1 Evaluate the following integrals:

$$\int \frac{-2x}{\sqrt{4-x^2}}\,dx$$

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UCADEMY

Power Rule

$$\int f^n(x)\cdot f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c,\quad n \neq -1$$

$$\int \frac{-2x}{\sqrt{4-x^2}}\,dx$$

$$= \int (4-x^2)^{-1/2}(-2x)\,dx$$

$$f(x) = 4-x^2$$

$$f'(x) = -2x$$

$$= \frac{(4-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c.$$

$$= \frac{(4-x^2)^{1/2}}{\frac{1}{2}} + c.$$

$$= 2\sqrt{4-x^2} + c.$$

2 Evaluate the following integrals:

$$\int \frac{dx}{x^2+4x+13}$$

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UCADEMY

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\mathrm{Tan}^{-1}\left(\frac{x}{a}\right)+c$$

$$\int \frac{dx}{x^2+4x+13}$$

$$= \int \frac{dx}{(x^2+4x+4)+9} = \int \frac{dx}{(x+2)^2+3^2}$$

Let $u = x+2$ ✓

$du = dx$ ✓

$$= \int \frac{du}{u^2+3^2}$$

$$= \frac{1}{3}\mathrm{Tan}^{-1}\frac{u}{3} + C$$

$$= \frac{1}{3}\mathrm{Tan}^{-1}\left(\frac{x+2}{3}\right) + C$$

3 Evaluate the following integrals:

$$\int \frac{x^2}{4+x^2}\,dx$$

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Ex 3.3

UCADEMY

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\text{Tan}^{-1}\left(\frac{x}{a}\right)+c$$

$$\int \frac{x^2}{4+x^2}\,dx$$

$$= \int \frac{x^2}{x^2+4}\,dx$$

$$\begin{array}{r|l} & \quad 1 \\ x^2+4 & \cancel{x^2} \\ & \pm\cancel{x^2} \pm 4 \\ \hline & \quad -4 \end{array}$$

$$= \int \left(1 - \frac{4}{x^2+4}\right) dx$$

$$= \int dx - 4\int \frac{dx}{x^2+4}$$

$$= \int dx - 4\int \frac{dx}{x^2+2^2}$$

$$= x - \overset{2}{\cancel{4}} \cdot \frac{1}{\cancel{2}}\text{Tan}^{-1}\left(\frac{x}{2}\right)+c = x - 2\,\text{Tan}^{-1}\left(\frac{x}{2}\right)+c.$$

4 Evaluate the following integrals:

$$\int \frac{1}{x \ln x} dx$$

Quotient Rule $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

Let $I = \int \frac{1}{x \ln x} dx$

Let $u = \ln x$
$du = \frac{1}{x} dx$

$$I = \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$$

$$= \int \frac{1}{u} \cdot du = \ln|u| + C.$$

$$= \ln|\ln x| + C.$$

5 Evaluate the following integrals:

$$\int \frac{e^x}{e^x+3}\,dx$$

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$$\int \frac{e^x}{e^x+3}\,dx$$

$$= \ln|e^x+3| + C.$$

$f(x) = e^x + 3$

$f'(x) = e^x$

Quotient Rule

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

6 Evaluate the following integrals:

$$\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}\, dx$$

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UCADEMY

Power Rule $\int f^n(x) \cdot f'(x)\, dx = \frac{f^{n+1}(x)}{n+1} + c,\ n \neq -1$

Let $I = \int \frac{x+b}{(x^2+2bx+c)^{1/2}}\, dx$

Let $u = x^2+2bx+c$

$du = (2x+2b)\, dx$

$du = 2(x+b)\, dx$

$\frac{1}{2} du = (x+b)\, dx$

$$I = \int \frac{\frac{1}{2} du}{u^{1/2}} = \frac{1}{2} \int u^{-1/2}\, du$$

$$= \frac{1}{2} \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C_1 = \frac{1}{\cancel{2}} \cdot \frac{u^{1/2}}{\frac{1}{\cancel{2}}} + C_1$$

