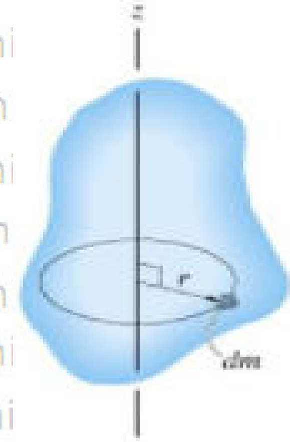
Moment of Inertia



Definition: Moment of inertia of a particle of mass m about a line (called axis of rotation) is defined as

$$I = m\,r^2,$$

where, r is the perpendicular distance of particle from line.

Definition: Moment of inertia of a system of a number of particles with masses m_i, about a line (called axis of rotation) is defined as

$$I = \sum_i m_i {r_i}^2,$$

where, r_i is the perpendicular distance of i-th particle of mass m_i from line.

Definition: Moment of inertia of a continuous distribution of mass, such as the solid rigid body (shown in the figure), having mass M and constant density ρ, about a line is defined as

$$I = \int_M r^2\,\mathrm{d}m = \rho \int_M r^2\,\mathrm{d}V,$$

where, r is the perpendicular distance of point mass element dm of the body and dV is its elementary volume.

Moments of inertia with respect to Cartesian coordinate axes are defined in the following table:

Moment of inertia	**Moment of inertia of a particle with respect to 3-dimensioal Cartesian coordinate system**	**Moment of inertia of a set of particles with respect to 3-dimensioal Cartesian coordinate system**	**Moment of inertia of a continuous rigid body with respect to 3-dimensioal Cartesian coordinate system**
About x-axis $I_{xx} = I_{11}$	$m(y^2+z^2)$	$\sum_i m_i(y_i^2+z_i^2)$	$\int_M (y^2+z^2)\mathrm{d}m$
About y-axis $I_{yy} = I_{22}$	$m(x^2+z^2)$	$\sum_i m_i(x_i^2+z_i^2)$	$\int_M (x^2+z^2)\mathrm{d}m$
About z-axis $I_{zz} = I_{33}$	$m(x^2+y^2)$	$\sum_i m_i(x_i^2+y_i^2)$	$\int_M (x^2+y^2)\mathrm{d}m$

Products of inertia with respect to Cartesian coordinate axes are defined in the following table:

Product of inertia	**Product of inertia of a particle with respect to 3-dimensioal Cartesian coordinate system**	**Product of inertia of a set of particles with respect to 3-dimensioal Cartesian coordinate system**	**Product of inertia of a continuous rigid body with respect to 3-dimensioal Cartesian coordinate system**
$I_{xy} = I_{yx} = I_{12} = I_{21}$	$-\,mxy$	$-\sum_i m_i\, x_i y_i$	$-\int_M xy\,\mathrm{d}m$
$I_{yz} = I_{zy} = I_{23} = I_{32}$	$-\,myz$	$-\sum_i m_i y_i z_i$	$-\int_M yz\,\mathrm{d}m$
$I_{xz} = I_{zx} = I_{13} = I_{31}$	$-\,mxz$	$-\sum_i m_i x_i z_i$	$-\int_M xz\,\mathrm{d}m$

Definition: Radius of gyration k of a rigid body of mass M with respect to a line l is defined as

$$k = \sqrt{I/M},$$

where, I is the moment of inertia of the body with respect to l.

Problem: Prove in matrix notation that $[\mathbf{L}] = [\,\mathbf{I}\,][\boldsymbol{\omega}]$, where, all the notations used have their usual meanings.

Proof: The angular momentum of a rigid body, in the form of a set of particles, about an instantaneous axis through a fixed point, is given by

$$\mathbf{L} = \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i) = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i) = \sum_i m_i\big(\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)\big) \qquad \because \mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

$$= \sum_i m_i(\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)) = \sum_i m_i[(\mathbf{r}_i \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i]$$

Let, $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$ and $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$

$$\Rightarrow \quad \mathbf{r}_i \cdot \mathbf{r}_i = x_i^2 + y_i^2 + z_i^2 \quad \text{and} \quad \mathbf{r}_i \cdot \boldsymbol{\omega} = x_i\omega_1 + y_i\omega_2 + z_i\omega_3$$

$$\Rightarrow \quad L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k} = \sum_i m_i\left[\left(x_i^2 + y_i^2 + z_i^2\right)(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) - (x_i\omega_1 + y_i\omega_2 + z_i\omega_3)(x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k})\right]$$

Comparing corresponding components on both sides of above vector equation, we get

$$L_1 = \sum_i m_i\left[\left(x_i^2 + y_i^2 + z_i^2\right)\omega_1 - (x_i\omega_1 + y_i\omega_2 + z_i\omega_3)x_i\right] \quad \text{-----------}\rightarrow \quad (1)$$

$$L_2 = \sum_i m_i\left[\left(x_i^2 + y_i^2 + z_i^2\right)\omega_2 - (x_i\omega_1 + y_i\omega_2 + z_i\omega_3)y_i\right] \quad \text{-----------}\rightarrow \quad (2)$$

$$L_3 = \sum_i m_i\left[\left(x_i^2 + y_i^2 + z_i^2\right)\omega_3 - (x_i\omega_1 + y_i\omega_2 + z_i\omega_3)z_i\right] \quad \text{-----------}\rightarrow \quad (3)$$

From Eq. (1), we get, $L_1 = \sum_i m_i\left[x_i^2\omega_1 + \left(y_i^2 + z_i^2\right)\omega_1 - x_i^2\omega_1 - x_iy_i\omega_2 - x_iz_i\omega_3\right]$

$$= \omega_1\sum_i m_i\left(y_i^2 + z_i^2\right) - \omega_2\sum_i m_ix_iy_i - \omega_3\sum_i m_ix_iz_i$$

$$= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \quad \text{---------}\rightarrow \quad (4)$$

Similarly, from (2) and (3), we get $\quad L_2 = I_{12}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \quad \text{---------}\rightarrow \quad (5)$

and $\quad L_3 = I_{13}\omega_1 + I_{23}\omega_2 + I_{33}\omega_3 \quad \text{---------}\rightarrow \quad (6)$

Writing Eqs. (4), (5)and (6)in matrix form, we get, $\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

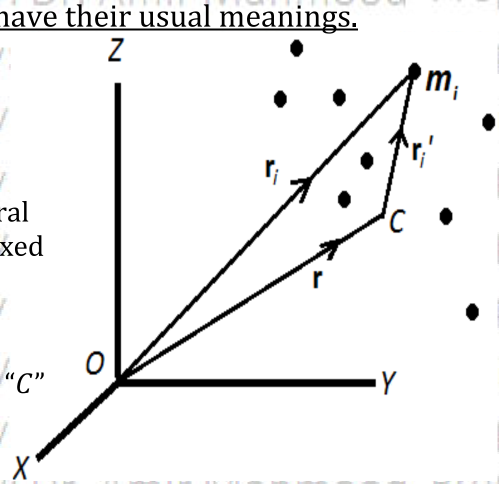
$$\boxed{\Rightarrow \quad [\mathbf{L}] = [\,\mathbf{I}\,][\boldsymbol{\omega}] \qquad \textbf{Hence proved.}}$$

<u>Problem:</u> <u>Prove that $T = \frac{1}{2}M\mathbf{v}^2 + \frac{1}{2}\boldsymbol{\omega}.\mathbf{L}$, where all the notations used have their usual meanings.</u>

<u>(or) prove that</u> $T = T_{tr} + T_{rot}$

where, $T_{tr} = \frac{1}{2}M\mathbf{v}^2 =$ total translational kinetic energy of the body

and $T_{rot} = \frac{1}{2}\boldsymbol{\omega}.\mathbf{L} =$ total rotational kinetic energy of the body.



Proof: Consider a rigid body, in the form of a set of particles, which is in general state of motion (i.e., having both translation and rotation) with respect to a fixed (inertial) frame of reference $Oxyz$.

Let, $M =$ total mass of the body

$\mathbf{r}_i =$ position vector of i-th particle of mass m_i with respect to origin "O"

$\mathbf{r}_i' =$ position vector of i-th particle of mass m_i with respect to centre of mass "C"

$\mathbf{r} =$ position vector of centre of mass "C" with respect to origin "O"

$\mathbf{v}_i =$ velocity of i-th particle of mass m_i with respect to origin "O"

$\mathbf{v}_i' =$ velocity of i-th particle of mass m_i with respect to centre of mass "C"

$\mathbf{v} =$ velocity of centre of mass "C" with respect to origin "O"

$\boldsymbol{\omega} =$ instantaneous angular velocity of body about instantaneous axis through centre of mass "C"

From figure, $\quad \mathbf{r}_i = \mathbf{r} + \mathbf{r}_i'$

Differentiating both sides with respect to time "t", we get

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}} + \dot{\mathbf{r}}_i'$$

$$\Rightarrow \quad \mathbf{v}_i = \mathbf{v} + \mathbf{v}_i' = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i' \qquad \because \quad \mathbf{v}_i' = \boldsymbol{\omega} \times \mathbf{r}_i'$$

Kinetic energy of the i-th particle is $\quad T_i = \frac{1}{2}m_i\mathbf{v}_i^2$

Kinetic energy of the whole body is

$$T = \sum_i T_i = \frac{1}{2}\sum_i m_i\mathbf{v}_i^2 = \frac{1}{2}\sum_i m_i\,(\mathbf{v}_i \cdot \mathbf{v}_i) = \frac{1}{2}\sum_i m_i\,\{(\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i') \cdot (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i')\} \qquad \because \quad \mathbf{v}_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i'$$

$$= \frac{1}{2}\sum_i m_i\,\{\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}_i') + (\boldsymbol{\omega} \times \mathbf{r}_i') \cdot \mathbf{v} + (\boldsymbol{\omega} \times \mathbf{r}_i') \cdot (\boldsymbol{\omega} \times \mathbf{r}_i')\}$$

$$T=\frac{1}{2}\sum_i m_i\{\mathbf{v}^2+2\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}_i')+\boldsymbol{\omega}\cdot\mathbf{r}_i'\times(\boldsymbol{\omega}\times\mathbf{r}_i')\}$$

$$=\frac{1}{2}\left(\sum_i m_i\right)\mathbf{v}^2+\sum_i m_i\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}_i')+\frac{1}{2}\sum_i m_i\{\boldsymbol{\omega}\cdot\mathbf{r}_i'\times(\boldsymbol{\omega}\times\mathbf{r}_i')\}$$

$$=\frac{1}{2}M\mathbf{v}^2+\mathbf{v}\cdot\left(\boldsymbol{\omega}\times\sum_i m_i\mathbf{r}_i'\right)+\frac{1}{2}\boldsymbol{\omega}\cdot\sum_i m_i\mathbf{r}_i'\times(\boldsymbol{\omega}\times\mathbf{r}_i'),\quad \text{where,}\quad M=\sum_i m_i=\text{total mass of the body}$$

Also, $\sum_i m_i\mathbf{r}_i'=\mathbf{0}$, as $\mathbf{r}_i'$ is the position vector of ith particle of mass m_i with respect to centre of mass "C"

$$\Rightarrow\quad T=\frac{1}{2}M\mathbf{v}^2+\frac{1}{2}\boldsymbol{\omega}\cdot\sum_i m_i\mathbf{r}_i'\times(\boldsymbol{\omega}\times\mathbf{r}_i')\quad ---------(1)$$

But, angular momentum $\mathbf{L}$ of the body with respect to centre of mass "C" is given by

$$\mathbf{L}=\sum_i \mathbf{r}_i'\times(m_i\mathbf{v}_i')=\sum_i \mathbf{r}_i'\times\{m_i(\boldsymbol{\omega}\times\mathbf{r}_i')\}=\sum_i m_i\mathbf{r}_i'\times(\boldsymbol{\omega}\times\mathbf{r}_i')\quad ---------(2)$$

Using (2) in (1), we get,

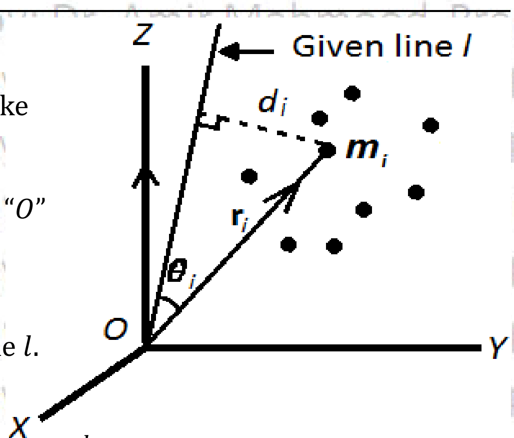
$$\boxed{T=\frac{1}{2}M\mathbf{v}^2+\frac{1}{2}\boldsymbol{\omega}.\mathbf{L}}$$

$$\boxed{T=T_{tr}+T_{rot}}$$

where, $T_{tr}=\frac{1}{2}M\mathbf{v}^2=$ total translational kinetic energy of the body

and $T_{rot}=\frac{1}{2}\boldsymbol{\omega}.\mathbf{L}=$ total rotational kinetic energy of the body

Problem: Find moment of inertia of a rigid body about a given line passing through the origin and having direction cosines are (λ,μ,ν).



Solution: Consider a rigid body, in the form of a set of particles. And let us take given line as z-axis, as shown in the figure.

Let, M = total mass of the body

$\mathbf{r}_i=x_i\mathbf{i}+y_i\mathbf{j}+z_i\mathbf{k}=$ position vector of i-th particle of mass m_i w.r.t. origin "O"

$d_i=$ perpendicular distance of i-th particle of mass m_i from given line l

$\theta_i=$ angle between position vector $\mathbf{r}_i$ and given line l

$\mathbf{e}=$ unit vector in the direction of given line l

Then, $\mathbf{e}=\lambda\mathbf{i}+\mu\mathbf{j}+\nu\mathbf{k}$, where, (λ,μ,ν) are direction cosines of the given line l.

The required moment of inertia I_l is given by

$$I_l=\sum_i m_i d_i^2=\sum_i m_i(|\mathbf{r}_i|\sin\theta_i)^2=\sum_i m_i(|\mathbf{e}\times\mathbf{r}_i|)^2 ---\rightarrow(1)\quad \because\ \sin\theta_i=\frac{d_i}{|\mathbf{r}_i|}\ \text{and}\ |\mathbf{r}_i|\sin\theta_i=|\mathbf{e}\times\mathbf{r}_i|$$

Now,
$$\mathbf{e}\times\mathbf{r}_i=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \lambda&\mu&\nu\\ x_i&y_i&z_i\end{vmatrix}=(\mu z_i-\nu y_i)\mathbf{i}+(\nu x_i-\lambda z_i)\mathbf{j}+(\lambda y_i-\mu x_i)\mathbf{k}$$

$$\Rightarrow(|\mathbf{e}\times\mathbf{r}_i|)^2=(\mu z_i-\nu y_i)^2+(\nu x_i-\lambda z_i)^2+(\lambda y_i-\mu x_i)^2 ------\rightarrow(2)$$

Using (2) in (1), we get

$$I_l=\sum_i m_i[(\mu z_i-\nu y_i)^2+(\nu x_i-\lambda z_i)^2+(\lambda y_i-\mu x_i)^2]$$

$$=\sum_i m_i[(\mu^2 z_i{}^2+\nu^2 y_i{}^2-2\mu\nu y_i z_i)+(\nu^2 x_i{}^2+\lambda^2 z_i{}^2-2\lambda\nu x_i z_i)+(\lambda^2 y_i{}^2+\mu^2 x_i{}^2-2\lambda\mu x_i y_i)]$$

$$=\lambda^2\sum_i m_i(y_i{}^2+z_i{}^2)+\mu^2\sum_i m_i(x_i{}^2+z_i{}^2)+\nu^2\sum_i m_i(x_i{}^2+y_i{}^2)+2\lambda\mu\left(-\sum_i m_i x_i y_i\right)$$

$$+2\mu\nu\left(-\sum_i m_i y_i z_i\right)+2\lambda\nu\left(-\sum_i m_i x_i z_i\right)$$

$$\boxed{\Rightarrow\quad I_l=\lambda^2 I_{xx}+\mu^2 I_{yy}+\nu^2 I_{zz}+2\lambda\mu I_{xy}+2\mu\nu I_{yz}+2\lambda\nu I_{xz}}$$

This is required moment of inertia.

Problem: Find the equation of "**ellipsoid of inertia**" or "**momental ellipsoid**" of a rigid body.

Solution: As we know that moment of inertia of a rigid body about a given line l having direction cosines (λ, μ, ν) with respect to a coordinate system $Oxyz$, whose origin "O" lies on the line l, is given by

$$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu\nu I_{yz} + 2\lambda\nu I_{xz} ------\rightarrow (1)$$

On the line l, choose a point P such that $|\overrightarrow{OP}| = 1/\sqrt{I_l}$. If coordinates of P are (x, y, z), then

$$\frac{x}{|\overrightarrow{OP}|} = \lambda, \quad \frac{y}{|\overrightarrow{OP}|} = \mu, \quad \frac{z}{|\overrightarrow{OP}|} = \nu$$

$$\Rightarrow \quad \lambda = x\sqrt{I_l}\,, \quad \mu = y\sqrt{I_l}\,, \quad \nu = z\sqrt{I_l} ------\rightarrow (2)$$

Eliminating λ, μ and ν from (1) and (2), we get

$$I_l = I_l\left(I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{yz}yz + 2I_{xz}xz\right)$$

$$\boxed{I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{yz}yz + 2I_{xz}xz = 1}$$

Since, I_{xx}, I_{yy} and I_{zz} are all positive, therefore, above equation represents an ellipsoid called "ellipsoid of inertia" or "momental ellipsoid" of the rigid body.

