\0&x-1&0\\0&0&x-1\end{vmatrix}$$

$$L.H.S=(x+3)(x-1)^3\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}$$

$$L.H.S=(x+3)(x-1)^3(1)$$

$$L.H.S=(x+3)(x-1)^3=R.H.S$$

Q9. Find $|AA^t|$ and $|A^tA|$ if,

(i)

$$A=\begin{bmatrix}3&2&-1\\2&1&3\end{bmatrix}$$

$$A^t=\begin{bmatrix}3&2\\2&1\\-1&3\end{bmatrix}$$

Consider,

$$AA^t=\begin{bmatrix}3&2&-1\\2&1&3\end{bmatrix}\begin{bmatrix}3&2\\2&1\\-1&3\end{bmatrix}$$

$$AA^t=\begin{bmatrix}9+4+1&6+2-3\\6+2-3&4+1+9\end{bmatrix}$$

$$AA^t=\begin{bmatrix}14 & 5\\ 5 & 14\end{bmatrix}$$

$$|AA^t|=\begin{vmatrix}14 & 5\\ 5 & 14\end{vmatrix}$$

$$|AA^t|=14(14)-5(5)=171$$

Now,

$$A^tA=\begin{bmatrix}3 & 2\\ 2 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}3 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}$$

$$A^tA=\begin{bmatrix}9+4 & 6+2 & -3+6\\ 6+2 & 4+1 & -2+3\\ -3+6 & -2+3 & 1+9\end{bmatrix}$$

$$A^tA=\begin{bmatrix}13 & 8 & 3\\ 8 & 5 & 1\\ 3 & 1 & 10\end{bmatrix}$$

$$|A^tA|=\begin{vmatrix}13 & 8 & 3\\ 8 & 5 & 1\\ 3 & 1 & 10\end{vmatrix}$$

$$|A^tA|=13\begin{vmatrix}5 & 1\\ 1 & 10\end{vmatrix}-8\begin{vmatrix}8 & 1\\ 3 & 10\end{vmatrix}+3\begin{vmatrix}8 & 5\\ 3 & 1\end{vmatrix}$$

$$|A^tA|=13(49)-8(77)+3(-7)=0$$

(ii).

$$A=\begin{bmatrix}3 & 4\\ 2 & 1\\ 1 & 1\\ 2 & 3\end{bmatrix}$$

$$A^t=\begin{bmatrix}3 & 2 & 1 & 2\\ 4 & 1 & 1 & 3\end{bmatrix}$$

$$AA^t=\begin{bmatrix}3 & 4\\ 2 & 1\\ 1 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 2 & 1 & 2\\ 4 & 1 & 1 & 3\end{bmatrix}$$

$$AA^t=\begin{bmatrix}25 & 10 & 7 & 18\\ 10 & 5 & 3 & 7\\ 7 & 3 & 2 & 5\\ 8 & 7 & 5 & 13\end{bmatrix}$$

$$|AA^t|=\begin{vmatrix}25 & 10 & 7 & 18\\ 10 & 5 & 3 & 7\\ 7 & 3 & 2 & 5\\ 8 & 7 & 5 & 13\end{vmatrix}$$

$$|AA^t|=\begin{vmatrix}25 & 10 & 7 & 18\\ 5 & 5 & 3 & 7\\ 4 & 3 & 2 & 5\\ 1 & 7 & 5 & 13\end{vmatrix}C_1-C_2$$

$$|AA^t|=\begin{vmatrix}0 & -165 & -118 & -307\\ 0 & -30 & -22 & -58\\ 0 & -25 & -18 & -47\\ 1 & 7 & 5 & 13\end{vmatrix}\begin{matrix}R_1-15R_4\\ R_2-5R_4\\ R_3-4R_4\end{matrix}$$

$$|AA^t|=-\begin{vmatrix}-165 & -118 & -307\\ -30 & -22 & -58\\ -25 & -18 & -47\end{vmatrix}$$

$$|AA^t|=-165\begin{vmatrix}-22 & -58\\ -18 & -47\end{vmatrix}-(-118)\begin{vmatrix}-30 & -58\\ -25 & -47\end{vmatrix}-307\begin{vmatrix}-30 & -22\\ -25 & -18\end{vmatrix}$$