$$= \sqrt{x^2+2bx+c} + C.$$

7 Evaluate the following integrals:

$$\int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx$$

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UCADEMY

Let $I = \int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx$

$\boxed{\text{Let } \quad u = \tan x \quad du = \sec^2 x\, dx}$

Power Rule $\int f^n(x) \cdot f'(x)\, dx = \frac{f^{n+1}(x)}{n+1} + c, \; n \neq -1$

$$I = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\, du$$

$$= \frac{u^{-1/2+1}}{-1/2+1} + C. = \frac{u^{1/2}}{1/2} + C.$$

$$= 2\sqrt{\tan x} + C.$$

8 Evaluate the following indefinite integrals:

(a) Show that $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x+\sqrt{x^2-a^2})+c$

L.H.S Let $I = \int \frac{dx}{\sqrt{x^2-a^2}}$

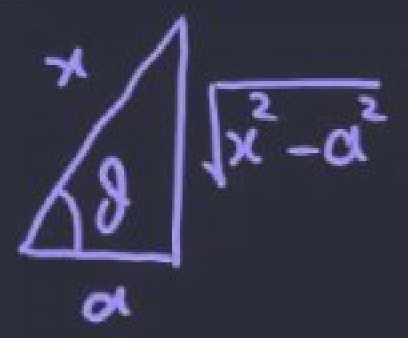
Let $x = a\sec\theta \Rightarrow \sec\theta = \frac{x}{a}$

$dx = a\sec\theta\tan\theta\, d\theta$

$$I = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}\, d\theta$$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2(\tan^2\theta)}}\, d\theta$$

$$= \int \frac{\cancel{a}\sec\theta\,\cancel{\tan\theta}}{\cancel{a}\,\cancel{\tan\theta}}\, d\theta$$



$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

$$I = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c_1$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1$$

$$= \ln|x+\sqrt{x^2-a^2}| - \ln a + c_1$$

$$= \ln|x+\sqrt{x^2-a^2}$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + c$$

$$\sec^2\theta - 1 = \tan^2\theta$$

8 Evaluate the following indefinite integrals:

(b) show that $\int \sqrt{x^2-a^2}\,dx = \frac{a^2}{2}\text{Sin}^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{x^2-a^2}) + c$

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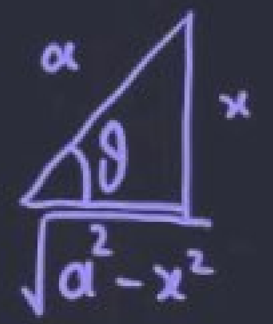
UCADEMY

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Let $I = \int \sqrt{a^2-x^2}\,dx$

Let $x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$

$dx = a\cos\theta\,d\theta \qquad \theta = \sin^{-1}\frac{x}{a}$



$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$I = \int \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

$I = \int \sqrt{a^2(\cos^2\theta)}\cdot a\cos\theta\,d\theta$

$= \int a\cos\theta\cdot a\cos\theta\,d\theta$

$= a^2\int \cos^2\theta\,d\theta$

$I = a^2\int \frac{1+\cos 2\theta}{2}\,d\theta$

$= \frac{a^2}{2}\int d\theta + \frac{a^2}{2}\int \cos 2\theta\,d\theta$

$= \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\frac{\sin 2\theta}{2} + C$

$= \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + C$

$= \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\sin\theta\cos\theta + C$

$= \frac{a^2}{2}\text{Sin}^{-1}\frac{x}{a} + \frac{\cancel{a^2}}{2}\cdot\frac{x}{\cancel{a}}\cdot\frac{\sqrt{a^2-x^2}}{\cancel{a}} + C$

$= \frac{a^2}{2}\text{Sin}^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x}$

9 Evaluate the following indefinite integrals:

$$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}\, dx$$

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Let $I = \int \frac{dx}{(1+x^2)^{3/2}}$