Note:

(i) The momental ellipsoid of a rigid body contains information about moments and product of inertia of that body.

(ii) The centre of momental ellipsoid lies at the origin of the coordinate system.

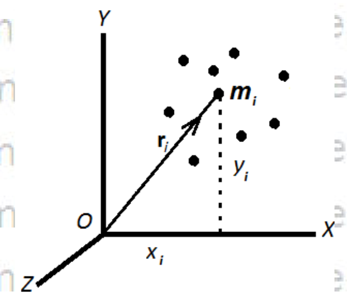
(iii) If P is any point on momental ellipsoid, then

$$|\overrightarrow{OP}| = \frac{1}{\sqrt{I_l}} \Rightarrow I_l = \frac{1}{|\overrightarrow{OP}|^2},$$

showing that moment of inertia about line $\overleftrightarrow{OP}$ is equal to the reciprocal of square of distance of point P from origin O.

Problem: State and prove perpendicular axis theorem for a discrete mass distribution.

Statement: The moment of inertia of a plane rigid body in the form of discrete mass distribution (i.e., a set of particles) about a given axis perpendicular to the plane of the body is equal to the sum of moments of inertia about two mutually perpendicular axes lying in the plane of the body and meeting at a common point on the given axis.



Proof: We choose Cartesian coordinate system $Oxyz$ such that xy-plane lies in the plane of the body, while z-axis lies perpendicular to it, which is assumed to the given axis.

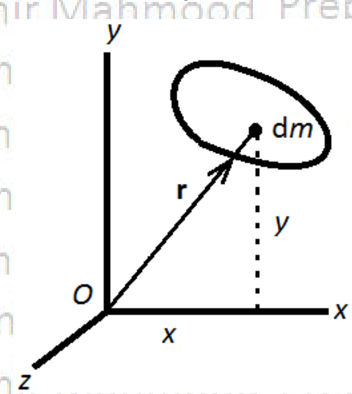
Let, $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j}$ be the position vector of i-th particle of mass m_i w.r.t. origin "O". Then moment of inertia of the body about z-axis is

$$I_{zz} = \sum_i m_i|\mathbf{r}_i|^2 = \sum_i m_i({x_i}^2 + {y_i}^2) = \sum_i m_i{x_i}^2 + \sum_i m_i{y_i}^2 = I_{xx} + I_{yy}$$

$$\boxed{\Rightarrow \quad I_{zz} = I_{xx} + I_{yy} \qquad \textbf{Hence proved.}}$$

Problem: State and prove perpendicular axis theorem for a continuous mass distribution.

Statement: The moment of inertia of a plane rigid body in the form of continuous mass distribution about a given axis perpendicular to the plane of the body is equal to the sum of moments of inertia of same body about two mutually perpendicular axes lying in the plane of body and meeting at a common point on the given axis.



Proof: We choose Cartesian coordinate system $Oxyz$ such that xy-plane lies in the plane of the body having mass M, while z-axis lies perpendicular to it, which is assumed to the given axis.

Let, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector of elementary particle of body of mass dm w.r.t. origin "O".

Then moment of inertia of the body about z-axis is

$$I_{zz} = \int_M |\mathbf{r}|^2 dm = \int_M (x^2 + y^2)dm = \int_M x^2 dm + \int_M y^2 dm = I_{xx} + I_{yy}$$

$$\boxed{\Rightarrow \quad I_{zz} = I_{xx} + I_{yy} \qquad \textbf{Hence proved.}}$$

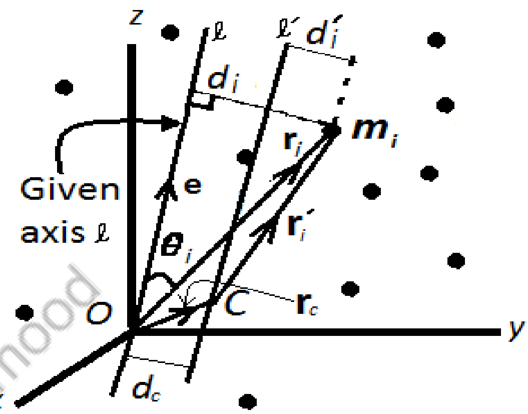
Problem: State and prove parallel axis theorem for the case of moment of inertia for a discrete mass

<u>distribution.</u>

Statement: The moment of inertia of a rigid body in the form of discrete mass distribution (i.e., a set of particles) about a given axis is equal to the sum of moment of inertia of same body about a parallel axis (to the given axis) through the centre of mass of the body and the moment of inertia due to the total mass of the body placed at its centre of mass, about given axis.

Proof: Consider a rigid body, in the form of a set of particles. Let l be the given and l' be an axis which is parallel to l and passing through centre of mass of the body. Let, $M =$ total mass of the body

$\mathbf{r}_i =$ position vector of i-th particle of mass m_i with respect to origin "O"
$\mathbf{r}_i' =$ position vector of i-th particle of mass m_i with respect to centre of mass "C"
$\mathbf{r}_c =$ position vector of centre of mass "C" with respect to origin "O"
$\theta_i =$ angle between position vector $\mathbf{r}_i$ and given line l
$d_i =$ perpendicular distance of i-th particle of mass m_i from given axis l
$d_i' =$ perpendicular distance of i-th particle of mass m_i from parallel axis l'
$d_c =$ perpendicular distance of centre of mass C from given axis l
$\quad =$ perpendicular distance between l and l'
$\mathbf{e} =$ unit vector in the direction of given line l



From figure, $\sin\theta_i = \frac{d_i}{|\mathbf{r}_i|} \Rightarrow d_i = |\mathbf{r}_i|\sin\theta_i = |\mathbf{e}\times\mathbf{r}_i|$

Similarly, $d_i' = |\mathbf{e}\times\mathbf{r}_i'|$ and $d_c = |\mathbf{e}\times\mathbf{r}_c|$

Moment of inertia of the body about given axis l is given by

$$I_l = \sum_i m_i d_i^2 = \sum_i m_i(|\mathbf{e}\times\mathbf{r}_i|)^2 = \sum_i m_i(\mathbf{e}\times\mathbf{r}_i)\cdot(\mathbf{e}\times\mathbf{r}_i)$$

$$= \sum_i m_i[\mathbf{e}\times(\mathbf{r}_c+\mathbf{r}_i')]\cdot[\mathbf{e}\times(\mathbf{r}_c+\mathbf{r}_i')] \qquad \because \mathbf{r}_i = \mathbf{r}_c + \mathbf{r}_i' \text{ (from figure)}$$

$$= \sum_i m_i(\mathbf{e}\times\mathbf{r}_c + \mathbf{e}\times\mathbf{r}_i')\cdot(\mathbf{e}\times\mathbf{r}_c + \mathbf{e}\times\mathbf{r}_i')$$

$$= \sum_i m_i[(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}_c) + 2(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}_i') + (\mathbf{e}\times\mathbf{r}_i')\cdot(\mathbf{e}\times\mathbf{r}_i')]$$

$$= \sum_i m_i[(|\mathbf{e}\times\mathbf{r}_c|)^2 + 2(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}_i') + (|\mathbf{e}\times\mathbf{r}_i'|)^2]$$

$$= \left(\sum_i m_i\right)(|\mathbf{e}\times\mathbf{r}_c|)^2 + 2(\mathbf{e}\times\mathbf{r}_c)\cdot\sum_i m_i\,(\mathbf{e}\times\mathbf{r}_i') + \sum_i m_i(|\mathbf{e}\times\mathbf{r}_i'|)^2$$

$$= M{d_c}^2 + 2(\mathbf{e}\times\mathbf{r}_c)\cdot\left(\mathbf{e}\times\sum_i m_i\,\mathbf{r}_i'\right) + \sum_i m_i {d_i'}^2, \quad \text{where,} \quad M = \sum_i m_i = \text{total mass of the body}$$

Also, $\sum_i m_i\mathbf{r}_i' = \mathbf{0}$, as $\mathbf{r}_i'$ is the position vector of ith particle of mass m_i with respect to centre of mass "C" and

$$I_{l'} = \sum_i m_i {d_i'}^2 = \text{moment of inertia of the body axis } l'$$

$$\boxed{\Rightarrow \quad I_l = I_{l'} + M{d_c}^2 \qquad \textbf{Hence proved.}}$$

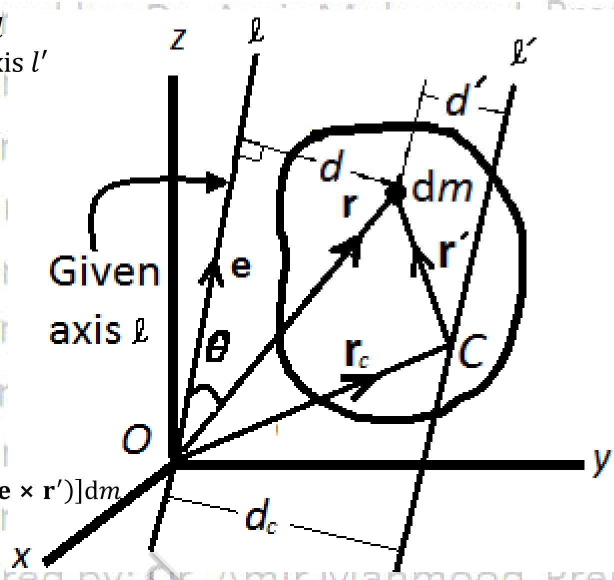
<u>**Problem:** State and prove parallel axis theorem for the case of moment of inertia for a continuous mass distribution.</u>

Statement: The moment of inertia of a rigid body in the form of a continuous mass distribution about a given axis is equal to the sum of moment of inertia of same body about a parallel axis (to the given axis) through the centre of mass of the body and the moment of inertia due to the total mass of the body placed at its centre of mass, about given axis.

Proof: Consider a rigid body, in the form of a continuous mass distribution. Let l be the given and l' be an axis which is parallel to l and passing through centre of mass of the body.

Let, $M =$ total mass of the body
$\mathbf{r} =$ position vector of elementary mass dm with respect to origin "O"
$\mathbf{r}' =$ position vector of elementary mass dm with respect to centre of mass "C"

$\mathbf{r}_c =$ position vector of centre of mass "C" with respect to origin "O"
$\theta =$ angle between position vector $\mathbf{r}$ and given line l
$d =$ perpendicular distance of elementary mass dm from given axis l
$d' =$ perpendicular distance of elementary mass dm from parallel axis l'
$d_c =$ perpendicular distance of centre of mass C from given axis l
$=$ perpendicular distance between l and l'
$\mathbf{e} =$ unit vector in the direction of given line l



From figure, $\sin\theta = d/|\mathbf{r}| \Rightarrow d = |\mathbf{r}|\sin\theta = |\mathbf{e}\times\mathbf{r}|$
Similarly, $d' = |\mathbf{e}\times\mathbf{r}'|$ and $d_c = |\mathbf{e}\times\mathbf{r}_c|$
Moment of inertia of the body about given axis l is given by

$$I_l = \int_M d^2 dm = \int_M (|\mathbf{e}\times\mathbf{r}|)^2 dm = \int_M (\mathbf{e}\times\mathbf{r})\cdot(\mathbf{e}\times\mathbf{r})dm$$

$$= \int_M [\mathbf{e}\times(\mathbf{r}_c+\mathbf{r}')]\cdot[\mathbf{e}\times(\mathbf{r}_c+\mathbf{r}')]dm \qquad \because \mathbf{r} = \mathbf{r}_c + \mathbf{r}'$$

$$= \int_M [(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}_c) + 2(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}') + (\mathbf{e}\times\mathbf{r}')\cdot(\mathbf{e}\times\mathbf{r}')]dm$$

$$= \int_M [(|\mathbf{e}\times\mathbf{r}_c|)^2 + 2(\mathbf{e}\times\mathbf{r}_c)\cdot(\mathbf{e}\times\mathbf{r}') + (|\mathbf{e}\times\mathbf{r}'|)^2]dm$$

$$= \left(\int_M dm\right)(|\mathbf{e}\times\mathbf{r}_c|)^2 + 2(\mathbf{e}\times\mathbf{r}_c)\cdot\int_M(\mathbf{e}\times\mathbf{r}')dm + \int_M (|\mathbf{e}\times\mathbf{r}'|)^2 dm$$

$$= Md_c^{\,2} + 2(\mathbf{e}\times\mathbf{r}_c)\cdot\left(\mathbf{e}\times\int_M \mathbf{r}'dm\right) + \int_M d'^2 dm, \quad \text{where,} \quad M = \int_M dm = \text{total mass of the body}$$

Also, $\int_M \mathbf{r}'dm = \mathbf{0}$, as $\mathbf{r}'$ is the position vector of mass element dm with respect to centre of mass "C",
and $I_{l'} = \int_M d'^2 dm =$ moment of inertia of the body axis l'

$$\boxed{\Rightarrow \quad I_l = I_{l'} + Md_c^{\,2} \qquad \textbf{Hence proved.}}$$

Problem: <u>Prove in matrix notation that $[\dot{\mathbf{L}}] = [\boldsymbol{\omega}\times\mathbf{L}] + [\,\mathbf{I}\,][\dot{\boldsymbol{\omega}}]$, where, all the notations used have their usual meanings.</u>

Proof: As we know that the angular momentum of a system of particles is given by

$$\mathbf{L} = \sum_i \mathbf{r}_i\times(m_i\mathbf{v}_i) = \sum_i m_i\mathbf{r}_i\times\mathbf{v}_i$$

Differentiating both sides with respect to time "t", we get

$$\dot{\mathbf{L}} = \sum_i m_i\dot{\mathbf{r}}_i\times\mathbf{v}_i + \sum_i m_i\mathbf{r}_i\times\dot{\mathbf{v}}_i = \sum_i m_i\mathbf{v}_i\times\mathbf{v}_i + \sum_i m_i\mathbf{r}_i\times\dot{\mathbf{v}}_i$$

$$= \sum_i m_i\mathbf{r}_i\times\frac{d}{dt}(\boldsymbol{\omega}\times\mathbf{r}_i) \qquad \because \quad \mathbf{v}_i\times\mathbf{v}_i = \mathbf{0} \quad \text{and} \quad \dot{\mathbf{v}}_i = \frac{d\mathbf{v}_i}{dt} = \frac{d}{dt}(\boldsymbol{\omega}\times\mathbf{r}_i)$$

$$= \sum_i m_i\mathbf{r}_i\times[(\boldsymbol{\omega}\times\dot{\mathbf{r}}_i) + (\dot{\boldsymbol{\omega}}\times\mathbf{r}_i)] = \sum_i m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\dot{\mathbf{r}}_i) + \sum_i m_i\mathbf{r}_i\times(\dot{\boldsymbol{\omega}}\times\mathbf{r}_i)$$

Writing in matrix form, we get

$$[\dot{\mathbf{L}}] = \left[\sum_i m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\dot{\mathbf{r}}_i)\right] + \left[\sum_i m_i\mathbf{r}_i\times(\dot{\boldsymbol{\omega}}\times\mathbf{r}_i)\right] \longrightarrow (1)$$

As we know that $\quad [\mathbf{L}] = [\,\mathbf{I}\,][\boldsymbol{\omega}] \quad \Rightarrow \quad \left[\sum_i \mathbf{r}_i\times(m_i\mathbf{v}_i)\right] = [\,\mathbf{I}\,][\boldsymbol{\omega}] \qquad \because \mathbf{L} = \sum_i \mathbf{r}_i\times(m_i\mathbf{v}_i)$

$$\Rightarrow \quad \left[\sum_i m_i\mathbf{r}_i\times\mathbf{v}_i\right] = [\,\mathbf{I}\,][\boldsymbol{\omega}]$$

$$\Rightarrow \quad \left[\sum_i m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i)\right] = [\,\mathbf{I}\,][\boldsymbol{\omega}] \qquad \because \mathbf{v}_i = \boldsymbol{\omega}\times\mathbf{r}_i$$

Replace $\boldsymbol{\omega}$ by $\dot{\boldsymbol{\omega}}$ on both sides, we get

$$\left[\sum_i m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)\right] = [\,\mathbf{I}\,][\dot{\boldsymbol{\omega}}] - - - - \longrightarrow (2)$$

Now consider,

$$\sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i) = \sum_i m_i \mathbf{r}_i \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = \sum_i m_i \mathbf{r}_i \times [(\boldsymbol{\omega} \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{r}_i]$$

$$= \sum_i m_i[(\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_i \times \mathbf{r}_i)] = \sum_i m_i (\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) - - - - \longrightarrow (3) \quad \because \mathbf{r}_i \times \mathbf{r}_i = \mathbf{0}$$

Further consider that

$$\boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{v}_i) = \boldsymbol{\omega} \times [\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = \boldsymbol{\omega} \times [(\mathbf{r}_i \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i] = (\mathbf{r}_i \cdot \mathbf{r}_i)(\boldsymbol{\omega} \times \boldsymbol{\omega}) - (\mathbf{r}_i \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{r}_i)$$
$$= -(\mathbf{r}_i \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{r}_i) = (\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) - - - - - - - \longrightarrow (4) \qquad \because \boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$$

Using (4) in (3), we get

$$\sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \sum_i m_i\, \boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{v}_i) = \boldsymbol{\omega} \times \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i) = \boldsymbol{\omega} \times \mathbf{L} \qquad \because \mathbf{L} = \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i)$$

Writing above equation in matrix form, we get, $\left[\sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i)\right] = [\boldsymbol{\omega} \times \mathbf{L}] - - - - - - - \longrightarrow (5)$