$$|AA^t|=-165(-10)-(-118)(-40)-307(-10)=0$$

Now,

$$A^tA=\begin{bmatrix}3 & 2 & 1 & 2\\ 4 & 1 & 1 & 3\end{bmatrix}\begin{bmatrix}3 & 4\\ 2 & 1\\ 1 & 1\\ 2 & 3\end{bmatrix}$$

$$A^tA=\begin{bmatrix}9+4+1+4 & 12+2+1+6\\ 12+2+1+6 & 16+1+1+9\end{bmatrix}=\begin{bmatrix}18 & 21\\ 21 & 27\end{bmatrix}$$

$$|A^tA|=\begin{vmatrix}18 & 21\\ 21 & 27\end{vmatrix}$$

$$|A^tA|=18(27)-21(21)$$

$$|A^tA|=18(27)-21(21)=45$$

Q10. If A is a square matrix of order 3 then show that $|kA|=k^3|A|$.

Let,

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$kA=k\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}ka_{11} & ka_{12} & ka_{13}\\ ka_{21} & ka_{22} & ka_{23}\\ ka_{31} & ka_{32} & ka_{33}\end{bmatrix}$$

$$|kA|=\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

$$|kA|=k.k.k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|kA|=k^3|A|$$

Hence, proved.

Q11. Find the value of λ if A and B are singular:

$$A=\begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$$

$$|A|=\begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix}$$

$$|A|=4(3-18)-\lambda(7-12)+3(21-6)=-60+5\lambda+45$$

$$|A|=-15+5\lambda$$

Since A is singular so,

$$|A|=0$$

$$-15+5\lambda=0\Rightarrow\lambda=3$$

$$B=\begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$$

$$|B|=\begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix}$$

$$|B|=\begin{vmatrix} 5 & 1 & 2 & 0 \\ 5 & 0 & 5 & 0 \\ 3 & 2 & 0 & 1 \\ -7 & \lambda-6 & -1 & 0 \end{vmatrix}\begin{matrix} \\ R_2-R_3 \\ R_4-3R_3 \\ \\ \end{matrix}$$

$$|B|=\begin{vmatrix} 5 & 1 & 2 \\ 5 & 0 & 5 \\ -7 & \lambda-6 & -1 \end{vmatrix}$$

$$|B|=5(0-5(\lambda-6))-1(-5+35)+2(5(\lambda-6)-0)$$

$$|B|=-25(\lambda-6)-1(30)+10(\lambda-6)=-25\lambda+150-30+10\lambda-60$$

Since B is singular so,

$$|B|=0$$

$$-15\lambda+60=0\Rightarrow\lambda=4$$

Q12. Which of the following matrices are singular and which of them are non-singular:

(i).

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

Let,

$$A=\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

to check the A is singular we have,

$$|A|=\begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$$

$$|A|=1(6)-0(12)+3(6)=21\neq0$$

So A is non-singular.

(ii).

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$

Let,

$$A=\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$

to check the A is singular we have,

$$|A|=\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$|A|=2(5)-3(5)-1(-5)=0$$

So A is singular.

(iii).

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{bmatrix}$$

Let,

$$A=\begin{bmatrix}1 & 1 & 2 & -1\\ 1 & 2 & -1 & -3\\ 2 & 3 & 1 & 2\\ 3 & -1 & 3 & 4\end{bmatrix}$$

to check the A is singular we have,

$$|A|=\begin{vmatrix}1 & 1 & 2 & -1\\ 1 & 2 & -1 & -3\\ 2 & 3 & 1 & 2\\ 3 & -1 & 3 & 4\end{vmatrix}$$

$$|A|=\begin{vmatrix}1 & 1 & 2 & -1\\ 0 & 1 & -3 & -2\\ 0 & 1 & -3 & 4\\ 0 & -4 & -3 & 7\end{vmatrix}\begin{matrix}R_2-R_1\\ R_3-2R_1\\ R_4-3R_1\end{matrix}$$

$$|A|=\begin{vmatrix}1 & -3 & -2\\ 1 & -3 & 4\\ -4 & -3 & 7\end{vmatrix}$$

$$|A|=1(-9)+3(23)-2(-15)=90\neq 0$$

So, A is non-singular.

Q13. Find the inverse of $A=\begin{bmatrix}2 & 1 & 0\\ 1 & -1 & 3\\ 2 & -4 & 1\end{bmatrix}$ and show that $A^{-1}A=I_3$.