Let $x = \tan\theta$

$dx = \sec^2\theta \cdot d\theta$

$$I = \int \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^{3/2}}$$

$$= \int \frac{\sec^2\theta}{(\sec^{\cancel{2}}\theta)^{3/\cancel{2}}} d\theta = \int \frac{\cancel{\sec^2\theta}}{\sec^{3}\theta} d\theta$$

$$= \int \cos\theta \, d\theta = \sin\theta + C.$$

$$= \frac{x}{\sqrt{1+x^2}} + C.$$

$\sin\theta = \frac{x}{\sqrt{1+x^2}}$

10 Evaluate the following indefinite integrals: $\int \frac{1}{(1+x^2)\text{Tan}^{-1}x}\,dx$

Quotient Rule $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

Let $\quad I = \int \frac{1}{(1+x^2)\,\text{Tan}^{-1}x}\,dx$

$$\boxed{\text{Let} \quad u = \text{Tan}^{-1}x, \quad du = \frac{1}{1+x^2}\,dx}$$

$$I = \int \frac{1}{\text{Tan}^{-1}x}\cdot\frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{u}\cdot du = \ln|u| + c$$

$$= \ln|\text{Tan}^{-1}x| + c.$$

11 Evaluate the following indefinite integrals: $\int \sqrt{\frac{1+x}{1-x}}\,dx$

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UCADEMY

Let $I = \int \sqrt{\frac{1+x}{1-x}}\,dx$

$I = \int \sqrt{\frac{1+x}{1-x} \times \frac{1+x}{1+x}}\,dx = \int \sqrt{\frac{(1+x)^2}{1-x^2}}\,dx$

$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= \int \frac{1}{\sqrt{1^2-x^2}}\,dx - \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$

$= \sin^{-1}\left(\frac{x}{1}\right) - \frac{1}{2}\,\frac{(1-x^2)^{1/2}}{\cancel{1/2}} + c$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

Power Rule $\int f^n(x)\cdot f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c,\; n \neq -1$

12 Evaluate the following indefinite integrals: $\int \frac{\sin\theta}{1+\cos^2\theta}\,dx$

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UCADEMY

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\text{Tan}^{-1}\left(\frac{x}{a}\right)+c$$

Let $I = \int \frac{\sin\theta}{1+\cos^2\theta}\,d\theta$

Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$$I = \int \frac{-du}{1+u^2} = -\int \frac{du}{1^2+u^2}$$

$$= -\frac{1}{1}\text{Tan}^{-1}\left(\frac{u}{1}\right) + C.$$

17 Evaluate the following indefinite integrals:

$$\int \frac{x\,dx}{4+2x+x^2}\,dx$$

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UCADEMY

Let

$$I = \int \frac{x\,dx}{4+2x+x^2}$$

$f(x) = x^2+2x+4$

$f'(x) = 2x+\underline{2}$

Quotient Rule

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$I = \frac{1}{2}\int \frac{2x\,dx}{x^2+2x+4}$$

$$= \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+4}dx = \frac{1}{2}\int\left(\frac{2x+2}{x^2+2x+4} - \frac{2}{x^2+2x+4}\right)dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}dx - \frac{\cancel{2}}{\cancel{2}}\int \frac{dx}{x^2+2x+4}$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\text{Tan}^{-1}\left(\frac{x}{a}\right) + c$$

$$= \frac{1}{2}\ln|x^2+2x+4| - \int \frac{dx}{x^2+2x+1+3} = \frac{1}{2}\ln|x^2+2x+4| - \int \frac{dx}{(x+1)^2+(\sqrt{3})^2}$$

$$= \frac{1}{2}\ln|x^2+2x+4| - \frac{1}{\sqrt{3}}\text{Tan}^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c.$$

18 Evaluate the following indefinite integrals: $\int \frac{x}{x^4+2x^2+5}\,dx$

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$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\text{Tan}^{-1}\left(\frac{x}{a}\right)+c$$

