

Scalar:

A quantity which has magnitude only is called scalar. e. g time, volume, speed and work e.tc the scalars are denoted by letters.

Vectors:

A quantity which has both magnitude and direction is called vector. e. g Velocity, displacement, force and torque e. t. c. A vector say V is denoted by \vec{V} or \underline{V} e.t.c. or by bold face letter v .

Geometrical interpretation of Vector:

Geometrically, A vector is represented by a line segment AB with A its initial point and B its terminal point. It is often found convenient to denoted a vector by an arrow and written either as \vec{AB} or as a bold face symbol like v or in underlined form \underline{v} .

Magnitude of a vector.

Let v be a vector, then its magnitude is denoted by $|\vec{v}|$ or simply v . it is also called norm or length of vector if the line \vec{AB} represent a vector, then distance from Point A to point B will be magnitude of \vec{AB} and is Denoted by $|\vec{AB}|$

Unit vector:

A vector whose magnitude is one (unity) is called unit vector. Unit vector of \underline{v} is written as \hat{v} (read as v hat) and is defined as $\hat{v} = \frac{v}{|v|}$

Null vector:

A vector whose magnitude is zero but no specific direction is called null or zero vector. It is denoted by \vec{o}

Negative Vectors:

Two vectors are said to be negative of each other if they have the same magnitude but opposite direction.

If $\vec{AB} = \underline{v}$ then $\vec{BA} = -\vec{AB} = -\underline{v}$ and $|\vec{BA}| = |-\vec{AB}|$
 (∵ the magnitude of a vector is a non negative member)

Multiplication of vector by a scalar: let k be a scalar Number ($k \in R$ and \underline{v} be a vector. then $k\underline{v}$ is a vector Which is k times to vector \underline{v} .

(i) \underline{v} and $k\underline{v}$ are in same direction if $k > 0$.

(ii) \underline{v} and $k\underline{v}$ are in opposite direction if $k < 0$

Equal Vectors:

Two vectors \vec{AB} and \vec{CD} are said to be equal, if they have same magnitude and same direction i. e $|\vec{AB}| = |\vec{CD}|$

Parallel vectors:

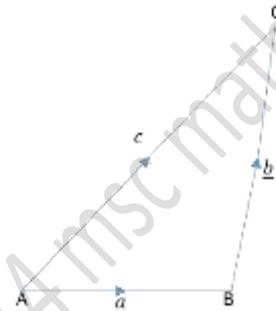
Two vectors \vec{u} , and \vec{v} are said to be parallel if $\vec{u} \times k \vec{v}$ Or $\vec{v} \times k \vec{u}$ if $\vec{u} \times k \vec{v}$ then \vec{u} and \vec{v} are same direction if $k > 0$ also \vec{u} and \vec{v} are in opposite direction if $k < 0$

Addition of two vectors:

Addition of two vectors is explained by following two laws.

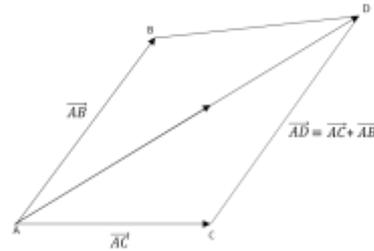
i. Triangle law of addition:

If two vectors \underline{a} and \underline{b} are represented by two Sides AB and BC of ΔABC s. that the terminal point \underline{a} coincide with the initial point of \underline{b} . Then the third side AC of the triangle gives Vector sum AC of the triangle gives vector sum $\underline{a} + \underline{b}$ i. e $\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \underline{a} + \underline{b} = \vec{AC}$



ii. Parallelogram law of addition:

If two vectors \underline{u} and \underline{v} are represented by two Sides AB and AC of ||gram as shown in figure. then Diagonal AD give the sum of \vec{AB} and \vec{AC} i. e $\vec{AD} = \vec{AB} + \vec{AC} = \underline{u} + \underline{v}$



Subtraction of two vectors:

let ABCD be a ||gram.

$$\vec{AB} = \vec{DC} = \underline{a}$$

$$\vec{BC} = \vec{AD} = \underline{b}$$

In ΔABC

$$\vec{AC} = \vec{AB} = \vec{BC}$$

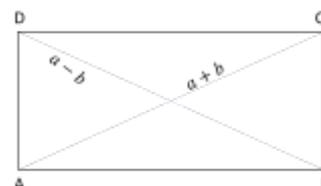
$$\vec{AC} = \underline{a} = \underline{b}$$

In ΔABD , $\vec{AB} + \vec{BD} = \vec{AD}$

$$\Rightarrow \vec{BD} = \vec{AD} - \vec{AB}$$

$$\Rightarrow \vec{BD} = \underline{b} - \underline{a} \Rightarrow \vec{DB} = -(\underline{b} - \underline{a})$$

$$\Rightarrow \vec{DB} = \underline{a} - \underline{b}$$

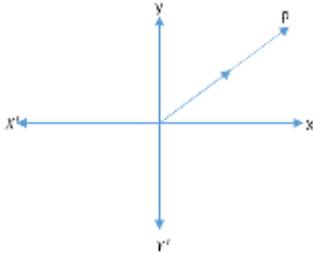


Position vector:

The vector whose initial point is at called position vector of point p.

Position vector of $p = \vec{OP}$

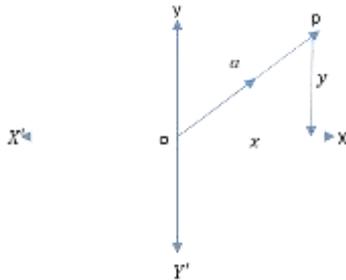
$$\vec{AB} = \vec{DC} = \vec{a}$$
$$\vec{AB} = \vec{DC} = \vec{a}$$



Vectors in plane:

let $\vec{OP} = \vec{a}$ be a vector in $xy = \text{plane}$. we resolve x and y be the horizontal and vertical components of \vec{a} respectively. Then \vec{a}

$= [x, y]$ is a vector in $xy - \text{plane}$



Addition:

let $\vec{a} = [x_1, y_1]$, $\vec{b} = [x_2, y_2]$ be two vectors in plane then $\vec{a} + \vec{b} = [x_1, y_1] + [x_2, y_2], [x_1 + x_2, y_1 + y_2]$

Subtraction:

let $\vec{a} = [x_1, y_1]$, $\vec{b} = [x_2, y_2]$ be two vectors in plane then $\vec{a} - \vec{b} = [x_1, y_1] - [x_2, y_2], [x_1 - x_2, y_1 - y_2]$

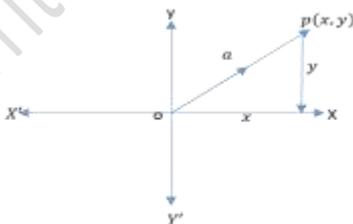
Magnitude of a vector in plane:

Let $\vec{a} = [x, y]$ be vector in plane as shown in fig.

Then by pythagoras theorem

$$|\vec{OP}|^2 = |\vec{OA}|^2 + |\vec{AP}|^2$$
$$|\vec{a}|^2 = x^2 + y^2$$

$\Rightarrow a = \sqrt{x^2 + y^2}$ which is magnitude of \vec{a} .



Another notation for represent vectors in plane:

let \vec{OA} and \vec{OB} be two unit vectors along $x - \text{axis}$ and $y - \text{axis}$ respectively. then $|\vec{OA}| = 1$ $|\vec{OB}| = 1$

as $\vec{OA} = [1, 0]$, $\vec{OB} = [0, 1]$

Then $\hat{i} = [1, 0]$, $\hat{j} = [0, 1]$

The ratio formula

Statement:

let A and B be two points whose position vectors \vec{a} and \vec{b} respectively. if a point p divides AB in the ratio $p:q$ then the position vector of p is given

By

$$\vec{r} = \frac{q\vec{a} + p\vec{b}}{p + q}$$

Proof:

In fig.

$$\vec{OA} = \vec{a}, \quad \vec{OB} = \vec{b}$$
$$\vec{OP} = \vec{r}$$

Given that

$$\vec{AP} : \vec{PB} = p : q \Rightarrow \frac{\vec{AP}}{\vec{PB}} = \frac{p}{q} \rightarrow (i)$$

$$\Delta OAP, \quad \vec{OP} = \vec{OA} + \vec{AP}$$

$$\vec{AP} = \vec{OA} - \vec{OP} = \vec{r} - \vec{a}$$

$$\text{In } \Delta OAB, \quad \vec{OB} = \vec{OP} + \vec{PB}$$

$$\vec{PB} = \vec{OB} - \vec{OP} = \vec{b} - \vec{r}$$

By (i)

$$\Rightarrow \frac{\vec{r} - \vec{a}}{\vec{b} - \vec{r}} = \frac{p}{q} \Rightarrow q(\vec{r} - \vec{a}) = p(\vec{b} - \vec{r})$$

$$\Rightarrow q\vec{r} - q\vec{a} = p\vec{b} - p\vec{r} \Rightarrow q\vec{r} + p\vec{r} = p\vec{b} + q\vec{a}$$

$$\Rightarrow (q + p)\vec{r} = p\vec{b} + q\vec{a}$$

$$\Rightarrow \vec{r} = \frac{p\vec{b} + q\vec{a}}{(q + p)}$$

$$\text{Hence } \Rightarrow \vec{r} = \frac{q\vec{b} + p\vec{a}}{(q + p)}$$

Corollary:

if p is the mid point of AB then $p:q = 1:1$

$$\text{So, } \vec{r} = \frac{\vec{a} + \vec{b}}{2}$$

Exercise NO 7.1

Question No.1

Write the vector \vec{PQ} in the form $x\hat{i} + y\hat{j}$.

(i) $P(2, 3)$, $Q(6, -2)$

$$\vec{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$$\vec{PQ} = (6-2)\hat{i} + (-2-3)\hat{j}$$

$$\vec{PQ} = 4\hat{i} - 5\hat{j}$$

(ii) $P(0, 5)$, $Q(-1, -6)$

$$\overrightarrow{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$

$$\overrightarrow{PQ} = (-1-0)\underline{i} + (-6-5)\underline{j}$$

$$\overrightarrow{PQ} = -1\underline{i} - 11\underline{j}$$

Q.2: Find the magnitude of the vector u:

(i) $\underline{u} = 2\underline{i} - 7\underline{j}$

$$|\underline{u}| = \sqrt{(x)^2 + (y)^2}$$

$$|\underline{u}| = \sqrt{(2)^2 + (-7)^2}$$

$$|\underline{u}| = \sqrt{4 + 49}$$

$$|\underline{u}| = \sqrt{53}$$

(ii) $\underline{u} = \underline{i} + \underline{j}$

$$|\underline{u}| = \sqrt{(x)^2 + (y)^2}$$

$$|\underline{u}| = \sqrt{(1)^2 + (1)^2}$$

$$|\underline{u}| = \sqrt{1 + 1}$$

$$|\underline{u}| = \sqrt{2}$$

(iii) $\underline{u} = [3, -4]$

$$\underline{u} = 3\underline{i} - 4\underline{j}$$

$$|\underline{u}| = \sqrt{(x)^2 + (y)^2}$$

$$|\underline{u}| = \sqrt{(3)^2 + (-4)^2}$$

$$|\underline{u}| = \sqrt{9 + 16}$$

$$|\underline{u}| = \sqrt{25}$$

$$|\underline{u}| = 5$$

Q.3: if $\underline{u} = 2\underline{i} - 7\underline{j}$, $\underline{v} = \underline{i} - 6\underline{j}$, $\underline{w} = -\underline{i} + \underline{j}$. Find the following vectors.

(i) $\underline{u} + \underline{v} - \underline{w}$

$$= (2\underline{i} - 7\underline{j}) + (\underline{i} - 6\underline{j}) - (-\underline{i} + \underline{j})$$

$$= 2\underline{i} - 7\underline{j} + \underline{i} - 6\underline{j} + \underline{i} - \underline{j}$$

$$= 4\underline{i} - 14\underline{j}$$

(ii) $2\underline{u} - 3\underline{v} + 4\underline{w}$

$$= 2(2\underline{i} - 7\underline{j}) - 3(\underline{i} - 6\underline{j}) + 4(-\underline{i} + \underline{j})$$

$$= 4\underline{i} - 14\underline{j} - 3\underline{i} + 18\underline{j} - 4\underline{i} + 4\underline{j}$$

$$= -3\underline{i} + 8\underline{j}$$

(iii) $\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} + \frac{1}{2}\underline{w}$

$$= \frac{1}{2}(\underline{u} + \underline{v} + \underline{w}) = \frac{1}{2}(2\underline{i} - 7\underline{j} + \underline{i} - 6\underline{j} - \underline{i} + \underline{j})$$

$$= \frac{1}{2}(2\underline{i} - 12\underline{j}) = \underline{i} - 6\underline{j}$$

Q.4: Find the sum of the vectors AB and CD, given the four points A(1, -1), B(2,0), C(-1,3), D(-2,2).

SOLUTION: Given: A(1, -1), B(2,0), C(-1,3), D(-2,2)

$$\overrightarrow{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$

$$\overrightarrow{AB} = (2-1)\underline{i} + (0+1)\underline{j} = \underline{i} + \underline{j}$$

$$\overrightarrow{CD} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}$$

$$\overrightarrow{CD} = (-2+1)\underline{i} + (2-3)\underline{j} = -\underline{i} - \underline{j}$$

$$\text{NOW } \overrightarrow{AB} + \overrightarrow{CD} = \underline{i} + \underline{j} - \underline{i} - \underline{j} = 0\underline{i} + 0\underline{j} = \underline{0} \text{ (Null Vector)}$$

Written by Amir Shehzad 03434443214 msc mathematics

Q.5: find the vector from the point A to the origin where $\underline{AB} = 4\underline{i} - 2\underline{j}$ and B is the point (-2,5).

SOLUTION: $\overrightarrow{AO} = ?$ here o = (0,0)

$$\text{Given that: } \overrightarrow{AB} = 4\underline{i} - 2\underline{j}, \quad B(-2,5) \Rightarrow \overrightarrow{OB} = -2\underline{i} + 5\underline{j}$$

AS we know that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\Rightarrow \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{AB}$$

$$\Rightarrow \overrightarrow{OA} = (-2\underline{i} + 5\underline{j}) - (4\underline{i} - 2\underline{j})$$

$$= -2\underline{i} + 5\underline{j} - 4\underline{i} + 2\underline{j}$$

$$= -6\underline{i} + 7\underline{j}$$

$$\Rightarrow -\overrightarrow{AO} = -6\underline{i} + 7\underline{j} \Rightarrow \overrightarrow{AO} = 6\underline{i} - 7\underline{j} \quad (\because \overrightarrow{OA} = -\overrightarrow{AO})$$

Q.9: if "O" is the origin and $\overrightarrow{OP} = \overrightarrow{AB}$, find the point when A and B are (-3,7) and (1,0) respectively. (2019 I S.Q)

SOLUTION: NOW we find point P(x, Y). Given that (-3,7) and (1,0) and here O(0,0).

$$\text{As } \overrightarrow{OP} = \overrightarrow{AB}$$

$$\Rightarrow (x - 0)\underline{i} + (y - 0)\underline{j} = (1 + 3)\underline{i} + (0 - 7)\underline{j}$$

$$x\underline{i} + y\underline{j} = 4\underline{i} - 7\underline{j}$$

Comparing both sides, we have

$$x = 4 \text{ and } y = -7$$

$$\text{Thus } P(x, y) = P(4, -7)$$

Q.6: Find a unit vector in the direction of the vector given below.

(i) $\underline{v} = 2\underline{i} - \underline{j}$

$$|\underline{v}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$$

$$\text{NOW } \hat{v} = \frac{\underline{v}}{|\underline{v}|}$$

$$= \frac{2\underline{i} - \underline{j}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\underline{i} - \frac{1}{\sqrt{5}}\underline{j}$$

$$= \frac{1}{\sqrt{5}}(2\underline{i} - \underline{j})$$

(ii) $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

$$|\underline{v}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\text{NOW } \hat{v} = \frac{\underline{v}}{|\underline{v}|}$$

$$= \frac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}}{1} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$$

(iii) $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

$$|\underline{v}| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

$$\text{NOW } \hat{v} = \frac{\underline{v}}{|\underline{v}|}$$

$$= \frac{-\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$$

Q.7: If A, B, C are respectively the points (2,-4), (4,0), (1,6). Use vector method to find the coordinates of the point D

(i) ABCD is a parallelogram

SOLUTION: we find $D(x, y)$

Given: A (2, -4), B (4,0), C (1,6)

Since ABCD is a parallelogram, so

$$\overline{AB} = \overline{DC} \text{ and } \overline{AB} \parallel \overline{DC}$$

$$\overline{AB} = [4,0] - [2, -4] = [4 - 2, 0 + 4] = [2,4]$$

$$\overline{DC} = [1,6] - [x, y] = [1 - x, 6 - y]$$

$$\because \overline{AB} = \overline{DC} \Rightarrow [2,4] = [1 - x, 6 - y]$$

$$\Rightarrow 2 = 1 - x, \quad 4 = 6 - y \Rightarrow x = 1 - 2, y = 6 - 4$$

$$\Rightarrow x = -1, y = 2 \text{ so } D(x, y) = (-1, 2)$$



(ii) $\because A(2, -4), B(4,0)$ and $C(1,6)$

Let $D(x, y)$ be the req. pt

as $ADBC$ is a parallelogram.

