# Exercise 4.5

**Polynomial function:-** A polynomial in $x$ is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad \text{— (1)}$$

where $n$ is a non-negative integer and $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$.

If $a_n \neq 0$, then (1) is a polynomial of degree $n$ and $a_n$ is leading coefficient.

**Rule** dividend = (divisor)(quotient) + remainder.

**Remainder Theorem** If a polynomial $f(x)$ is divided by $x - a$, then

$$\text{Remainder} = f(a).$$

**Factor Theorem** $x - a$ is a factor of the polynomial $f(x)$ if and only if

$$f(a) = 0.$$

Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial.

1. $x^2+3x+7$, $x+1$

Let $f(x) = x^2+3x+7$, Let $x+1=0$

$x=-1$.

$$\text{Remainder} = f(-1) = (-1)^2+3(-1)+7$$
$$= 1-3+7$$
$$= 5$$

Ans.

2- $x^3-x^2+5x+4$, $x-2$

Let $f(x) = x^3-x^2+5x+4$, let $x-2=0$

$x=2$

$$\text{Remainder} = f(2) = 2^3-2^2+5(2)+4$$
$$= 8-\cancel{4}+10+\cancel{4} = 18$$

Ans.

3. $3x^4+4x^3+x-5$, $x+1$

Let $f(x) = 3x^4+4x^3+x-5$, let $x+1=0$

$x=-1$

So $\text{Remainder} = f(-1) = 3(-1)^4+4(-1)^3+(-1)-5$
$$= 3-4-1-5$$
$$= -7$$

Ans.

4. $x^3-2x^2+3x+3$, $x-3$

Let $f(x) = x^3-2x^2+3x+3$, let $x-3=0$

$x=3$

So $\text{Remainder} = f(3) = 3^3-2(3)^2+3(3)+3$
$$= 27-18+9+3$$
$$= 21$$

Ans.

Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.

5. $x-1$ , $x^2+4x-5$

Remainder $f(a) = 0$

$\Rightarrow$ $x-a$ is a factor of $f(x)$.

Let $f(x) = x^2 + 4x - 5$ ✓ , let $x-1=0$

$x=1$

Remainder $= f(1) = 1^2 + 4(1) - 5$

$= 1 + 4 - 5 = 0$

So $x-1$ is a factor of $x^2+4x-5$.

6. $x-2$ , $x^3+x^2-7x+1$

Let $f(x) = x^3 + x^2 - 7x + 1$ , let $x-2=0$

$x=2$

Remainder $= f(2) = 2^3 + 2^2 - 7(2) + 1$

$= 8 + 4 - 14 + 1$

$= -1 \neq 0$

So $x-2$ is not a factor of $x^3+x^2-7x+1$.

7. $w+2$ , $2w^3+w^2-4w+7$

Let $f(w) = 2w^3 + w^2 - 4w + 7$ , let $w+2=0$

$w=-2$

Remainder $= f(-2) = 2(-2)^3 + (-2)^2 - 4(-2) + 7$

$= 2(-8) + 4 + 8 + 7 = -16 + 4 + 8 + 7 = 3 \neq 0$

So $w+2$ is not a factor of $2w^3+w^2-4w+7$.

8. $x-a$, $x^n - a^n$ , n is a positive integer.

Let $f(x) = x^n - a^n$ ✓, let $x-a=0$

$x=a$

Remainder $= f(a) = a^n - a^n = 0$

So $x-a$ is a factor of $x^n-a^n$.

9. $x+a$, $x^n+a^n$, $n$ is an odd integer.

Let $f(x) = x^n + a^n$ ; let $x+a=0$

$x=-a$

Remainder $= f(-a) = (-a)^n + a^n$ $\qquad (-)^{odd} = -$

$= -a^n + a^n$

$= 0$

So $x+a$ is a factor of $x^n+a^n$.

10. When $x^4+2x^3+kx^2+3$ is divided by $x-2$, the remainder is 1. Find the value of $k$.

Let $f(x) = x^4+2x^3+kx^2+3$ Given Remainder = 1.

Let $x-2=0$

$x=2$

So Remainder $= f(2) = 2^4+2(2)^3+k(2)^2+3$

$1 = 16+2(8)+k(4)+3$

$1 = 16+16+4k+3$

$1 = 35+4k$

$1-35 = 4k$

$-34 = 4k$

$-\frac{\cancel{34}^{17}}{\cancel{4}_{2}} = k$

$k = -\frac{17}{2}$ Ans.

11. When the polynomial $x^3+2x^2+kx+4$ is divided by $x-2$, the remainder is 14. Find the value of $k$.

Let $f(x) = x^3+2x^2+kx+4$, Given Remainder = 14

Let $x-2=0$

$x=2$

So Remainder $= f(2) = 2^3+2(2)^2+k(2)+4$

$$14 = 8+2(4)+2k+4$$

$$14 = 8+8+2k+4$$

$$14 = 20+2k$$

$$14-20 = 2k$$

$$-6 = 2k$$

$$k = -3$$ Ans

Use synthetic division to show that $x$ is the solution of the polynomial and use the result to factorize the polynomial completely.