Using (2) and (5) in (1), we get

$$\boxed{[\dot{\mathbf{L}}] = [\boldsymbol{\omega} \times \mathbf{L}] + [\,\mathbf{I}\,][\dot{\boldsymbol{\omega}}] \qquad \textbf{Hence proved.}}$$

<u>Problem:</u> <u>Show that inertia matrix [**I**] is a Cartesian tensor of rank 2.</u>

Proof: As we know that the angular momentum of a system of particles is given by

$$\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times (m_\alpha \mathbf{v}_\alpha) = \sum_\alpha m_\alpha (\mathbf{r}_\alpha \times \mathbf{v}_\alpha) = \sum_\alpha m_\alpha \big(\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)\big) \qquad \because \mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

$$= \sum_\alpha m_\alpha [(\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha)\boldsymbol{\omega} - (\mathbf{r}_\alpha \cdot \boldsymbol{\omega})\mathbf{r}_\alpha] = \sum_\alpha m_\alpha [\mathbf{r}_\alpha^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)\mathbf{r}_\alpha] - - - - - - - - - \longrightarrow (1)$$

Let, $\mathbf{L} = (L_1,\, L_2,\, L_3)$, $\boldsymbol{\omega} = (\omega_1,\, \omega_2,\, \omega_3)$ and $\mathbf{r}_\alpha = \left(x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3}\right)$

Then, $\boldsymbol{\omega} \cdot \mathbf{r}_\alpha = \omega_1 x_{\alpha,1} + \omega_2 x_{\alpha,2} + \omega_3 x_{\alpha,3} = \sum_{j=1}^{3} \omega_j x_{\alpha,j}$

So (1) can be written as: $(L_1,\, L_2,\, L_3) = \sum_\alpha m_\alpha \left[\mathbf{r}_\alpha^2 (\omega_1,\, \omega_2,\, \omega_3) - \left(\sum_{j=1}^{3} \omega_j x_{j,\alpha}\right)\left(x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3}\right)\right]$

$$\Rightarrow \; L_i = \sum_\alpha m_\alpha \left[\mathbf{r}_\alpha^2 \omega_i - \left(\sum_{j=1}^{3} \omega_j x_{\alpha,j}\right) x_{\alpha,i}\right], \qquad i = 1, 2, 3$$

$$= \sum_\alpha m_\alpha \left[\mathbf{r}_\alpha^2 \sum_{j=1}^{3} \omega_j \delta_{ij} - \left(\sum_{j=1}^{3} \omega_j x_{\alpha,j}\right) x_{\alpha,i}\right] \qquad \because \omega_i = \sum_{j=1}^{3} \omega_j \delta_{ij}$$

$$= \sum_\alpha m_\alpha \sum_{j=1}^{3} [\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i}]\omega_j = \sum_{j=1}^{3} \omega_j \sum_\alpha m_\alpha [\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}] = \sum_{j=1}^{3} \omega_j I_{ij} - - - - - - \longrightarrow (2)$$

where, $I_{ij} = \sum_\alpha m_\alpha [\mathbf{r}_\alpha^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}] = ij'$th component of inertia tensor

Since, both the angular velocity $\boldsymbol{\omega} = (\omega_j)$ and the angular momentum $\mathbf{L} = (L_i)$ are known to be vectors (i.e., Cartesian tensors of rank 1), it follows from equation (2) and quotient theorem that the inertia tensor $[\,\mathbf{I}\,] = \left(I_{ij}\right)$ is a Cartesian tensor of rank 2.

<u>Problem:</u> <u>Express angular momentum in tensor notation.</u>

Solution: As we know that the angular momentum of a system of particles is given by

$$\mathbf{L}=\sum_{\alpha}\mathbf{r}_\alpha\times(m_\alpha\mathbf{v}_\alpha)=\sum_{\alpha}m_\alpha(\mathbf{r}_\alpha\times\mathbf{v}_\alpha)=\sum_{\alpha}m_\alpha\big(\mathbf{r}_\alpha\times(\boldsymbol{\omega}\times\mathbf{r}_\alpha)\big)\qquad\because\mathbf{v}_\alpha=\boldsymbol{\omega}\times\mathbf{r}_\alpha$$

$$\mathbf{L}=\sum_{\alpha}m_\alpha[(\mathbf{r}_\alpha\cdot\mathbf{r}_\alpha)\boldsymbol{\omega}-(\mathbf{r}_\alpha\cdot\boldsymbol{\omega})\mathbf{r}_\alpha]=\sum_{\alpha}m_\alpha[\mathbf{r}_\alpha^2\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)\mathbf{r}_\alpha]-------\rightarrow(1)$$

Let, $\mathbf{L}=(L_1,\,L_2,\,L_3)$, $\boldsymbol{\omega}=(\omega_1,\,\omega_2,\,\omega_3)$ and $\mathbf{r}_\alpha=\left(x_{\alpha,1},\,x_{\alpha,2},\,x_{\alpha,3}\right)$

Then, $\boldsymbol{\omega}\cdot\mathbf{r}_\alpha=\omega_1x_{\alpha,1}+\omega_2x_{\alpha,2}+\omega_3x_{\alpha,3}=\sum_{j=1}^{3}\omega_jx_{\alpha,j}$

So, (1) can be written as

$$(L_1,\,L_2,\,L_3)=\sum_{\alpha}m_\alpha\left[\mathbf{r}_\alpha^2(\omega_1,\,\omega_2,\,\omega_3)-\left(\sum_{j=1}^{3}\omega_jx_{j,\alpha}\right)(x_{\alpha,1},\,x_{\alpha,2},\,x_{\alpha,3})\right]$$

$$\Rightarrow L_i=\sum_{\alpha}m_\alpha\left[\mathbf{r}_\alpha^2\omega_i-\left(\sum_{j=1}^{3}\omega_jx_{\alpha,j}\right)x_{\alpha,i}\right],\qquad i=1,2,3$$

$$=\sum_{\alpha}m_\alpha\left[\mathbf{r}_\alpha^2\sum_{j=1}^{3}\omega_j\delta_{ij}-\left(\sum_{j=1}^{3}\omega_jx_{\alpha,j}\right)x_{\alpha,i}\right]\qquad\because\omega_i=\sum_{j=1}^{3}\omega_j\delta_{ij}$$

$$=\sum_{\alpha}m_\alpha\sum_{j=1}^{3}[\mathbf{r}_\alpha^2\delta_{ij}-x_{\alpha,j}x_{\alpha,i}]\omega_j=\sum_{j=1}^{3}\omega_j\sum_{\alpha}m_\alpha[\mathbf{r}_\alpha^2\delta_{ij}-x_{\alpha,i}x_{\alpha,j}]=\sum_{j=1}^{3}\omega_jI_{ij}------\rightarrow(2)$$

where, $I_{ij}=\sum_{\alpha}m_\alpha[\mathbf{r}_\alpha^2\delta_{ij}-x_{\alpha,i}x_{\alpha,j}]=ij$th component of inertia tensor

Equation (2) is required tensor form of angular momentum.

Problem: <u>Express rotational kinetic energy in tensor notation.</u>

Solution: As we know that the rotational kinetic energy of a system is given by

$$T_{rot}=\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$$

Let, $\boldsymbol{\omega}=(\omega_1,\,\omega_2,\,\omega_3)$, $\mathbf{L}=(L_1,\,L_2,\,L_3)$

$$\Rightarrow T_{rot}=\frac{1}{2}(\omega_1L_1+\omega_2L_2+\omega_3L_3)=\frac{1}{2}\sum_{i=1}^{3}\omega_iL_i$$

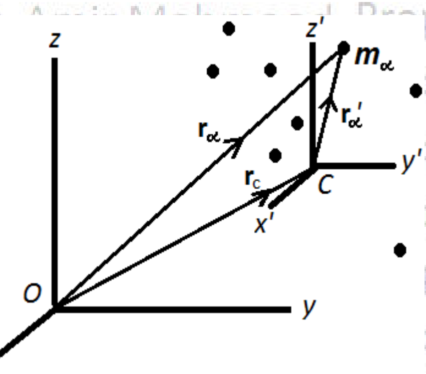
$$T_{rot}=\frac{1}{2}\sum_{i=1}^{3}\omega_i\left(\sum_{j=1}^{3}\omega_j\sum_{\alpha}m_\alpha[\mathbf{r}_\alpha^2\delta_{ij}-x_{\alpha,i}x_{\alpha,j}]\right)\qquad\because L_i=\sum_{j=1}^{3}\omega_j\sum_{\alpha}m_\alpha[\mathbf{r}_\alpha^2\delta_{ij}-x_{\alpha,i}x_{\alpha,j}]$$

$$=\frac{1}{2}\sum_{i=1}^{3}\omega_i\sum_{j=1}^{3}\omega_j\left(\sum_{\alpha}m_\alpha[\mathbf{r}_\alpha^2\delta_{ij}-x_{\alpha,i}x_{\alpha,j}]\right)=\frac{1}{2}\sum_{i,j=1}^{3}\omega_i\omega_jI_{ij}------\rightarrow(1)$$

where, $I_{ij}=\sum_{\alpha}m_\alpha[\mathbf{r}_\alpha^2\delta_{ij}-x_{\alpha,i}x_{\alpha,j}]=ij$th component of inertia tensor

Equation (1) is required tensor form of rotational kinetic energy.

Problem: <u>Express parallel axis theorem in tensor notation for a discrete mass distribution.</u>



Solution: Consider a rigid body in the form of discrete mass distribution (i.e., a set of particles). Let, C be the centre of mass of the body. We consider two parallel coordinate systems $Oxyz$ and $Cx'y'z'$, as shown in the figure.

Let, $M=$ total mass of the body

$\mathbf{r}_\alpha=$ position vector of α-th particle of mass m_α with respect to origin "O"

$\mathbf{r}'_\alpha=$ position vector of α-th particle of mass m_α with respect to centre of mass "C"

$\mathbf{r}_c=$ position vector of centre of mass "C" with respect to origin "O"

From figure, $\mathbf{r}_\alpha=\mathbf{r}_c+\mathbf{r}'_\alpha--------\rightarrow(1)$

Let, $\mathbf{r}_\alpha = (x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3})$, $\mathbf{r}_c = (x_{c,1},\, x_{c,2},\, x_{c,3})$ and $\mathbf{r}'_\alpha = (x'_{\alpha,1},\, x'_{\alpha,2},\, x'_{\alpha,3})$

Equation (1) becomes

$$(x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3}) = (x_{c,1},\, x_{c,2},\, x_{c,3}) + (x'_{\alpha,1},\, x'_{\alpha,2},\, x'_{\alpha,3})$$

$$\Rightarrow \quad x_{\alpha,i} = x_{c,i} + x'_{\alpha,i}, \qquad i = 1,2,3 \quad ----------\rightarrow (2)$$

As we know that

$$I_{ij} = \sum_\alpha m_\alpha[\mathbf{r}_\alpha^2\delta_{ij} - x_{\alpha,i}x_{\alpha,j}] = \sum_\alpha m_\alpha[(\mathbf{r}_\alpha\cdot\mathbf{r}_\alpha)\delta_{ij} - x_{\alpha,i}x_{\alpha,j}]$$

$$= \sum_\alpha m_\alpha[((\mathbf{r}_c + \mathbf{r}'_\alpha)\cdot(\mathbf{r}_c + \mathbf{r}'_\alpha))\delta_{ij} - (x_{c,i} + x'_{\alpha,i})(x_{c,j} + x'_{\alpha,j})] \qquad \text{(by using (1) and (2))}$$

$$= \sum_\alpha m_\alpha[(\mathbf{r}_c\cdot\mathbf{r}_c + 2\,\mathbf{r}_c\cdot\mathbf{r}'_\alpha + \mathbf{r}'_\alpha\cdot\mathbf{r}'_\alpha)\delta_{ij} - x_{c,i}x_{c,j} - x_{c,i}x'_{\alpha,j} - x_{c,j}x'_{\alpha,i} - x'_{\alpha,i}x'_{\alpha,j}]$$

$$= \sum_\alpha m_\alpha[(\mathbf{r}_c^2 + 2\,\mathbf{r}_c\cdot\mathbf{r}'_\alpha + \mathbf{r}'^2_\alpha)\delta_{ij} - x_{c,i}x_{c,j} - x_{c,i}x'_{\alpha,j} - x_{c,j}x'_{\alpha,i} - x'_{\alpha,i}x'_{\alpha,j}]$$

$$= \sum_\alpha m_\alpha[\mathbf{r}'^2_\alpha\delta_{ij} - x'_{\alpha,i}x'_{\alpha,j}] + 2\,\mathbf{r}_c\cdot\left(\sum_\alpha m_\alpha\mathbf{r}'_\alpha\right)\delta_{ij} + \left(\sum_\alpha m_\alpha\right)\mathbf{r}_c^2\delta_{ij} - \left(\sum_\alpha m_\alpha\right)x_{c,i}x_{c,j}$$

$$- \left(\sum_\alpha m_\alpha\, x'_{\alpha,j}\right)x_{c,i} - \left(\sum_\alpha m_\alpha\, x'_{\alpha,i}\right)x_{c,j} \quad ---------------\rightarrow (3)$$

Now, $\sum_\alpha m_\alpha[\mathbf{r}'^2_\alpha\delta_{ij} - x'_{\alpha,i}x'_{\alpha,j}] = I'_{ij}$ = ijth component of inertia tensor with respect to $Cx'y'z'$ system

Also, $\sum_\alpha m_\alpha\mathbf{r}'_\alpha = \mathbf{0}$, ($\because$ $\mathbf{r}'_\alpha$ is the position vector of α-th particle of mass m_α with respect to centre of mass "C")

$$\Rightarrow \sum_\alpha m_\alpha(x'_{\alpha,1},\, x'_{\alpha,2},\, x'_{\alpha,3}) = (0,0,0) \;\Rightarrow\; \sum_\alpha m_\alpha x'_{\alpha,i} = 0, \quad i = 1,2,3 \quad \because\ \mathbf{r}'_\alpha = (x'_{\alpha,1},\, x'_{\alpha,2},\, x'_{\alpha,3})$$

And, $\sum_\alpha m_\alpha = M$ = total mass of the body

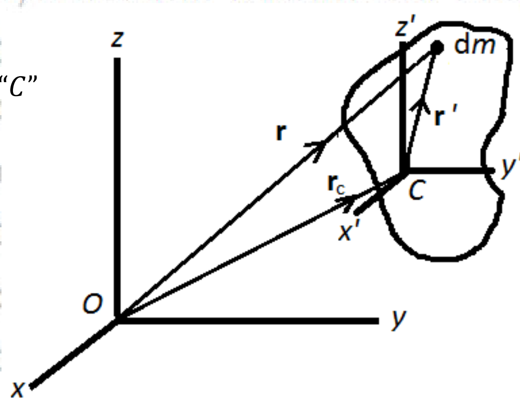
So equation (3) becomes

$$\boxed{I_{ij} = I'_{ij} + M\mathbf{r}_c^2\delta_{ij} - Mx_{c,i}x_{c,j}}$$

This is required tensor form of parallel axis theorem for discrete mass distribution.

<u>Problem:</u> <u>Express parallel axis theorem in tensor notation for a continuous mass distribution.</u>

Solution: Consider a rigid body in the form of continuous mass distribution. Let, C be the centre of mass of the body. We consider two parallel coordinate systems $Oxyz$ and $Cx'y'z'$, as shown in the figure.



Let, M = total mass of the body

$\mathbf{r}$ = position vector of elementary mass dm with respect to origin "O"

$\mathbf{r}'$ = position vector of elementary mass dm with respect to centre of mass "C"

$\mathbf{r}_c$ = position vector of centre of mass "C" with respect to origin "O"

From figure, $\mathbf{r} = \mathbf{r}_c + \mathbf{r}'$ $---------\rightarrow (1)$

Let, $\mathbf{r} = (x_1,\, x_2,\, x_3)$, $\mathbf{r}_c = (x_{c,1},\, x_{c,2},\, x_{c,3})$ and $\mathbf{r}' = (x'_1,\, x'_2,\, x'_3)$

So, equation (1) becomes $(x_1,\, x_2,\, x_3) = (x_{c,1},\, x_{c,2},\, x_{c,3}) + (x'_1,\, x'_2,\, x'_3)$

$$\Rightarrow \quad x_i = x_{c,i} + x'_i, \qquad i = 1,2,3 \quad ---------\rightarrow (2)$$

$$I_{ij} = \int_M [\mathbf{r}^2\delta_{ij} - x_{\alpha,i}x_{\alpha,j}]\mathrm{d}m = \int_M [(\mathbf{r}\cdot\mathbf{r})\delta_{ij} - x_{\alpha,i}x_{\alpha,j}]\mathrm{d}m$$

$$= \int_M [((\mathbf{r}_c + \mathbf{r}')\cdot(\mathbf{r}_c + \mathbf{r}'))\delta_{ij} - (x_{c,i} + x'_i)(x_{c,j} + x'_j)]\mathrm{d}m \qquad \text{(by using (1) and (2))}$$

$$= \int_M [(\mathbf{r}_c\cdot\mathbf{r}_c + 2\,\mathbf{r}_c\cdot\mathbf{r}' + \mathbf{r}'\cdot\mathbf{r}')\delta_{ij} - x_{c,i}x_{c,j} - x_{c,i}x'_j - x_{c,j}x'_i - x'_ix'_j]\mathrm{d}m$$

$$I_{ij} = \int_M [(\mathbf{r}_c^2 + 2\,\mathbf{r}_c\cdot\mathbf{r}' + \mathbf{r}'^2)\delta_{ij} - x_{c,i}x_{c,j} - x_{c,i}x'_j - x_{c,j}x'_i - x'_ix'_j]\mathrm{d}m$$

$$= \int_M [\mathbf{r}'^2\delta_{ij} - x'_ix'_j]\mathrm{d}m + 2\mathbf{r}_c\cdot\left(\int_M \mathbf{r}'\mathrm{d}m\right)\delta_{ij} + \left(\int_M \mathrm{d}m\right)\mathbf{r}_c^2\delta_{ij} - \left(\int_M \mathrm{d}m\right)x_{c,i}x_{c,j} - \left(\int_M x'_j\mathrm{d}m\right)x_{c,i}$$

$$-\left(\int_M x'_i\mathrm{d}m\right)x_{c,j} \;\longrightarrow (3)$$

Now, $\int_M [\mathbf{r}'^2\delta_{ij} - x'_ix'_j]\mathrm{d}m = I'_{ij} = ij$th component of inertia tensor with respect to $Cx'y'z'$ system

Also, $\int_M \mathbf{r}'\mathrm{d}m = \mathbf{0}$, ($\because$ $\mathbf{r}'$ is the position vector of mass element dm with respect to centre of mass "C")

$$\Rightarrow \int_M (x'_1,\, x'_2,\, x'_3)\mathrm{d}m = (0,0,0) \quad\Rightarrow\quad \int_M x'_i\mathrm{d}m = 0,\quad i = 1,2,3 \qquad \because\ \mathbf{r}' = (x'_1,\, x'_2,\, x'_3)$$

And, $\int_M \mathrm{d}m = M =$ total mass of the body

So equation (3) becomes

$$\boxed{I_{ij} = I'_{ij} + M\mathbf{r}_c^2\delta_{ij} - Mx_{c,i}x_{c,j}}$$

This is required tensor form of parallel axis theorem for continuous mass distribution.