So from fig $\because \overline{AD} = \overline{CB}$ and $\overline{AD} \parallel \overline{CB}$

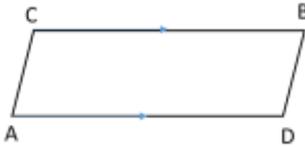
$$\overline{AD} = [x, y] - [2, -4] = [x - 2, y + 4]$$

$$\overline{CB} = [4,0] - [1,6] = [4 - 1, 0 - 6] = [3, -6]$$

$$\because \overline{AD} = \overline{CB} \Rightarrow [x - 2, y + 4] = [3, -6]$$

$$\Rightarrow x - 2 = 3, \quad y + 4 = -6 \Rightarrow x = 5, y = -10$$

$$\Rightarrow \text{so } D(x, y) = (5, -10)$$



Q.8: If B, C, D are respectively the points (4,1), (-2,3), (-8,0). Use vector method to find the coordinates of the point:

(i) A, if ABCD is a parallelogram

SOLUTION: we find $A(x, y)$

Given: B (4,1), C (-2,3), D (-8,0)

Since ABCD is a parallelogram, so

$$\overline{AB} = \overline{DC}$$

$$(x_2 - x_1) \underline{i} + (y_2 - y_1) \underline{j} = (x_2 - x_1) \underline{i} + (y_2 - y_1) \underline{j}$$

$$(4-x) \underline{i} + (1-y) \underline{j} = (-2+8) \underline{i} + (3-0) \underline{j}$$

$$(4-x) \underline{i} + (1-y) \underline{j} = 6 \underline{i} + 3 \underline{j}$$

Comparing both sides

$$4 - x = 6 \quad \& \quad 1 - y = 3$$

$$\Rightarrow x = -2 \quad \& \quad y = -2$$

$$\text{Thus } A(-2, -2)$$

(ii) E, AEBC is a parallelogram

SOLUTION: we find $E(x, y)$

Given: A (-2, -2), B (4,1), D (-8,0)

Since ADCE is a parallelogram, so

$$\overline{AE} = \overline{BD}$$



$$(x_2 - x_1) \underline{i} + (y_2 - y_1) \underline{j} = (x_2 - x_1) \underline{i} + (y_2 - y_1) \underline{j}$$

$$(x+2) \underline{i} + (y+2) \underline{j} = (4+8) \underline{i} + (1-0) \underline{j}$$

$$(x+2) \underline{i} + (y+2) \underline{j} = 12 \underline{i} + \underline{j}$$

Comparing both sides

$$x+2 = 12 \quad \& \quad y+2 = 1$$

$$\Rightarrow x = 10 \quad \& \quad y = -1 \quad \text{Thus } E(10, -1)$$

Q.10: Use vectors, to show that ABCD is a parallelogram, when the points A, B, C and D are respectively (0,0), (a,0), (b,c), (b-a,c).

SOLUTION: Given that: (0,0), (a,0), (b,c), (b-a,c)

$$\overline{AB} = (a-0) \underline{i} + (0-0) \underline{j} = a \underline{i} + 0 \underline{j}$$

$$\overline{DC} = (b - b + a) \underline{i} + (c - c) \underline{j} = a \underline{i} + 0 \underline{j}$$

$$\overline{AD} = (b-a-0) \underline{i} + (c-0) \underline{j} = (b-a) \underline{i} + c \underline{j}$$

$$\overline{BC} = (b-a) \underline{i} + (c-0) \underline{j} = (b-a) \underline{i} + c \underline{j}$$

$$\text{As } \overline{AB} = \overline{DC} \text{ \& } \overline{AD} = \overline{BC}$$

So ABCD is a parallelogram.

Q.11: if $\overline{AB} = \overline{CD}$. Find the coordinates of point A, when points B, C, D are (1,2), (-2,5), (4,11) respectively.

SOLUTION: NOW we find point A(x, Y). Given that B(1,2), C(-2,5), D(4,11)

As $\overline{AB} = \overline{CD}$

$$\Rightarrow (1-x) \underline{i} + (2-y) \underline{j} = (4+2) \underline{i} + (11-5) \underline{j}$$

$$(1-x) \underline{i} + (2-y) \underline{j} = 6 \underline{i} + 6 \underline{j}$$

Comparing both sides, we have

$$1 - x = 6 \quad \text{and} \quad 2 - y = 6$$

$$x = 5 \quad \text{and} \quad y = -4$$

$$\text{Thus } A(x, y) = A(5, -4)$$

Q.12: Find the position vectors of the point of division of the line segments joining the following pair of points, in the given ratio:

(i) point "C" with position vector $2 \underline{i} - 3 \underline{j}$ and point D with position vector $3 \underline{i} + 2 \underline{j}$ in the ratio 4 : 3

SOLUTION: Given that $\overline{OC} = a = 2 \underline{i} - 3 \underline{j}$, $\overline{OD} = b = 3 \underline{i} + 2 \underline{j}$,
p : q = 4 : 3

Let $\overline{OP} = \underline{r} = ?$, Let P be the point which divides CD in ratio 4 : 3

Using ratio formula, we have

$$\underline{r} = \frac{qa + pb}{p + q} = \frac{3(2 \underline{i} - 3 \underline{j}) + 4(3 \underline{i} + 2 \underline{j})}{3 + 4} = \frac{6 \underline{i} - 9 \underline{j} + 12 \underline{i} + 8 \underline{j}}{7}$$

$$= \frac{18 \underline{i} - \underline{j}}{7} = \frac{18}{7} \underline{i} - \frac{1}{7} \underline{j}$$

(ii) point "E" with position vector $5 \underline{i}$ and point F with position vector $4 \underline{i} + \underline{j}$ in the ratio 2 : 5.

SOLUTION: Given that $\overline{OE} = a = 5 \underline{j}$, $\overline{OF} = b = 4 \underline{i} + \underline{j}$, p : q = 2 : 5

Let $\overline{OP} = \underline{r} = ?$

Let P be the point which divides CD in ratio 2 : 5 Using ratio formula, we have

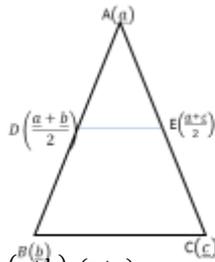
$$\begin{aligned} \underline{r} &= \frac{qa + pb}{p + q} = \frac{5(5\mathbf{j}) + 2(4\mathbf{i} + \mathbf{j})}{2 + 5} \\ &= \frac{25\mathbf{j} + 8\mathbf{i} + 2\mathbf{j}}{7} = \frac{8\mathbf{i} + 27\mathbf{j}}{7} \\ &= \frac{8}{7}\mathbf{i} + \frac{27}{7}\mathbf{j} \end{aligned}$$

Q.14: Prove that line segment joining the mid points of two sides of a triangle is parallel to third side & half as long.

SOLUTION: consider a triangle ABC, such that $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, $\overrightarrow{OC} = \underline{c}$

Let D & E be the mid points of sides \overline{AB} & \overline{AC} respectively.

$$\overrightarrow{OD} = \frac{\underline{a} + \underline{b}}{2}; \quad \overrightarrow{OE} = \frac{\underline{a} + \underline{c}}{2}$$



$$\text{NOW } \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \underline{c} - \underline{b} \rightarrow (i)$$

$$\text{AND } \overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = \frac{\underline{a} + \underline{c}}{2} - \frac{\underline{a} + \underline{b}}{2} = \frac{(\underline{a} + \underline{c}) - (\underline{a} + \underline{b})}{2} =$$

$$\frac{\underline{a} + \underline{c} - \underline{a} - \underline{b}}{2} = \frac{\underline{c} - \underline{b}}{2} = \frac{1}{2}(\underline{c} - \underline{b}) \quad (ii)$$

From (i) & (ii) it is clear that BC & DE are parallel and \overrightarrow{DE} is half to \overrightarrow{BC} .

Q.15: prove that line segments joining the mid points of sides of a quadrilateral taken in order form a parallelogram.

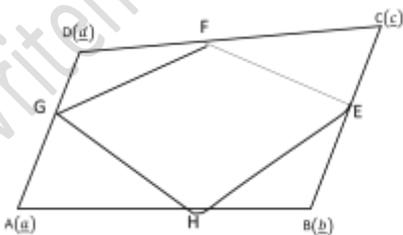
Q.15: prove that line segments joining the mid points of sides of a quadrilateral taken in order form a parallelogram.

SOLUTION: consider the quadrilateral ABCD, such that $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, $\overrightarrow{OC} = \underline{c}$, $\overrightarrow{OD} = \underline{d}$

Let E, F, G, H be the mid points of sides \overline{AB} , \overline{BC} , \overline{CD} , \overline{AD} respectively. D(d)

$$\overrightarrow{OE} = \frac{\underline{a} + \underline{b}}{2}, \overrightarrow{OF} = \frac{\underline{b} + \underline{c}}{2}, \overrightarrow{OG} = \frac{\underline{c} + \underline{d}}{2}, \overrightarrow{OH} = \frac{\underline{a} + \underline{d}}{2}$$

$$\text{NOW } \overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{\underline{b} + \underline{c}}{2} - \frac{\underline{a} + \underline{b}}{2}$$



$$\begin{aligned} &= \frac{(\underline{b} + \underline{c}) - (\underline{a} + \underline{b})}{2} = \frac{\underline{b} + \underline{c} - \underline{a} - \underline{b}}{2} \\ &= \frac{\underline{c} - \underline{a}}{2} \end{aligned}$$

$$\overrightarrow{HG} = \overrightarrow{OG} - \overrightarrow{OH} = \frac{\underline{c} + \underline{d}}{2} - \frac{\underline{a} + \underline{d}}{2}$$

$$= \frac{(\underline{c} + \underline{d}) - (\underline{a} + \underline{d})}{2} = \frac{\underline{c} + \underline{d} - \underline{a} - \underline{d}}{2} = \frac{\underline{c} - \underline{a}}{2}$$

$$\overrightarrow{EH} = \overrightarrow{OH} - \overrightarrow{OE} = \frac{\underline{d} + \underline{d}}{2} - \frac{\underline{a} + \underline{b}}{2}$$

$$= \frac{(\underline{a} + \underline{d}) - (\underline{a} + \underline{b})}{2} = \frac{\underline{a} + \underline{d} - \underline{a} - \underline{b}}{2} = \frac{\underline{d} - \underline{b}}{2}$$

$$\overrightarrow{FG} = \overrightarrow{OG} - \overrightarrow{OF} = \frac{\underline{c} + \underline{d}}{2} - \frac{\underline{b} + \underline{c}}{2}$$

$$= \frac{(\underline{c} + \underline{d}) - (\underline{b} + \underline{c})}{2} = \frac{\underline{c} + \underline{d} - \underline{b} - \underline{c}}{2} = \frac{\underline{d} - \underline{b}}{2}$$

$$\text{As } \overrightarrow{EF} = \overrightarrow{HG} \quad \text{and} \quad \overrightarrow{EH} = \overrightarrow{FG}$$

$$\Rightarrow \overrightarrow{EF} \parallel \overrightarrow{HG} \quad \text{and} \quad \overrightarrow{EH} \parallel \overrightarrow{FG}$$

Thus EFGH is a parallelogram.

Concept of vector in space

If three dimensional space XOX' and ZOZ' are called Coordinate axis.

The planes made by XOY , YOZ and ZOX are called XY Plane, YZ - plane and ZX plane, respectively

These planes are mutually orthogonal to each other.

Note:

If P is point in space then it will have three coordinates along x - axis, y - axis and z - axis respectively.

if a, b, c are distance along x - axis, y - axis and z - axis respectively

Then coordinates of point p are $p(a, b, c)$ as shown in fig. let \vec{u} be vector in space whose position vector is

$$\overrightarrow{OP} \quad \text{then} \quad \vec{u} = [x, y, z]$$

Now (i) Addition:

$$\text{let } \vec{u} = [x_1, y_1, z_1], \quad \vec{v} = [x_2, y_2, z_2]$$

$$\vec{u} + \vec{v} = [x_1, y_1, z_1] + [x_2, y_2, z_2]$$

$$\vec{u} + \vec{v} = [x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

(ii) Difference:

$$\text{let } \vec{u} = [x_1, y_1, z_1], \quad \vec{v} = [x_2, y_2, z_2]$$

$$\vec{u} - \vec{v} = [x_1, y_1, z_1] - [x_2, y_2, z_2]$$

$$\vec{u} - \vec{v} = [x_1 - x_2, y_1 - y_2, z_1 - z_2]$$

(iii) Scalar multiplication:

if k be any scalar and $\vec{u} = [x, y, z]$ then $k\vec{u} = k[x, y, z]$

$$= [kx, ky, kz]$$

Magnitude of vector (in space)

let $\vec{u} = [x, y, z]$ be a vector in space then its length or norm or magnitude denoted by $|\vec{u}|$ and defines as

$$|\vec{u}| = \sqrt{x^2 + y^2 + z^2}$$

Properties of a vector:

(i) Commutative property:

for any two vectors \vec{u} and \vec{v}

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

Proof:

$$\begin{aligned} \text{let } \vec{u} &= [x_1, y_1, z_1], \vec{v} = [x_2, y_2, z_2] \\ \vec{u} + \vec{v} &= [x_1, y_1, z_1] + [x_2, y_2, z_2] \\ \vec{u} + \vec{v} &= [x_1 + x_2, y_1 + y_2, z_1 + z_2] \\ &= [x_1 + x_2, y_1 + y_2, z_1 + z_2] \\ &= [x_2 + x_1, y_2 + y_1, z_2 + z_1] \\ &= [x_2, y_2, z_2] + [x_1, y_1, z_1] \\ &= \vec{v} + \vec{u} \end{aligned}$$

(in real numbers commutatives law hold)

Thus $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(ii) Associative Property:

for any three vectors \vec{u}, \vec{v} and \vec{w}
 $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Proof:

Let

$$\begin{aligned} (\vec{u} + \vec{v}) + \vec{w} &= (x_1, y_1, z_1 + x_2, y_2, z_2) + (x_3, y_3, z_3) \\ &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) + (x_3, y_3, z_3) \\ &= (x_1, y_1, z_1) + (x_2 + x_3, y_2 + y_3, z_2 + z_3) \\ &= (x_1, y_1, z_1) + ((x_2, y_2, z_2) + (x_3, y_3, z_3)) \\ &= \vec{u} + (\vec{v} + \vec{w}) \end{aligned}$$

Hence $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Inverse property:

For a vector \vec{u} , $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = 0$

Proof:

$$\begin{aligned} \text{let } \vec{u} &= [x, y, z] \text{ then} \\ -\vec{u} &= [-x, -y, -z] \end{aligned}$$

$$\begin{aligned} \text{then } \vec{u} + (-\vec{u}) &= [x, y, z] + [-x, -y, -z] \\ &= [x - x, y - y, z - z] \\ &= (0, 0, 0) \end{aligned}$$

Similarly:

$$\begin{aligned} (-\vec{u}) + \vec{u} &= [-x, -y, -z] + [x, y, z] \\ &= [-x + x, -y + y, -z + z] = [0, 0, 0] = 0 \end{aligned}$$

Thus for \vec{u} its additive inverse is $-\vec{u}$

(iv) Distributive Property:

if k be any scalar and \vec{u}, \vec{v} be two vectors then
 $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

Proof:

$$\begin{aligned} \text{let } \vec{u} &= [x_1, y_1, z_1], \vec{v} = [x_2, y_2, z_2] \\ k(\vec{u} + \vec{v}) &= k[(x_1, y_1, z_1) + (x_2, y_2, z_2)] \\ &= k[x_1 + x_2, y_1 + y_2, z_1 + z_2] \\ &= [kx_1 + kx_2, ky_1 + ky_2, kz_1 + kz_2] \\ &= k(x_1, y_1, z_1) + k(x_2, y_2, z_2) \\ &= k\vec{u} + k\vec{v} \end{aligned}$$

(v) Scalar Multiplication property:

for a, b being scalars and \vec{u} be a vector. then
 $a(b\vec{u}) = (ab)\vec{u}$

Proof:

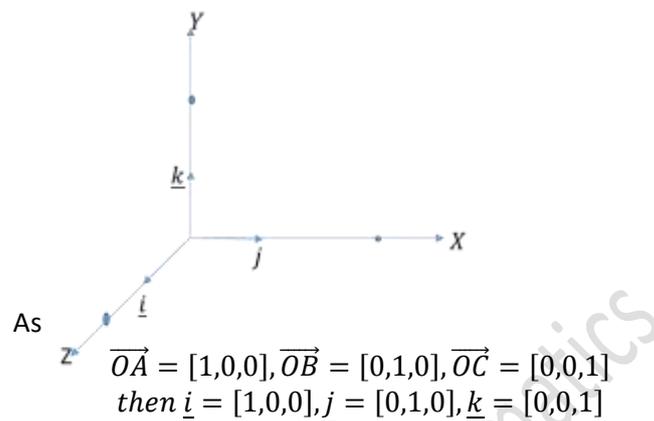
Let $\vec{u} = [x, y, z]$

$$\begin{aligned} \Rightarrow a(b\vec{u}) &= a(b[x, y, z]) = a[bx, by, bz] \\ \Rightarrow &= [abx, aby, abz] = (ab)\vec{u} \end{aligned}$$

Another Notation for represented vectors in space:

Let \vec{OA} , \vec{OB} and \vec{OC} be vectors along x -axis, y -axis and z -axis respectively.

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where $\underline{i}, \underline{j}$, and \underline{k} are called unit vectors along x -axis, y -axis and z -axis.

