Exercise 4.4

Resolve into partial fractions.

1. $\frac{x^3}{(x^2+4)^2}$

Let $\frac{x^3}{(x^2+4)^2} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+4)^2}$ - - - - - - - - - (i)

Multiplying eq (i) by $(x^2 + 4)^2$ on b/s

$(x^2 + 4)^2 \frac{x^3}{(x^2+4)^2} = \frac{Ax+B}{(x^2+4)}(x^2 + 4)^2 + \frac{Cx+D}{(x^2+4)^2}(x^2 + 4)^2$

$x^3 = (Ax + B)(x^2 + 4) + (Cx + D)$ - - - - - - - - -(ii)

$x^3 = Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$

$x^3 = Ax^3 + Bx^2 + (4A + C)x + 4B + D$ - - - - - - - - -(iii)

Equating coefficients of x^3 on b/s of eq(iii)

$A = 1$

Equating coefficients of x^2 on b/s of eq(iii)

B=0

Equating coefficients of x on b/s of eq (iii)

$4A + C = 0$

$C = -4A$

$C = -4(1)$ $(\because A = 1)$

$C = -4$

Equating constants on b/s of eq(iii)

$4B + D = 0$

$D = -4B$

$D = -4(0)$ $(\because B = 0)$

$D = 0$

By putting the values of A, B, C & D in eq(i)

$\frac{x^3}{(x^2+4)^2} = \frac{(1)x+0}{(x^2+4)} + \frac{-4x+0}{(x^2+4)^2}$

Thus $\frac{x^3}{(x^2+4)^2} = \frac{x}{(x^2+4)} - \frac{4x}{(x^2+4)^2}$ are the required partial fractions.

2. $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$

$\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$ - - - - - - - - -(i)

Multiplying eq (i) by $(x+1)(x^2+1)^2$ on b/s

$(x+1)(x^2+1)^2 \frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{A}{(x+1)}(x+1)(x^2+1)^2 + \frac{Bx+C}{(x^2+1)}(x+1)(x^2+1)^2 + \frac{Dx+E}{(x^2+1)^2}(x+1)(x^2+1)^2$

$x^4+3x^2+x+1 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1)$ - - - - - - - - (ii)

$x^4+3x^2+x+1 = A(x^4+2x^2+1) + (Bx+C)(x^3+x+x^2+1) + Dx^2+Dx+Ex+E$

$x^4+3x^2+x+1 = Ax^4+2Ax^2+A+Bx^4+Bx^2+Bx^3+Bx+Cx^3+Cx+Cx^2+C+Dx^2+Dx+Ex+E$

$x^4+3x^2+x+1 = (A+B)x^4+(B+C)x^3+(2A+B+C+D)x^2+(B+C+D+E)x+A+C+E$ - - - - - - - - - -(iii)

Eq(ii) is an identity which holds good for all x & hence for $x=-1$

Put $x=-1$ $i.e.,$ $x+1=0$ in eq(ii)

$(-1)^4+3(-1)^2-1+1 = A((-1)^2+1)^2+(Bx+C)(0)(x^2+1)+(Dx+E)(0)$

$1+3+0 = A(2^2)+0$

$4 = 4A$

$A = 1$

Equating coefficients of x^4 on b/s of eq(iii)

$1 = A+B$

$B = 1-A$

$B = 1-1 \qquad (\because A=1)$

$B = 0$

Equating coefficients of x^3 on b/s of eq(iii)

$0 = B + C$

$C = -B$

$C = 0 \qquad (\because B = 0)$

Equating coefficients of x^2 on b/s of eq (iii)

$3 = 2A + B + C + D$

$D = 3 - 2A - B - C$

$D = 3 - 2(1) - 0 - 0 \quad (\because A = 1, B = 0, C = 0)$

$D = 3 - 2$

$D = 1$

Equating coefficients of x on b/s of eq(iii)

$1 = B + C + D + E$

$E = 1 - B - C - D$

$E = 1 - 0 - 0 - 1 \qquad (\because B = 0, C = 0, D = 1)$

$E = 0$

By putting the values of A, B, C, D & E in e q(i)

$\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{1}{(x+1)} + \frac{0x+0}{(x^2+1)} + \frac{(1)x+0}{(x^2+1)^2}$

Thus $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{1}{(x+1)} + \frac{x}{(x^2+1)^2}$ are the required partial fractions.