<u>Problem:</u> <u>State and prove parallel axis theorem for the case of products of inertia for a discrete mass distribution.</u>

Statement: Consider a rigid body, in the form of discrete mass distribution (i.e., a set of particles). Let, C be the centre of mass of the body. If $Oxyz$ and $Cx'y'z'$ be two parallel coordinate systems as shown in figure, then we have

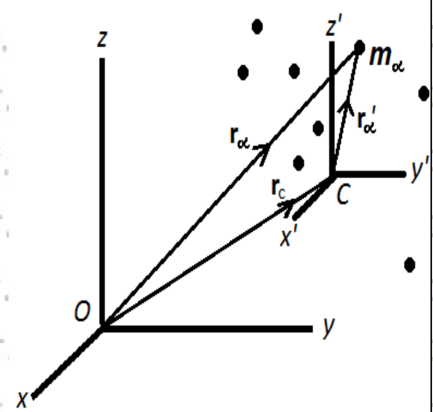
$I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}$, $\quad i \neq j$, $\quad i,j \in \{1,2,3\}$

I_{ij} = product of inertia with respect to $Oxyz$-system

I'_{ij} = product of inertia with respect to $Cx'y'z'$-system

$(x_{c,1},\, x_{c,2},\, x_{c,3}) =$ position vector of centre of mass "C" with respect to origin "O"

M = total mass of the body



Proof: Consider a rigid body, in the form of a set of particles.

Let, $\mathbf{r}_\alpha$ = position vector of α-th particle of mass m_α with respect to origin "O"

$\mathbf{r}'_\alpha$ = position vector of α-th particle of mass m_α with respect to centre of mass "C"

$\mathbf{r}_c$ = position vector of centre of mass "C" with respect to origin "O"

From figure, $\mathbf{r}_\alpha = \mathbf{r}_c + \mathbf{r}'_\alpha \longrightarrow (1)$

Let, $\mathbf{r}_\alpha = (x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3})$, $\quad \mathbf{r}_c = (x_{c,1},\, x_{c,2},\, x_{c,3})$ and $\mathbf{r}'_\alpha = (x'_{\alpha,1},\, x'_{\alpha,2},\, x'_{\alpha,3})$

So, equation (1) becomes $\quad (x_{\alpha,1},\, x_{\alpha,2},\, x_{\alpha,3}) = (x_{c,1},\, x_{c,2},\, x_{c,3}) + (x'_{\alpha,1},\, x'_{\alpha,2},\, x'_{\alpha,3})$

$$\Rightarrow\ x_{\alpha,i} = x_{c,i} + x'_{\alpha,i},\qquad i = 1,2,3 \longrightarrow (2)$$

Now consider for $i \neq j$, $\quad I_{ij} = -\sum_\alpha m_\alpha\, x_{\alpha,i}x_{\alpha,j} = -\sum_\alpha m_\alpha\,(x_{c,i} + x'_{\alpha,i})(x_{c,j} + x'_{\alpha,j})$

$$= -\left(\sum_\alpha m_\alpha\right)x_{c,i}x_{c,j} - \left(\sum_\alpha m_\alpha\, x'_{\alpha,j}\right)x_{c,i} - \left(\sum_\alpha m_\alpha\, x'_{\alpha,i}\right)x_{c,j} - \sum_i m_\alpha\, x'_{\alpha,i}x'_{\alpha,j} \longrightarrow (3)$$

where, $\sum_\alpha m_\alpha = M =$ total mass of the body,

Also, $-\sum_i m_\alpha\, x'_{\alpha,i}x'_{\alpha,j} = I'_{ij} =$ product of inertia with respect to $Cx'y'z'$-system

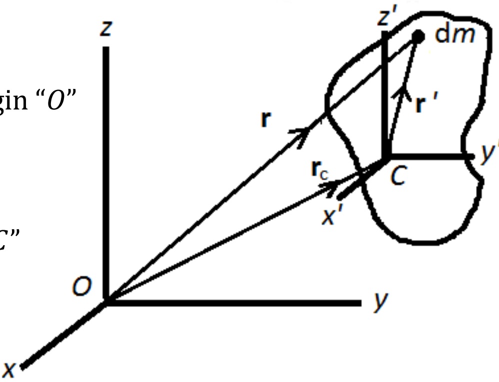
$$\sum_\alpha m_\alpha\mathbf{r}'_\alpha = \mathbf{0} \quad\Rightarrow\quad \sum_\alpha m_\alpha(x'_{\alpha,1},\, x'_{\alpha,2},\, x'_{\alpha,3}) = (0,0,0) \quad\Rightarrow\quad \sum_\alpha m_\alpha x'_{\alpha,i} = 0,\quad i = 1,2,3$$

$\Rightarrow \quad I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}$, $\quad i \neq j$, $\quad i,j \in \{1,2,3\}$ **Hence proved.**

Problem: State and prove parallel axis theorem for the case of products of inertia for a continuous mass distribution.

Statement: Consider a rigid body, in the form of a continuous mass distribution. Let, C be the centre of mass of the body. If $Oxyz$ and $Cx'y'z'$ be two parallel coordinate systems as shown in the figure, then we have

$$I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}, \quad i \neq j, \quad i,j \in \{1,2,3\}$$



I_{ij} = product of inertia with respect to $Oxyz$-system

I'_{ij} = product of inertia with respect to $Cx'y'z'$-system

$(x_{c,1}, x_{c,2}, x_{c,3})$ = position vector of centre of mass "C" with respect to origin "O"

M = total mass of the body

Proof: Consider a rigid body, in the form of a set of particles.

$\mathbf{r}$ = position vector of elementary mass dm with respect to origin "O"

$\mathbf{r}'$ = position vector of elementary mass dm with respect to centre of mass "C"

$\mathbf{r}_c$ = position vector of centre of mass "C" with respect to origin "O"

From figure, $\mathbf{r} = \mathbf{r}_c + \mathbf{r}'$ ---------→ (1)

Let, $\mathbf{r} = (x_1, x_2, x_3)$, $\mathbf{r}_c = (x_{c,1}, x_{c,2}, x_{c,3})$ and $\mathbf{r}' = (x'_1, x'_2, x'_3)$

So, equation (1) becomes $(x_1, x_2, x_3) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_1, x'_2, x'_3)$

$\Rightarrow \quad x_i = x_{c,i} + x'_i, \quad i = 1,2,3$ ---------→ (2)

Now consider for $i \neq j$, $I_{ij} = -\int\limits_M x_i x_j \mathrm{d}m = -\int\limits_M (x_{c,i} + x'_i)(x_{c,j} + x'_j)\mathrm{d}m$

$$= -\left(\int\limits_M \mathrm{d}m\right)x_{c,i}x_{c,j} - \left(\int\limits_M x'_j \mathrm{d}m\right)x_{c,i} - \left(\int\limits_M x'_i \mathrm{d}m\right)x_{c,j} - \int\limits_M x'_i x'_j \mathrm{d}m$$

where, $\int\limits_M dm = M$ = total mass of the body,

Also, $-\int\limits_M x'_i x'_j \mathrm{d}m = I'_{ij}$ = product of inertia with respect to $Cx'y'z'$-system

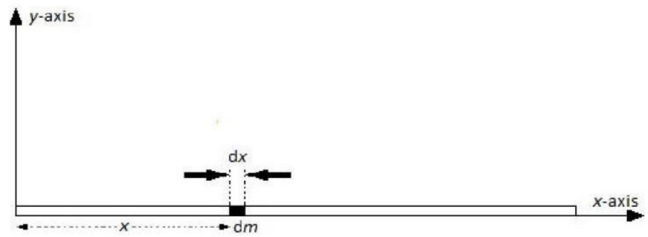
And $\int\limits_M \mathbf{r}' \mathrm{d}m = \mathbf{0} \Rightarrow \int\limits_M (x'_1, x'_2, x'_3)\mathrm{d}m = \left(\int\limits_M x'_1 \mathrm{d}m, \int\limits_M x'_2 \mathrm{d}m, \int\limits_M x'_2 \mathrm{d}m\right) = (0,0,0) \Rightarrow \int\limits_M x'_i \mathrm{d}m = 0, \quad i = 1,2,3$

So equation (3) gives

$$\boxed{I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}, \quad i \neq j, \quad i,j \in \{1,2,3\} \quad \textbf{Hence proved.}}$$

Example 1: ***Find the moment of inertia*** of a (uniform) rod of length l ***about*** an axis perpendicular to the rod and passing through one of its end points.

Solution: Let M, l and λ, respectively, be the mass, length and linear mass density of the rod. Choose x-axis and y-axis as shown in the figure, so that we have to find moment of inertia of the rod about y-axis. We divide rod into large number of elements of infinitesimal lengths. One typical element of mass $\mathrm{d}m$ and length $\mathrm{d}x$, at distance x from the origin, is shown in the figure.



Moment of inertia of typical mass element about y-axis is given by

$$\mathrm{d}I_{yy} = x^2\mathrm{d}m$$

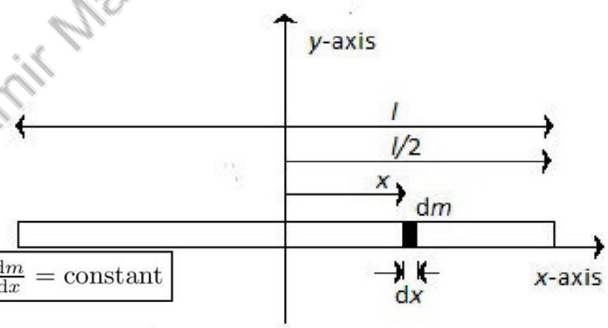
Thus, moment of inertia of rod about y-axis is

$$I_{yy} = \int_{\text{Rod}} x^2\mathrm{d}m = \lambda\int_{\text{Rod}} x^2\mathrm{d}x \qquad \boxed{\because\ \ \lambda = \tfrac{\mathrm{d}m}{\mathrm{d}x} = \text{constant}}$$

$$= \frac{M}{l}\int_{x=0}^{l} x^2\mathrm{d}x = \frac{M}{l}\left(\frac{l^3}{3}\right) = \frac{1}{3}Ml^2 \qquad \boxed{\because\ \ \lambda = \tfrac{M}{l}\ \ \text{(for rod)}}$$

Example 2: ***Find the moment of inertia*** of a (uniform) rod of length l ***about*** an axis perpendicular to the rod and passing through its centre.

Solution: Let M, l and λ, respectively, be the mass, length and linear mass density of the rod. Choose x-axis and y-axis as shown in the figure. We divide rod into large number of elements of infinitesimal lengths. One typical element of mass $\mathrm{d}m$ and length $\mathrm{d}x$, at distance x from the origin, is shown in the figure.



Moment of inertia of typical element about y-axis is given by

$$\mathrm{d}I_{yy} = x^2\mathrm{d}m$$

Thus, moment of inertia of rod about y-axis is

$$I_{yy} = \int_{\text{Rod}} x^2\mathrm{d}m = \lambda\int_{\text{Rod}} x^2\mathrm{d}x \qquad \boxed{\because\ \ \lambda = \tfrac{\mathrm{d}m}{\mathrm{d}x} = \text{constant}}$$

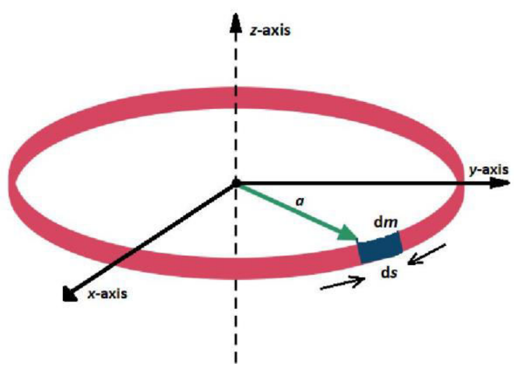
$$= \frac{M}{l}\int_{x=-l/2}^{l/2} x^2\mathrm{d}x = \frac{M}{l}\left(\frac{l^3}{12}\right) = \frac{1}{12}Ml^2 \qquad \boxed{\because\ \ \lambda = \tfrac{M}{l}\ \ \text{(for rod)}}$$

Example 3: ***Find the moment of inertia*** of a (uniform) circular ring of radius a ***about***

(i) an axis passing through its centre and perpendicular to its plane,

(ii) its diameter.

Solution: (i) **<u>Moment of inertia about central axis</u>:** Let M, a and λ, respectively, be the mass, radius and linear mass density of the ring. Choose coordinate axes as shown in the figure. We divide ring into large number of elements of infinitesimal lengths. One typical element of mass $\mathrm{d}m$ and length $\mathrm{d}s$ is shown in the figure.



Moment of inertia of typical element about z-axis is given by

$$\mathrm{d}I_{zz} = a^2\mathrm{d}m$$

Thus, moment of inertia of ring about z-axis is

$$I_{zz} = a^2\int_{\text{Ring}} \mathrm{d}m = \lambda a^2\int_{\text{Ring}} \mathrm{d}s \qquad \boxed{\because\ \ \lambda = \tfrac{\mathrm{d}m}{\mathrm{d}s} = \text{constant}}$$

$$= \frac{Ma}{2\pi}\int_{s=0}^{2\pi a} \mathrm{d}s = \frac{Ma}{2\pi}(2\pi a) = Ma^2 \qquad \boxed{\because\ \ \lambda = \tfrac{M}{2\pi a}\ \ \text{(for ring)}}$$

(ii) **<u>Moment of inertia about diameter</u>:**

By perpendicular axis theorem

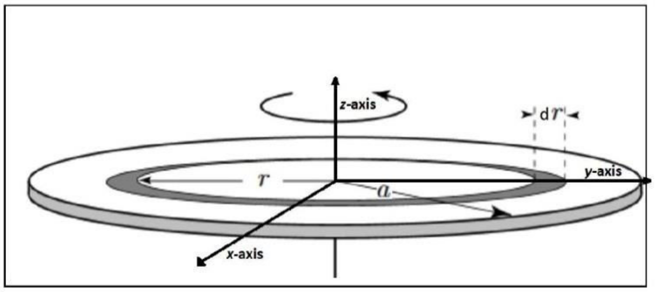
$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx}, \qquad \boxed{\because\ \ I_{xx} = I_{yy}\ \ \text{(by symmetry)}}$$

$$\Rightarrow I_{xx} = \frac{1}{2}Ma^2$$

Example 4: ***Find the moment of inertia*** of a (uniform) circular disc of Mass M and radius a ***about***
(i) an axis passing through its centre and perpendicular to its plane,
(ii) its diameter.

Solution: (i) **Moment of inertia about central axis:** Let M, a and σ, respectively, be the mass, radius and surface (areal) mass density of the disc. Choose coordinate axes as shown in figure.

We divide disc into large number of concentric circular rings of infinitesimal widths. One typical elementary ring of mass $\mathrm{d}m$, radius r, width $\mathrm{d}r$ and area $\mathrm{d}A$ is shown in the figure.