Now if $\vec{u} = [x, y, z]$ then \vec{u} can be written as

$$\begin{aligned} \vec{u} &= [x, 0, 0] + [0, y, 0] + [0, 0, z] \\ &= x[1, 0, 0] + y[0, 1, 0] + z[0, 0, 1] \\ \vec{u} &= x\underline{i} + y\underline{j} + z\underline{k} \end{aligned}$$

Distance between two points in space:

if \vec{OP}_1 and \vec{OP}_2 are the position vectors of the points $p_1(x_1, y_1, z_1)$ and $p_2(x_2, y_2, z_2)$

The vector $\vec{p_1p_2}$ is given by

$$\begin{aligned} \vec{p_1p_2} &= \vec{OP}_2 - \vec{OP}_1 \\ &= [x_2, y_2, z_2] - [x_1, y_1, z_1] \\ &= [x_2 - x_1, y_2 - y_1, z_2 - z_1] \end{aligned}$$

By

Distance b/w p_1 and $p_2 = |\vec{p_1p_2}|$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is called distance formula b/w two points

p_1 and p_2 in R

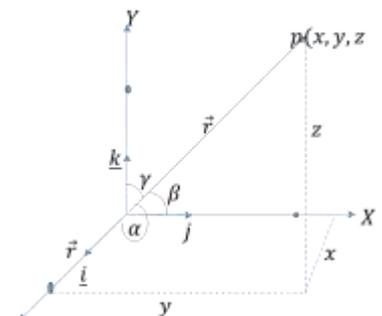
Distance angles and direction cosine of a vector:

Let $C = \vec{OP} = [x, y, z] = x\underline{i} + y\underline{j} + z\underline{k}$ be a vector s . that it makes angles α, β and γ along coordinates

axes the α, β and γ are called direction angles

Of vectors

\vec{r} while $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called direction cosines of \vec{r}



Important result

Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

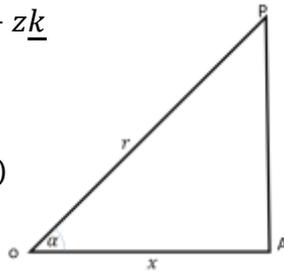
Proof:

$$\text{let } \vec{r} = [x_1, y_1, z_1] = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2 \rightarrow (i)$$



in right ΔOAP

$$\cos\alpha = \frac{OA}{OP} = \frac{x}{r}$$

$$\text{similarly } \cos\beta = \frac{y}{r} = \cos\gamma = \frac{z}{r}$$

$$\text{So } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \text{ hence proved.}$$

Exercise NO 7.2

Q.1: Let $A = (2,5)$, $B = (-1,1)$, $C = (2, -6)$. Find

(i) $\vec{AB} = ?$

$$\vec{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$$

$$\vec{AB} = (-1-2)\vec{i} + (1-5)\vec{j}$$

$$\vec{AB} = -3\vec{i} - 4\vec{j}$$

(ii) $2\vec{AB} - \vec{CB} = ?$

$$2\vec{AB} - \vec{CB} = 2[(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}] - [(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}]$$

$$2\vec{AB} - \vec{CB} = 2[(-1-2)\vec{i} + (1-5)\vec{j}] - [(-1-2)\vec{i} + (1+6)\vec{j}]$$

$$2\vec{AB} - \vec{CB} = 2[-3\vec{i} - 4\vec{j}] - [-3\vec{i} + 7\vec{j}] = -6\vec{i} - 8\vec{j} + 3\vec{i} - 7\vec{j}$$

$$= -3\vec{i} - 15\vec{j}$$

(iii) $2\vec{CB} - 2\vec{CA} = ?$

$$2\vec{CB} - 2\vec{CA} = 2(\vec{CB} - \vec{CA}) = 2\{[(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}] - [(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}]\}$$

$$2\vec{CB} - 2\vec{CA} = 2(\vec{CB} - \vec{CA}) = 2\{[(-1-2)\vec{i} + (1+6)\vec{j}] - [(-2-2)\vec{i} + (5+6)\vec{j}]\} = 2\{[-3\vec{i} + 7\vec{j}] - [0\vec{i} + 11\vec{j}]\}$$

$$2\vec{CB} - 2\vec{CA} = 2(\vec{CB} - \vec{CA}) = -6\vec{i} + 14\vec{j} - 0\vec{i} - 11\vec{j} = -6\vec{i} - 3\vec{j}$$

Q.2: Let $\vec{u} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{v} = 3\vec{i} - 2\vec{j} + 2\vec{k}$, $\vec{w} = 5\vec{i} - \vec{j} + 3\vec{k}$.

Find the indicated vectors or numbers:

(i) $\vec{u} + 2\vec{v} + \vec{w} = ?$

$$\vec{u} + 2\vec{v} + \vec{w} = (\vec{i} + 2\vec{j} - \vec{k}) + 2(3\vec{i} - 2\vec{j} + 2\vec{k}) + (5\vec{i} - \vec{j} + 3\vec{k})$$

$$= \vec{i} + 2\vec{j} - \vec{k} + 6\vec{i} - 4\vec{j} + 4\vec{k} + 5\vec{i} - \vec{j} + 3\vec{k} = 12\vec{i} - 3\vec{j} + 6\vec{k}$$

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(ii) $\vec{v} - 3\vec{w} = ?$

$$\vec{v} - 3\vec{w} = (3\vec{i} - 2\vec{j} + 2\vec{k}) - 3(5\vec{i} - \vec{j} + 3\vec{k}) = 3\vec{i} - 2\vec{j} + 2\vec{k} - 15\vec{i} + 3\vec{j} - 9\vec{k} = -12\vec{i} + \vec{j} - 7\vec{k}$$

(iii) $|3\vec{v} + \vec{w}| = ?$

$$3\vec{v} + \vec{w} = 3(3\vec{i} - 2\vec{j} + 2\vec{k}) + 5\vec{i} - \vec{j} + 3\vec{k} = 9\vec{i} - 6\vec{j} + 6\vec{k} + 5\vec{i} - \vec{j} + 3\vec{k} = 14\vec{i} - 7\vec{j} + 9\vec{k}$$

$$|3\vec{v} + \vec{w}| = \sqrt{(14)^2 + (-7)^2 + (9)^2} = \sqrt{196 + 49 + 81} = \sqrt{326}$$

Q.3: Find the magnitude of the vector \vec{v} and write the direction cosine of \vec{v} :

(i) $\vec{v} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

MAGNITUDE

$$|\vec{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$$

$$|\vec{v}| = \sqrt{(2)^2 + (3)^2 + (4)^2}$$

$$|\vec{v}| = \sqrt{4 + 9 + 16} = \sqrt{29}$$

DIRECTION COSINE

$$\text{Direction cosine} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\therefore \text{direction ratio} = 2, 3, 4$$

$$\text{So, Direction cosine} = \frac{2, 3, 4}{\sqrt{29}}$$

$$\text{Direction cosine} = \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

(ii) $\vec{v} = \vec{i} - \vec{j} - \vec{k}$

MAGNITUDE

$$|\vec{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$$

$$|\vec{v}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2}$$

$$|\vec{v}| = \sqrt{1 + 1 + 1} = \sqrt{3}$$

DIRECTION COSINE

$$\text{Direction cosine} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\therefore \text{direction ratio} = 1, -1, -1$$

$$\text{So, Direction cosine} = \frac{1, -1, -1}{\sqrt{3}}$$

$$\text{Direction cosine} = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$$

(iii) $\vec{v} = 4\vec{i} - 5\vec{j}$

MAGNITUDE

$$|\vec{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$$

$$|\vec{v}| = \sqrt{(4)^2 + (-5)^2 + (0)^2}$$

$$|\vec{v}| = \sqrt{16 + 25 + 0} = \sqrt{41}$$

DIRECTION COSINE

$$\text{Direction cosine} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\therefore \text{direction ratio} = 4, -5, 0$$

$$\text{So, Direction cosine} = \frac{4, -5, 0}{\sqrt{41}}$$

$$\text{Direction cosine} = \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}, \frac{0}{\sqrt{29}}$$

Q.4: Find α , so that

$$|\alpha \underline{i} + (\alpha + 1)\underline{j} + 2\underline{k}| = 3$$

SOLUTION:

$$|\alpha \underline{i} + (\alpha + 1)\underline{j} + 2\underline{k}| = 3$$

$$\sqrt{(\alpha)^2 + (\alpha + 1)^2 + (2)^2} = 3$$

$$\sqrt{\alpha^2 + \alpha^2 + 1 + 2\alpha + 4} = 3$$

Squaring on both side

$$\alpha^2 + \alpha^2 + 1 + 2\alpha + 4 = 9$$

$$2\alpha^2 + 2\alpha + 5 = 9$$

$$2\alpha^2 + 2\alpha + 5 - 9 = 0$$

$$2\alpha^2 + 2\alpha - 4 = 0$$

Dividing by 2

$$\alpha^2 + \alpha - 2 = 0$$

$$\alpha^2 + 2\alpha - \alpha - 2 = 0$$

$$\alpha(\alpha+2) - 1(\alpha+2) = 0$$

$$(\alpha+2)(\alpha-1) = 0$$

Either $\alpha+2=0$ or $\alpha-1=0$

Q.5: Find a unit vector in the direction of $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

SOLUTION:

Let \hat{V} be unit vector in the direction of \underline{v}

$$\text{Then, } \underline{v} = \underline{i} + 2\underline{j} - \underline{k}$$

$$|\underline{v}| = \sqrt{(1)^2 + (2)^2 + (1)^2}$$

$$|\underline{v}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

Required unit vector is:

$$\hat{V} = \frac{\underline{v}}{|\underline{v}|} = \frac{\underline{i} + 2\underline{j} - \underline{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\underline{i} + \frac{2}{\sqrt{6}}\underline{j} - \frac{1}{\sqrt{6}}\underline{k}$$

Q.6: if $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$, $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$, $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$. find a unit vector parallel to $3\underline{a} - 2\underline{b} + 4\underline{c}$.

SOLUTION:

$$\text{Let } \underline{v} = 3\underline{a} - 2\underline{b} + 4\underline{c} = 3(3\underline{i} - \underline{j} - 4\underline{k}) - 2(-2\underline{i} - 4\underline{j} - 3\underline{k}) + 4(\underline{i} + 2\underline{j} - \underline{k})$$

$$\underline{v} = 9\underline{i} - 3\underline{j} - 12\underline{k} + 4\underline{i} + 8\underline{j} + 6\underline{k} + 4\underline{i} + 8\underline{j} - 4\underline{k} = 17\underline{i} + 13\underline{j} - 10\underline{k}$$

$$|\underline{v}| = \sqrt{(17)^2 + (13)^2 + (-10)^2} =$$

$$\sqrt{289 + 169 + 100} = \sqrt{558}$$

Let \underline{u} be a unit vector parallel to \underline{v} , then

$$\underline{u} = \frac{\underline{v}}{|\underline{v}|} = \frac{17\underline{i} + 13\underline{j} - 10\underline{k}}{\sqrt{558}} = \frac{17}{\sqrt{558}}\underline{i} + \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}$$

Q.7: Find a vector whose:

i) Magnitude is 4 and is parallel to $2\underline{i} - 3\underline{j} + 6\underline{k}$ (2019 I.S.Q)

SOLUTION:

$$\underline{u} = ? \text{ Given: } |\underline{u}| = 4, \underline{v} = 2\underline{i} - 3\underline{j} + 6\underline{k}$$

$$\text{As } \underline{u} \parallel \underline{v}$$

$$\text{So } \underline{u} = \underline{v}$$

$$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = |\underline{u}| \frac{\underline{v}}{|\underline{v}|} = (4) \frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{4[2\underline{i} - 3\underline{j} + 6\underline{k}]}{\sqrt{49}} = \frac{8\underline{i} - 12\underline{j} + 24\underline{k}}{7}$$

$$\underline{u} = \frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$$

ii) magnitude is 2 and is parallel to $-\underline{i} + \underline{j} + \underline{k}$

SOLUTION:

$$\underline{u} = ? \text{ Given: } |\underline{u}| = 2, \underline{v} = -\underline{i} + \underline{j} + \underline{k}$$

$$\text{As } \underline{u} \parallel \underline{v}$$

$$\text{So } \underline{u} = \underline{v}$$

$$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = |\underline{u}| \frac{\underline{v}}{|\underline{v}|} = (2) \frac{-\underline{i} + \underline{j} + \underline{k}}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{2[-\underline{i} + \underline{j} + \underline{k}]}{\sqrt{49}} = \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{7}$$

$$\underline{u} = -\frac{2}{7}\underline{i} + \frac{2}{7}\underline{j} + \frac{2}{7}\underline{k}$$

Q.8: if $\underline{u} = 2\underline{i} + 3\underline{j} + 4\underline{k}$, $\underline{v} = -\underline{i} + 3\underline{j} - \underline{k}$,

$\underline{w} = \underline{i} + 6\underline{j} + \underline{z}\underline{k}$ represents the sides of a triangle. Find the value of \underline{z} .

SOLUTION:

As $\underline{u}, \underline{v}, \underline{w}$ represents the sides of triangle

$$\text{So } \underline{u} + \underline{v} = \underline{w}$$

$$2\underline{i} + 3\underline{j} + 4\underline{k} - \underline{i} + 3\underline{j} - \underline{k} = \underline{i} + 6\underline{j} + \underline{z}\underline{k}$$

$$\underline{i} + 6\underline{j} + 8\underline{k} = \underline{i} + 6\underline{j} + \underline{z}\underline{k}$$

By comparing

$$\Rightarrow \underline{z} = 3$$

Q.9: The position vectors of the points A, B, C and D are $2\underline{i} - \underline{j} + \underline{k}$, $3\underline{i} + \underline{j}$, $2\underline{i} + 4\underline{j} - 2\underline{k}$ and $-\underline{i} - 2\underline{j} + \underline{k}$ respectively. Show that AB is parallel to CD .

SOLUTION:

$$\text{Given that: } OA = 2\underline{i} - \underline{j} + \underline{k}, OB = 3\underline{i} + \underline{j}, OC = 2\underline{i} + 4\underline{j} - 2\underline{k}, OD = -\underline{i} - 2\underline{j} + \underline{k}$$

$$AB = OB - OA = (3\underline{i} + \underline{j}) - (2\underline{i} - \underline{j} + \underline{k}) = 3\underline{i} + \underline{j} - 2\underline{i} + \underline{j} - \underline{k} = \underline{i} + 2\underline{j} - \underline{k}$$

$$CD = OD - OC = (-\underline{i} - 2\underline{j} + \underline{k}) - (2\underline{i} + 4\underline{j} - 2\underline{k}) = -\underline{i} - 2\underline{j} + \underline{k} - 2\underline{i} - 4\underline{j} + 2\underline{k} = -3\underline{i} - 6\underline{j} + 3\underline{k} = -3AB$$

$$\Rightarrow AB \parallel CD \text{ Hence } AB \text{ is parallel to } CD.$$

Q.10: We say that vectors \underline{v} and \underline{w} are parallel if there is a scalar c such that $\underline{v} = c\underline{w}$. The vector point in the same direction if $c > 0$ and the vector point in the opposite direction if $c < 0$.

a) Find two vectors of length 2 parallel to vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$

SOLUTION:

$$\text{Given: } |\underline{u}| = 2, \underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k} \underline{u} = ?$$

$$\text{As } \underline{u} \parallel \underline{v}$$

$$\text{So } \underline{u} = \pm \underline{v}$$

$$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = \pm \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = \pm |\underline{u}| \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = \pm(2) \frac{2i-4j+4k}{\sqrt{2^2+(-4)^2+4^2}}$$

$$\underline{u} = \pm \frac{2[2i-4j+4k]}{6}$$

$$\underline{u} = \pm \frac{2i-4j+4k}{3}$$

$$\underline{u} = \pm \left(\frac{2}{3}i - \frac{4}{3}j + \frac{4}{3}k \right)$$

b) Find the constant a so that the vectors $\underline{v} = i - 3j + 4k$ & $\underline{w} = ai + 9j - 12k$ are parallel.

SOLUTION:

$$\text{As } \underline{v} \parallel \underline{w}$$

$$\therefore \frac{a}{1} = \frac{9}{-3} \quad (\text{Direction ratio are same})$$

$$\Rightarrow a = -3$$

c) Find a vectors of length 5 in the direction opposite that of $\underline{v} = i - 2j + 3k$ (2019 S.Q)

SOLUTION: $\underline{u} = ?$

$$\text{Given: } |\underline{u}| = 5, \quad \underline{v} = i - 2j + 3k$$

$$\text{As } \underline{u} \parallel \underline{v}$$

$$\text{So } \underline{u} = -\underline{v}$$

$$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = -\frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = -|\underline{u}| \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = -(5) \frac{i-2j+3k}{\sqrt{1^2+(-2)^2+3^2}}$$

$$\underline{u} = -\frac{5[i-2j+3k]}{\sqrt{14}}$$

$$\underline{u} = \frac{-5i+10j-15k}{\sqrt{14}}$$

$$\underline{u} = \frac{-5}{\sqrt{14}}i + \frac{10}{\sqrt{14}}j - \frac{15}{\sqrt{14}}k$$

d) Find a & b so that the vectors

$3i - j + 4k, ai + bj - 2k$ are parallel.

SOLUTION: As $\underline{v} \parallel \underline{w}$

$$\therefore \frac{a}{3} = \frac{b}{-1} = \frac{-2}{4} \Rightarrow \frac{a}{3} = \frac{b}{-1} = \frac{-1}{2}$$

$$\Rightarrow \frac{a}{3} = \frac{-1}{2} \text{ and } \frac{b}{-1} = \frac{-1}{2}$$

$$\Rightarrow a = -\frac{3}{2} \text{ and } b = \frac{1}{2}$$

Q.11: Find the direction cosines for the given vector.

i) $3i - j + 2k$

SOLUTION:

$$\text{Let } \underline{v} = 3i - j + 2k$$

$$|\underline{v}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$$

$$|\underline{v}| = \sqrt{9+1+4} = \sqrt{14}$$

Direction ratio of v are 3, -1, 2

$$\text{Direction cosines of v} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\text{Direction cosines of v} = \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}$$

ii) $6i - 2j + k$ (2017S.Q)

SOLUTION:

$$\text{Let } \underline{v} = 6i - 2j + k$$

$$|\underline{v}| = \sqrt{(6)^2 + (-2)^2 + (1)^2}$$

$$|\underline{v}| = \sqrt{36+4+1} = \sqrt{41}$$

Direction ratio of v are 6, -2, 1

$$\text{Direction cosines of v} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\text{Direction cosines of v} = \frac{6}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{1}{\sqrt{41}}$$

iii) \overrightarrow{PQ} , where $p = (2,1,5)$ & $Q = (1,3,1)$

$$\overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

$$\overrightarrow{PQ} = (1-2)i + (3-1)j + (1-5)k$$

$$\overrightarrow{PQ} = -i + 2j - 4k$$

$$|\overrightarrow{PQ}| = \sqrt{(-1)^2 + (2)^2 + (-4)^2}$$

$$|\overrightarrow{PQ}| = \sqrt{1+4+16} = \sqrt{21}$$

Direction ratio of v are -1, 2, -4

$$\text{Direction cosines of v} = \frac{\text{Direction ratio}}{\text{magnitude}}$$

$$\text{Direction cosines of v} = \frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}$$

Q.12: Which of the following triples can be the direction angles of a single vector:

i) $45^\circ, 45^\circ, 60^\circ$ (2018S.Q)

SOLUTION:

$$\text{Let } \alpha = 45^\circ, \beta = 45^\circ, \gamma = 60^\circ$$

If α, β, γ are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$= \cos^2 45^\circ + \cos^2 45^\circ + \cos^2 60^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{2+2+1}{4} = \frac{5}{4} \neq \text{R.H.S}$$

Thus, β, γ are not the direction angles of a single vector.

ii) $30^\circ, 45^\circ, 60^\circ$

SOLUTION:

$$\text{Let } \alpha = 30^\circ, \beta = 45^\circ, \gamma = 60^\circ$$

If α, β, γ are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$= \cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ$$

$$= \left(\frac{3}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3+2+1}{4} = \frac{6}{4} \neq \text{R.H.S}$$

Thus, β, γ are not the direction angles of a single vector.

iii) $45^\circ, 60^\circ, 60^\circ$

SOLUTION:

$$\text{Let } \alpha = 45^\circ, \beta = 60^\circ, \gamma = 60^\circ$$

If α, β, γ are direction angles of a single vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{L.H.S} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$= \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 60^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{2+1+1}{4} = \frac{4}{4} = 1 \text{ R.H.S}$$

Thus, β , are the direction angles of a single vector.

