CHAPTER 06

THE LAPLACE TRANSFORM AND ITS APPLICATIONS

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SOURCE: 0301-4695644

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Exercise Chap # 6

Pg # 232

Q # 1

Compute Laplace transforms of the following functions

(a) $\sin^2 \omega t$

Solution:

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$$= \mathcal{L}[\sin^2(\omega t)]$$

$$= \mathcal{L}\left(\frac{1-\cos 2\omega t}{2}\right)$$

$$= \frac{1}{2}\left[\mathcal{L}(1) - \mathcal{L}(\cos 2\omega t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+(2\omega)^2}\right]$$

(b) $\cos^2 \omega t$

$$\cos^2 \omega t = \frac{1+\cos 2\omega t}{2}$$

$$\mathcal{L}\left[\frac{1+\cos 2\omega t}{2}\right]$$

$$= \frac{1}{2}\mathcal{L}[1+\cos 2\omega t]$$

$$= \frac{1}{2}\mathcal{L}(1) + \frac{1}{2}\mathcal{L}(\cos 2\omega t)$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{s}{s^2+4\omega^2}\right)$$

(c) $\sin(\omega t-\phi)$

Solution.

$\sin(\omega t-\phi)=\sin\omega t\cos\phi-\sin\phi\cos\omega t$

$\mathcal{L}[\sin(\omega t-\phi)]=\mathcal{L}[\sin\omega t\cos\phi-\sin\phi\cos\omega t]$

$=\mathcal{L}[\sin\omega t\cos\phi)-\mathcal{L}[\sin\phi\cos\omega t]$

$=\cos\phi\,\mathcal{L}(\sin\omega t)-\sin\phi\,\mathcal{L}(\cos\omega t)$

$=\cos\phi\left(\frac{\omega}{s^2+\omega^2}\right)-\sin\phi\left(\frac{s}{s^2+\omega^2}\right)$

(d) $\cos(\omega t-\phi)$

Solution:

$\cos(\omega t-\phi)=\cos\omega t\cos\phi+\sin\omega t\sin\phi$

$\mathcal{L}[\cos(\omega t-\phi)]=\mathcal{L}[\cos\omega t\cos\phi+\sin\omega t\sin\phi]$

$=\cos\phi\,\mathcal{L}(\cos\omega t)+\sin\phi\,\mathcal{L}(\sin\omega t)$

$=\cos\phi\left(\frac{s}{s^2+\omega^2}\right)+\sin\phi\left(\frac{\omega}{s^2+\omega^2}\right)$

(e) $e^{2(t+1)}$

Solution:

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$=e^{2t+2}$

$=\mathcal{L}(e^{2t}\cdot e^{2})$

$=e^{2}\mathcal{L}(e^{2t})$

$=e^{2}\left(\frac{1}{s-2}\right)$

$=\frac{e^{2}}{s-2}$

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Q#2: 03014696644

Find the Laplace transforms of the following functions

(a) $\sin\omega t\, e^{-2t}$

Solution:

$$\mathcal{L}(\sin\omega t e^{-2t}) = \mathcal{L}(\sin\omega t)_{s\to s+2}$$

$$= \frac{\omega}{s^2+\omega^2}\Big|_{s\to s+2}$$

$$= \frac{\omega}{(s+2)^2+\omega^2}$$

(b) $\cos 3\omega t\, e^{4t}$

Solution:

$$\mathcal{L}(\cos 3\omega t e^{4t}) = \mathcal{L}(\cos 3\omega t)_{s\to s-4}$$

$$= \frac{s}{s^2+(3\omega)^2}\Big|_{s\to s-4}$$

$$= \frac{s}{s^2+9\omega^2}\Big|_{s\to s-4}$$

$$= \frac{s-4}{(s-4)^2+9\omega^2}$$

(c) $e^{3t} t^4$

Solution:

$$\mathcal{L}(e^{3t} t^4) = \cancel{e^{3t}}\mathcal{L}(t^4)_{s\to s-3}$$

$$= \frac{4!}{(s)^5}\Big|_{s\to s-3}$$

$$= \frac{4!}{(s-3)^5}$$

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Q#3:

Derive the Laplace transforms of $t^{1/2}$ and $t^{-1/2}$ from definition.