Moment of inertia of typical elementary ring about z-axis is given by

$$\mathrm{d}I_{zz} = r^2 \mathrm{d}m$$

Thus, moment of inertia of disc about z-axis is

$$I_{zz} = \int_{\text{Disc}} r^2 \mathrm{d}m$$

$$= 2\pi\sigma \int_{\text{Disc}} r^3 \mathrm{d}r \qquad \boxed{\because \;\; \sigma = \tfrac{\mathrm{d}m}{\mathrm{d}A} = \tfrac{\mathrm{d}m}{(2\pi r)\mathrm{d}r} = \text{constant}}$$

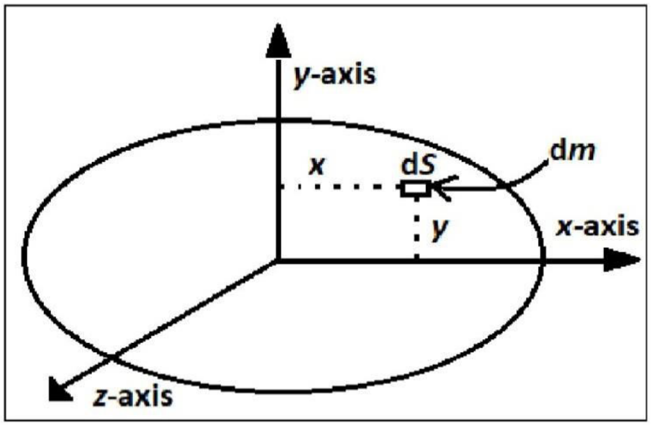
$$= \frac{2M}{a^2}\int_{r=0}^{a} r^3 \mathrm{d}r = \frac{2M}{a^2}\left(\frac{a^4}{4}\right) = \frac{1}{2}Ma^2 \qquad \boxed{\because \;\; \sigma = \tfrac{M}{\pi a^2} \;\; \text{(for disc)}} \qquad (1)$$

(ii) **Moment of inertia about diameter:** By perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx}, \qquad \boxed{\because \;\; I_{xx} = I_{yy} \;\; \text{(by symmetry)}}$$

$$\Rightarrow I_{xx} = \frac{1}{4}Ma^2 \qquad (2)$$

Example 5: ***Find the moment of inertia*** of a (uniform) elliptical plate with semi-major axis and semi minor axis a and b, respectively ***about***
(i) major axis,
(ii) minor axis,
(iii) an axis passing through centre of plate and perpendicular to its plane.



Solution: Consider an elliptical plate in xy-plane whose boundary curve is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b$$

Let M and σ, respectively, be the mass and surface (areal) mass density of the elliptical plate. To find moment of inertia about major axis (x-axis), we proceed as follows. We divide plate into large number of elementary rectangular pieces of infinitesimal areas with sides parallel to x and y axis. One typical area element, located at point (x, y), having mass $\mathrm{d}m$, area $\mathrm{d}S$, length $\mathrm{d}x$ and width $\mathrm{d}y$ is shown in the figure.
Moment of inertia of typical area element about x-axis is given by

$$\mathrm{d}I_{xx} = y^2 \mathrm{d}m$$

Thus, moment of inertia of elliptical plate about x-axis is

$$I_{xx} = \int_{\text{Elliptical plate}} y^2 \mathrm{d}m = \sigma \int_{\text{Elliptical plate}} y^2 \,\mathrm{d}x\,\mathrm{d}y \qquad \boxed{\because \;\; \sigma = \tfrac{\mathrm{d}m}{\mathrm{d}S} = \tfrac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y} = \text{constant}}$$

$$= \frac{M}{\pi ab}\int_{\text{Elliptical plate}} y^2 \mathrm{d}x\,\mathrm{d}y \qquad \boxed{\because \;\; \sigma = \tfrac{M}{\pi a b} \;\; \text{(for elliptical plate)}}$$

$$= \frac{M}{\pi ab}\int_{x=-a}^{a}\left(\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 \mathrm{d}y\right)\mathrm{d}x = \frac{M}{\pi ab}\left(\frac{2b^3}{3a^3}\right)\int_{x=-a}^{a}(a^2-x^2)^{3/2}\mathrm{d}x = \frac{4Mb^2}{3\pi a^4}\int_{x=0}^{a}(a^2-x^2)^{3/2}\mathrm{d}x$$

Put $x = a\sin\theta \Rightarrow \mathrm{d}x = a\cos\theta\mathrm{d}\theta,\ x = 0 \Rightarrow \theta = 0,\ x = a \Rightarrow \theta = \pi/2$

$$I_{xx} = \frac{4Mb^2}{3\pi a^4}\int_{x=0}^{\pi/2} a^4\cos^4\theta\mathrm{d}\theta$$

Using Wallis cosine formula, we get, $$I_{xx} = \frac{4Mb^2}{3\pi}\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{1}{4}Mb^2 \tag{3}$$

Similarly, moment of inertia about minor axis is $I_{yy} = \frac{1}{4}Ma^2$

By perpendicular axis theorem, the moment of inertia about the axis passing through centre of the elliptical plate and perpendicular to its plane, is

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{4}M(a^2 + b^2) \tag{4}$$

Corollary: ***The moment of inertia*** of a (uniform) circular disc of radius a ***about*** (i) its diameter and (ii) an axis passing through its centre and perpendicular to its plane can be obtained by putting $b = a$ in (3) and (4), to give (respectively)

$$I_{xx} = \frac{1}{4}Ma^2 \tag{5}$$

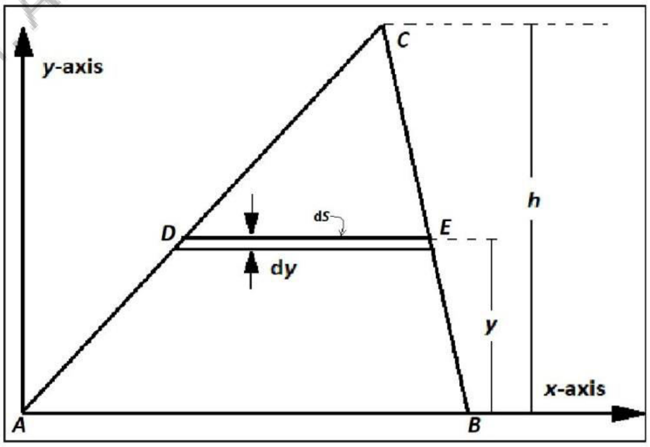
and

$$I_{zz} = \frac{1}{2}Ma^2 \tag{6}$$

Note that, the results obtained in (5) and (6) are in accordance (as they should be) with the results, obtained in (2) and (1), respectively.

Example 6: ***Find the moment of inertia*** of a (uniform) triangular lamina (i.e., two dimensional triangular plate) of mass M ***about*** one of its sides.

Solution: Let M and σ, respectively, be the mass and surface (areal) mass density of the triangular lamina lying in xy-plane. Choose x-axis and y-axis as shown in figure. We divide lamina into large number of strips of infinitesimal widths parallel to the base AB of lamina. One typical elementary strip DE of mass dm, width dy and area dS is shown in the figure.



Moment of inertia of typical elementary strip about side AB (x-axis) is given by

$$\mathrm{d}I_{xx} = y^2\mathrm{d}m$$

Thus, moment of inertia of triangular lamina about x-axis is

$$I_{xx} = \int_{\text{Triangular lamina}} y^2\mathrm{d}m = \sigma\int_{\text{Triangular lamina}} y^2|DE|\mathrm{d}y \qquad \boxed{\because\ \sigma = \frac{\mathrm{d}m}{\mathrm{d}S} = \frac{\mathrm{d}m}{|DE|\mathrm{d}y} = \text{constant}}$$

$$= \frac{2M}{h}\int_{\text{Triangular lamina}} y^2\frac{|DE|}{|AB|}\mathrm{d}y \qquad \boxed{\because\ \sigma = \frac{M}{\frac{1}{2}|AB|h}\quad \text{(for tiangular lamina)}}$$

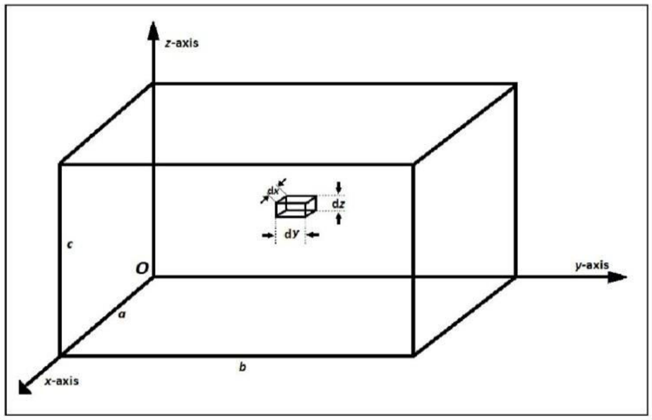
From equivalent triangles ABC and DEC, we have

$$\frac{|DE|}{|AB|} = \frac{\text{height of triangle } DEC}{\text{height of triangle } ABC} = \frac{h-y}{h}$$

$$\Rightarrow \quad I_{xx} = \frac{2M}{h}\int_{\text{Triangular lamina}} y^2\left(\frac{h-y}{h}\right)dy$$

$$= \frac{2M}{h^2}\int_{y=0}^{h} y^2(h-y)dy = \frac{2M}{h^2}\left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{1}{6}Mh^2$$

Example 7: ***Calculate the inertia matrix*** of a (uniform solid) rectangular box (rectangular parallelopiped or cuboid) of mass M ***at*** one of its corners, by taking coordinate axes along its edges.



Solution: Let M and ρ, respectively, be the mass and volume mass density of the rectangular box. Let the lengths of adjacent edges be a, b and c. Choose coordinate axis along the edges of box, as shown in figure. We divide box into large number of elementary rectangular boxes of infinitesimal volumes. One typical elementary volume element of mass $\mathrm{d}m$, volume $\mathrm{d}V$ and dimensions $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, is shown in the figure.

Moment of inertia of typical elementary volume element (or elementary box) about x-axis is given by

$$\mathrm{d}I_{xx} = (y^2+z^2)\mathrm{d}m$$

Thus, moment of inertia of rectangular box about x-axis is

$$I_{xx} = \int_{\text{Rectangular box}} (y^2+z^2)\mathrm{d}m = \rho \int_{\text{Rectangular box}} (y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \boxed{\because\ \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \frac{\mathrm{d}m}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} = \text{constant}}$$

$$= \frac{M}{a\,b\,c}\int_{\text{Rectangular box}} (y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \boxed{\because\ \rho = \frac{M}{a\,b\,c} \quad \text{(for rectangular box)}}$$

$$= \frac{M}{a\,b\,c}\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a}(y^2+z^2)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{M}{a\,b\,c}\left[\int_{x=0}^{a}\mathrm{d}x\right]\left[\int_{z=0}^{c}\int_{y=0}^{b}(y^2+z^2)\,\mathrm{d}y\,\mathrm{d}z\right]$$

$$= \frac{M}{b\,c}\int_{z=0}^{c}\int_{y=0}^{b}(y^2+z^2)\,\mathrm{d}y\,\mathrm{d}z = \frac{M}{b\,c}\int_{z=0}^{c}\left(\frac{b^3}{3}+bz^2\right)\mathrm{d}z = \frac{M}{b\,c}\left(\frac{b^3c}{3}+\frac{b\,c^3}{3}\right) = \frac{M}{3}\left(b^2+c^2\right)$$

Similarly, $$I_{yy} = \frac{M}{3}\left(a^2+c^2\right) \quad \text{and} \quad I_{zz} = \frac{M}{3}\left(a^2+b^2\right)$$

For product of inertia

$$I_{xy} = \int_{\text{Rectangular box}} x\,y\,\mathrm{d}m = -\frac{M}{abc}\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} x\,y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -\frac{M}{abc}\left(\frac{a^2}{2}\right)\left(\frac{b^2}{2}\right)c = -\frac{1}{4}Mab$$

Similarly, $$I_{yz} = -\frac{1}{4}Mbc \quad \text{and} \quad I_{xz} = -\frac{1}{4}Mac$$

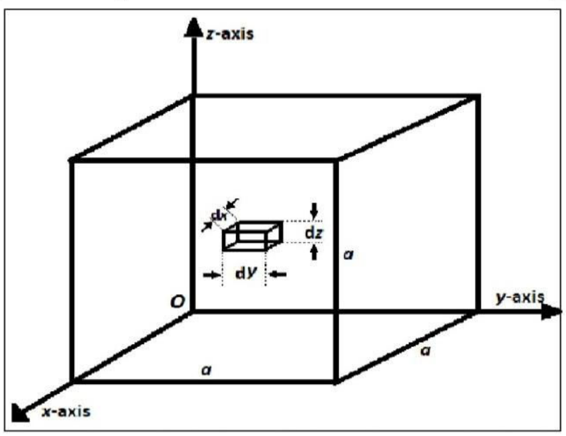
The required inertia matrix is given by

$$[I_O] = \begin{bmatrix} (1/3)M(b^2+c^2) & -(1/4)Mab & -(1/4)Mac \\ -(1/4)Mab & (1/3)M(a^2+c^2) & -(1/4)Mbc \\ -(1/4)Mac & -(1/4)Mbc & (1/3)M(a^2+b^2) \end{bmatrix} = \frac{1}{12}M\begin{bmatrix} 4(b^2+c^2) & -3ab & -3ac \\ -3ab & 4(a^2+c^2) & -3bc \\ -3ac & -3bc & 4(a^2+b^2) \end{bmatrix}$$

Example 8: ***Calculate the inertia matrix*** of a (uniform solid) cube of mass M ***at*** one of its corners, by taking coordinate axes along its edges.

Solution: Repeat example 7 for $a=b=c$ and get

$$[I_O] = \frac{1}{12}Ma^2\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$



Example 9: ***Find the moment of inertia*** of a (uniform solid) hemisphere of mass M ***about***
(i) its axis of symmetry
(ii) an axis perpendicular to the axis of symmetry and passing through the centre of the base.

Solution: (i) **Moment of inertia about axis of symmetry:**
Let M, a and ρ, respectively, be the mass, base radius and volume mass density of the hemisphere. Choose coordinate axes as shown in figure.
Moment of inertia of typical volume element of hemisphere, having mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of hemisphere about z-axis is

$$I_{zz} = \int_{\text{Hemisphere}} (x^2 + y^2)\mathrm{d}m = \rho \int_{\text{Hemisphere}} (x^2 + y^2)\,\mathrm{d}V \qquad \boxed{\because \;\; \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \text{constant}}$$

$$= \frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} (x^2 + y^2)\,\mathrm{d}V \qquad \boxed{\because \;\; \rho = \frac{M}{(2/3)\pi a^3} \;\; \text{(for hemisphere)}}$$

To make the computation simpler, we transform the problem from Cartesian coordinates $(x,\, y,\, z)$ to spherical coordinates $(r,\, \theta,\, \phi)$ by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

where, volume element in spherical coordinates is given by

$$\mathrm{d}V = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$\Rightarrow \quad x^2 + y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\sin^2\theta$$

For hemisphere, $\qquad 0 \le r \le a, \quad 0 \le \theta \le \pi/2, \quad 0 \le \phi < 2\pi$

$$\Rightarrow \quad I_{zz} = \frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4\,\mathrm{d}r \int_{\theta=0}^{\pi/2} \sin^3\theta\,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi \tag{7}$$

where, $\displaystyle \int_{\theta=0}^{\pi/2} \sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2} (3\sin\theta - \sin 3\theta) \qquad \boxed{\because \;\; \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi/2} = \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{2}{3} \tag{8}$$

Using (8) in (7), we get

$$I_{zz} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

(ii) **Moment of inertia about a diameter of the base:**

$$I_{xx} = \int_{\text{Hemisphere}} (y^2 + z^2)\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} (y^2 + z^2)\,\mathrm{d}V$$

Transforming problem in spherical coordinates $(r,\, \theta,\, \phi)$, we get

$$I_{xx} = \frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4(\sin^3\theta\sin^2\phi + \cos^2\theta\sin\theta)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4\,\mathrm{d}r \left(\int_{\theta=0}^{\pi/2} \sin^3\theta\,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin^2\phi\,\mathrm{d}\phi + \int_{\theta=0}^{\pi/2} \cos^2\theta\sin\theta\,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi\right), \tag{9}$$

where,

$$\int_{\phi=0}^{2\pi} \sin^2\phi\,\mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi} (1 - \cos 2\phi)\,\mathrm{d}\phi = \frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi \tag{10}$$

and

$$\int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta \, \mathrm{d}\theta = -\frac{1}{3}\cos^3\theta \Big|_{\theta=0}^{\pi/2} = \frac{1}{3} \tag{11}$$

Using (8), (10) and (11) in (9), we get

$$I_{xx} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right) = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4\pi}{3}\right) = \frac{2}{5}Ma^2$$

Example 10: ***Find three products of inertia*** of a (uniform) solid hemisphere of mass M ***with respect to*** coordinate axes as in figure of example 9.

Solution:

$$I_{xy} = -\int_{\text{Hemisphere}} x\,y\,\mathrm{d}m = -\frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} x\,y\,\mathrm{d}V = -\frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = 0 \qquad \boxed{\because\ \int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \tfrac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0}$$

Now,

$$I_{xz} = -\int_{\text{Hemisphere}} x\,z\,\mathrm{d}m = -\frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} x\,z\,\mathrm{d}V = -\frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\cos\theta\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin^2\theta\cos\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi = 0 \qquad \boxed{\because\ \int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi = \sin\phi\Big|_{\phi=0}^{2\pi} = 0}$$

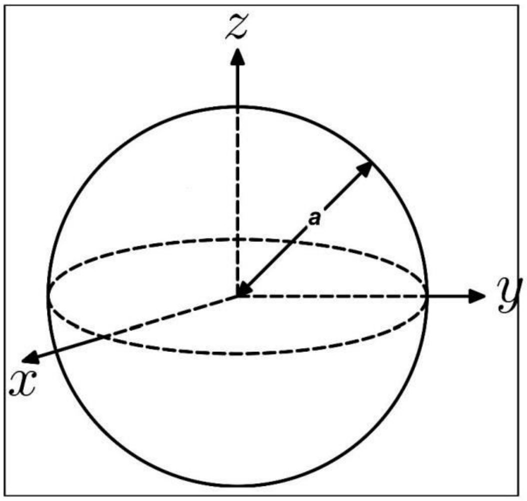
Thus,

$$I_{xy} = I_{xz} = I_{yz} = 0, \qquad \boxed{\because\ I_{xz} = I_{yz}\ \text{(by symmetry)}}$$

Example 11: ***Find the moments and products of inertia*** of a (uniform solid) sphere of mass M and radius a ***with respect to*** its axes of symmetry.