12. $1x^3-7x+6=0$, $x=2$.

$$\begin{array}{c|cccc} 2 & 1 & 0 & -7 & 6 \\ & & 2 & 4 & -6 \\ \hline & 1 & 2 & -3 & \boxed{0} \end{array}$$ → remainder.

Since remainder = 0, so $x=2$ is a solution of $x^3-7x+6=0$.

Here $x^3-7x+6 = (x-2)(x^2+2x-3)$

$= (x-2)(x^2+3x-x-3)$

$= (x-2)(x(x+3)-1(x+3))$

$= (x-2)(x-1)(x+3)$

13. $x^3-28x-48=0$ , $x=-4$

$$\begin{array}{r|rrrr}
-4 & 1 & 0 & -28 & -48 \\
 & & -4 & 16 & 48 \\
\hline
 & 1 & -4 & -12 & \boxed{0}
\end{array}$$

Since remainder $=0$, so $x=-4$ is a solution of $x^3-28x-48=0$.

$$x^3-28x-48=(x+4)(x^2-4x-12)$$
$$=(x+4)(x^2-6x+2x-12)$$
$$=(x+4)[x(x-6)+2(x-6)]$$
$$=(x+4)(x+2)(x-6).$$

14. $2x^4+7x^3-4x^2-27x-18=0$ , $x=2$, $x=-3$.

$$\begin{array}{r|rrrrr}
2 & 2 & 7 & -4 & -27 & -18 \\
 & & 4 & 22 & 36 & 18 \\
\hline
-3 & 2 & 11 & +18 & 9 & \boxed{0} \\
 & & -6 & -15 & -9 & \\
\hline
 & 2 & 5 & 3 & \boxed{0} &
\end{array}$$

So $x=2$ and $x=-3$ are solutions of $2x^4+7x^3-4x^2-27x-18=0$.

$$2x^4+7x^3-4x^2-27x-18=(x-2)(x+3)(2x^2+5x+3)$$
$$=(x-2)(x+3)[2x^2+3x+2x+3]$$
$$=(x-2)(x+3)[x(2x+3)+1(2x+3)]$$
$$=(x-2)(x+3)(x+1)(2x+3).$$

15. Use synthetic division to find the values of $p$ and $q$ if $x+1$ and $x-2$ are the factors of $x^3+px^2+qx+6$.

Let $f(x) = x^3+px^2+qx+6$

Let $x+1=0$ and $x-2=0$

$x=-1$ and $x=2$

$$\begin{array}{r|rrrr}
-1 & 1 & p & q & 6 \\
 & & -1 & -p+1 & p-q-1 \\
\hline
2 & 1 & p-1 & -p+q+1 & \boxed{p-q+5} \\
 & & 2 & 2p+2 & \\
\hline
 & 1 & p+1 & \boxed{p+q+3} &
\end{array}$$

Since $x+1$ and $x-2$ are factors, so

$p-q+5=0$ and $p+q+3=0$

$p=q-5$

Put $p=q-5$

$q-5+q+3=0$

$2q-2=0$

$2q=2$

$\boxed{q=1}$

Put $q=1$

$p=1-5$

$\boxed{p=-4}$

16- Find the values of $a$ and $b$ if $-2$ and $2$ are the roots of the polynomial $x^3-4x^2+ax+b$.

Let $f(x) = x^3 - 4x^2 + ax + b$ $\quad x=-2, \quad x=2$

| | | | | |
|---|---|---|---|---|
| $-2$ | 1 | $-4$ | $a$ | $b$ |
| | | $-2$ | $12$ | $-2a-24$ |
| $2$ | 1 | $-6$ | $a+12$ | $\boxed{-2a+b-24}$ |
| | | $2$ | $-8$ | |
| | 1 | $-4$ | $\boxed{a+4}$ | |

Since $-2$ and $2$ are roots, so

$-2a+b-24=0$ and $a+4=0$

$\boxed{a=-4}$

Put $a=-4$

$-2(-4)+b-24=0$

$8+b-24=0$

$b-16=0$

$\boxed{b=16}$

2nd Method.

Let $f(x) = x^3 - 4x^2 + ax + b$ $\quad x=-2, \quad x=2.$

$f(-2)=0$ and $f(2)=0$

$(-2)^3-4(-2)^2+a(-2)+b=0$

$-8-16-2a+b=0$

$-24-2a+b=0$

$-2a+b=24 \quad -①$

$2^3-4(2)^2+a(2)+b=0$

$8-16+2a+b=0$

$-8+2a+b=0$

$2a+b=8 \quad -②$

Add ① and ②

$b+b=24+8$

$2b=32$

$\boxed{b=16}$

Put in ①

$-2a+16=24$

$-2a=8$

$\boxed{a=-4}$