3. $\frac{x^2}{(x+1)(x^2+1)^2}$

Let $\frac{x^2}{(x+1)(x^2+1)^2} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$ - - - - - - - - (i)

Multiplying eq(i) by $(x+1)(x^2+1)^2$ on b/s

$(x+1)(x^2+1)^2 \frac{x^2}{(x+1)(x^2+1)^2} = \frac{A}{(x+1)}(x+1)(x^2+1)^2 + \frac{Bx+C}{(x^2+1)}(x+1)(x^2+1)^2 + \frac{Dx+E}{(x^2+1)^2}(x+1)(x^2+1)^2$

$x^2 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1)$ - - - - - - -(ii)

$x^2 = A(x^4+2x^2+1) + (Bx+C)(x^3+x^2+x+1) + (Dx+C)(x+1)$

$x^2 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^3 + Bx^2 + Bx + Cx^3 + Cx^2 + Cx + C + Dx^2 + Dx + Ex + E$

$x^2 = (A+B)x^4 + (B+C)x^3 + (2A+B+C+D)x^2 + (B+C+D+E)x + (A+C+E)$ - - - - - - - - - (iii)

Eq(ii) is an identity which holds good for all x and hence for x=-1

Put x=-1 i.e. , x+1=0 in eq(ii)

$(-1)^2 = A((-1)^2+1)^2 + (Bx+C)(0) + (Dx+E)(0)$

$1 = A(1+1)^2$

$1 = A(2)^2$

$1 = 4A$

$A = \frac{1}{4}$

Equating coefficients of x^4 on b/s of eq(iii)

$0 = A + B$

$B = -A$

$B = -\frac{1}{4}$ $\left(\because A = \frac{1}{4}\right)$

Equating coefficients of x^3 on b/s of eq (iii)

$0 = B + C$

$0 = -\frac{1}{4} + C$ $\left(\because B = -\frac{1}{4}\right)$

$C = \frac{1}{4}$

Equating coefficients of x^2 on b/s of eq (iii)

$1 = 2A + B + C + D$

$1 = \frac{2}{4} - \frac{1}{4} + \frac{1}{4} + D$ $\left(\because A = \frac{1}{4}, B = -\frac{1}{4}, C = \frac{1}{4}\right)$

$D = 1 - \frac{2}{4}$

$D = \frac{4-2}{4}$

$D = \frac{2}{4}$

$D = \frac{1}{2}$

Equating coefficients of x on b/s of eq(iii)

$0 = B + C + D + E$

$E = -B - C - D$

$E = \frac{1}{4} - \frac{1}{4} - \frac{1}{2}$ $\left(\because B = -\frac{1}{4}, C = \frac{1}{4}, D = \frac{1}{2}\right)$

$E = -\frac{1}{2}$

By putting the values of A, B, C, D & E in eq(i)

$\frac{x^2}{(x+1)(x^2+1)^2} = \frac{\frac{1}{4}}{(x+1)} + \frac{-\frac{1}{4}x+\frac{1}{4}}{(x^2+1)} + \frac{\frac{1}{2}x-\frac{1}{2}}{(x^2+1)^2}$

$\frac{x^2}{(x+1)(x^2+1)^2} = \frac{1}{4(x+1)} + \frac{-\frac{x-1}{4}}{(x^2+1)} + \frac{\frac{x-1}{2}}{(x^2+1)^2}$

$\frac{x^2}{(x+1)(x^2+1)^2} = \frac{1}{4(x+1)} - \frac{(x-1)}{4(x^2+1)} + \frac{(x-1)}{2(x^2+1)^2}$ are the required partial fractions.