The Scalar product of two vectors (Dot product)

Definition:

The scalar product of two non-zero vectors \vec{u} and \vec{v} is denoted by $\vec{u} \cdot \vec{v}$ (read as \vec{u} dot \vec{v}) and defined as ;

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos\theta$$

Where θ is angle from \vec{u} to \vec{v} and $0 \leq \theta \leq \pi$

Note: the dot product is also referred to the scalar product or the inner product.

Deduction of the important results:

For unit vector $\underline{i}, \underline{j}, \underline{k}$ we have.

$$(a) \underline{i} \cdot \underline{i} = |\underline{i}| |\underline{i}| \cos 0^\circ = 1$$

$$\underline{j} \cdot \underline{j} = |\underline{j}| |\underline{j}| \cos 0^\circ = 1$$

$$\underline{k} \cdot \underline{k} = |\underline{k}| |\underline{k}| \cos 0^\circ = 1$$

$$(b) \underline{i} \cdot \underline{j} = |\underline{i}| |\underline{j}| \cos 90^\circ = 0$$

$$\underline{j} \cdot \underline{k} = |\underline{j}| |\underline{k}| \cos 90^\circ = 0$$

$$\underline{k} \cdot \underline{i} = |\underline{k}| |\underline{i}| \cos 90^\circ = 0$$

(c) prove that $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos\theta$$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos(-\theta)$$

$$\underline{v} \cdot \underline{u} = |\underline{v}| |\underline{u}| \cos\theta$$

$$\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$$

Hence dot product of two vectors is commutative.

Perpendicular (orthogonal) vectors

Two non-zero vectors \underline{u} and \underline{v} are perpendicular if and only if $\underline{u} \cdot \underline{v} = 0$

$$(\because \text{angle between } \underline{u} \text{ and } \underline{v} \text{ is } \frac{\pi}{2} \text{ so } \cos \frac{\pi}{2} = 0)$$

$$\text{thus } \underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos\theta \cos \frac{\pi}{2} \Rightarrow \underline{u} \cdot \underline{v} = 0$$

Properties of Dot Product:

Let $\underline{u}, \underline{v}$ and \underline{w} be vectors and let $c \in \mathbb{R}$

then (i) $\underline{u} \cdot \underline{v} = 0 \Rightarrow \underline{u} = 0$ or $\underline{v} = 0$

(ii) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ (Commutative Property)

(iii) $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$ (dis. property)

(iv) $(c\underline{u}) \cdot \underline{v} = c(\underline{u} \cdot \underline{v})$ (c is scalar)

$$(v) \underline{u} \cdot \underline{u} = |\underline{u}|^2$$

Analytical Expression of dot product $\underline{u} \cdot \underline{v}$:

(Dot product of vectors in their components from)

$$\text{let } \underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k} \text{ and}$$

$$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}$$

Be two vectors

Now

$$\underline{u} \cdot \underline{v} = (a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}) \cdot (a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k})$$

$$= (a_1 \cdot a_2)(\underline{i} \cdot \underline{i}) + a_1 b_2 (\underline{i} \cdot \underline{j}) + a_1 c_2 (\underline{i} \cdot \underline{k}) + b_1 a_2 (\underline{j} \cdot \underline{i})$$

$$+ b_1 b_2 (\underline{j} \cdot \underline{j}) + b_1 c_2 (\underline{j} \cdot \underline{k}) + c_1 a_2 (\underline{k} \cdot \underline{i}) + c_1 b_2 (\underline{k} \cdot \underline{j})$$

$$+ c_1 c_2 (\underline{k} \cdot \underline{k})$$

$$\Rightarrow \underline{u} \cdot \underline{v} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

Angle between two vectors:

For two vectors \underline{u} and \underline{v}

$$(a) \underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos\theta \quad \because 0 \leq \theta \leq \pi$$

$$\therefore \cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

(b) if $\underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}$ and

$$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}$$

$$\Rightarrow \underline{u} \cdot \underline{v} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

$$|\underline{u}| = \sqrt{a_1^2 + b_1^2 + c_1^2}, \quad |\underline{v}| = \sqrt{a_2^2 + b_2^2 + c_2^2}$$

Since

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\Rightarrow \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Corollaries:

if $\theta = 0^\circ$ or π the vector \underline{u} and \underline{v} are collinear.

(ii) if $\theta = \frac{\pi}{2}$, $\cos\theta = 0 \Rightarrow \underline{u} \cdot \underline{v} = 0$

Then vector \underline{u} and \underline{v} are \perp or orthogonal.

Projection of a vector upon another vector:

Let $\vec{OA} = \underline{u}$ and $\vec{OB} = \underline{v}$

let θ be the angle between them such that

$0 \leq \theta \leq \pi$ Draw $\vec{BM} \perp \vec{OA}$ then \vec{OM} is called the projection of \underline{v} along \underline{u} . Now in $\triangle OMB$.

$$\cos\theta = \frac{|\vec{OM}|}{|\vec{OB}|} \Rightarrow |\vec{OM}| = |\vec{OB}| \cos\theta$$

$$|\vec{OM}| = |\underline{v}| \cos\theta \quad \because \cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\therefore \text{projection of } \underline{v} \text{ along } \underline{u} = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}|}$$

Similarly,

$$\therefore \text{projection of } \underline{u} \text{ along } \underline{v} = \frac{\underline{u} \cdot \underline{v}}{|\underline{v}|}$$

Exercise NO 7.3

Question No1

Find the cosine of angle θ between \underline{u} and \underline{v}

i) $\underline{u} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{v} = 2\underline{i} - \underline{j} + \underline{k}$ (2016 S.Q)

SOLUTION:

$$\underline{u} = 3\underline{i} + \underline{j} - \underline{k}, \quad \underline{v} = 2\underline{i} - \underline{j} + \underline{k}$$

If θ is the angle between \underline{u} and \underline{v} , then

$$\cos \theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\cos \theta = \frac{(3\underline{i} + \underline{j} - \underline{k}) \cdot (2\underline{i} - \underline{j} + \underline{k})}{\sqrt{3^2 + 1^2 + (-1)^2} \cdot \sqrt{2^2 + (-1)^2 + 1^2}}$$

$$\cos \theta = \frac{6 - 1 - 1}{\sqrt{9 + 1 + 1} \cdot \sqrt{4 + 1 + 1}}$$

$$\cos \theta = \frac{4}{\sqrt{11} \cdot \sqrt{6}} = \frac{4}{\sqrt{66}}$$

ii) $\underline{u} = \underline{i} - 3\underline{j} + 4\underline{k}$, $\underline{v} = 4\underline{i} - \underline{j} + 3\underline{k}$

SOLUTION:

$$\underline{u} = \underline{i} - 3\underline{j} + 4\underline{k}, \underline{v} = 4\underline{i} - \underline{j} + 3\underline{k}$$

If θ is the angle between \underline{u} and \underline{v} , then

$$\cos \theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\cos \theta = \frac{(\underline{i} - 3\underline{j} + 4\underline{k}) \cdot (4\underline{i} - \underline{j} + 3\underline{k})}{\sqrt{1^2 + (-3)^2 + 4^2} \cdot \sqrt{(4)^2 + (-1)^2 + 3^2}}$$

$$\cos \theta = \frac{4 + 3 + 12}{\sqrt{1 + 9 + 16} \cdot \sqrt{16 + 1 + 9}}$$

$$\cos \theta = \frac{19}{\sqrt{26} \cdot \sqrt{26}} = \frac{19}{26}$$

iii) $\underline{u} = [-3, 5]$, $\underline{v} = [6, -2]$

SOLUTION:

$$\underline{u} = -3\underline{i} + 5\underline{j}, \underline{v} = 6\underline{i} - 2\underline{j}$$

If θ is the angle between \underline{u} and \underline{v} , then

$$\cos \theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\cos \theta = \frac{(-3\underline{i} + 5\underline{j}) \cdot (6\underline{i} - 2\underline{j})}{\sqrt{(-3)^2 + (5)^2} \cdot \sqrt{(6)^2 + (-2)^2}}$$

$$\cos \theta = \frac{-18 - 10}{\sqrt{9 + 25} \cdot \sqrt{36 + 4}}$$

$$\cos \theta = \frac{-28}{\sqrt{34} \cdot \sqrt{40}} = \frac{-28}{\sqrt{1360}}$$

$$\cos \theta = \frac{-28}{\sqrt{16} \cdot \sqrt{85}} = \frac{-28}{4\sqrt{85}} = \frac{-7}{\sqrt{85}}$$

iv) $\underline{u} = [2, -3, 1]$, $\underline{v} = [2, 4, 1]$

SOLUTION:

$$\underline{u} = 2\underline{i} - 3\underline{j} + \underline{k}, \underline{v} = 2\underline{i} + 4\underline{j} + \underline{k}$$

If θ is the angle between \underline{u} and \underline{v} , then

$$\cos \theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$\cos \theta = \frac{(2\underline{i} - 3\underline{j} + \underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k})}{\sqrt{2^2 + (-3)^2 + 1^2} \cdot \sqrt{(2)^2 + (4)^2 + 1^2}}$$

$$\cos \theta = \frac{4 - 12 + 1}{\sqrt{4 + 9 + 1} \cdot \sqrt{4 + 16 + 1}}$$

$$\cos \theta = \frac{-7}{\sqrt{14} \cdot \sqrt{21}} = \frac{-7}{\sqrt{294}}$$

$$\cos \theta = \frac{-7}{\sqrt{49 \times 6}} = \frac{-7}{7\sqrt{6}} = \frac{-1}{\sqrt{6}}$$

Q.2: Calculate the projection of \underline{a} along \underline{b} and \underline{b} along \underline{a} when:

i) $\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$

SOLUTION:

$$\text{Projection of } \underline{a} \text{ along } \underline{b} = \underline{a} \cdot \hat{\underline{b}} = \underline{a} \cdot \frac{\underline{b}}{|\underline{b}|} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|} = \frac{(\underline{i} - \underline{k}) \cdot (\underline{j} + \underline{k})}{\sqrt{1^2 + 1^2}} = \frac{0 + 0 + 1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\text{Projection of } \underline{b} \text{ along } \underline{a} = \underline{b} \cdot \hat{\underline{a}} = \underline{b} \cdot \frac{\underline{a}}{|\underline{a}|} = \frac{\underline{b} \cdot \underline{a}}{|\underline{a}|} =$$

$$\frac{(\underline{j} + \underline{k}) \cdot (\underline{i} - \underline{k})}{\sqrt{1^2 + (-1)^2}} = \frac{0 + 0 - 1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

ii) $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = -2\underline{i} - \underline{j} + \underline{k}$ (2018 S.Q)

SOLUTION:

$$\text{Projection of } \underline{a} \text{ along } \underline{b} = \underline{a} \cdot \hat{\underline{b}} = \underline{a} \cdot \frac{\underline{b}}{|\underline{b}|} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|} =$$

$$\frac{(3\underline{i} + \underline{j} - \underline{k}) \cdot (-2\underline{i} - \underline{j} + \underline{k})}{\sqrt{(-2)^2 + 1^2 + 1^2}} = \frac{-6 - 1 - 1}{\sqrt{4 + 1 + 1}} = -\frac{8}{\sqrt{6}}$$

$$\text{Projection of } \underline{b} \text{ along } \underline{a} = \underline{b} \cdot \hat{\underline{a}} = \underline{b} \cdot \frac{\underline{a}}{|\underline{a}|} = \frac{\underline{b} \cdot \underline{a}}{|\underline{a}|} =$$

$$\frac{(-2\underline{i} - \underline{j} + \underline{k}) \cdot (3\underline{i} + \underline{j} - \underline{k})}{\sqrt{3^2 + (1)^2 + (-1)^2}} = \frac{-6 - 1 - 1}{\sqrt{9 + 1 + 1}} = -\frac{8}{\sqrt{11}}$$

Q.3(7.3): Find a real number ' α ' so that the vectors \underline{u} and \underline{v} are perpendicular.

i) $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$

SOLUTION:

Since \underline{u} and \underline{v} are perpendicular, so

$$\underline{u} \cdot \underline{v} = 0$$

$$(2\alpha\underline{i} + \underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 4\underline{k}) = 0$$

$$2\alpha + \alpha - 4 = 0$$

$$3\alpha = 4 \Rightarrow \alpha = \frac{4}{3}$$

ii) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$

SOLUTION:

Since \underline{u} and \underline{v} are perpendicular, so

$$\underline{u} \cdot \underline{v} = 0$$

$$(\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$$

$$\alpha + 2\alpha^2 - 3 = 0$$

$$2\alpha^2 + \alpha - 3 = 0$$

$$2\alpha^2 + 3\alpha - 2\alpha - 3 = 0$$

$$\alpha(2\alpha + 3) - 1(2\alpha + 3) = 0$$

$$(2\alpha + 3)(\alpha - 1) = 0$$

$$2\alpha + 3 = 0 \quad \text{or} \quad \alpha - 1 = 0$$

$$\alpha = -\frac{3}{2} \quad \text{or} \quad \alpha = 1$$

Q.4: Find the number ' Z ' so that the triangle with vertices $A(1, -1, 0)$, $B(-2, 2, 1)$, $C(0, 2, Z)$ is a right triangle with right angle at C .

SOLUTION:

$$A(1, -1, 0), B(-2, 2, 1), C(0, 2, Z)$$

$$\overrightarrow{CA} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{CA} = (1 - 0)\underline{i} + (-1 - 2)\underline{j} + (0 - Z)\underline{k}$$

$$\overrightarrow{CA} = \underline{i} - 3\underline{j} - Z\underline{k}$$

$$\overrightarrow{CB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$



$$\overline{CB} = (-2-0)\underline{i} + (2-2)\underline{j} + (1-Z)\underline{k}$$

$$\overline{CB} = -2\underline{i} + 0\underline{j} + (1-Z)\underline{k}$$

As $m\angle ACB = 90^\circ$, so $\overline{CB} \perp \overline{CA}$

$$\Rightarrow \overline{CA} \cdot \overline{CB} = 0$$

$$(\underline{i} - 3\underline{j} - Z\underline{k}) \cdot (-2\underline{i} + 0\underline{j} + (1-Z)\underline{k}) = 0$$

$$-2 - 0 - Z(1-Z) = 0$$

$$-2 - Z + Z^2 = 0$$

$$Z^2 - Z - 2 = 0$$

$$Z^2 - 2Z + Z - 2 = 0$$

$$Z(Z-2) + 1(Z-2) = 0$$

$$(Z-2)(Z+1) = 0$$

$$Z-2 = 0 \quad \text{or} \quad Z+1 \Rightarrow Z = -1$$

Q.5: if \underline{v} is a vector for which $\underline{v} \cdot \underline{i} = 0$, $\underline{v} \cdot \underline{j} = 0$, $\underline{v} \cdot \underline{k} = 0$.

Find \underline{v} . (2018 S.Q)

SOLUTION:

Let the required vector $\underline{v} = xi + yj + zk$ (i)

$$\text{As } \underline{v} \cdot \underline{i} = 0 \Rightarrow (xi + yj + zk) \cdot \underline{i} = 0 \Rightarrow x = 0$$

$$\text{As } \underline{v} \cdot \underline{j} = 0 \Rightarrow (xi + yj + zk) \cdot \underline{j} = 0 \Rightarrow y = 0$$

$$\text{As } \underline{v} \cdot \underline{k} = 0 \Rightarrow (xi + yj + zk) \cdot \underline{k} = 0 \Rightarrow z = 0$$

Putting the values in (i)

$$\underline{v} = 0i + 0j + 0k = 0 \text{ (Null vector)}$$

Q.6: Show that the vectors $3\underline{i} - 2\underline{j} + \underline{k}$, $\underline{i} - 3\underline{j} + 5\underline{k}$ and $2\underline{i} + \underline{j} - 4\underline{k}$ form a right triangle.

SOLUTION:

(i) Let $u = 3\underline{i} - 2\underline{j} + \underline{k}$, $v = \underline{i} - 3\underline{j} + 5\underline{k}$, $w = 2\underline{i} + \underline{j} - 4\underline{k}$

$$\text{Since } v + w = \underline{i} - 3\underline{j} + 5\underline{k} + 2\underline{i} - 4\underline{k} = 3\underline{i} - 2\underline{j} + \underline{k} = u$$

Therefore u , v and w are the sides of a triangle.

$$\text{Since } u \cdot w = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (2\underline{i} + \underline{j} - 4\underline{k}) = 6 - 2 - 4 = 0$$

$$\Rightarrow u \perp w = 0$$

$\therefore u$, v and w are the sides of a right triangle.

(ii) Show that the set of points $P = (1, 3, 2)$, $Q = (4, 1, 4)$, $R = (6, 5, 5)$ form a right triangle.

SOLUTION:

Let $P = (1, 3, 2)$, $Q = (4, 1, 4)$, $R = (6, 5, 5)$

$$\overline{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overline{PQ} = (4-1)\underline{i} + (1-3)\underline{j} + (4-2)\underline{k}$$

$$\overline{PQ} = 3\underline{i} - 2\underline{j} + 2\underline{k}$$

$$\overline{QR} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overline{QR} = (6-4)\underline{i} + (5-1)\underline{j} + (5-4)\underline{k}$$

$$\overline{QR} = 2\underline{i} + 4\underline{j} + \underline{k}$$

$$\overline{PR} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overline{PR} = (6-1)\underline{i} + (5-3)\underline{j} + (5-2)\underline{k}$$

$$\overline{PR} = 5\underline{i} + 2\underline{j} + 3\underline{k}$$

$$\therefore \overline{PQ} + \overline{QR} = 3\underline{i} - 2\underline{j} + 2\underline{k} + 2\underline{i} + 4\underline{j} + \underline{k} = 5\underline{i} + 2\underline{j} + 3\underline{k} = \overline{PR}$$

$\therefore P, Q, R$ are the vertices of a triangle.

$$\therefore \overline{PQ} \cdot \overline{QR} = (3\underline{i} - 2\underline{j} + 2\underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k}) = 6 - 8 + 2 = 0$$

$\therefore P, Q, R$ are the vertices of a right triangle.

Q.7: Show that mid point of hypotenuse of a right triangle is equidistance from its vertices.