Let $I = \int \frac{x}{x^4+2x^2+5}\,dx$

Let $u = x^2$, $du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$I = \int \frac{\frac{1}{2}du}{u^2+2u+5} = \frac{1}{2}\int \frac{du}{\underbrace{u^2+2u+1}+4}$$

$$= \frac{1}{2}\int \frac{du}{(u+1)^2+2^2} = \frac{1}{2}\cdot\frac{1}{2}\text{Tan}^{-1}\left(\frac{u+1}{2}\right)+C$$

$$= \frac{1}{4}\text{Tan}^{-1}\left(\frac{x^2+1}{2}\right)+C.$$

19 Evaluate the following indefinite integrals: $\int [\cos(\sqrt{x} - \frac{x}{2}) \times (\frac{1}{\sqrt{x}} - 1)]\, dx$

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UCADEMY

$$\int \cos dx = \sin x + c$$

Let

$$I = \int \left[\cos\left(\sqrt{x} - \frac{x}{2}\right)\right] \times \left(\frac{1}{\sqrt{x}} - 1\right) dx$$

Let

$$u = \sqrt{x} - \frac{x}{2} = x^{\frac{1}{2}} - \frac{x}{2}$$

$$du = \left(\frac{1}{2} x^{\frac{1}{2}-1} - \frac{1}{2}\right) dx = \frac{1}{2}\left(x^{-\frac{1}{2}} - 1\right) dx$$

$$2du = \left(\frac{1}{\sqrt{x}} - 1\right) dx \quad \checkmark$$

$$I = \int \cos u \,.\, 2du = 2\int \cos u \, du$$

$$= 2 \sin u + C.$$

$$= 2\sin\left(\sqrt{x} - \frac{x}{2}\right) + C.$$

20 Evaluate the following indefinite integrals: $\int \frac{x+2}{\sqrt{x+3}}\,dx$

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UCADEMY

$$\int \frac{x+2}{\sqrt{x+3}}\,dx$$

$$= \int \frac{x+3-1}{\sqrt{x+3}}\,dx = \int \left(\frac{x+3}{\sqrt{x+3}} - \frac{1}{\sqrt{x+3}}\right) dx$$

$$= \int \left[(x+3)^{1/2} - (x+3)^{-1/2}\right] dx$$

$$= \int (x+3)^{1/2}\,dx - \int (x+3)^{-1/2}\,dx$$

$$= \frac{(x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x+3)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c.$$

$$= \frac{2}{3}(x+3)^{3/2} - 2(x+3)^{1/2} + c.$$

Power Rule

$$\int f^n(x)\cdot f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c,\ n \neq -1$$

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UCADEMY

21 Evaluate the following indefinite integrals: $\int \frac{\sqrt{2}}{\sin x+\cos x}\,dx$

Let $I = \int \frac{\sqrt{2}}{\sin x + \cos x}\,dx$

$= \int \frac{\left(\frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}\right)}{\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}}}\,dx$

$= \int \frac{1}{\sin x \cdot \sin\frac{\pi}{4} + \cos x \cdot \cos\frac{\pi}{4}}\,dx = \int \frac{1}{\cos\left(x - \frac{\pi}{4}\right)}\,dx$

$= \int \sec\left(x - \frac{\pi}{4}\right)dx$

Let $u = x - \frac{\pi}{4}$, $du = dx$

$I = \int \sec u\, du = \ln|\sec u + \tan u| + c.$

$= \ln\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right| + c$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

22 Evaluate the following indefinite integrals: $\int \frac{dx}{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}\,dx$

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UCADEMY

Let

$$I = \int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$$

$$I = \int \frac{dx}{\sin x \cdot \cos\frac{\pi}{3} + \cos x \cdot \sin\frac{\pi}{3}}$$

$$= \int \frac{dx}{\sin\left(x+\frac{\pi}{3}\right)} = \int \operatorname{cosec}\left(x+\frac{\pi}{3}\right) dx$$

Let $u = x + \frac{\pi}{3}$, $du = dx$

$$I = \int \operatorname{cosec} u \, du = \ln|\operatorname{cosec} u - \cot u| + c.$$

$$= \ln\left|\operatorname{cosec}\left(x+\frac{\pi}{3}\right) - \cot\left(x+\frac{\pi}{3}\right)\right| + c$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c$$

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