Solution: (i) **Moment of inertia about axis of symmetry:**

Let M, a and ρ, respectively, be the mass, radius and volume mass density of the sphere. Choose coordinate axes as shown in figure. Moment of inertia of typical volume element of sphere, having mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by



$$\mathrm{d}I_{zz} = (x^2+y^2)\mathrm{d}m$$

Thus, moment of inertia of sphere about z-axis is

$$I_{zz} = \int_{\text{Sphere}}(x^2+y^2)\mathrm{d}m$$

$$= \rho\int_{\text{Sphere}}(x^2+y^2)\,\mathrm{d}V \qquad \boxed{\because\ \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \text{constant}}$$

$$= \frac{3M}{4\pi a^3}\int_{\text{Sphere}}(x^2+y^2)\,\mathrm{d}V \qquad \boxed{\because\ \rho = \tfrac{M}{(4/3)\pi a^3}\quad \text{(for sphere)}}$$

To make the computation simpler, we transform the problem from Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

where, volume element in spherical coordinates is given by

$$\mathrm{d}V = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$\Rightarrow\quad x^2+y^2 = r^2(\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = r^2\sin^2\theta$$

For sphere, $\qquad 0 \le r \le a, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$

$$\Rightarrow \quad I_{zz} = \frac{3M}{4\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{4\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi,$$

where,

$$\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta-\sin 3\theta) \qquad \boxed{\because\quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta+\frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right)-\left(-3+\frac{1}{3}\right)\right] = \frac{4}{3}$$

Thus,

$$I_{zz} = \frac{3M}{4\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

Similarly,

$$I_{xx} = I_{yy} = \frac{2}{5}Ma^2 \qquad \boxed{\because\quad I_{xx}=I_{yy}=I_{zz}\ \text{(by symmetry)}}$$

(ii) **Products of inertia with respect to axes of symmetry:**

$$I_{xy} = -\int_{\text{Sphere}} x\,y\,\mathrm{d}m = -\frac{3M}{4\pi a^3}\int_{\text{Sphere}} x\,y\,\mathrm{d}V = -\frac{3M}{4\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{3M}{4\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = 0 \qquad \boxed{\because\quad \int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \tfrac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0}$$

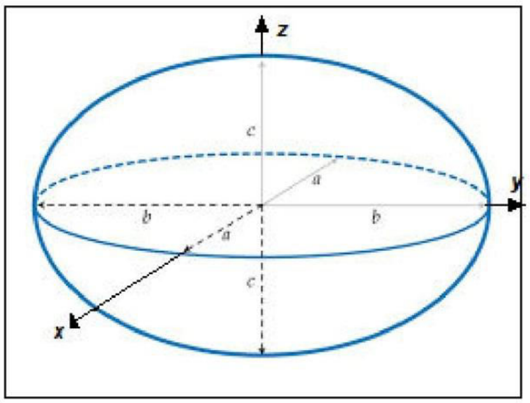
Similarly,

$$I_{yz} = I_{xz} = 0 \qquad \boxed{\because\quad I_{xy}=I_{yz}=I_{xz}\ \text{(by symmetry)}}$$

Example 12: ***Find the moments and products of inertia*** of a (uniform) solid ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ of mass M ***with respect to*** its axes of symmetry.

Solution: (i) **Moment of inertia about axis of symmetry:**

Let M and ρ, respectively, be the mass and volume mass density of the ellipsoid. Choose coordinate axes as shown in figure.



Moment of inertia of typical volume element of ellipsoid, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2+y^2)\mathrm{d}m$$

Thus, moment of inertia of ellipsoid about z-axis is

$$I_{zz} = \int_{\text{Ellipsoid}}(x^2+y^2)\mathrm{d}m$$

$$= \rho\int_{\text{Ellipsoid}}(x^2+y^2)\,\mathrm{d}V \qquad \boxed{\because\quad \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \text{constant}}$$

$$= \frac{3M}{4\pi a\,b\,c}\int_{\text{Ellipsoid}}(x^2+y^2)\,\mathrm{d}V \qquad \boxed{\because\quad \rho = \tfrac{M}{(4/3)\pi a\,b\,c}\quad \text{(for ellipsoid)}}$$

Let us substitute

$$x/a = x', \quad y/b = y', \quad z/c = z'$$

$$\Rightarrow \quad \mathrm{d}x/a = \mathrm{d}x', \quad \mathrm{d}y/b = \mathrm{d}y', \quad \mathrm{d}z/c = \mathrm{d}z', \qquad \mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

Under the above transformation, the given ellipsoid is transformed into the unit sphere

$$S: x'^2+y'^2+z'^2 = 1.$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{4\pi a\,b\,c}\int_S(a^2x'^2+b^2y'^2)(a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z') = \frac{3M}{4\pi}\int_S(a^2x'^2+b^2y'^2)\mathrm{d}V', \quad \text{where, } \mathrm{d}V' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

$$= \frac{3M(a^2+b^2)}{4\pi}\int_S x'^2\mathrm{d}V' \qquad \boxed{\because\quad \int_S x'^2\mathrm{d}V' = \int_S y'^2\mathrm{d}V' \quad \text{(by symmetry)}}$$

To make the computation simpler, we transform the problem from Cartesian coordinates (x', y', z') to spherical coordinates (r, θ, ϕ) by using

$$x' = r\sin\theta\cos\phi, \qquad y' = r\sin\theta\sin\phi, \qquad z' = r\cos\theta,$$

where, volume element in spherical coordinates is given by, $\quad dV' = dr\,(r\,d\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$

For unit sphere, $\qquad 0 \le r \le 1, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$

$$\Rightarrow \; I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\cos^2\phi\,dr\,d\theta\,d\phi = \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1} r^4\,dr\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta\int_{\phi=0}^{2\pi}\cos^2\phi\,d\phi,$$

where, $\displaystyle\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) = \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3}$

and $\displaystyle\int_{\phi=0}^{2\pi}\cos^2\phi\,d\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1+\cos 2\phi)\,d\phi = \frac{1}{2}\left(\phi + \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi$

$$\Rightarrow \; I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\left(\frac{1}{5}\right)\left(\frac{4}{3}\right)(\pi) = \frac{1}{5}M(a^2+b^2)$$

Similarly, $\quad I_{xx} = \frac{1}{5}M(b^2+c^2) \qquad$ and $\qquad I_{yy} = \frac{1}{5}M(a^2+c^2)$

(ii) **Products of inertia with respect to axes of symmetry:**

$$\begin{aligned}
I_{xy} &= -\int_{\text{Ellipsoid}} x\,y\,dm = -\frac{3M}{4\pi a\,b\,c}\int_{\text{Ellipsoid}} x\,y\,dV = -\frac{3M}{4\pi a\,b\,c}\int_{S}(a\,b\,x'\,y')(a\,b\,c\,dx'\,dy'\,dz') \\
&= -\frac{3a\,b\,M}{4\pi}\int_{S} x'\,y'\,dV' = -\frac{3\,a\,b\,M}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\sin\phi\cos\phi\,dr\,d\theta\,d\phi \\
&= -\frac{3\,a\,b\,M}{4\pi}\int_{r=0}^{1} r^4\,dr\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,d\phi = 0 \qquad \boxed{\because\; \int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,d\phi = \tfrac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0}
\end{aligned}$$

Similarly, it is not difficult to show that

$$I_{yz} = I_{xz} = 0$$

Corollary: ***The moment and product of inertia*** of a (uniform) solid sphere of mass M and radius a ***with respect to*** its axes of symmetry can be obtained by putting $a = b = c$ in results of above example 12. The obtained results are in accordance (as they should be) with the results of example 11.

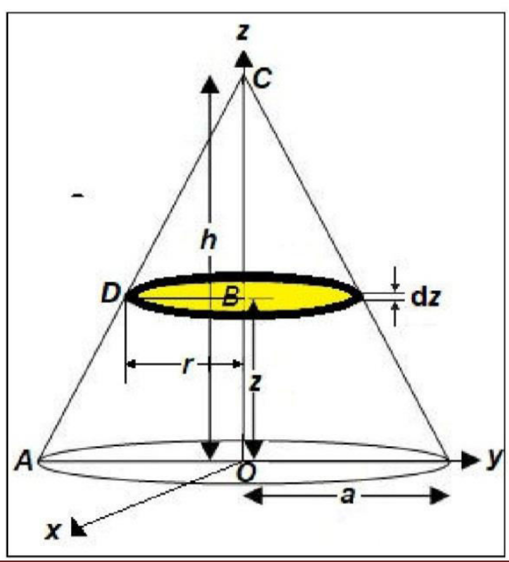
Example 13: ***Find the moment of inertia*** of a (uniform) right circular solid cone ***about***

(i) its axis of symmetry and

(ii) any diameter of the base.

Solution: (i) **Moment of inertia about axis of symmetry:**

Let M, h, a and ρ, respectively, be the mass, height, radius of base and volume mass density of a (uniform) right circular solid cone. Choose coordinate axes as shown in figure. Let us divide cone into large number of elementary solid discs parallel to the base of the cone. One such elementary disc of radius r, mass dm, thickness dz and volume dV is shown in the figure, which is located at a distance z from the base of the cone.



Moment of inertia of elementary disc about z-axis is given by

$$dI_{zz} = \frac{1}{2}r^2 dm$$

Thus, moment of inertia of cone about z-axis is

$$I_{zz} = \frac{1}{2}\int_{\text{Cone}} r^2 dm = \frac{\rho}{2}\int_{\text{Cone}} r^2\,(\pi r^2)\,dz \qquad \boxed{\because\; \rho = \frac{dm}{dV} = \frac{dm}{(\pi r^2)\,dz}}$$

$$= \frac{3M}{2a^2h}\int_{\text{Cone}} r^4\,dz \qquad \boxed{\because\; \rho = \frac{M}{(1/3)\pi a^2 h} \quad \text{(for cone)}}$$

From similar triangles AOC and DBC

$$\frac{r}{a} = \frac{h-z}{h} \qquad \text{or} \qquad r = \frac{a(h-z)}{h}$$

$$\Rightarrow I_{zz} = \frac{3M}{2a^2h}\int_{\text{Cone}}\left[\frac{a(h-z)}{h}\right]^4 \mathrm{d}z = \frac{3M\,a^2}{2\,h^5}\int_{z=0}^{h}(h-z)^4\,\mathrm{d}z = -\,\frac{3M\,a^2}{10\,h^5}\,(h-z)^5\Big|_{z=0}^{h} = \frac{3}{10}\,M\,a^2$$

(ii) **Moment of inertia about diameter of the base:** It is clear that, the moment of inertia of elementary disc of mass $\mathrm{d}m$ about its diameter, along DB, is given by

$$\mathrm{d}I_o = \frac{1}{4}\,r^2\mathrm{d}m$$

We note that the diameter (along DB) of elementary disc passes through its center B (which is also its centre of mass). Hence, by parallel axis theorem, the moment of inertia of the elementary disc about axis AO (parallel to DB and passing through the centre of the base O) is given by

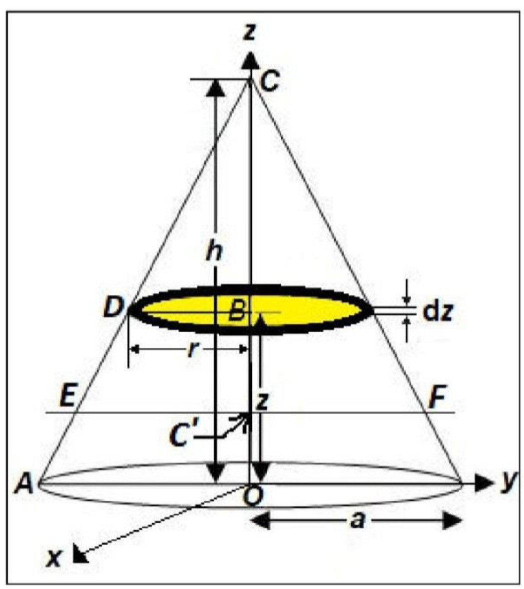
$$\mathrm{d}I_{yy} = \mathrm{d}I_o + (\mathrm{d}m)\,z^2 = \frac{1}{4}\,r^2\mathrm{d}m + (\mathrm{d}m)\,z^2 = \left(\frac{1}{4}\,r^2 + z^2\right)\mathrm{d}m = \frac{3M}{a^2h}\left(\frac{1}{4}\,r^4 + r^2z^2\right)\mathrm{d}z, \quad \boxed{\because \mathrm{d}m = \rho\,\mathrm{d}V = \tfrac{3M}{a^2h}(r^2\,\mathrm{d}z)}$$

Therefore, the moment of inertia of whole cone about diameter of the base is given by

$$\begin{aligned}
I_{yy} &= \frac{3M}{a^2h}\int_{z=0}^{h}\left[\frac{1}{4}\left(\frac{a(h-z)}{h}\right)^4 + \left(\frac{a(h-z)}{h}\right)^2 z^2\right]\mathrm{d}z \qquad \boxed{\because r = \tfrac{a(h-z)}{h}}\\
&= \frac{3M}{a^2h}\int_{z=0}^{h}\left[\frac{a^4}{4\,h^4}(h-z)^4 + \frac{a^2}{h^2}(h^2z^2 - 2h\,z^3 + z^4)\right]\mathrm{d}z\\
&= \frac{3M}{a^2h}\left[-\,\frac{a^4}{20\,h^4}(h-z)^5 + \frac{a^2}{h^2}\left(\frac{h^2z^3}{3} - \frac{h\,z^4}{2} + \frac{z^5}{5}\right)\right]\Bigg|_{z=0}^{h}\\
&= \frac{3M}{a^2h}\left[\frac{a^2}{h^2}\left(\frac{h^5}{3} - \frac{h^5}{2} + \frac{h^5}{5}\right) + \frac{a^4\,h}{20}\right] = \frac{3M}{a^2h}\left[a^2\,h^3\left(\frac{10-15+6}{30}\right) + \frac{a^4\,h}{20}\right]\\
&= \frac{3M}{a^2h}\left[\frac{a^2\,h^3}{30} + \frac{a^4\,h}{20}\right] = \frac{3M}{a^2h}\left[\frac{2a^2\,h^3 + 3a^4\,h}{60}\right] = \frac{1}{20}M(3a^2 + 2h^2)
\end{aligned}$$

Example 14: ***Find the moment of inertia*** of a (uniform) right circular solid cone of mass M, height h and base radius a ***about*** a line through its centre of mass and perpendicular to the axis of the cone.

Solution: Firstly, we find moment of inertia of solid cone about diameter of its base. Let M, h, a and ρ, respectively, be the mass, height, radius of base and volume mass density of a (uniform) right circular solid cone. Let C' be the centre of mass of the cone, which lies on its symmetry axis OC. Choose coordinate axes as shown in figure. Let us divide cone into large number of elementary solid discs parallel to the base of the cone. One such elementary disc of radius r, mass $\mathrm{d}m$, thickness $\mathrm{d}z$ and volume $\mathrm{d}V$ is shown in the figure, at a distance z from the base of the cone.



Moment of inertia of the elementary disc of mass $\mathrm{d}m$ about its diameter, along DB, is given by

$$\mathrm{d}I_o = \frac{1}{4}\,r^2\mathrm{d}m$$

We note that the diameter (along DB) of elementary disc passes through its center B (which is also its centre of mass). Hence, by parallel axis theorem, the moment of inertia of the elementary disc about axis AO (parallel to DB and passing through the centre of the base O) is given by

$$\mathrm{d}I_{yy} = \mathrm{d}I_o + (\mathrm{d}m)\,z^2 = \frac{1}{4}\,r^2\mathrm{d}m + (\mathrm{d}m)\,z^2 = \left(\frac{1}{4}\,r^2 + z^2\right)\mathrm{d}m = \frac{3M}{a^2h}\left(\frac{1}{4}\,r^4 + r^2z^2\right)\mathrm{d}z, \quad \boxed{\because \mathrm{d}m = \rho\,\mathrm{d}V = \tfrac{3M}{\pi a^2h}(\pi r^2\,\mathrm{d}z)}$$

From similar triangles AOC and DBC

$$\frac{r}{a} = \frac{h-z}{h} \qquad \Rightarrow \qquad r = \frac{a(h-z)}{h}$$

Therefore, the moment of inertia of whole cone about diameter of the base is given by

$$\begin{aligned}
I_{yy} &= \frac{3M}{a^2 h}\int_{z=0}^{h}\left[\frac{1}{4}\left(\frac{a(h-z)}{h}\right)^4 + \left(\frac{a(h-z)}{h}\right)^2 z^2\right]\mathrm{d}z \qquad \boxed{\because r = \tfrac{a(h-z)}{h}}\\
&= \frac{3M}{a^2 h}\int_{z=0}^{h}\left[\frac{a^4}{4\,h^4}(h-z)^4 + \frac{a^2}{h^2}(h^2 z^2 - 2h\,z^3 + z^4)\right]\mathrm{d}z\\
&= \frac{3M}{a^2 h}\left[-\frac{a^4}{20\,h^4}(h-z)^5 + \frac{a^2}{h^2}\left(\frac{h^2 z^3}{3} - \frac{h\,z^4}{2} + \frac{z^5}{5}\right)\right]\Bigg|_{z=0}^{h}\\
&= \frac{3M}{a^2 h}\left[\frac{a^2}{h^2}\left(\frac{h^5}{3} - \frac{h^5}{2} + \frac{h^5}{5}\right) + \frac{a^4\,h}{20}\right] = \frac{3M}{a^2 h}\left[a^2\,h^3\left(\frac{10-15+6}{30}\right) + \frac{a^4\,h}{20}\right]\\
&= \frac{3M}{a^2 h}\left[\frac{a^2\,h^3}{30} + \frac{a^4\,h}{20}\right] = \frac{3M}{a^2 h}\left[\frac{2a^2\,h^3 + 3a^4\,h}{60}\right] = \frac{1}{20}M(3a^2 + 2h^2)
\end{aligned}$$

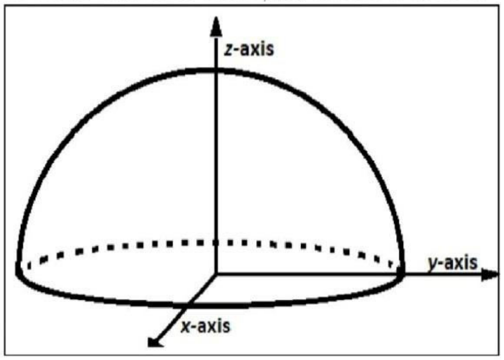
Using parallel axis theorem, the required moment of inertia about line EF, which is passing through centre of mass $C^{'}$ of the cone and perpendicular to its axis, is given by

$$\begin{aligned}
I_{EF} &= I_{yy} - M|OC^{'}|^2 = \frac{1}{20}M(3a^2 + 2h^2) - M\left(\frac{1}{4}h\right)^2 \qquad \boxed{\because |OC^{'}| = h/4}\\
&= \frac{1}{20}M(3a^2 + 2h^2) - \frac{1}{16}Mh^2\\
&\text{(Do simple calculation by yourself)}\\
&= \frac{3}{80}M(4a^2 + h^2)
\end{aligned}$$

Example 15: ***Find the moment of inertia*** of a (uniform) hemispherical shell of mass M ***about***
(i) its axis of symmetry
(ii) an axis perpendicular to the axis of symmetry and passing through the centre of the base.