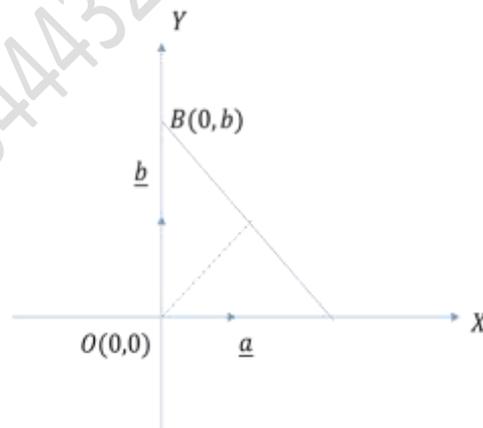
SOLUTION:

Consider a right triangle AOB such that $O(0,0)$, $A(a,0)$, $B(0,b)$.

Let M be the mid point of hypotenuse \overline{AB} such that M

$$\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = M\left(\frac{a}{2}, \frac{b}{2}\right)$$

We want to prove: $|\overline{AM}| = |\overline{BM}| = |\overline{OM}|$



$$\overline{AM} = \left(\frac{a}{2} - a\right)\underline{i} + \left(\frac{b}{2} - 0\right)\underline{j}$$

$$\overline{AM} = \left(-\frac{a}{2}\right)\underline{i} + \left(\frac{b}{2}\right)\underline{j}$$

$$|\overline{AM}| = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$|\overline{AM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2 + b^2}{4}}$$

$$|\overline{AM}| = \frac{1}{2}\sqrt{a^2 + b^2}$$

$$\overline{BM} = \left(\frac{a}{2} - 0\right)\underline{i} + \left(\frac{b}{2} - b\right)\underline{j}$$

$$\overline{BM} = \left(\frac{a}{2}\right)\underline{i} - \left(\frac{b}{2}\right)\underline{j}$$

$$|\overline{BM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$$

$$|\overline{BM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2 + b^2}{4}}$$

$$|\overline{BM}| = \frac{1}{2}\sqrt{a^2 + b^2}$$

$$\vec{OM} = \left(\frac{a}{2} - 0\right)\underline{i} + \left(\frac{b}{2} - 0\right)\underline{j}$$

$$\vec{OM} = \left(\frac{a}{2}\right)\underline{i} + \left(\frac{b}{2}\right)\underline{j}$$

$$|\vec{OM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$|\vec{OM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2+b^2}{4}}$$

$$|\vec{OM}| = \frac{1}{2}\sqrt{a^2+b^2}$$

Clearly $|\vec{AM}| = |\vec{BM}| = |\vec{OM}|$, which is the required result.

Q.8: Prove that the perpendicular bisectors of the sides of a triangle are concurrent.

SOLUTION:

Consider the triangle ABC such that D, E, F are the mid points of sides \vec{AB} , \vec{BC} , \vec{AC} respectively. **C (c)**

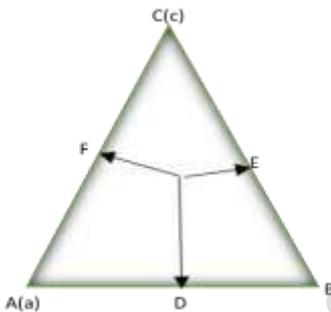
Then $\vec{OA} = \underline{a}$, $\vec{OB} = \underline{b}$, $\vec{OC} = \underline{c}$, $\vec{OD} = \frac{\underline{a}+\underline{b}}{2}$, $\vec{OE} = \frac{\underline{b}+\underline{c}}{2}$,

$$\vec{OF} = \frac{\underline{a}+\underline{c}}{2}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = \underline{b} - \underline{a}, \quad \vec{BC} = \vec{OC} - \vec{OB} = \underline{c} - \underline{b}, \quad \vec{AC} =$$

$$\vec{OC} - \vec{OA} = \underline{c} - \underline{a}$$

Let $\vec{OD} \perp \vec{AB}$, $\vec{OE} \perp \vec{BC}$ be the right bisectors which meet at 'O'



As $\vec{OD} \perp \vec{AB}$, so

$$\vec{OD} \cdot \vec{AB} = 0$$

$$\left(\frac{\underline{a}+\underline{b}}{2}\right) \cdot (\underline{b} - \underline{a}) = 0$$

$$(\underline{b}+\underline{a}) \cdot (\underline{b} - \underline{a}) = 0$$

$$b^2 - a^2 = 0 \rightarrow (i)$$

As $\vec{OE} \perp \vec{BC}$, so

$$\vec{OE} \cdot \vec{BC} = 0$$

$$\left(\frac{\underline{b}+\underline{c}}{2}\right) \cdot (\underline{c} - \underline{b}) = 0$$

$$(\underline{c}+\underline{b}) \cdot (\underline{c} - \underline{b}) = 0$$

$$c^2 - b^2 = 0 \rightarrow (ii)$$

Adding (i) and (ii)

$$b^2 - a^2 + c^2 - b^2 = 0$$

$$\Rightarrow c^2 - a^2 = 0$$

$$\Rightarrow (\underline{c}+\underline{a}) \cdot (\underline{c} - \underline{a}) = 0$$

Divide both side by 2

$$\Rightarrow \left(\frac{\underline{c}+\underline{a}}{2}\right) \cdot (\underline{c} - \underline{a}) = \frac{0}{2}$$

$$\Rightarrow \vec{OF} \cdot \vec{AC} = 0$$

$$\Rightarrow \vec{OF} \perp \vec{AC}$$

So, \vec{OF} is also a right bisector of \vec{AC} .

Thus all the right bisector are concurrent at 'O'

Q.9: Prove that the altitudes of a triangle are concurrent.

SOLUTION:

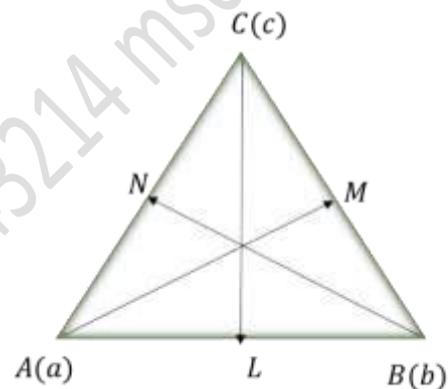
Consider the triangle ABC such that

Then $\vec{OA} = \underline{a}$, $\vec{OB} = \underline{b}$, $\vec{OC} = \underline{c}$

$$\vec{AB} = \vec{OB} - \vec{OA} = \underline{b} - \underline{a}, \quad \vec{BC} = \vec{OC} - \vec{OB} = \underline{c} - \underline{b}, \quad \vec{AC} =$$

$$\vec{OC} - \vec{OA} = \underline{c} - \underline{a}$$

Let $\vec{AM} \perp \vec{BC}$, $\vec{CL} \perp \vec{AB}$ be the altitudes of a triangle which meet at 'O'



As $\vec{AM} \perp \vec{BC}$

so $\vec{OA} \perp \vec{BC}$

$$\Rightarrow \vec{OA} \cdot \vec{BC} = 0$$

$$\underline{a} \cdot (\underline{c} - \underline{b}) = 0$$

$$\underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{b} = 0 \rightarrow (i)$$

As $\vec{CL} \perp \vec{AB}$

so $\vec{OC} \perp \vec{AB}$

$$\Rightarrow \vec{OC} \cdot \vec{AB} = 0$$

$$\underline{c} \cdot (\underline{b} - \underline{a}) = 0$$

$$\underline{c} \cdot \underline{b} - \underline{c} \cdot \underline{a} = 0 \rightarrow (ii)$$

Adding (i) and (ii)

$$\underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{b} + \underline{c} \cdot \underline{b} - \underline{c} \cdot \underline{a} = 0$$

$$\Rightarrow \underline{c} \cdot \underline{b} - \underline{a} \cdot \underline{b} = 0$$

$$\Rightarrow (\underline{c}-\underline{a}) \cdot \underline{b} = 0$$

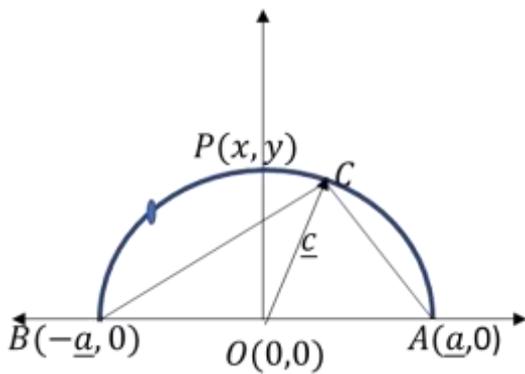
$$\Rightarrow \vec{AC} \cdot \vec{OB} = 0$$

$$\Rightarrow \vec{OB} \perp \vec{AC}$$

$\Rightarrow \vec{BN} \perp \vec{AC}$ & passing through; O'

So altitude of the triangle are concurrent at 'o'

Q.10: Prove that the angle in a semi circle is a right angle. (2017 L.Q)



Consider a semicircle APB with center at origin $O(0,0)$ and A, B are ends of diameter having position vectors $\underline{a}, -\underline{a}$ respectively.

Let C be any point on semicircle having position vector \underline{c} .

Clearly $|\underline{a}| = |-\underline{a}| = |\underline{c}|$ (radii of semicircle) (i)

Now $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \underline{c} - \underline{a}$

And $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \underline{c} - (-\underline{a}) = \underline{c} + \underline{a}$

Consider $\overrightarrow{AC} \cdot \overrightarrow{BC} = (\underline{c} - \underline{a}) \cdot (\underline{c} + \underline{a})$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = \underline{c} \cdot (\underline{c} + \underline{a}) - \underline{a} \cdot (\underline{c} + \underline{a})$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = \underline{c} \cdot \underline{c} + \underline{c} \cdot \underline{a} - \underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{a}$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = c^2 + \underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{c} - a^2 \quad \because \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a}$$

\underline{a}

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = c^2 - a^2$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = c^2 - a^2 \quad \because |\underline{a}| =$$

$|\underline{c}|$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$$

$$\Rightarrow \overrightarrow{AC} \perp \overrightarrow{BC}$$

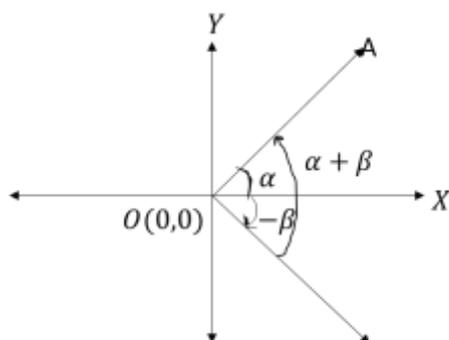
Thus the angle in a semicircle is a right angle

Q.11: Prove that $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

SOLUTION:

Consider two unit vectors \overrightarrow{OA} and \overrightarrow{OB} in XY-plane making angle $\alpha, -\beta$ with positive axis respectively

So that $m\angle AOB = \alpha + \beta$



$$\overrightarrow{OA} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$$

$$\overrightarrow{OB} = \cos(-\beta) \underline{i} + \sin(-\beta) \underline{j} = \cos \beta \underline{i} - \sin \beta \underline{j}$$

Taking dot product of \overrightarrow{OA} and \overrightarrow{OB}

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = (\cos \alpha \underline{i} + \sin \alpha \underline{j}) \cdot (\cos \beta \underline{i} - \sin \beta \underline{j})$$

$$|\overrightarrow{OA}| |\overrightarrow{OB}| \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$1 \cdot 1 \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad \because |\overrightarrow{OA}| =$$

$$|\overrightarrow{OB}| = 1$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Q.12: Prove that in any triangle ABC

i) $b = c \cos A + a \cos C$

SOLUTION:

Consider a triangle ABC such that

$$\overrightarrow{BC} = \underline{a}, \overrightarrow{CA} = \underline{b}, \overrightarrow{AB} = \underline{c}$$

$$\therefore \underline{a} + \underline{b} + \underline{c} = 0$$

$$\underline{b} = -\underline{a} - \underline{c}$$

Taking dot product with \underline{b}

$$\underline{b} \cdot \underline{b} = (-\underline{a} - \underline{c}) \cdot \underline{b}$$

$$b^2 = -\underline{a} \cdot \underline{b} - \underline{c} \cdot \underline{b}$$

$$b^2 = -|\underline{a}| |\underline{b}| \cos(\pi - C) - |\underline{c}| |\underline{b}| \cos(\pi - A)$$

$$\because \cos(\pi - C) = (-\cos C)$$

$$\because \cos(\pi - A) = (-\cos A)$$

$$b^2 = -ab(-\cos C) - cb(-\cos A)$$

$$b^2 = ab \cos C + cb \cos A$$

Taking b common

$$b^2 = b[a \cos C + c \cos A]$$

Cancel b on both sides

$$b = a \cos C + c \cos A$$

ii) $c = a \cos B + b \cos A$

SOLUTION:

Consider a triangle ABC such that

$$\overrightarrow{BC} = \underline{a}, \overrightarrow{CA} = \underline{b}, \overrightarrow{AB} = \underline{c}$$

$$\therefore \underline{a} + \underline{b} + \underline{c} = 0$$

$$\underline{c} = -\underline{a} - \underline{b}$$

Taking dot product with \underline{c}

$$\underline{c} \cdot \underline{c} = (-\underline{a} - \underline{b}) \cdot \underline{c}$$

$$c^2 = -\underline{a} \cdot \underline{c} - \underline{b} \cdot \underline{c}$$

$$\because \cos(\pi - B) = (-\cos B)$$

$$\because \cos(\pi - A) = (-\cos A)$$

$$c^2 = -|\underline{a}| |\underline{c}| \cos(\pi - B) - |\underline{b}| |\underline{c}| \cos(\pi - A)$$

$$c^2 = -ac(-\cos B) - bc(-\cos A)$$

$$c^2 = ac \cos B + bc \cos A$$

Taking c common

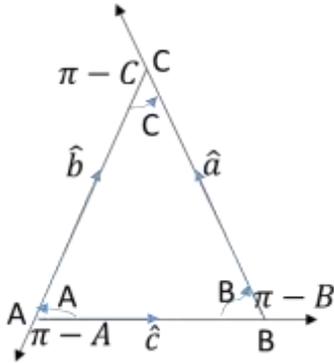
$$c^2 = c[a \cos C + c \cos A]$$

Cancel c on both sides

$$c = a \cos B + c \cos A$$

iii) $b^2 = c^2 + a^2 - 2ca \cos B$

SOLUTION:



Consider a triangle ABC such that

$$\vec{BC} = \underline{a}, \vec{CA} = \underline{b}, \vec{AB} = \underline{c}$$

$$\therefore \underline{a} + \underline{b} + \underline{c} = \underline{0}$$

$$\underline{b} = -\underline{a} - \underline{c} = -(\underline{a} + \underline{c})$$

Squaring on both sides

$$(\underline{b})^2 = [-(\underline{a} + \underline{c})]^2$$

$$(\underline{b})^2 = (\underline{a} + \underline{c})^2$$

$$\underline{b} \cdot \underline{b} = (\underline{a} + \underline{c}) \cdot (\underline{a} + \underline{c})$$

$$b^2 = \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{c} + \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{c}$$

$$b^2 = a^2 + c^2 + 2\underline{a} \cdot \underline{c}$$

$$b^2 = a^2 + c^2 + 2|\underline{a}||\underline{c}| \cos(\pi - B)$$

$$\therefore \cos(\pi - B) = (-\cos B)$$

$$b^2 = a^2 + c^2 + 2ca(-\cos B)$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

iii) $c^2 = a^2 + b^2 - 2ab \cos C$

SOLUTION:

Consider a triangle ABC such that

$$\vec{BC} = \underline{a}, \vec{CA} = \underline{b}, \vec{AB} = \underline{c}$$

$$\therefore \underline{a} + \underline{b} + \underline{c} = \underline{0}$$

$$\underline{c} = -\underline{a} - \underline{b} = -(\underline{a} + \underline{b})$$

squaring on both sides

$$(\underline{c})^2 = [-(\underline{a} + \underline{b})]^2$$

$$(\underline{c})^2 = (\underline{a} + \underline{b})^2$$

$$\underline{c} \cdot \underline{c} = (\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b})$$

$$c^2 = \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{a} + \underline{b} \cdot \underline{b}$$

$$c^2 = a^2 + b^2 + 2\underline{a} \cdot \underline{b}$$

$$c^2 = a^2 + b^2 + 2|\underline{a}||\underline{b}| \cos(\pi - C)$$

$$\therefore \cos(\pi - C) = (-\cos C)$$

$$c^2 = a^2 + b^2 + 2ab(-\cos C)$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

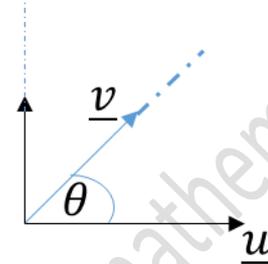
The cross product (vector product)

let \underline{u} and \underline{v} be two non-zero vectors. The cross or vector product of \underline{u} and \underline{v} written as $\underline{u} \times \underline{v}$ is define

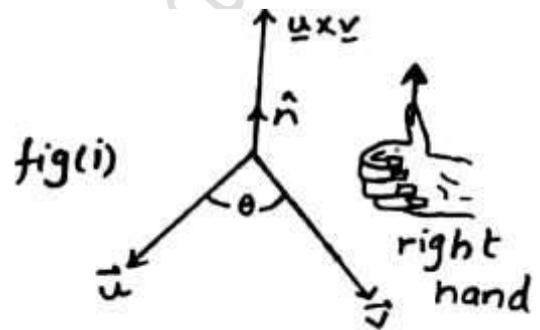
$$\text{as } \underline{u} \times \underline{v} = (|\underline{u}| |\underline{v}| \sin \theta) \hat{n}$$

Where θ is the angle b/w the vectors s . that

$0 \leq \theta \leq \pi$ and \hat{n} is a unit vector \perp ar to the plane of \underline{u} and \underline{v} written direction is given by the right hand rule.