(i).

$$A=\begin{bmatrix}2 & 1 & 0\\ 1 & -1 & 3\\ 2 & -4 & 1\end{bmatrix}$$

As we know that

$$A^{-1}=\frac{adjA}{|A|}$$

$$|A|=\begin{vmatrix}2 & 1 & 0\\ 1 & -1 & 3\\ 2 & -4 & 1\end{vmatrix}$$

$$|A|=2(11)-1(-5)+0(-2)=27\neq 0$$

Hence, A is singular so its inverse exist,

$$adjA=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}^t$$

$$a_{11}=(-1)^{1+1}\begin{vmatrix}-1 & 3\\ -4 & 1\end{vmatrix}=11 \qquad a_{12}=(-1)^{1+2}\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}=5$$

$$a_{13}=(-1)^{1+3}\begin{vmatrix}1 & -1\\ 2 & -4\end{vmatrix}=-2 \qquad a_{21}=(-1)^{2+1}\begin{vmatrix}1 & 0\\ -4 & 1\end{vmatrix}=-1$$

$$a_{22}=(-1)^{2+2}\begin{vmatrix}2 & 0\\ 2 & 1\end{vmatrix}=2 \qquad a_{23}=(-1)^{2+3}\begin{vmatrix}2 & 1\\ 2 & -4\end{vmatrix}=10$$

$$a_{31}=(-1)^{3+1}\begin{vmatrix}1 & 0\\ -1 & 3\end{vmatrix}=3 \qquad a_{32}=(-1)^{3+2}\begin{vmatrix}2 & 0\\ -1 & 3\end{vmatrix}=-6$$

$$a_{33}=(-1)^{3+3}\begin{vmatrix}2 & 1\\ -1 & -1\end{vmatrix}=-3$$

$$adjA=\begin{bmatrix}11 & 5 & -2\\ -1 & 2 & 10\\ 3 & -6 & -3\end{bmatrix}^t=\begin{bmatrix}11 & -1 & 3\\ 5 & 2 & -6\\ -2 & 10 & -3\end{bmatrix}$$

Putt values,

$$A^{-1}=\frac{\begin{bmatrix}11 & -1 & 3\\ 5 & 2 & -6\\ -2 & 10 & -3\end{bmatrix}}{27}$$

Now,

$$A^{-1}A=\frac{1}{27}\begin{bmatrix}11 & -1 & 3\\ 5 & 2 & -6\\ -2 & 10 & -3\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & -1 & 3\\ 2 & -4 & 1\end{bmatrix}$$

$$A^{-1}A=\frac{1}{27}\begin{bmatrix}22-1+6 & 11+1-12 & 0-3+3\\ 10+2-12 & 5-2+24 & 0+6-6\\ -4+10-6 & -2-10+12 & 0+30-3\end{bmatrix}$$

$$A^{-1}A=\frac{1}{27}\begin{bmatrix}27 & 0 & 0\\ 0 & 27 & 0\\ 0 & & 27\end{bmatrix}$$

$$A^{-1}A=\frac{27}{27}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & & 1\end{bmatrix}=I_3$$

Hence, proved.

Q14. Verify that $(AB)^{-1}=B^{-1}A^{-1}$ if,

(i).

$$A=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix};\ B=\begin{bmatrix}-3 & 1\\ 4 & -1\end{bmatrix}$$

we have to show that, $(AB)^{-1}=B^{-1}A^{-1}$,

Consider,

$$AB=\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}-3 & 1\\ 4 & -1\end{bmatrix}=\begin{bmatrix}-3+8 & 1-2\\ 3+0 & -1-0\end{bmatrix}$$

$$AB=\begin{bmatrix}5 & -1\\ 3 & -1\end{bmatrix}$$

and,

$$(AB)^{-1}=\frac{adj(AB)}{|AB|}$$

$$|AB|=\begin{vmatrix}5 & -1\\ 3 & -1\end{vmatrix}=-2$$

$$adj(AB)=\begin{bmatrix}-1 & 1\\ -3 & 5\end{bmatrix}$$

$$(AB)^{-1}=\frac{\begin{bmatrix}-1 & 1\\ -3 & 5\end{bmatrix}}{-2}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{3}{2} & -\frac{5}{2}\end{bmatrix}$$