Solution: (i) **Moment of inertia about axis of symmetry:**
Let M, a and σ, respectively, be the mass, radius of base and areal mass density of the hemispherical shell. Choose coordinate axes as shown in figure.
Moment of inertia of typical area element of hemispherical shell, with mass $\mathrm{d}m$ and area $\mathrm{d}S$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$



Thus, moment of inertia of hemisphere about z-axis is

$$\begin{aligned}
I_{zz} &= \int_S (x^2 + y^2)\mathrm{d}m, \qquad S:\ \text{hemispherical shell}\\
&= \sigma\int_S (x^2 + y^2)\,\mathrm{d}S \qquad \boxed{\because\ \ \sigma = \tfrac{\mathrm{d}m}{\mathrm{d}S} = \text{constant}}\\
&= \frac{M}{2\pi a^2}\int_S (x^2 + y^2)\,\mathrm{d}S \qquad \boxed{\because\ \ \sigma = \tfrac{M}{2\pi a^2}\quad \text{(for hemispherical shell)}}
\end{aligned}$$

To make the computation simpler, we use the parametric equations of hemispherical shell as follows

$$x = a\sin\theta\cos\phi, \qquad y = a\sin\theta\sin\phi, \qquad z = a\cos\theta$$

$$\text{For hemispherical shell}: \qquad 0 \le \theta \le \pi/2, \qquad 0 \le \phi < 2\pi$$

$$\mathrm{d}S = (a\,\mathrm{d}\theta)\,(a\sin\theta\,d\phi) = a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2 + y^2 = a^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = a^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = a^2\sin^2\theta$$

$$\Rightarrow \quad I_{zz} = \frac{M}{2\pi a^2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^4\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{Ma^2}{2\pi}\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi, \tag{12}$$

where,

$$\int_{\theta=0}^{\pi/2} \sin^3\theta\, \mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2} (3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi/2} = \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{2}{3} \tag{13}$$

Using (13) in (12), we get

$$I_{zz} = \frac{Ma^2}{2\pi}\left(\frac{2}{3}\right)(2\pi) = \frac{2}{3}Ma^2$$

(ii) **<u>Moment of inertia about a diameter of the base</u>:**

$$I_{xx} = \int_S (y^2 + z^2)\mathrm{d}m = \sigma\int_S (y^2+z^2)\mathrm{d}S = \frac{M}{2\pi a^2}\int_S (y^2+z^2)\,\mathrm{d}S, \qquad S: \text{ hemispherical shell}$$

Using parametric equations of hemispherical shell, we get

$$I_{xx} = \frac{M}{2\pi a^2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^4(\sin^3\theta\sin^2\phi + \cos^2\theta\sin\theta)\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \frac{Ma^2}{2\pi}\left(\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi + \int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi\right) \tag{14}$$

where,

$$\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1-\cos 2\phi)\,\mathrm{d}\phi = \frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi \tag{15}$$

and

$$\int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta = -\frac{1}{3}\cos^3\theta\Big|_{\theta=0}^{\pi/2} = \frac{1}{3} \tag{16}$$

Using (13), (15) and (16) in (14), we get

$$I_{xx} = \frac{Ma^2}{2\pi}\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{Ma^2}{2\pi}\left(\frac{4\pi}{3}\right) = \frac{2}{3}Ma^2$$

Example 16: ***Find three products of inertia*** of a (uniform) hemispherical shell of mass M ***with respect to*** coordinate axes as in figure of example 15.

Solution:

$$I_{xy} = -\int_S x\,y\,\mathrm{d}m = -\sigma\int_S x\,y\,\mathrm{d}S = -\frac{M}{2\pi a^2}\int_S x\,y\,\mathrm{d}S, \quad S: \text{ hemispherical shell}$$

Using parametric equations of hemispherical shell, we get

$$I_{xy} = -\frac{M}{2\pi a^2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^4\sin^3\theta\sin\phi\cos\phi\,\mathrm{d}\theta\,\mathrm{d}\phi = -\frac{Ma^2}{2\pi}\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

Now,

$$I_{xz} = -\int_S x\,z\,\mathrm{d}m = -\frac{M}{2\pi a^2}\int_S x\,z\,\mathrm{d}S = -\frac{M}{2\pi a^2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^4\sin^2\theta\cos\theta\cos\phi\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{Ma^2}{2\pi}\int_{\theta=0}^{\pi/2}\sin^2\theta\cos\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi$$

But

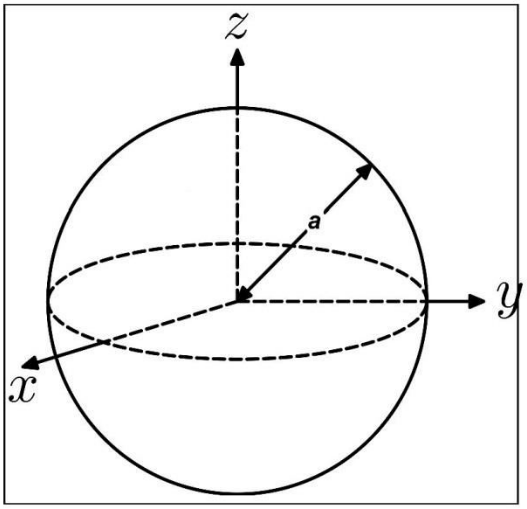
$$\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi = \sin\phi\Big|_{\phi=0}^{2\pi} = 0 \implies I_{xz} = 0 = I_{yz}, \qquad \boxed{\because \; I_{xz} = I_{yz} \text{ (by symmetry)}}$$

Thus,

$$I_{xy} = I_{xz} = I_{yz} = 0$$

Example 17: ***Find the moments and products of inertia*** of a (uniform) spherical shell of mass M and radius a ***with respect to*** its axes of symmetry.

Solution: (i) **Moment of inertia about axis of symmetry:**
Let M, a and σ, respectively, be the mass, radius of base and areal mass density of the spherical shell. Choose coordinate axes as shown in figure.



Moment of inertia of typical area element of spherical shell, with mass $\mathrm{d}m$ and area $\mathrm{d}S$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2+y^2)\mathrm{d}m$$

Thus, moment of inertia of spherical shell about z-axis is

$$\begin{aligned} I_{zz} &= \int_S (x^2+y^2)\mathrm{d}m, \qquad S: \text{ spherical shell} \\ &= \sigma \int_S (x^2+y^2)\,\mathrm{d}S \qquad \boxed{\because \quad \sigma = \tfrac{\mathrm{d}m}{\mathrm{d}S} = \text{constant}} \\ &= \frac{M}{4\pi a^2}\int_S (x^2+y^2)\,\mathrm{d}S \qquad \boxed{\because \quad \sigma = \tfrac{M}{4\pi a^2} \quad \text{(for spherical shell)}} \end{aligned}$$

To make the computation simpler, we use the parametric equations of spherical shell as follows

$$x = a\sin\theta\cos\phi, \qquad y = a\sin\theta\sin\phi, \qquad z = a\cos\theta$$

$$\text{For spherical shell}: \quad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$$

$$\mathrm{d}S = (a\,\mathrm{d}\theta)\,(a\sin\theta\,d\phi) = a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2+y^2 = a^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = a^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = a^2\sin^2\theta$$

$$\Rightarrow \quad I_{zz} = \frac{M}{4\pi a^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a^4\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{Ma^2}{4\pi}\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi, \tag{17}$$

where,

$$\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3} \tag{18}$$

Using (18) in (17), we get

$$I_{zz} = \frac{Ma^2}{4\pi}\left(\frac{4}{3}\right)(2\pi) = \frac{2}{3}Ma^2$$

(ii) **Products of inertia with respect to axes of symmetry:**

$$I_{xy} = -\int_S x\,y\,\mathrm{d}m = -\sigma\int_S x\,y\,\mathrm{d}S = -\frac{M}{4\pi a^2}\int_S x\,y\,\mathrm{d}S, \quad S: \text{ spherical shell}$$

Using parametric equations of spherical shell, we get

$$\begin{aligned} I_{xy} &= -\frac{M}{4\pi a^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a^4\sin^3\theta\sin\phi\cos\phi\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -\frac{Ma^2}{4\pi}\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi \end{aligned}$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

Similarly,

$$I_{yz} = I_{xz} = 0 \qquad \boxed{\because \quad I_{xy} = I_{yz} = I_{xz} \ \text{(by symmetry)}}$$

Definition: A set of three mutually perpendicular axes having origin O which are fixed in the rigid body and rotating with it and which are such that the product of inertia with respect to them are zero are called "principal axes of inertia" or simply "principal axes" of body at point O.

Definition: An axis is called "principal axis of inertia" or simply "principal axis" of a rigid body if directions of angular momentum $\mathbf{L}$ and angular velocity $\boldsymbol{\omega}$ are same, when rigid body is rotating about this axis.

Theorem: Above two definitions of principal axes are equivalent.

Proof: Suppose that for a rigid body we have three mutually concurrent and mutually perpendicular axes for which first definition holds. Choosing these axes as Cartesian coordinate axes, the inertia matrix with respect to this coordinate system is given by

$$[\mathbf{I}] = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

If rigid body rotates about $x-$ axis, then its angular velocity has the form $\boldsymbol{\omega}_x = \begin{pmatrix} \omega_{x1} \\ 0 \\ 0 \end{pmatrix}$

As we know that $[\mathbf{L}_x] = [\mathbf{I}][\boldsymbol{\omega}_x]$

$$\Rightarrow \begin{pmatrix} L_{x1} \\ L_{x2} \\ L_{x3} \end{pmatrix} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}\begin{pmatrix} \omega_{x1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{11}\omega_{x1} \\ 0 \\ 0 \end{pmatrix} = I_{11}\begin{pmatrix} \omega_{x1} \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \mathbf{L}_x = I_{11}\boldsymbol{\omega}_x$$

This shows that angular momentum is parallel to angular velocity. Similarly, we can show that when body rotates about y or z axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes.

Conversely, suppose that for a rigid body we have three mutually concurrent and mutually perpendicular axes for which second definition holds. Choosing these axes as Cartesian coordinate axes, and assuming that body rotates about $x-$ axis, we have, by supposition, angular momentum and angular velocity are parallel

$$\Rightarrow \quad \mathbf{L}_x = \lambda_1\boldsymbol{\omega}_x, \quad \text{where } \lambda \text{ is constant}$$

$$\Rightarrow L_{x1}\mathbf{i} + L_{x2}\mathbf{j} + L_{x3}\mathbf{k} = \lambda_1(\omega_{x1}\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})$$

$$\Rightarrow \begin{pmatrix} L_{x1} \\ L_{x2} \\ L_{x3} \end{pmatrix} = \begin{pmatrix} \lambda_1\omega_{x1} \\ 0 \\ 0 \end{pmatrix} -----\rightarrow (1)$$

As we know that $[\mathbf{L}_x] = [\mathbf{I}][\boldsymbol{\omega}_x]$

$$\begin{pmatrix} L_{x1} \\ L_{x2} \\ L_{x3} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}\begin{pmatrix} \omega_{x1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{11}\omega_{x1} \\ I_{12}\omega_{x1} \\ I_{13}\omega_{x1} \end{pmatrix} \rightarrow (2)$$

From (1) and (2), we have

$$\begin{pmatrix} \lambda_1\omega_{x1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{11}\omega_{x1} \\ I_{12}\omega_{x1} \\ I_{13}\omega_{x1} \end{pmatrix}$$

$$\Rightarrow \quad I_{12} = I_{13} = 0 \quad \because \quad \omega_{x1} \neq 0$$

Similarly, assuming the rotation of body about $y-$ axis ($\mathbf{L}_y = \lambda_2\boldsymbol{\omega}_y$), we get, $I_{12} = I_{23} = 0$.

$\Rightarrow$ All product of inertia are zero. Hence first definition also holds for given axes. (Note: $\lambda_i = I_{ii}, \quad i = 1,2,3$)

Definition: The moment of inertia with respect to a principal axis is called "principal moment of inertia".

Theorem: Prove that for a rigid body a set of three mutually perpendicular principal axes exists at given point.

Proof: As we know from the definition of principal axis that if a rigid body rotates bout principal axes, passing through a point O, then the angular momentum $\mathbf{L}$ and the angular velocity $\boldsymbol{\omega}$ of the body are in same direction. So we can write, $\mathbf{L} = \lambda\boldsymbol{\omega}$, where, λ is constant

Let, $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, $\quad \boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$

Then, $L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k} = \lambda(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k})$

Comparing corresponding components on both sides of above vector equation, we get

$$L_1 = \lambda\omega_1, \quad L_2 = \lambda\omega_2, \quad L_3 = \lambda\omega_3 \; -------\rightarrow (1)$$

As we know that, $[\mathbf{L}] = [\mathbf{I}][\boldsymbol{\omega}]$

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} -------\rightarrow (2)$$

From (1) and (2), we get,

$$I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = \lambda\omega_1$$
$$I_{12}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 = \lambda\omega_2$$
$$I_{13}\omega_1 + I_{23}\omega_2 + I_{33}\omega_3 = \lambda\omega_3$$

This system can be written as,

$$\left.\begin{aligned} (I_{11}-\lambda)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0 \\ I_{12}\omega_1 + (I_{22}-\lambda)\omega_2 + I_{23}\omega_3 = 0 \\ I_{13}\omega_1 + I_{23}\omega_2 + (I_{33}-\lambda)\omega_3 = 0 \end{aligned}\right\} -------\rightarrow (3)$$

This is homogeneous system of three equations in three unknowns ω_1, ω_2 and ω_3. This system will have non trivial solution if an only if

$$\begin{vmatrix} I_{11}-\lambda & I_{12} & I_{13} \\ I_{12} & I_{22}-\lambda & I_{23} \\ I_{13} & I_{23} & I_{33}-\lambda \end{vmatrix} = 0$$

This is cubic equation in I which is called characteristic equation of inertia matrix $[\,\mathbf{I}\,]$. It has three roots, say, λ_1, λ_2 and λ_3, which are, in fact, principal moments of inertia. By substituting $\lambda = \lambda_1$ in system (3), we can obtain the ratios $\omega_1:\omega_2:\omega_3$, which give direction of principal axes relative to which moment of inertia is λ_1. Similarly, we can find direction of other two principal axes corresponding to moments of inertia λ_2 and λ_3. We can always find three mutually perpendicular principal axes because $[\,\mathbf{I}\,]$ is symmetric. This shows that there exists three mutually perpendicular principal axes passing through given point O.

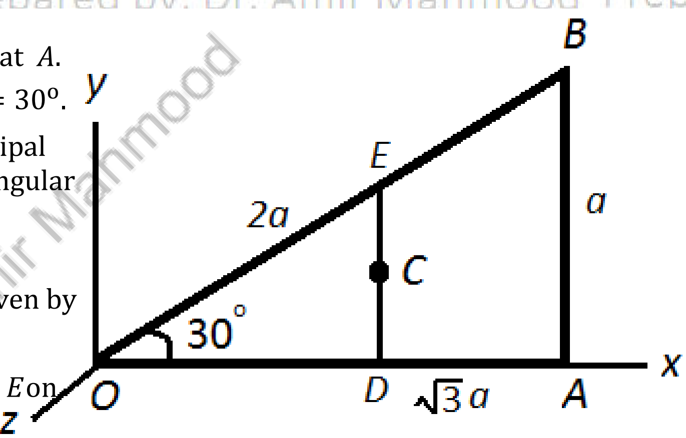
Problem: A triangular plate is made of uniform material and has sides of lengths a, $2a$ and $\sqrt{3}a$. Determine the (direction of) principal axes and corresponding principal moments of inertia at 30^{o} corner (or vertex).

Solution: Let M and σ, respectively, be the mass and surface (areal) mass density of triangular plate OAB lying in xy-plane, as shown in the figure, with $|OA| = \sqrt{3}a$, $|AB| = a$ and $|OB| = 2a$.