Right hand rule:



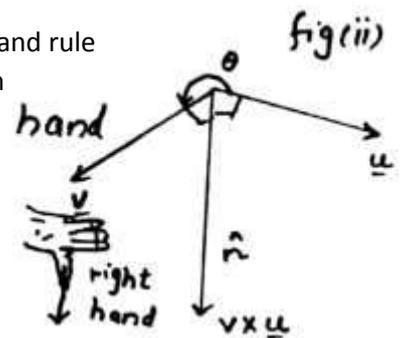
if the figure right hand point along the vectors \underline{u} and the curl towards the vector \underline{v} , then the thumbs will give

The direction of \hat{n} which is $\underline{u} \times \underline{v}$

In fig.(ii) the right hand rule

Shows the direction

Of $\underline{v} \times \underline{u}$



Derivative of uses full results of cross products:

For unit vector $\underline{i}, \underline{j}, \underline{k}$ we have.

$$(a) \underline{i} \times \underline{i} = |\underline{i}| |\underline{i}| \sin 0 \hat{n} = 0$$

$$\underline{j} \times \underline{j} = |\underline{j}| |\underline{j}| \sin 0 \hat{n} = 0$$

$$\underline{k} \times \underline{k} = |\underline{k}| |\underline{k}| \sin 0 \hat{n} = 0$$

$$(b) \underline{i} \times \underline{j} = |\underline{i}| |\underline{j}| \sin 90^\circ \hat{k} =$$

$$\underline{j} \times \underline{k} = |\underline{j}| |\underline{k}| \sin 90^\circ \hat{i} = \hat{i}$$

$$\underline{k} \times \underline{i} = |\underline{k}| |\underline{i}| \sin 90^\circ \hat{j} = \hat{j}$$

$$(c) \text{ Prove that } \underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

$$\underline{u} \times \underline{v} = |\underline{u}| |\underline{v}| \sin \theta \hat{n}$$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \sin(-\theta) \hat{n}$$

$$\underline{u} \cdot \underline{v} = |\underline{v}| |\underline{u}| \sin \theta \hat{n}$$

$$\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

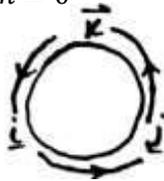
$$\underline{u} \times \underline{u} = |\underline{u}| |\underline{u}| \sin 0^\circ \hat{n} = 0$$

Note:

The cross product of \underline{i} , \underline{j} and \underline{k}

are written in cyclic pattern

It is helpful to remember.



Properties of cross product:

$$(i) \underline{u} \times \underline{v} = 0 \text{ if } \underline{u} = 0 \text{ or } \underline{v} = 0$$

$$(ii) \underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

$$(iii) \underline{u} \times (\underline{v} \times \underline{w}) = \underline{u} \times \underline{v} + \underline{u} \times \underline{w} \text{ (distributive property)}$$

$$(iv) \underline{u} \times (k\underline{v}) = k(\underline{u} \times \underline{v}) = k(\underline{u} \times \underline{v})$$

$$(v) \underline{u} \times \underline{u} = 0$$

Analytical Expression of $\underline{u} \times \underline{v}$

Let

$$\underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k} \text{ and}$$

$$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}$$

$$\underline{u} \times \underline{v} = (a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}) \times (a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k})$$

$$= (a_1 a_2) (\underline{i} \times \underline{i}) + a_1 b_2 (\underline{i} \times \underline{j}) + a_1 c_2 (\underline{i} \times \underline{k})$$

$$+ b_1 a_2 (\underline{j} \times \underline{i})$$

$$+ b_1 b_2 (\underline{j} \times \underline{j}) + b_1 c_2 (\underline{j} \times \underline{k}) + c_1 a_2 (\underline{k} \times \underline{i})$$

$$+ c_1 b_2 (\underline{k} \times \underline{j}) + c_1 c_2 (\underline{k} \times \underline{k})$$

$$\Rightarrow a_1 b_2 \underline{k} + a_1 c_2 \underline{j} - b_1 a_2 \underline{k} + b_1 a_2 \underline{i} + c_1 a_2 \underline{j} - c_1 b_2 \underline{i}$$

Rearrange we have,

$$\Rightarrow \underline{u} \times \underline{v} = (b_1 c_2 - c_1 b_2) \underline{i} - (a_1 c_2 - c_1 a_2) \underline{j} +$$

$$(a_1 b_2 - a_2 b_1) \underline{k}$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1 c_2 - c_1 b_2) \underline{i} - (a_1 c_2 - c_1 a_2) \underline{j} +$$

$$+ (a_1 b_2 - a_2 b_1) \underline{k}$$

$$\Rightarrow (b_1 c_2 - c_1 b_2) \underline{i} - (a_1 c_2 - c_1 a_2) \underline{j} +$$

$$(a_1 b_2 - a_2 b_1) \underline{k}$$

Parallel vectors:

If \underline{u} and \underline{v} are ||vectors $\theta = 0$, $\sin 0^\circ = 0$ then

$$\underline{u} \times \underline{v} = |\underline{u}| |\underline{v}| \sin \theta \hat{n} = 0 \Rightarrow \underline{u} \times \underline{v} = 0$$

and if $\underline{u} \times \underline{v} = 0$ then either $\sin \theta = 0$

or $\underline{u} = 0$ or $\underline{v} = 0$

(i) if $\sin \theta = 0 \Rightarrow \theta = 180^\circ$ or 0° which shows

That vectors \underline{u} and \underline{v} are parallel.

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if $\underline{u} = 0$ or \underline{v} then since zero vector has no specific direction, so zero vector is ||to every vector.

Note:

Zero vector is both ||to \perp ar to every vector.

Area of parallelogram:

Exercise NO 7.4

Q.1: Compute the cross product $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$. Check your answer by showing that each \underline{a} and \underline{b} is perpendicular to $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$.

$$i) \underline{a} = 2\underline{i} + \underline{j} - \underline{k}, \underline{b} = \underline{i} - \underline{j} + \underline{k}$$

SOLUTION:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$$

$$= 0\underline{i} - 3\underline{j} - 3\underline{k}$$

$$\text{Now } \underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \underline{i}(1-1) - \underline{j}(-1-2) + \underline{k}(1+2)$$

$$= 0\underline{i} + 3\underline{j} + 3\underline{k}$$

As

$$\underline{a} \cdot \underline{a} \times \underline{b} = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} - 3\underline{j} - 3\underline{k})$$

$$= 0 - 3 + 3 = 0$$

So $\underline{a} \perp \underline{a} \times \underline{b}$

As

$$\underline{a} \cdot \underline{b} \times \underline{a} = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} + 3\underline{j} + 3\underline{k})$$

$$= 0 + 3 - 3 = 0$$

So $\underline{a} \perp \underline{b} \times \underline{a}$

As

$$\underline{b} \cdot \underline{a} \times \underline{b} = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} - 3\underline{j} - 3\underline{k})$$

$$= 0 - 3 + 3 = 0$$

So $\underline{b} \perp \underline{a} \times \underline{b}$

As

$$\underline{b} \cdot \underline{b} \times \underline{a} = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} + 3\underline{j} + 3\underline{k})$$

$$= 0 - 3 + 3 = 0$$

So $\underline{b} \perp \underline{b} \times \underline{a}$

$$ii) \underline{a} = \underline{i} + \underline{j}, \underline{b} = \underline{i} - \underline{j}$$

SOLUTION:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \underline{i}(0+0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$= 0\underline{i} - 0\underline{j} - 2\underline{k}$$

$$\text{Now } \underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(1+1)$$

$$= 0\underline{i} - 0\underline{j} + 2\underline{k}$$

As

$$\underline{a} \cdot \underline{a} \times \underline{b} = (\underline{i} + \underline{j}) \cdot (0\underline{i} - 0\underline{j} - 2\underline{k})$$

$$= 0 - 0 - 0 = 0$$

So $\underline{a} \perp \underline{a} \times \underline{b}$

As

$$\underline{a} \cdot \underline{b} \times \underline{a} = (\underline{i} + \underline{j}) \cdot (0\underline{i} - 0\underline{j} + 2\underline{k})$$

$$= 0 + 0 + 0 = 0$$

So $\underline{a} \perp \underline{b} \times \underline{a}$

As

$$\underline{b} \cdot \underline{a} \times \underline{b} = (\underline{i} - \underline{j}) \cdot (0\underline{i} - 0\underline{j} - 2\underline{k})$$

$$= 0 + 0 - 0 = 0$$

So $\underline{b} \perp \underline{a} \times \underline{b}$

As

$$\underline{b} \cdot \underline{b} \times \underline{a} = (\underline{i} - \underline{j}) \cdot (0\underline{i} - 0\underline{j} + 2\underline{k})$$

$$= 0 + 0 - 0 = 0$$

So $\underline{b} \perp \underline{b} \times \underline{a}$

iii) $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}, \underline{b} = \underline{i} + \underline{j}$

SOLUTION:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \underline{i}(0-1) - \underline{j}(0-1) + \underline{k}(3+2)$$

$$= -1\underline{i} + \underline{j} + 5\underline{k}$$

Now $\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$

$$= \underline{i}(1+0) - \underline{j}(1-0) + \underline{k}(-2-3)$$

$$= 1\underline{i} - \underline{j} - 5\underline{k}$$

As

$$\underline{a} \cdot \underline{a} \times \underline{b} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (-1\underline{i} + \underline{j} + 5\underline{k})$$

$$= -3 - 2 + 5 = 0$$

So $\underline{a} \perp \underline{a} \times \underline{b}$

As

$$\underline{a} \cdot \underline{b} \times \underline{a} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (1\underline{i} - \underline{j} - 5\underline{k})$$

$$= 3 + 2 - 5 = 0$$

So $\underline{a} \perp \underline{b} \times \underline{a}$

As

$$\underline{b} \cdot \underline{a} \times \underline{b} = (\underline{i} + \underline{j}) \cdot (-1\underline{i} + \underline{j} + 5\underline{k})$$

$$= -1 + 1 + 0 = 0$$

So $\underline{b} \perp \underline{a} \times \underline{b}$

As

$$\underline{b} \cdot \underline{b} \times \underline{a} = (\underline{i} + \underline{j}) \cdot (1\underline{i} - \underline{j} - 5\underline{k})$$

$$= 1 - 1 + 0 = 0$$

So $\underline{b} \perp \underline{b} \times \underline{a}$

Q.2: Find a unit vector perpendicular to the plane containing \underline{a} and \underline{b} . Also find sine of the angle between them:

i) $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}, \underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$ (2019 S.Q)

SOLUTION:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$

$$= \underline{i}(6+9) - \underline{j}(-2+12) + \underline{k}(6+24)$$

$$= 15\underline{i} - 10\underline{j} + 30\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{15^2 + (-10)^2 + 30^2}$$

$$|\underline{a} \times \underline{b}| = \sqrt{225 + 100 + 900} = \sqrt{1225} = 35$$

Required unit vector = $\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$$\hat{n} = \frac{15\underline{i} - 10\underline{j} + 30\underline{k}}{35} = \frac{15}{35}\underline{i} + \frac{10}{35}\underline{j} + \frac{30}{35}\underline{k} = \frac{3}{7}\underline{i} + \frac{2}{7}\underline{j} + \frac{6}{7}\underline{k}$$

If θ is the angle between \underline{a} and \underline{b} , then

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{35}{\sqrt{2^2 + (-6)^2 + (-3)^2} \cdot \sqrt{4^2 + 3^2 + (-1)^2}}$$

$$\sin \theta = \frac{35}{\sqrt{4+36+9} \cdot \sqrt{16+9+1}} = \frac{35}{\sqrt{49} \cdot \sqrt{26}} = \frac{35}{7 \cdot \sqrt{26}} = \frac{5}{\sqrt{26}}$$

ii) $\underline{a} = -\underline{i} - \underline{j} - \underline{k}, \underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

SOLUTION:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$$= \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2)$$

$$= -7\underline{i} + 2\underline{j} + 5\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + 2^2 + 5^2}$$

$$|\underline{a} \times \underline{b}| = \sqrt{49 + 4 + 25} = \sqrt{78}$$

Required unit vector = $\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$$\hat{n} = \frac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}} = \frac{-7}{\sqrt{78}}\underline{i} + \frac{2}{\sqrt{78}}\underline{j} + \frac{5}{\sqrt{78}}\underline{k}$$

If θ is the angle between \underline{a} and \underline{b} , then

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{\sqrt{78}}{\sqrt{(-1)^2 + (-1)^2 + (-1)^2} \cdot \sqrt{2^2 + (-3)^2 + 4^2}}$$

$$\sin \theta = \frac{\sqrt{78}}{\sqrt{1+1+1} \cdot \sqrt{4+9+16}} = \frac{\sqrt{78}}{\sqrt{3} \cdot \sqrt{29}} = \sqrt{\frac{78}{3}} \cdot \frac{1}{\sqrt{29}} = \sqrt{26} \cdot \frac{1}{\sqrt{29}} = \sqrt{\frac{26}{29}}$$

iii) $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k}$, $\underline{b} = -\underline{i} + \underline{j} - 2\underline{k}$ (2016 S.Q)

SOLUTION:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= \underline{i}(4-4) - \underline{j}(-4+4) + \underline{k}(2-2)$$

$$= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0} \quad (\text{null vector})$$

$$|\underline{a} \times \underline{b}| = \sqrt{0^2 + 0 + 0^2}$$

$$|\underline{a} \times \underline{b}| = 0$$

As $\underline{a} \times \underline{b} = 0$

$\Rightarrow \underline{a}$ and \underline{b} are parallel.

$$\text{Also } \underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k} = -2(-\underline{i} + \underline{j} - 2\underline{k}) = -2\underline{b}$$

$\Rightarrow \underline{a}$ and \underline{b} are parallel but opposite in direction.

Thus the angle between them is 180°

$$\therefore \theta = 180^\circ$$

$$\Rightarrow \sin \theta = 0$$

iv) $\underline{a} = \underline{i} + \underline{j}$, $\underline{b} = \underline{i} - \underline{j}$

SOLUTION:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \underline{i}(0+0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$= 0\underline{i} - 0\underline{j} - 2\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{0^2 + (-0)^2 + (-2)^2}$$

$$|\underline{a} \times \underline{b}| = \sqrt{0+0+4} = \sqrt{4} = 2$$

Required unit vector = $\frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$$\hat{n} = \frac{0\underline{i} - 0\underline{j} - 2\underline{k}}{2} = \frac{0}{2}\underline{i} + \frac{0}{2}\underline{j} + \frac{-2}{2}\underline{k} = 0\underline{i} - 0\underline{j} - \underline{k} = -\underline{k}$$

If θ is the angle between \underline{a} and \underline{b} , then

$$\sin \theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{2}{\sqrt{1^2+1^2+0} \cdot \sqrt{1^2+(-1)^2+0^2}}$$

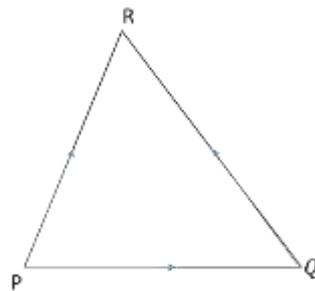
$$\sin \theta = \frac{2}{\sqrt{1+1+0} \cdot \sqrt{1+1+0}} = \frac{2}{\sqrt{2} \cdot \sqrt{2}} = \frac{2}{(\sqrt{2})^2} = \frac{2}{2} = 1$$

Q.3: Find the area of the triangle, determined by the points P, Q, R.

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(i) P(0,0,0) ; Q(2,3,2) ; R(-1,1,4)

SOLUTION:



$$\overrightarrow{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PQ} = (2-0)\underline{i} + (3-0)\underline{j} + (2-0)\underline{k}$$

$$\overrightarrow{PQ} = 2\underline{i} + 3\underline{j} + 2\underline{k}$$

$$\overrightarrow{PR} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PR} = (-1-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k}$$

$$\overrightarrow{PR} = -\underline{i} + \underline{j} + 4\underline{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

$$= \underline{i}(12-2) - \underline{j}(8+2) + \underline{k}(2+3)$$

$$= 10\underline{i} - 10\underline{j} + 5\underline{k}$$

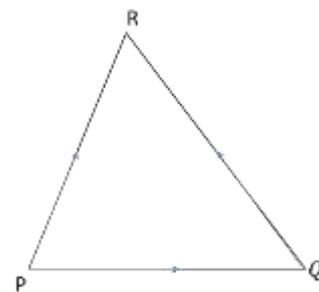
$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{10^2 + (-10)^2 + 5^2}$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{100 + 100 + 25} = \sqrt{225} = 15$$

$$\text{Area of } \Delta PQR = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \cdot 15 = \frac{15}{2} \text{ square unit}$$

(ii) P(1,-1,-1) ; Q(2,0,-1) ; R(0,2,1)

SOLUTION:



$$\overrightarrow{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PQ} = (2-1)\underline{i} + (0+1)\underline{j} + (-1+1)\underline{k}$$

$$\overrightarrow{PQ} = \underline{i} + \underline{j} + 0\underline{k}$$

$$\overrightarrow{PR} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{PR} = (0-1)\underline{i} + (2+1)\underline{j} + (1+1)\underline{k}$$

$$\overrightarrow{PR} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

$$= \underline{i}(2-0) - \underline{j}(2+0) + \underline{k}(3+1)$$

$$= 2\underline{i} - 2\underline{j} + 4\underline{k}$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{2^2 + (-2)^2 + 4^2}$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{4 + 4 + 16} = \sqrt{24} = \sqrt{2 \times 6} = 2\sqrt{6}$$

$$\text{Area of } \Delta PQR = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \cdot 2\sqrt{6} = \sqrt{6} \text{ square unit}$$

Q.4: Find the area of the parallelogram whose vertices are:

i) $A(0,0,0)$; $B(1,2,3)$; $C(2, -1,1)$; $D(3,1,4)$

SOLUTION:

$$\overrightarrow{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{AB} = (1 - 0)\underline{i} + (2 - 0)\underline{j} + (3 - 0)\underline{k}$$

$$\overrightarrow{AB} = \underline{i} + 2\underline{j} + 3\underline{k}$$

$$\overrightarrow{AD} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{AD} = (3 - 0)\underline{i} + (1 - 0)\underline{j} + (4 - 0)\underline{k}$$

$$\overrightarrow{AD} = 3\underline{i} + \underline{j} + 4\underline{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{vmatrix}$$

$$= \underline{i}(8-3) - \underline{j}(4-9) + \underline{k}(1-6)$$

$$= 5\underline{i} + 5\underline{j} - 5\underline{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{5^2 + 5^2 + (-5)^2}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{25 + 25 + 25} = \sqrt{75} = 5\sqrt{3}$$

$$\text{Area of } \Delta PQR = |\overrightarrow{AB} \times \overrightarrow{AD}| = 5\sqrt{3} \text{ sq. unit}$$

ii) $A(1,2, -1)$; $B(4,2, -3)$; $C(6, -5,2)$; $D(9, -5,0)$

SOLUTION:

$$\overrightarrow{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{AB} = (4 - 1)\underline{i} + (2 - 2)\underline{j} + (-3 + 1)\underline{k}$$

$$\overrightarrow{AB} = 3\underline{i} + 0\underline{j} - 2\underline{k}$$

$$\overrightarrow{AD} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{AD} = (9 - 1)\underline{i} + (-5 - 2)\underline{j} + (0 + 1)\underline{k}$$

$$\overrightarrow{AD} = 8\underline{i} - 7\underline{j} + \underline{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 8 & -7 & 1 \end{vmatrix}$$

$$= \underline{i}(0-14) - \underline{j}(3+16) + \underline{k}(-21-0)$$

$$= -14\underline{i} - 19\underline{j} - 21\underline{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(-14)^2 + (-19)^2 + (-21)^2}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{196 + 361 + 441} = \sqrt{998}$$

$$\text{Area of } \Delta PQR = |\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{998} \text{ sq. unit}$$

iii) $A(-1,1,1)$; $B(-1,2,2)$; $C(-3, 4, -5)$; $D(-3,5, -4)$

SOLUTION:

$$\overrightarrow{AB} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{AB} = (-1 + 1)\underline{i} + (2 - 1)\underline{j} + (2 - 1)\underline{k}$$

$$\overrightarrow{AB} = 0\underline{i} + \underline{j} + \underline{k}$$

$$\overrightarrow{AD} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\overrightarrow{AD} = (-3 + 1)\underline{i} + (5 - 1)\underline{j} + (4 - 1)\underline{k}$$

$$\overrightarrow{AD} = -2\underline{i} + 4\underline{j} - 5\underline{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 4 & -5 \end{vmatrix}$$

$$= \underline{i}(-5-4) - \underline{j}(0+2) + \underline{k}(0+2)$$

$$= -9\underline{i} - 2\underline{j} + 2\underline{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(-9)^2 + (-2)^2 + 2^2}$$