Now,

$$A^{-1}=\frac{adjA}{|A|};\ B^{-1}=\frac{adjB}{|B|}$$

$$|A|=\begin{vmatrix}1 & 2\\ -1 & 0\end{vmatrix}=2;\ |B|=\begin{vmatrix}-3 & 1\\ 4 & -1\end{vmatrix}=-1$$

$$adjA=\begin{bmatrix}0 & -2\\ 1 & 1\end{bmatrix};\ adjB=\begin{bmatrix}-1 & -1\\ -4 & -3\end{bmatrix}$$

putt values,

$$A^{-1}=\frac{\begin{bmatrix}0 & -2\\ 1 & 1\end{bmatrix}}{2};\ B^{-1}=\frac{\begin{bmatrix}-1 & -1\\ -4 & -3\end{bmatrix}}{-1}$$

$$B^{-1}A^{-1}=\frac{\begin{bmatrix}-1 & -1\\ -4 & -3\end{bmatrix}}{-1}\frac{\begin{bmatrix}0 & -2\\ 1 & 1\end{bmatrix}}{2}=\frac{\begin{bmatrix}-1 & 1\\ -3 & 5\end{bmatrix}}{-2}$$

$$B^{-1}A^{-1}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{3}{2} & -\frac{5}{2}\end{bmatrix}$$

Hence, proved.

Q15. Verify that $(AB)^t = B^tA^t$ if,

$$A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix};\ B=\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$$

we have to show that, $(AB)^t=B^tA^t$,

$$AB=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}=\begin{bmatrix}1-3+0 & 1-2-2\\ 0+9+0 & 0+6-1\end{bmatrix}$$

$$AB=\begin{bmatrix}-2 & -3\\ 9 & 5\end{bmatrix}\Rightarrow (AB)^t=\begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}$$

Now,

$$A^t=\begin{bmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{bmatrix};\ B^t=\begin{bmatrix}1 & 3 & 0\\ 1 & 0 & -1\end{bmatrix}$$

$$B^tA^t=\begin{bmatrix}1 & 3 & 0\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{bmatrix}$$

$$B^tA^t=\begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}$$

Hence, proved.

Q16. If $A=\begin{bmatrix}2 & -1\\ 3 & 1\end{bmatrix}$ then verify that $(A^{-1})^t=(A^t)^{-1}$.

$$|A|=\begin{vmatrix}2 & -1\\ 3 & 1\end{vmatrix}=5$$

$$adjA=\begin{bmatrix}1 & 1\\ -3 & 2\end{bmatrix}$$

$$A^{-1}=\frac{1}{5}\begin{bmatrix}1 & 1\\ -3 & 2\end{bmatrix}\Rightarrow (A^{-1})^t=\frac{1}{5}\begin{bmatrix}1 & -3\\ 1 & 2\end{bmatrix}$$

Now,

$$A^t=\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}\Rightarrow |A^t|=\begin{vmatrix}2 & 3\\ -1 & 1\end{vmatrix}=5$$

$$adjA^t=\begin{bmatrix}1 & -3\\ 1 & 2\end{bmatrix}$$

$$(A^t)^{-1}=\frac{1}{5}\begin{bmatrix}1 & -3\\ 1 & 2\end{bmatrix}$$

Hence, proved.

Q17. If A and B are non-singular matrices. Then show that,

(i) $(AB)^{-1}=B^{-1}A^{-1}$

Given that A and B are non-singular matrices then the inverse of A and B are exists.
Consider,

$$(AB)(B^{-1}A^{-1})=A(BB^{-1}A^{-1})$$
$$(AB)(B^{-1}A^{-1})=A(BB^{-1})A^{-1}$$
$$(AB)(B^{-1}A^{-1})=AIA^{-1}$$
$$(AB)(B^{-1}A^{-1})=AA^{-1}$$
$$(AB)(B^{-1}A^{-1})=I$$

Now, consider,

$$(B^{-1}A^{-1})(AB)=B^{-1}(A^{-1}A)B$$

$$(B^{-1}A^{-1})(AB) = B^{-1}IB$$
$$(B^{-1}A^{-1})(AB) = B^{-1}B$$
$$(B^{-1}A^{-1})(AB) = I$$
hence, AB is the inverse of $B^{-1}A^{-1}$, so,
$$(AB)^{-1} = B^{-1}A^{-1}$$
(ii). $(A^{-1})^{-1} = A$

Given that A is non-singular so, A^{-1} exist, consider,
$$A^{-1}A = I$$
and,
$$AA^{-1} = 1$$
which shows that A^{-1} is the inverse of A, then,
$$(A^{-1})^{-1} = A$$