4. $\frac{x^2}{(x-1)(1+x^2)^2}$

Let $\frac{x^2}{(x-1)(1+x^2)^2} = \frac{A}{(x-1)} + \frac{Bx+C}{(1+x^2)} + \frac{Dx+E}{(1+x^2)^2}$ - - - - - - - - (i)

Multiplying eq (i) by $(x-1)(1+x^2)^2$ on b/s

$(x-1)(1+x^2)^2 \frac{x^2}{(x-1)(1+x^2)^2} = \frac{A}{(x-1)}(x-1)(1+x^2)^2 + \frac{Bx+C}{(1+x^2)}(x-1)(1+x^2)^2 + \frac{Dx+E}{(1+x^2)^2}(x-1)(1+x^2)^2$

$x^2 = A(1+x^2)^2 + (Bx+C)(x-1)(1+x^2) + (Dx+E)(x-1)$- - - - - -(ii)

$x^2 = A(1+2x^2+x^4) + (Bx+C)(x+x^3-1-x^2) + Dx^2 - Dx + Ex - E$

$x^2 = A + 2Ax^2 + Ax^4 + Bx^2 + Bx^4 - Bx - Bx^3 + Cx + Cx^3 - C - Cx^2 + Dx^2 - Dx + Ex - E$

$x^2 = (A+B)x^4 + (C-B)x^4 + (2A+B-C+D)x^2 + (C-B-D+E)x + A-C-E$- - - - (iii)

Eq(ii) is an identity which holds good for all x & hence for $x = 1$

Put $x = 1$ $i.e., x - 1 = 0$ I eq(ii)

$(1)^2 = A(1+(1)^2)^2 + (Bx+C)(0)(1+x^2) + (Dx+E)(0)$

$1 = A(2)^2$

$1 = 4A$

$A = \frac{1}{4}$

Equating coefficients of x^4 on b/s of eq (iii)

$0 = A + B$

$B = -A$

$B = -\frac{1}{4}$ $\left(\because A = \frac{1}{4}\right)$

Equating coefficients of x^3 on b/s of eq(iii)

$0 = C - B$

$C = B$

$C = -\frac{1}{4}$ $\left(\because B = -\frac{1}{4}\right)$

Equating coefficients of x^2 on b/s of eq(iii)

$1 = 2A + B - C + D$

$D = 1 - 2A - B + C$

$D = 1 - 2\left(\frac{1}{4}\right) - \left(\frac{-1}{4}\right) + \left(-\frac{1}{4}\right)$ $\left(\because A = \frac{1}{4}, B = -\frac{1}{4}, C = -\frac{1}{4}\right)$

$D = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{4}$

$D = \frac{2-1}{2}$

$D = \frac{1}{2}$

Equating coefficients of x on b/s of eq(iii)

$0 = C - B - D + E$

$E = B - C + D$

$E = -\frac{1}{4} - (-\frac{1}{4}) + \frac{1}{2}$ $\qquad (\because B = -\frac{1}{4}, C = -\frac{1}{4}, D = \frac{1}{2})$

$E = -\frac{1}{4} + \frac{1}{4} + \frac{1}{2}$

$E = \frac{1}{2}$

By putting the values of A, B, C, D & E in eq (i)

$\frac{x^2}{(x-1)(1+x^2)^2} = \frac{\frac{1}{4}}{(x-1)} + \frac{-\frac{1}{4}x - \frac{1}{4}}{(1+x^2)} + \frac{\frac{1}{2}x + \frac{1}{2}}{(1+x^2)^2}$

Thus $\frac{x^2}{(x-1)(1+x^2)^2} = \frac{1}{4(x-1)} - \frac{x+1}{4(1+x^2)} + \frac{x+1}{2(1+x^2)^2}$ the required partial fractions.