Solution:

$$\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$t^{1/2}$

$\alpha = 1/2$

$$\Rightarrow \mathcal{L}\{t^{1/2}\} = \frac{\Gamma(1/2+1)}{s^{1/2+1}}$$

$$\mathcal{L}\{t^{1/2}\} = \frac{1}{s^{1/2+1}} \cdot \frac{1}{2}\Gamma(1/2) \qquad \because \Gamma(n+1) = n\Gamma(n)$$

$$\mathcal{L}\{t^{1/2}\} = \frac{1}{2s} \cdot \frac{1}{s^{1/2}} \cdot \Gamma(1/2)$$

$$\mathcal{L}\{t^{1/2}\} = \frac{1}{2s}\sqrt{\frac{\pi}{s}}$$

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$t^{-1/2}$

$\alpha = -1/2$

$$\mathcal{L}\{t^{-1/2}\} = \frac{1}{s^{-1/2+1}}\Gamma(-1/2+1)$$

$$\mathcal{L}\{t^{-1/2}\} = \frac{1}{s^{1/2}}\Gamma(1/2)$$

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\frac{\pi}{s}}$$

Q#4:

$$\frac{s}{(s^2+2as+b^2)}$$

Solution:

$$\frac{s}{s^2+2as+b^2} = \frac{s}{s^2+2as+a^2-a^2+b^2}$$

$$= \frac{s}{(s+a)^2+(b^2-a^2)}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+(b^2-a^2)}\right\} - \mathcal{L}^{-1}\left\{\frac{a}{(s+a)^2+(b^2-a^2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+(b^2-a^2)}\right\} - \frac{a}{\sqrt{b^2-a^2}} e^{-at} \mathcal{L}^{-1}\left\{\frac{\sqrt{b^2-a^2}}{(s+a)^2+(\sqrt{b^2-a^2})^2}\right\}$$

$$= e^{-at}\cos\sqrt{b^2-a^2}\,t - \frac{a}{\sqrt{b^2-a^2}} e^{-at}\sin\sqrt{b^2-a^2}\,t$$

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Q#5:

$$\frac{1}{s^2+2s+10}$$

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Solution:

$$\frac{1}{s^2+2s+10} = \frac{1}{s^2+2s+1-1+10}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+1-1+10}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(3)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+(3)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \frac{1}{3}e^{-t}\sin 3t$$

Q#6:

$$\frac{1}{s^2-4s+8}$$

Solution:

$$\frac{1}{s^2-4s+8} = \frac{1}{s^2-4s+4+4}$$

$$\frac{1}{s^2-4s+8} = \frac{1}{(s-2)^2+(2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+(2)^2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+(2)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+8}\right\} = \frac{1}{2} e^{2t} \mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+8}\right\} = \frac{1}{2} e^{2t} \sin 2t$$

Q#7.

$$\frac{s}{s^2+6s+13}$$

Solution:

$$\frac{s}{s^2+6s+13} = \frac{s}{s^2+6s+(3)^2-(3)^2+13}$$

$$\frac{s}{s^2+6s+13} = \frac{s}{(s+3)^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)^2+(2)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3-3}{(s+3)^2+(2)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+(2)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{(s+3)^2+(2)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+13}\right\} = e^{-3t} \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{3}{2} \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+(2)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+13}\right\} = e^{-3t} \cos 2t - \frac{3}{2} e^{-3t} \sin 2t$$

Q#8:

$$\frac{(2s+3)}{(s+4)^3}$$

Solution.

$$\mathcal{L}^{-1}\frac{(2s+3)}{(s+3)^3} = \mathcal{L}^{-1}\frac{(2s+8-8+3)}{(s+4)^3}$$

$$\mathcal{L}^{-1}\frac{(2s+3)}{(s+3)^3} = \mathcal{L}^{-1}\frac{(2s+8-5)}{(s+3)^3}$$

$$L^{-1}\frac{(2s+3)}{(s+4)^3} = L^{-1}\left\{\frac{2(s+4)}{(s+4)^3}\right\} - L^{-1}\left\{\frac{5}{(s+4)^3}\right\}$$

$$L^{-1}\frac{(2s+3)}{(s+4)^3} = L^{-1}\left\{\frac{2}{(s+4)^2}\right\} - \frac{5}{2!}L^{-1}\left\{\frac{2!}{(s+4)^{2+1}}\right\}$$

$$L^{-1}\frac{(2s+3)}{(s+4)^3} = 2L^{-1}\left\{\frac{2}{(s+4)^{1+1}}\right\} - \frac{5}{2!}L^{-1}\left\{\frac{2!}{(s+4)^{2+1}}\right\}$$

$$L^{-1}\frac{(2s+3)}{(s+4)^3} = e^{-4t}2t^1 - e^{-4t}\frac{5}{2}t^2$$

Q#9:

Solution: $\frac{s^2}{(s-1)^4}$

$$L^{-1}\left\{\frac{s^2}{(s-1)^4}\right\} = L^{-1}\left\{\frac{s^2-1+1}{(s-1)^4}\right\}$$

$$= L^{-1}\left\{\frac{(s+1)(s-1)}{(s-1)^4}\right\} + L^{-1}\left\{\frac{1}{(s-1)^4}\right\}$$

$$= L^{-1}\left\{\frac{(s+1)}{(s-1)^3}\right\} + L^{-1}\left\{\frac{1}{(s-1)^4}\right\}$$

$$= L^{-1}\left\{\frac{s-1+1+1}{(s-1)^3}\right\} + L^{-1}\left\{\frac{1}{(s-4)^4}\right\}$$

$$= L^{-1}\left\{\frac{(s-1)}{(s-1)^3}\right\} + L^{-1}\left\{\frac{2}{(s-1)^3}\right\} + L^{-1}\left\{\frac{1}{(s-4)^4}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{2}{(s-1)^3}\right\} + L^{-1}\left\{\frac{1}{(s-4)^4}\right\}$$

$$= e^t L^{-1}\left\{\frac{1}{s^2}\right\} + e^t L^{-1}\left\{\frac{2!}{(s)^3}\right\} + \frac{e^t}{3!}L^{-1}\left\{\frac{3!}{(s)^4}\right\}$$

$$= e^t t + e^t t^2 + \frac{e^t}{6}t^3$$

Q#10:

Solution: $\frac{2s-3}{s^2-4s+8}$

$$\mathcal{L}^{-1}\left(\frac{2s-3}{s^2-4s+8}\right) = \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-4s+4-4+8}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{(2s-3)}{s^2-4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{(s-2)^2+2^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{2s-4-3+4}{(s-2)^2+2^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s-2)}{(s-2)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+2^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-4s+8}\right\} = 2e^{2t}\cos 2t + \frac{1}{2}e^{2t}\sin 2t$$

Q#11: & Q#12 (same)

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Solution: $\frac{1}{s^2-4}$

$$\frac{1}{s^2-4} = \frac{1}{(s-2)(s+2)}$$

$$\frac{1}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s-2)$$

put $s-2=0 \Rightarrow s=2$

$1 = A(2+2) + 0 \Rightarrow 1 = 4A$

$A = \frac{1}{4}$

put $s+2=0 \Rightarrow s=-2$

$1 = 0 + B(-2-2)$

$B = -\frac{1}{4}$

$$\frac{1}{s^2-4} = \frac{1}{4(s-2)} + \frac{(-1)}{4(s+2)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\} = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$$

Q#13:

Solution: $\frac{s+3}{s(s^2+2)}$

$$\frac{s+3}{s(s^2+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$$

$$s+3 = A(s^2+2) + (Bs+C)s$$

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put $s=0$

$$3 = 2A + 0$$

$$A = \frac{3}{2}$$

By comparing coefficient of s^2, s

$$0 = A + B$$

$$0 = \frac{3}{2} + B$$

$$B = -\frac{3}{2}$$

$$C = 1$$

$$\frac{s+3}{s(s^2+2)} = \frac{3}{2} \cdot \frac{1}{s} + \frac{(-\frac{3}{2})s+1}{s^2+2}$$

$$\frac{s+3}{s(s^2+2)} = \frac{3}{2} \cdot \frac{1}{s} + \frac{(-3s)+1}{2(s^2+2)}$$

$$\frac{s+3}{s(s^2+2)} = \frac{3}{2} \cdot \frac{1}{s} + \left(\frac{-3}{2}\right)\frac{s}{s^2+2} + \frac{1}{s^2+2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s(s^2+2)}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{2})^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s(s^2+2)}\right\} = \frac{3}{2}(1) - \frac{3}{2}\cos\sqrt{2}t + \sin\sqrt{2}t$$

Q#14:

Solution: $\frac{4}{s(s+1)}$

$$\frac{4}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$4 = A(s+1) + B(s)$$

put $s=0$

$A=4$

put $s+1=0 \Rightarrow s=-1$

$4=0-B$

$B=-4$

$$\frac{4}{s(s+1)}=\frac{4}{s}+\frac{(-4)}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s(s+1)}\right\}=4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{s(s+1)}\right\}=4(1)-4e^{-t}$$

$$=4-4e^{-t}$$

Q#15:

Solve the I.V.Ps:

(a) $u'+2u=0$, $u(0)=1$

Solution:

$$\mathcal{L}\{u'+2u\}=\mathcal{L}\{0\}$$

$$\mathcal{L}\{u'\}+2\mathcal{L}\{u\}=0$$

$$sU(s)-u(0)+2U(s)=0$$

$$(s+2)U(s)-1=0 \qquad \because u(0)=1$$

$$U(s)=\frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{U(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$u(t)=e^{-2t}$$

(b) $u''+9u=0$, $u(0)=0$, $u'(0)=1$

Solution:

$$\mathcal{L}\{u''+9u\}=\mathcal{L}\{0\}$$

$$s^2U(s)-su(0)-u'(0)+9U(s)=0$$

$$(s^2+9)U(s)-0-1=0$$

$$U(s)=\frac{1}{s^2+3^2}$$

$$\mathcal{L}^{-1} U(s) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$\mathcal{L}^{-1} U(s) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+(3)^2}\right\}$$

$$u(t) = \frac{1}{3}\sin 3t$$