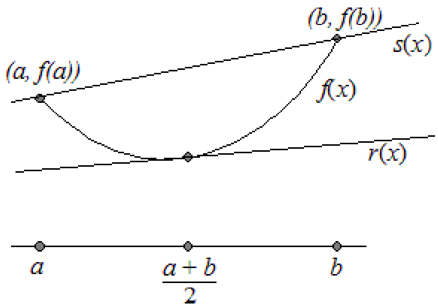
Hermite-Hadamard integral inequality

If $f : [a,b] \to \mathbb{R}$ is convex, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}.$$

Proof: First of all, let's recall that a convex function on a open interval (a,b) is continuous on (a,b) and admits left and right derivative $f'_+(x)$ and $f'_-(x)$ for any $x \in (a,b)$. For this reason, it's always possible to construct at least one supporting line for $f(x)$ at any $x_0 \in (a,b)$: if $f(x_0)$ is differentiable in x_0, one has $r(x) = f(x_0) + f'(x_0)(x - x_0)$; if not, it's obvious that all $r(x) = f(x_0) + c(x - x_0)$ are supporting lines for any $c \in [f'_-(x_0), f'_+(x_0)]$.

Let now $r(x) = f\left(\frac{a+b}{2}\right) + c\left(x - \frac{a+b}{2}\right)$ be a supporting line of $f(x)$ in $x = \frac{a+b}{2} \in (a,b)$. Then, $r(x) \leq f(x)$. On the other side, by convexity definition, having defined $s(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ the line connecting the points $(a, f(a))$ and $(b, f(b))$, one has $f(x) \leq s(x)$. Shortly,



$$r(x) \leq f(x) \leq s(x)$$

Integrating both inequalities between a and b

$$\int_a^b r(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b s(x)dx. \tag{1}$$

Now

$$\begin{aligned}\int_a^b r(x)\,dx &= \int_a^b \left[f\left(\frac{a+b}{2}\right) + c\left(x - \frac{a+b}{2}\right)\right]dx \\ &= f\left(\frac{a+b}{2}\right)(b-a) + c\int_a^b \left(x - \frac{a+b}{2}\right)dx \\ &= f\left(\frac{a+b}{2}\right)(b-a),\end{aligned}$$

and

$$\begin{aligned}\int_a^b s(x)dx &= \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]dx \\ &= f(a)(b-a) + \frac{f(b)-f(a)}{b-a}\int_a^b (x-a)dx \\ &= \frac{f(a)+f(b)}{2}(b-a).\end{aligned}$$

Using above value in (1), we have

$$f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f(x)\,dx \leq, \frac{f(a)+f(b)}{2}(b-a)$$

which is the thesis.