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{81 + 4 + 4} = \sqrt{89}$$

$$\text{Area of } \Delta PQR = |\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{89} \text{ sq. unit}$$

Q.5: Which vectors if any, are perpendicular or parallel:

i) $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}$; $\underline{v} = \underline{j} - 5\underline{k}$; $\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

SOLUTION:

$$\text{As } \underline{u} \cdot \underline{v} = (5\underline{i} - \underline{j} + \underline{k}) \cdot (\underline{j} - 5\underline{k}) = 0 - 1 - 5 = -6 \neq 0$$

So \underline{u} and \underline{v} are not perpendicular.

$$\text{As } \underline{u} \cdot \underline{w} = (5\underline{i} - \underline{j} + \underline{k}) \cdot (-15\underline{i} + 3\underline{j} - 3\underline{k}) = -75 - 3 - 3 = -81 \neq 0$$

So \underline{u} and \underline{w} are not perpendicular.

$$\text{As } \underline{v} \cdot \underline{w} = (\underline{j} - 5\underline{k}) \cdot (-15\underline{i} + 3\underline{j} - 3\underline{k}) = 0 + 3 + 15 = 18 \neq 0$$

So \underline{u} and \underline{w} are not perpendicular.

$$\text{Now } \underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$$

$$\underline{w} = -3(5\underline{i} - \underline{j} + \underline{k})$$

$$\underline{w} = -3\underline{u}$$

$$\Rightarrow \underline{u} \parallel \underline{w}$$

$$\text{ii) } \underline{u} = \underline{i} + 2\underline{j} - \underline{k} ; \underline{v} = -\underline{i} + \underline{j} + \underline{k} ; \underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$$

SOLUTION:

$$\text{As } \underline{u} \cdot \underline{v} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (-\underline{i} + \underline{j} + \underline{k}) = -1 + 2 - 1 = 0$$

So \underline{u} and \underline{v} are perpendicular.

$$\text{As } \underline{u} \cdot \underline{w} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}) = -\frac{\pi}{2} - 2\pi - \frac{\pi}{2} = -3\pi \neq 0$$

So \underline{u} and \underline{w} are not perpendicular.

$$\text{As } \underline{v} \cdot \underline{w} = (-\underline{i} + \underline{j} + \underline{k}) \cdot (-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}) = \frac{\pi}{2} - \pi + \frac{\pi}{2} = \frac{\pi - 2\pi + \pi}{2} = 0$$

So \underline{u} and \underline{w} are perpendicular.

$$\text{Now } \underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k} = \frac{-\pi i - 2\pi j + \pi k}{2}$$

$$\underline{w} = -\frac{\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$$

$$\underline{w} = -\frac{\pi}{2}\underline{u}$$

$$\Rightarrow \underline{u} \parallel \underline{w} \text{ but opposite in direction.}$$

$$\text{Q.6: Prove that } \underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = \underline{0}$$

SOLUTION: (2015 S.Q)

$$\text{L.H.S} = \underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$$

$$= (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c}) + (\underline{b} \times \underline{a}) + (\underline{c} \times \underline{a}) + (\underline{c} \times \underline{b})$$

$$\because \underline{b} \times \underline{a} = -(\underline{a} \times \underline{b}) \text{ \& } \underline{c} \times \underline{a} = -(\underline{a} \times \underline{c}) \text{ \& } \underline{c} \times \underline{b} = -(\underline{b} \times \underline{c})$$

$$= (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c}) - (\underline{a} \times \underline{b}) - (\underline{a} \times \underline{c}) - (\underline{b} \times \underline{c})$$

$$= \underline{0} \text{ Q.7: If } \underline{a} + \underline{b} + \underline{c} = \underline{0}, \text{ then prove that } \underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$$

SOLUTION: As $\underline{a} + \underline{b} + \underline{c} = \underline{0}$

$$\underline{0} = \text{R.H.S}$$

Taking cross product with \underline{a}

$$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{0} \times \underline{a}$$

$$\underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = \underline{0}$$

$$\because \underline{a} \times \underline{a} = \underline{0}$$

$$\underline{0} + \underline{a} \times \underline{b} - (\underline{c} \times \underline{a}) = \underline{0}$$

$$\underline{a} \times \underline{b} = \underline{c} \times \underline{a} \quad (1)$$

Taking cross product with \underline{b}

$$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{0} \times \underline{b}$$

$$\underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = \underline{0}$$

$$\because \underline{b} \times \underline{b} = \underline{0}$$

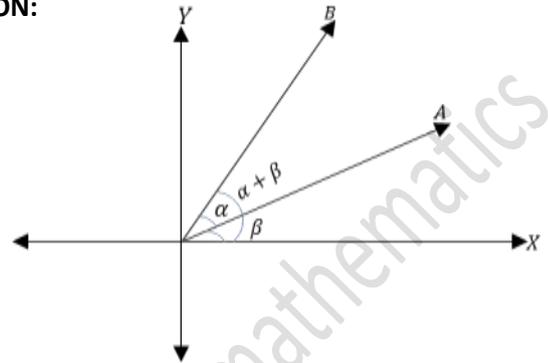
$$\underline{b} \times \underline{a} + \underline{0} - (\underline{c} \times \underline{b}) = \underline{0}$$

$$\underline{a} \times \underline{b} = \underline{c} \times \underline{a} \quad (2)$$

From (1) & (2), we conclude that $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

Q.8: Prove that $\sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ (2015 L.Q)

SOLUTION:



Consider two unit vectors \overrightarrow{OA} and \overrightarrow{OB} in xy -plane making angle α, β with positive x -axis respectively.

Such that $m\angle AOB = \alpha - \beta$

$$\overrightarrow{OA} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$$

$$\overrightarrow{OB} = \cos \beta \underline{i} + \sin \beta \underline{j}$$

Taking cross product of \overrightarrow{OB} and \overrightarrow{OA}

$$\overrightarrow{OB} \times \overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix}$$

$$|\overrightarrow{OB}| |\overrightarrow{OA}| \sin(\alpha - \beta) \underline{k} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(\sin \alpha \cos \beta - \cos \alpha \sin \beta)$$

$$1.1 \sin(\alpha - \beta) \underline{k} = \underline{k}(\sin \alpha \cos \beta - \cos \alpha \sin \beta)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Q.9: if $\underline{a} \times \underline{b} = \underline{0}$ and $\underline{a} \cdot \underline{b} = \underline{0}$ what conclusion can be drawn about \underline{a} or \underline{b}

SOLUTION:

$$\because \underline{a} \times \underline{b} = \underline{0}$$

$$\Rightarrow |\underline{a}| |\underline{b}| \sin \theta = 0 \Rightarrow \sin \theta = 0$$

$$\Rightarrow \theta = \sin^{-1} 0 = 0$$

$$\Rightarrow \theta = 0, \pi \text{ so } \underline{a} \text{ and } \underline{b} \text{ are parallel.}$$

$$\text{And } \underline{a} \cdot \underline{b} = 0$$

$$\Rightarrow |\underline{a}| |\underline{b}| \cos \theta = 0 \Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = \cos^{-1}(0) = 90^\circ \Rightarrow \theta = 90^\circ$$

so \underline{a} and \underline{b} are perpendicular. at the same time $||$ and linear not possible so one vector should be zero or null.

Triple product of vectors:

There are two types of triple product of vectors:

(a) Scalar triple product:

$$(\underline{u} \times \underline{v}) \cdot \underline{w} \text{ or } \underline{u} \cdot (\underline{v} \times \underline{w})$$

(b) Vector triple product:

$$\underline{u} \times (\underline{v} \times \underline{w})$$

in this section we shall study the scalar triple product only.

Definition:

Let $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$

$$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k} \text{ and}$$

$\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k}$ be three vectors, the scalar

Triple product of vectors $\underline{u}, \underline{v}$ and \underline{w} is defined by

$$(\underline{u} \times \underline{v}) \cdot \underline{w} \text{ or } \underline{u} \cdot (\underline{v} \times \underline{w}) \text{ or } \underline{w} \cdot (\underline{u} \times \underline{v}). \text{ the scalar}$$

product $\underline{u} \cdot (\underline{v} \times \underline{w})$ is written as;

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = [\underline{u}, \underline{v}, \underline{w}]$$

Analytic Expression of $\underline{u} \cdot (\underline{v} \times \underline{w})$

$$\text{let } \underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$$

$$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k} \text{ and}$$

$$\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k}$$

$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \underline{i}(b_2c_3 - c_2b_3) - \underline{j}(a_2c_3 - c_2a_3) + \underline{k}(a_2b_3 - b_2a_3)$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Which determine formula for scalar triple product of $\underline{u}, \underline{v}$ and \underline{w}

Prove that:

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$$

Proof: we know that

$$\text{let } \underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$$

$$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k} \text{ and}$$

$$\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k}$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ interchange } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix} \text{ interchange } R_1 \text{ and } R_2$$

$$\therefore \underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u})$$

Now

$$\underline{v} \cdot (\underline{w} \times \underline{u}) = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$= - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ interchange } R_1 \text{ and } R_2$$

$$= \begin{vmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \text{ interchange } R_1 \text{ and } R_2$$

$$\therefore \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$$

$$\text{Hence } \underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$$

Note: (i) Dot and cross may be interchanged with out altering the value

$$\text{i.e. } (\underline{u} \times \underline{v}) \cdot \underline{w} = \underline{u} \cdot (\underline{v} \times \underline{w}) = [\underline{u}, \underline{v}, \underline{w}]$$

$$(\underline{v} \times \underline{w}) \cdot \underline{u} = \underline{v} \cdot (\underline{w} \times \underline{u}) = [\underline{v}, \underline{w}, \underline{u}]$$

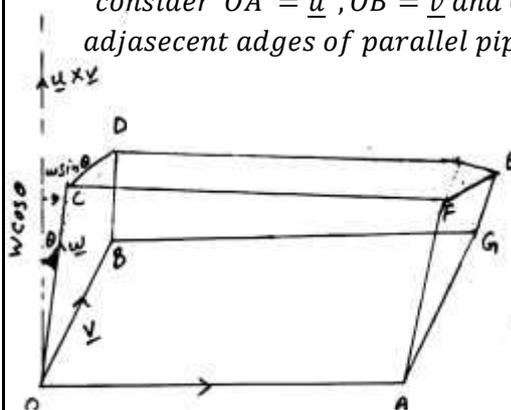
$$(\underline{w} \times \underline{u}) \cdot \underline{v} = \underline{w} \cdot (\underline{u} \times \underline{v}) = [\underline{w}, \underline{u}, \underline{v}]$$

(ii) the value of the product changes if the order is non cycle.

(iii) $\underline{u} \cdot \underline{v} \cdot \underline{w}$ and $\underline{u} \cdot (\underline{v} \times \underline{w})$ are meaningless.

the volume of parallelepiped

consider $\overrightarrow{OA} = \underline{u}$, $\overrightarrow{OB} = \underline{v}$ and $\overrightarrow{OC} = \underline{w}$ be the adjacent edges of parallel piped OAFCEGB



let θ be the angle between \underline{w} and $(\underline{u} \times \underline{v})$

As $|\underline{u} \times \underline{v}| = \text{area of } \parallel\text{gram } OAGB$
area of base of $\parallel\text{parallelepiped}$

$$\text{Resolve } \underline{w} \text{ in components in } \Delta COM, \cos\theta = \frac{|\overrightarrow{OM}|}{|\overrightarrow{OC}|}$$

$$\Rightarrow |\overrightarrow{OM}| = |\overrightarrow{OC}| \cos\theta$$

$$\Rightarrow |\overrightarrow{OM}| = w \cos\theta$$

As $|\overrightarrow{OM}| = \text{height of } \parallel\text{pipid}$

Exercise NO 7.5

As Volume of piped = (area of base)(height)
 $= |\underline{u} \times \underline{v}| \cdot |\underline{w}| \cos\theta$

$$\Rightarrow \text{Volume of piped} = \underline{w} \cdot (\underline{u} \times \underline{v})$$

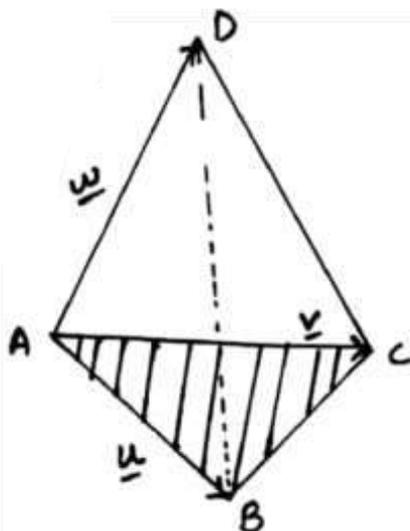
$$\therefore \underline{w} \cdot (\underline{u} \times \underline{v}) = \underline{u} \cdot (\underline{v} \times \underline{w}) \quad \left(\begin{array}{l} \because \underline{w} \cdot (\underline{u} \times \underline{v}) \\ |\underline{u} \times \underline{v}| \cdot |\underline{w}| \cos\theta \end{array} \right)$$

so Volume of piped = $\underline{u} \cdot (\underline{v} \times \underline{w})$

The Volume of Tetrahedron:

Volume of tetrahedron ABCD

$$= \frac{1}{3} (\Delta ABC) (\text{height of } D \text{ above the place } ABC)$$



Thus, $\text{Volume} = \frac{1}{6} (\underline{u} \times \underline{v}) \cdot \underline{w} = \frac{1}{6} [\underline{u} \ \underline{v} \ \underline{w}]$

Properties of scalar triple product:

1. if \underline{u} , \underline{v} and \underline{w} are coplaner, then the volume of piped so formed is zero. i.e the vector \underline{u} , \underline{v} , \underline{w} are coplaner $\Leftrightarrow (\underline{u} \times \underline{v}) \cdot \underline{w} = 0$

if any two vectors of scalar triple product are equal, then its value is zero. i.e

$$[\underline{u} \ \underline{v} \ \underline{w}] = [\underline{u} \ \underline{v} \ \underline{u}] = 0$$

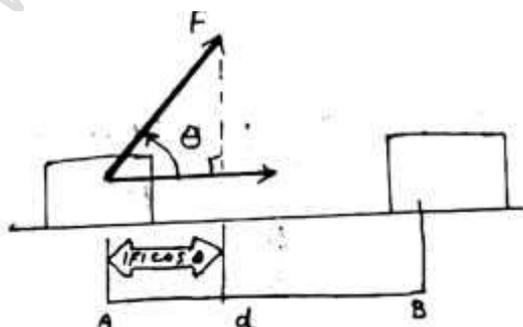
Application of vectors in physics and Engineering

(a) **Work done:**

if \underline{d} is displacement from A to B when force

\vec{F} is applied on the particle then,
Workdone = (force)(Displacement)

$$\Rightarrow \underline{W} = \vec{F} \cdot \vec{AB} = \underline{F} \cdot \underline{d} \quad \underline{d} = \vec{AB}$$



Q.1: find the volume of the parallelepiped for which the given vectors are three edges:

i) $\underline{u} = 3\underline{i} + 2\underline{k}$

$$\underline{v} = \underline{i} + 2\underline{j} + \underline{k}$$

$$\underline{w} = -\underline{j} + 4\underline{k}$$

Volume of a parallelepiped = $[\underline{u} \ \underline{v} \ \underline{w}]$

$$= \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$$

$$= 3(8+1) - 0(4-0) + 2(-1-0)$$

$$= 27 - 0 - 2 = 25 \quad \text{cubic unit}$$

ii) $\underline{u} = 3\underline{i} - 4\underline{j} - \underline{k}$

$$\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$$

$$\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$$

Volume of a parallelepiped = $[\underline{u} \ \underline{v} \ \underline{w}]$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -2 \\ -3 & 1 \end{vmatrix} - (-4) \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= 1(-1-6) + 4(1+4) - 1(-3+2)$$

$$= -7 + 20 + 1 = 14 \quad \text{cubic unit}$$

iii) $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$

$$\underline{v} = 2\underline{i} - \underline{j} - \underline{k}$$

$$\underline{w} = \underline{j} + \underline{k}$$

volume of a parallelepiped = $[\underline{u} \ \underline{v} \ \underline{w}]$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(-1+1) + 2(2+0) + 3(2+0)$$

$$= 0 + 4 + 6 = 10 \quad \text{cubic unit}$$

Q.2: verify that $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$. if

$$\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}; \underline{b} = 4\underline{i} - 2\underline{j} + 4\underline{k}; \underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= 3(3+10) + 1(4+4) + 5(20-6)$$

$$= 39+8+70 = 117$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 5 & 1 \\ -1 & 5 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix}$$

$$= 4(25+1) - 3(10-3) - 2(-2-15)$$

$$= 104-21+34 = 117$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} -1 & 5 \\ 3 & -2 \end{vmatrix} - 5 \begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 2(2-15) - 5(-6-20) + 1(9+4)$$

$$= -26+130+13 = 117$$

Q.3: Prove that the vectors $\underline{i} - 2\underline{j} + 3\underline{k}$; $-2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{i} - 3\underline{j} + 5\underline{k}$ are coplanar.