5. $\frac{x^4}{(x^2+2)^2}$

By long division,

$\frac{x^4}{(x^2+2)^2} = 1 - \frac{4x^2+4}{(x^2+2)^2}$ ---------(A)

Let $\frac{4x^2+4}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2}$ ---------(i)

Multiplying eq(i) by $(x^2+2)^2$ on b/s

$(x^2+2)^2 \frac{4x^2+4}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)}(x^2+2)^2 + \frac{Cx+D}{(x^2+2)^2}(x^2+2)^2$

$4x^2 + 4 = (Ax+B)(x^2+2) + Cx + D$

$4x^2 + 4 = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$

$4x^2 + 4 = Ax^3 + Bx^2 + (2A + C)x + 2B + D$

Equating coefficients of x^3 on b/s of eq(iii)

$A = 0$

Equating coefficients of x^2 on b/s of eq (iii)

$B = 4$

Equating coefficients of x on b/s of eq (iii)

$0 = 2A + C$

$C = -2A \qquad (\because A = 0)$

C=0

Equating constants on b/s of eq(iii)

$4 = 2B + D$

$D = 4 - 2B$

$D = 4 - 2(4) \qquad (\because B = 4)$

$D = 4 - 8$

$D = -4$

By putting the values of $A, B, C\ \&\ D\ in\ eq\ (i)$

$\frac{4x^2+4}{(x^2+2)^2} = \frac{(0)x+4}{(x^2+2)} + \frac{(0)x-4}{(x^2+2)^2}$

$\frac{4x^2+4}{(x^2+2)^2} = \frac{4}{(x^2+2)} - \frac{4}{(x^2+2)^2}$

By putting this value in eq (A)

$\frac{x^4}{(x^2+2)^2} = 1 - \left(\frac{4}{(x^2+2)} - \frac{4}{(x^2+2)^2}\right)$

Thus $\frac{x^4}{(x^2+2)^2} = 1 - \frac{4}{(x^2+2)} + \frac{4}{(x^2+2)^2}$ are the required partial fractions.

6.$\frac{x^5}{(x^2+1)^2}$

By long division

$\frac{x^5}{(x^2+1)^2} = x - \frac{2x^3+x}{(x^2+1)^2}$ --------(A)

Let $\frac{2x^3+x}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2}$ -------- (i)

Multiply eq(i) by $(x^2 + 1)^2$ on b/s

$(x^2 + 1)^2 \frac{2x^3+x}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)}(x^2 + 1)^2 + \frac{Cx+D}{(x^2+1)^2}(x^2 + 1)^2$

$2x^3 + x = (Ax + B)(x^2 + 1) + Cx + D$

$2x^3 + x = Ax^3 + Ax + Bx^2 + B + Cx + D$

$2x^3 + x = Ax^3 + Bx^2 + (A + C)x + B + D$ --------(ii)

Equating coefficients of x^3 on b/s of eq(ii)

$A = 2$

Equating coefficients of x^2 on b/s of eq(ii)

$B = 0$

Equating coefficients of x on b/s of eq(ii)

$1 = A + C$

$C = 1 - A \qquad (A = 2)$

$C = 1 - 2$

$C = -1$

Equating constants ob b/s of eq (ii)

$0 = B + D$

$D = -B \qquad (\because B = 0)$

$D = 0$

By putting the values of A, B, C & D in eq (i)

$\frac{2x^3+x}{(x^2+1)^2} = \frac{2x+(0)}{(x^2+1)} + \frac{(-1)x+0}{(x^2+1)^2}$

$\frac{2x^3+x}{(x^2+1)^2} = \frac{2x}{(x^2+1)} - \frac{x}{(x^2+1)^2}$

By putting this value in eq(A)

$\frac{x^5}{(x^2+1)^2} = x - \left(\frac{2x}{(x^2+1)} - \frac{x}{(x^2+1)^2}\right)$

Thus $\frac{x^5}{(x^2+1)^2} = x - \frac{2x}{(x^2+1)} + \frac{x}{(x^2+1)^2}$ are the required partial fractions.