SOLUTION:

Let $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$; $\underline{v} = -2\underline{i} + 3\underline{j} - 4\underline{k}$; $\underline{w} = \underline{i} - 3\underline{j} + 5\underline{k}$

$$[\underline{u} \underline{v} \underline{w}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1 \begin{vmatrix} 3 & -4 \\ -3 & 5 \end{vmatrix} - (-2) \begin{vmatrix} -2 & -4 \\ 1 & 5 \end{vmatrix} + 3 \begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix}$$

$$= 1(15-12) + 2(-10+4) + 3(6-3) = 3-12+9 = 0$$

Thus the vectors are perpendicular.

Q.4: Find the constant 'α' such that the vectors are coplanar.

i) $\underline{i} - \underline{j} + \underline{k}$; $\underline{i} - 2\underline{j} - 3\underline{k}$ and $3\underline{i} - \alpha\underline{j} + 5\underline{k}$

SOLUTION: (2017 S.Q)

Let $\underline{u} = \underline{i} - \underline{j} + \underline{k}$; $\underline{v} = \underline{i} - 2\underline{j} - 3\underline{k}$; $\underline{w} = 3\underline{i} - \alpha\underline{j} + 5\underline{k}$

The vectors $\underline{u}, \underline{v}, \underline{w}$ are coplanar if $[\underline{u} \underline{v} \underline{w}] = 0$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

$$1(-10-3\alpha) + 1(5+9) + 1(-\alpha+6) = 0$$

$$-10-3\alpha+14-\alpha+6 = 0$$

$$-4\alpha+10 = 0$$

$$\Rightarrow \alpha = \frac{10}{4} \quad \Rightarrow \alpha = \frac{5}{2}$$

ii) $\underline{i} - 2\alpha\underline{j} - \underline{k}$; $\underline{i} - \underline{j} + 2\underline{k}$ and $\alpha\underline{i} - 2\underline{j} + \underline{k}$

SOLUTION:

Let $\underline{u} = \underline{i} - 2\alpha\underline{j} - \underline{k}$; $\underline{v} = \underline{i} - \underline{j} + 2\underline{k}$; $\underline{w} = \alpha\underline{i} - 2\underline{j} + \underline{k}$

The vectors $\underline{u}, \underline{v}, \underline{w}$ are coplanar if $[\underline{u} \underline{v} \underline{w}] = 0$

$$\begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix} = 0$$

$$1(-1+2) - 1(2\alpha-1) - \alpha(4\alpha-1) = 0$$

$$\Rightarrow 1 + 2\alpha + 1 - 4\alpha^2 - \alpha = 0 \Rightarrow -4\alpha^2 + \alpha + 2 = 0$$

$$4\alpha^2 - \alpha - 2 = 0$$

Using quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\alpha = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-2)}}{2(4)} = \frac{1 \pm \sqrt{1+32}}{8}$$

$$\Rightarrow \alpha = \frac{1 \pm \sqrt{33}}{8}$$

Q.5: (a) Find the value of i) $2\underline{i} \times 2\underline{j} \cdot \underline{k}$ ii) $3\underline{j} \cdot \underline{k} \times \underline{i}$ iii)

$[\underline{k} \underline{i} \underline{j}]$ (2015 S.Q) iv) $[\underline{i} \underline{i} \underline{k}]$

SOLUTION:

i) $2\underline{i} \times 2\underline{j} \cdot \underline{k} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2(2-0) - 0(0-0) + 0(0-0) = 4 - 0 - 0 = 4$

ii) $3\underline{j} \cdot \underline{k} \times \underline{i} = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0(0-0) - 3(0-1) + 0(0-1) = 0 + 3 + 0 = 3$

iii) $[\underline{k} \underline{i} \underline{j}] = \underline{k} \cdot \underline{j} \times \underline{i} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0(0-0) - 0(0-0) + 1(1-0) = 0 - 0 + 1 = 1$

iv) $[\underline{i} \underline{i} \underline{k}] = \underline{i} \cdot \underline{i} \times \underline{k} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(0-0) - 0(1-0) + 0(0-0) = 0 - 0 - 0 = 0$

(b) Prove that

$$\underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{v} \cdot (\underline{w} \times \underline{u}) + \underline{w} \cdot (\underline{u} \times \underline{v}) = 3\underline{u} \cdot (\underline{v} \times \underline{w})$$

SOLUTION:

L.H.S

$$= \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{v} \cdot (\underline{w} \times \underline{u}) + \underline{w} \cdot (\underline{u} \times \underline{v})$$

$$\because \underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$$

$$= \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{u} \cdot (\underline{v} \times \underline{w})$$

$$= 3\underline{u} \cdot (\underline{v} \times \underline{w})$$

= R.H.S

Q.6: Find the volume of the tetrahedron with the vertices.

i) $(0,1,2)$, $(3,2,1)$, $(1,2,1)$, and $(5,5,6)$

SOLUTION:

Let $A(0,1,2)$, $B(3,2,1)$, $C(1,2,1)$, $D(5,5,6)$

$$\overrightarrow{AB} = (3-0)\underline{i} + (2-1)\underline{j} + (1-2)\underline{k} = 3\underline{i} + \underline{j} - \underline{k}$$

$$\overrightarrow{AC} = (1-0)\underline{i} + (2-1)\underline{j} + (1-2)\underline{k} = \underline{i} + \underline{j} - \underline{k}$$

$$\overrightarrow{AD} = (5-0)\underline{i} + (5-1)\underline{j} + (6-2)\underline{k} = 5\underline{i} + 4\underline{j} + 4\underline{k}$$

$$\text{Volume of the tetrahedron ABCD} = \frac{1}{6} [\overrightarrow{AB} \overrightarrow{AC} \overrightarrow{AD}]$$

$$= \frac{1}{6} \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$$

$$= \frac{1}{6} [3(4+4) - 1(4+5) + (-1)(4-5)]$$

$$= \frac{1}{6} [24 - 9 + 1] = \frac{1}{6} \cdot 16 = \frac{8}{3} \text{ Cubic unit}$$

ii) $(2,1,8)$, $(3,2,9)$, $(2,1,4)$, and $(3,3,10)$ (2015 S.Q)(2018,19L.Q)

SOLUTION:

Let $A(2,1,8)$, $B(3,2,9)$, $C(2,1,4)$, $D(3,3,10)$

$$\overrightarrow{AB} = (3-2)\underline{i} + (2-1)\underline{j} + (9-8)\underline{k} = \underline{i} + \underline{j} + \underline{k}$$

$$\overrightarrow{AC} = (2-2)\underline{i} + (1-1)\underline{j} + (4-8)\underline{k} = 0\underline{i} + 0\underline{j} - 4\underline{k}$$

$$\overrightarrow{AD} = (3-2)\underline{i} + (3-1)\underline{j} + (10-8)\underline{k} = \underline{i} + 2\underline{j} + 2\underline{k}$$

$$\text{Volume of the tetrahedron ABCD} = \frac{1}{6} [\overrightarrow{AB} \overrightarrow{AC} \overrightarrow{AD}]$$

$$= \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= \frac{1}{6} [1(0+8) - 1(0+4) + 1(0-0)]$$

$$= \frac{1}{6} [8 - 4 + 0] = \frac{1}{6} \cdot 4 = \frac{2}{3} \text{ Cubic unit}$$

Q.7: Find the work done, if the point at which the constant force $\vec{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$ is applied to an object, moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$.

SOLUTION: Given: $\vec{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$, $P_1(3,1,-2)$, $P_2(2,4,6)$

Displacement: $\underline{d} = \overrightarrow{P_1P_2} = (2-3)\underline{i} + (4-1)\underline{j} + (6+2)\underline{k} = -\underline{i} + 3\underline{j} + 8\underline{k}$

We know that Work done = $\vec{F} \cdot \underline{d} = (4\underline{i} + 3\underline{j} + 5\underline{k}) \cdot (-\underline{i} + 3\underline{j} + 8\underline{k}) = -4 + 9 + 40 = 45$ Units

Q.8: A particle, acted by constant forces $4\underline{i} + \underline{j} - \underline{k}$ and $3\underline{i} - \underline{j} - \underline{k}$ is displaced from $A(1,2,3)$ to $B(5,4,1)$. Find the work done.

SOLUTION: Given: let $\vec{F}_1 = 4\underline{i} + \underline{j} - \underline{k}$; $\vec{F}_2 = 3\underline{i} - \underline{j} - \underline{k}$ and $A(1,2,3)$, $B(5,4,1)$

Total force: $\vec{F} = \vec{F}_1 + \vec{F}_2 = 4\underline{i} + \underline{j} - \underline{k} + 3\underline{i} - \underline{j} - \underline{k} = 7\underline{i} + 0\underline{j} - 4\underline{k}$

Displacement: $\underline{d} = \overrightarrow{AB} = (5-1)\underline{i} + (4-2)\underline{j} + (1-3)\underline{k} = 4\underline{i} + 2\underline{j} - 2\underline{k}$

We know that Work done = $\vec{F} \cdot \underline{d} = (3\underline{i} - \underline{j} - \underline{k}) \cdot (4\underline{i} + 2\underline{j} - 2\underline{k}) = 28 + 0 + 8 = 36$ units

Q.9: A particle is displaced from the point $A(5,-5,-7)$, to the point $B(6,2,-2)$ under the action of constant forces defined by

$10\underline{i} - 6\underline{j} + 11\underline{k}$, $4\underline{i} + 5\underline{j} + 9\underline{k}$, $-2\underline{i} + \underline{j} - 9\underline{k}$. Show that the total work done by the forces is 67 units.

SOLUTION: Given: Let $\vec{F}_1 = 10\underline{i} - 6\underline{j} + 11\underline{k}$, $\vec{F}_2 = 4\underline{i} + 5\underline{j} + 9\underline{k}$, $\vec{F}_3 = -2\underline{i} + \underline{j} - 9\underline{k}$ and $A(5,-5,-7)$, $B(6,2,-2)$

Total force: $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 10\underline{i} - 6\underline{j} + 11\underline{k} + 4\underline{i} + 5\underline{j} + 9\underline{k} + (-2\underline{i} + \underline{j} - 9\underline{k}) = 12\underline{i} + 0\underline{j} + 11\underline{k}$

Displacement: $\underline{d} = \overrightarrow{AB} = (6-5)\underline{i} + (2+5)\underline{j} + (-2+7)\underline{k} = \underline{i} + 7\underline{j} + 5\underline{k}$

We know that Work done = $\vec{F} \cdot \underline{d} = (12\underline{i} + 0\underline{j} + 11\underline{k}) \cdot (\underline{i} + 7\underline{j} + 5\underline{k}) = 12 + 0 + 55 = 67$ units

Q.10: A force of magnitude 6 units acting parallel to $2\underline{i} - 2\underline{j} + \underline{k}$ displaces the point of application from $(1,2,3)$ to $(5,3,7)$. Find the work done.

SOLUTION: Let $\vec{F} = ?$, $|\vec{F}| = 6$ Given vector: $\underline{v} = 2\underline{i} - 2\underline{j} + \underline{k}$ and let $A(1,2,3)$, $B(5,3,7)$

As $\vec{F} \parallel \underline{v}$

So $\hat{F} = \hat{v} \Rightarrow \frac{\vec{F}}{|\vec{F}|} = \frac{\underline{v}}{|\underline{v}|} \Rightarrow \vec{F} = |\vec{F}| \frac{\underline{v}}{|\underline{v}|} =$

$$6 \cdot \frac{2\underline{i} - 2\underline{j} + \underline{k}}{\sqrt{4+4+1}} = 6 \cdot \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3} = 2(2\underline{i} - 2\underline{j} + \underline{k}) = 4\underline{i} - 4\underline{j} + 2\underline{k}$$

Displacement: $\underline{d} = \overrightarrow{AB} = (5-1)\underline{i} + (3-2)\underline{j} + (7-3)\underline{k} = 4\underline{i} + \underline{j} + 4\underline{k}$

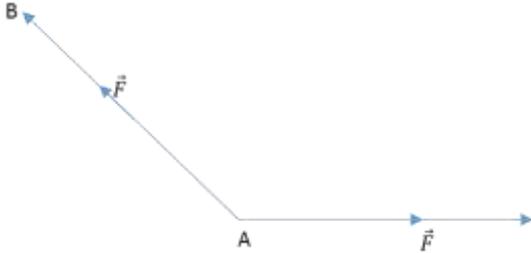
We know that

$$\begin{aligned} \text{Work done} &= \vec{F} \cdot \underline{d} \\ &= (4\underline{i} - 4\underline{j} + 2\underline{k}) \cdot (4\underline{i} + \underline{j} + 4\underline{k}) \end{aligned}$$

$$= 16 - 4 + 8 = 20 \quad \text{Units}$$

Q.11: A force $\vec{F} = 3\vec{i} + 2\vec{j} - 4\vec{k}$ is applied at the point $(1, -1, 2)$. Find the moment of the force about the point $(2, -1, 3)$.

SOLUTION: Given: $\vec{F} = 3\vec{i} + 2\vec{j} - 4\vec{k}$ and Let A $(1, -1, 2)$, B $(2, -1, 3)$



$$\underline{r} = \overline{AB} = (1 - 2)\underline{i} + (-1 + 1)\underline{j} + (2 - 3)\underline{k} = -\underline{i} + 0\underline{j} - \underline{k}$$

$$\text{Moment of force about A} = \underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = \underline{i}(0 + 2) - \underline{j}(4 + 3) + \underline{k}(-2 - 0) = 2\underline{i} - 7\underline{j} - 2\underline{k}$$

Q.12: A force $\vec{F} = 4\vec{i} - 3\vec{k}$ passes through the point $A(2, -2, 5)$. Find the moment of \underline{F} about the point $B(1, -3, 1)$.

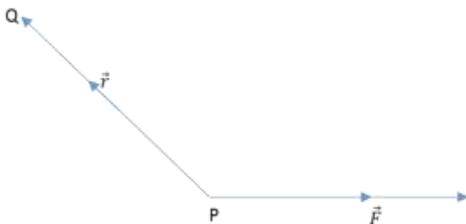
SOLUTION: Given: $\vec{F} = 4\vec{i} - 3\vec{k}$ and A $(2, -2, 5)$, B $(1, -3, 1)$

$$\underline{r} = \overline{AB} = (2 - 1)\underline{i} + (-2 + 3)\underline{j} + (5 - 1)\underline{k} = \underline{i} + \underline{j} + 4\underline{k}$$

$$\text{Moment of force about B} = \underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix} = \underline{i}(-3 - 0) - \underline{j}(-3 - 16) + \underline{k}(0 - 4) = -3\underline{i} + 19\underline{j} - 4\underline{k}$$

Q.13: Given a force $\vec{F} = 2\vec{i} + \vec{j} - 3\vec{k}$ acting at a point $A(1, -2, 1)$. Find the moment of \underline{F} about the point $B(2, 0, -2)$. (19 s.Q)

SOLUTION: Given: $\vec{F} = 2\vec{i} + \vec{j} - 3\vec{k}$ and A $(1, -2, 1)$, B $(2, 0, -2)$

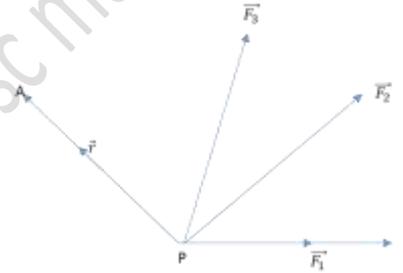


$$\underline{r} = \overline{AB} = (1 - 2)\underline{i} + (-2 - 0)\underline{j} + (1 + 2)\underline{k} = -\underline{i} - 2\underline{j} + 3\underline{k}$$

$$\text{Moment of force about B} = \underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix} = \underline{i}(6 - 3) - \underline{j}(3 - 6) + \underline{k}(-1 + 4) = 3\underline{i} + 3\underline{j} + 3\underline{k}$$

Q.14: Find the moment about $A(1, 1, 1)$ of each of the concurrent forces $\underline{i} - 2\underline{j}$, $3\underline{i} + 2\underline{j} - \underline{k}$, $5\underline{j} + 2\underline{k}$, where $P(2, 0, 1)$ is their point of concurrency.

SOLUTION: Given:



Let $\vec{F}_1 = \underline{i} - 2\underline{j}$, $\vec{F}_2 = 3\underline{i} + 2\underline{j} - \underline{k}$, $\vec{F}_3 = 5\underline{j} + 2\underline{k}$ and A $(1, 1, 1)$, P $(2, 0, 1)$

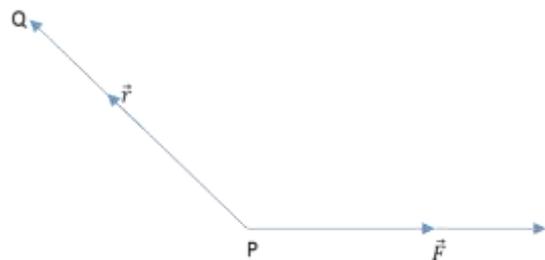
$$\text{Total force: } \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \underline{i} - 2\underline{j} + 3\underline{i} + 2\underline{j} - \underline{k} + 5\underline{j} + 2\underline{k} = 4\underline{i} + 5\underline{j} + \underline{k}$$

$$\underline{r} = \overline{AP} = (2 - 1)\underline{i} + (0 - 1)\underline{j} + (1 - 1)\underline{k} = \underline{i} - \underline{j} + 0\underline{k}$$

$$\text{Moment of force about B} = \underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 4 & 5 & 1 \end{vmatrix} = \underline{i}(-1 - 0) - \underline{j}(1 - 0) + \underline{k}(5 + 4) = -\underline{i} - \underline{j} + 9\underline{k}$$

Q.15: A force $\vec{F} = 7\vec{i} + 4\vec{j} - 3\vec{k}$ is applied at $P(1, -2, 3)$. Find its moment about the point $Q(2, 1, 1)$.

SOLUTION: $\vec{F} = 7\vec{i} + 4\vec{j} - 3\vec{k}$



$$\underline{r} = \overrightarrow{QP} = (1 - 2)\underline{i} + (-2 - 1)\underline{j} + (3 - 1)\underline{k}$$
$$= -\underline{i} - 3\underline{j} + 2\underline{k}$$

$$\text{Moment of force about B} = \underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix} =$$
$$\underline{i}(9 - 8) - \underline{j}(3 - 14) + \underline{k}(-4 + 21) = \underline{i} + 11\underline{j} + 17\underline{k}$$