Clearly, $\quad |OB|^2 = (2a)^2 = \left(\sqrt{3}a\right)^2 + a^2 = |OA|^2 + |AB|^2$.

This shows that OAB is right angled triangle with right angle at A.

Also, $\tan(m\angle AOB) = \frac{|AB|}{|OA|} = \frac{a}{\sqrt{3}a} \Rightarrow m\angle AOB = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^{o}$.



Thus, we have to find principal axes and corresponding principal moments of inertia at vertex O. The moment of inertia of triangular plate about side OA (x-axis)is given by

$$I_{xx} = I_{11} = I_{OA} = \tfrac{1}{6}M|AB|^2 = \tfrac{1}{6}Ma^2$$

The moment of inertia of triangular plate about side AB is given by

$$I_{AB} = \tfrac{1}{6}M|OA|^2 = \tfrac{1}{6}M\left(\sqrt{3}a\right)^2 = \tfrac{1}{2}Ma^2$$

Let C be the centre of mass of the plate and take D on OA and E on OB such that DE is passing through C and parallel to AB.

Then moment of inertia of plate about DE is given by (using parallel axis theorem)

$$I_{DE} = I_{AB} - M|AD|^2 = \tfrac{1}{2}Ma^2 - M|AD|^2 \longrightarrow (1)$$

From figure, $\quad |AD| = |OA| - |OD| = \sqrt{3}a - (x\text{-coordinate of centre of mass C}) = \sqrt{3}a - \frac{1}{3}(x_O + x_A + x_B)$

$= \sqrt{3}a - \frac{1}{3}\left(0 + \sqrt{3}a + \sqrt{3}a\right) = \sqrt{3}a - \frac{2\sqrt{3}a}{3} = \frac{3\sqrt{3}a - 2\sqrt{3}a}{3} = \frac{\sqrt{3}a}{3} = \frac{a}{\sqrt{3}} \longrightarrow (2)$

Using (2) in (1), we get, $\quad I_{DE} = \frac{1}{2}Ma^2 - M\left(\frac{a}{\sqrt{3}}\right)^2 = \frac{1}{2}Ma^2 - \frac{1}{3}Ma^2 = \frac{3Ma^2 - 2Ma^2}{6} = \frac{1}{6}Ma^2$

Then moment of inertia of plate about y-axis is given by (using parallel axis theorem), as follows,

$$I_{yy} = I_{22} = I_{DE} + M|OD|^2 = \frac{1}{6}Ma^2 + M(x\text{-coordinate of centre of mass } C)^2 = \frac{1}{6}Ma^2 + M\left(\frac{0 + \sqrt{3}a + \sqrt{3}a}{3}\right)^2$$

$$= \frac{1}{6}Ma^2 + M\left(\frac{2\sqrt{3}a}{3}\right)^2 = \frac{1}{6}Ma^2 + \frac{4}{3}Ma^2 = \frac{Ma^2 + 8Ma^2}{6} = \frac{9}{6}Ma^2 = \frac{3}{2}Ma^2$$

Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows,

$$I_{zz} = I_{33} = I_{xx} + I_{yy} = \tfrac{1}{6}Ma^2 + \tfrac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$$

$$I_{xy} = I_{12} = -\int xy\,\mathrm{d}m = -\sigma\int xy\,\mathrm{d}x\mathrm{d}y = -\sigma\int_{x=0}^{\sqrt{3}a}\left(\int_{y=0}^{\frac{x}{\sqrt{3}}} xy\,\mathrm{d}y\right)\mathrm{d}x = -\sigma\int_{x=0}^{\sqrt{3}a}\left(x\left(\frac{y^2}{2}\right)\Big|_{y=0}^{\frac{x}{\sqrt{3}}}\right)\mathrm{d}x \quad \because \mathrm{d}m = \sigma\mathrm{d}x\mathrm{d}y$$

$$= -\frac{\sigma}{6}\int_{x=0}^{\sqrt{3}a} x^3\mathrm{d}x = -\frac{1}{6}\left(\frac{2M}{\sqrt{3}a^2}\right)\left(\frac{x^4}{4}\right)\Big|_{x=0}^{\sqrt{3}a} = -\frac{1}{6}\left(\frac{2M}{\sqrt{3}a^2}\right)\left(\frac{9a^4}{4}\right) = -\frac{\sqrt{3}}{4}Ma^2 \quad \because \sigma = \frac{M}{\frac{1}{2}|OA||AB|} = \frac{M}{\frac{1}{2}(\sqrt{3}a)(a)} = \frac{2M}{\sqrt{3}a^2}$$

As $z = 0$ in xy-plane, therefore, $\quad I_{xz} = I_{13} = -\int xz\,\mathrm{d}m = 0$ and $I_{yz} = I_{23} = -\int yz\,\mathrm{d}m = 0$

The inertia matrix at point O, with respect to coordinate system $Oxyz$, is given by

$$[\mathbf{I}_O]=\begin{pmatrix} I_{11} & I_{12} & I_{13}\\ I_{12} & I_{22} & I_{23}\\ I_{13} & I_{23} & I_{33}\end{pmatrix}=\begin{pmatrix} \frac{1}{6}Ma^2 & -\frac{\sqrt{3}}{4}Ma^2 & 0\\ -\frac{\sqrt{3}}{4}Ma^2 & \frac{3}{2}Ma^2 & 0\\ 0 & 0 & \frac{5}{3}Ma^2\end{pmatrix}=\begin{pmatrix} 2\alpha & -3\sqrt{3}\alpha & 0\\ -3\sqrt{3}\alpha & 18\alpha & 0\\ 0 & 0 & 20\alpha\end{pmatrix}, \quad \text{where} \quad \alpha=\frac{1}{12}Ma^2$$

To find the eigenvalues, we have the characteristic equation $\det([\mathbf{I}_O]-\lambda[I_3])=0$, where $[I_3]$ is unit matrix of order 3.

$$\det([\mathbf{I}_O]-\lambda[I_3])=0 \ \Rightarrow \begin{vmatrix} 2\alpha-\lambda & -3\sqrt{3}\alpha & 0\\ -3\sqrt{3}\alpha & 18\alpha-\lambda & 0\\ 0 & 0 & 20\alpha-\lambda\end{vmatrix}=0$$

On expanding by third row, we get,

$$(20\alpha-\lambda)\left[(2\alpha-\lambda)(18\alpha-\lambda)-\left(-3\sqrt{3}\alpha\right)^2\right]=0 \ \Rightarrow \ (20\alpha-\lambda)[36\alpha^2-2\alpha\lambda-18\alpha\lambda+\lambda^2-27\alpha^2]=0$$

$$\Rightarrow (20\alpha-\lambda)[\lambda^2-20\alpha\lambda+9\alpha^2]=0$$

Either $20\alpha-\lambda=0 \Rightarrow \lambda=20\alpha$	or, $\lambda^2-20\alpha\lambda+9\alpha^2=0 \Rightarrow \lambda=\frac{20\alpha\pm\sqrt{(20\alpha)^2-4(1)(9\alpha^2)}}{2(1)}$ $\Rightarrow \lambda=\frac{20\alpha\pm\sqrt{400\alpha^2-36\alpha^2}}{2}=\frac{20\alpha\pm\sqrt{364\alpha^2}}{2}=\frac{20\alpha\pm2\sqrt{91}\alpha}{2}$ $=(10\pm\sqrt{91})\alpha$

Thus, $\lambda_1=20\alpha$, $\lambda_2=(10+\sqrt{91})\alpha$, and $\lambda_3=(10-\sqrt{91})\alpha$

These eigenvalues gives principal moments of inertia at point O. To find the direction of corresponding principal axes, we find eigenvectors corresponding to each eigenvalue.

For $\lambda_1=20\alpha$: Let $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_1=20\alpha$, then

$$([\mathbf{I}_O]-\lambda_1[I_3])X=\mathbf{0}\Rightarrow\begin{pmatrix} 2\alpha-20\alpha & -3\sqrt{3}\alpha & 0\\ -3\sqrt{3}\alpha & 18\alpha-20\alpha & 0\\ 0 & 0 & 20\alpha-20\alpha\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix} -18\alpha & -3\sqrt{3}\alpha & 0\\ -3\sqrt{3}\alpha & -2\alpha & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow\begin{cases}-18\alpha x_1-3\sqrt{3}\alpha x_2=0\\ -3\sqrt{3}\alpha x_1-2\alpha x_2=0\end{cases}\Rightarrow\begin{cases}6x_1+\sqrt{3}x_2=0 \ ------(3)\\ 3\sqrt{3}x_1+2x_2=0 \ -----(4)\end{cases}$$

From Eq. (3), we have $x_1=-\frac{\sqrt{3}}{6}x_2$ and putting it in (4), we get, $3\sqrt{3}\left(\frac{\sqrt{3}}{6}x_2\right)-2x_2=0\Rightarrow\frac{3}{2}x_2-2x_2=0\Rightarrow x_2=0.$ Put $x_2=0$ in (3), we get, $x_1=0$

Thus, $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\r\end{pmatrix}$, where, $r\in\mathbb{R}$, $r\neq0$ $\Rightarrow$ For $r=1$, we get, $X=\begin{pmatrix}0\\0\\1\end{pmatrix}=0\mathbf{i}+0\mathbf{j}+\mathbf{k}=\mathbf{k}$

For $\lambda_2=(10+\sqrt{91})\alpha$: Let $Y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_2=(10+\sqrt{91})\alpha$, then

$$([\mathbf{I}_O]-\lambda_2[I_3])Y=\mathbf{0}\Rightarrow\begin{pmatrix} -(8+\sqrt{91})\alpha & -3\sqrt{3}\alpha & 0\\ -3\sqrt{3}\alpha & (8-\sqrt{91})\alpha & 0\\ 0 & 0 & (10-\sqrt{91})\alpha\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow\begin{cases}-(8+\sqrt{91})\alpha y_1-3\sqrt{3}\alpha y_2=0\\ -3\sqrt{3}\alpha y_1+(8-\sqrt{91})\alpha y_2=0\\ (10-\sqrt{91})\alpha y_3=0\end{cases}\Rightarrow\begin{cases}(8+\sqrt{91})y_1+3\sqrt{3}y_2=0 \ ------(5)\\ 3\sqrt{3}y_1-(8-\sqrt{91})y_2=0 \ -----(6)\\ y_3=0\end{cases}$$

From Eq. (5), we have $\frac{y_1}{y_2}=\frac{-3\sqrt{3}}{8+\sqrt{91}}$ and from Eq. (6), we have $\frac{y_1}{y_2}=\frac{8-\sqrt{91}}{3\sqrt{3}}=\frac{8-\sqrt{91}}{3\sqrt{3}}\cdot\frac{8+\sqrt{91}}{8+\sqrt{91}}=\frac{-27}{3\sqrt{3}(8+\sqrt{91})}=\frac{-3\sqrt{3}}{8+\sqrt{91}}$

Thus, Eq. (5) and Eq. (6) are mutually identical, therefore, last system of equations can be written as

$$\begin{cases}(8+\sqrt{91})y_1+3\sqrt{3}y_2=0\\ y_3=0\end{cases}$$

Let, $y_2=s$, where, $s\in\mathbb{R}$, $s\neq0$ $\Rightarrow$ $y_1=\frac{-3\sqrt{3}}{8+\sqrt{91}}s$

Therefore, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{-3\sqrt{3}}{8+\sqrt{91}} s \\ s \\ 0 \end{pmatrix} \Rightarrow$ For $s = -(8+\sqrt{91})$, we get, $Y = \begin{pmatrix} 3\sqrt{3} \\ -(8+\sqrt{91}) \\ 0 \end{pmatrix} = 3\sqrt{3}\mathbf{i} - (8+\sqrt{91})\mathbf{j}$

<u>**For $\lambda_3 = (10 - \sqrt{91})\alpha$:**</u> Let $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_3 = (10 - \sqrt{91})\alpha$, then

$$([\mathbf{I}_O] - \lambda_2[I_3])Z = \mathbf{0} \quad \Rightarrow \begin{pmatrix} -(8-\sqrt{91})\alpha & -3\sqrt{3}\alpha & 0 \\ -3\sqrt{3}\alpha & (8+\sqrt{91})\alpha & 0 \\ 0 & 0 & (10+\sqrt{91})\alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -(8-\sqrt{91})\alpha z_1 - 3\sqrt{3}\alpha z_2 = 0 \\ -3\sqrt{3}\alpha z_1 + (8+\sqrt{91})\alpha z_2 = 0 \\ (10+\sqrt{91})\alpha z_3 = 0 \end{cases} \Rightarrow \begin{cases} (8-\sqrt{91})z_1 + 3\sqrt{3} z_2 = 0 \quad ------(7) \\ 3\sqrt{3} z_1 - (8+\sqrt{91}) z_2 = 0 \quad ------(8) \\ z_3 = 0 \end{cases}$$

From Eq. (7), we have $\frac{z_1}{z_2} = \frac{-3\sqrt{3}}{8-\sqrt{91}}$ and from Eq. (8), we have $\frac{z_1}{z_2} = \frac{8+\sqrt{91}}{3\sqrt{3}} = \frac{8+\sqrt{91}}{3\sqrt{3}} \cdot \frac{8-\sqrt{91}}{8-\sqrt{91}} = \frac{-27}{3\sqrt{3}(8-\sqrt{91})} = \frac{-3\sqrt{3}}{8-\sqrt{91}}$

Thus, Eq. (7) and Eq. (8) are mutually identical, therefore, last system of equations can be written as

$$\begin{cases} (8-\sqrt{91})z_1 + 3\sqrt{3} z_2 = 0 \\ z_3 = 0 \end{cases}$$

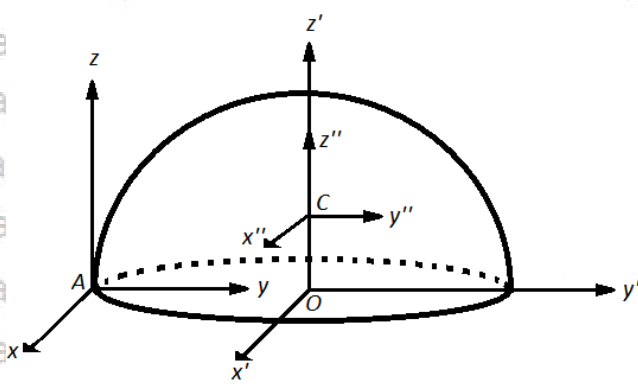
Let, $z_2 = t$, where, $t \in \mathbb{R}$, $t \neq 0$ $\quad \Rightarrow \quad z_1 = \frac{-3\sqrt{3}}{8-\sqrt{91}} t$

Therefore, $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{-3\sqrt{3}}{8-\sqrt{91}} t \\ t \\ 0 \end{pmatrix} \Rightarrow$ For $t = -(8-\sqrt{91})$, we get, $Z = \begin{pmatrix} 3\sqrt{3} \\ -(8-\sqrt{91}) \\ 0 \end{pmatrix} = 3\sqrt{3}\mathbf{i} - (8-\sqrt{91})\mathbf{j}$

Principal moment of inertia	Principal axis	Normalized principal axis
$\lambda_1 = 20\alpha$	$X = \mathbf{k}$	$\hat{X} = \mathbf{k}$
$\lambda_2 = (10+\sqrt{91})\alpha$	$Y = 3\sqrt{3}\mathbf{i} - (8+\sqrt{91})\mathbf{j}$	$\hat{Y} = \frac{1}{\sqrt{182+16\sqrt{91}}}[3\sqrt{3}\mathbf{i} - (8+\sqrt{91})\mathbf{j}]$
$\lambda_3 = (10-\sqrt{91})\alpha$	$Z = 3\sqrt{3}\mathbf{i} - (8-\sqrt{91})\mathbf{j}$	$\hat{Z} = \frac{1}{\sqrt{182+16\sqrt{91}}}[3\sqrt{3}\mathbf{i} - (8-\sqrt{91})\mathbf{j}]$

<u>**Problem:** Determine the (direction of) principal axes and corresponding principal moments of inertia of a uniform solid hemisphere at a point on its rim.</u>

Solution: Let M, a and ρ, respectively, be the mass, radius of the base and volume mass density of a uniform solid hemisphere. Let A, O and C, respectively, be point on the rim, centre of the base and centre of mass of the hemisphere. Choose three coordinate axes $Axyz$, $Ox'y'z'$ and $Cx''y''z''$ as shown in the figure.



As we know that, the moments and product of inertia with respect to coordinate system $Ox'y'z'$ are given by $I_{O11} = I_{O22} = I_{O33} = \frac{2}{5}Ma^2$ and $I_{O12} = I_{O23} = I_{O13} = 0$.
Therefore, the inertia matrix with respect to coordinate system $Ox'y'z'$ is given by

$$[\mathbf{I}_O] = (I_{Oij}) = \begin{pmatrix} I_{O11} & I_{O12} & I_{O13} \\ I_{O12} & I_{O22} & I_{O23} \\ I_{O13} & I_{O23} & I_{O33} \end{pmatrix} = \begin{pmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix}$$

Next, we apply parallel axis theorem in tensor notation to find inertia tensor $[\mathbf{I}_C]$ with respect to coordinate system $Cx''y''z''$, as follows

$$I_{Oij} = I_{Cij} + M\mathbf{r}_c^2\delta_{ij} - Mx_{c,i}x_{c,j}$$
$$\Rightarrow \; I_{Cij} = I_{Oij} - M\mathbf{r}_c^2\delta_{ij} + Mx_{c,i}x_{c,j}$$

$$\Rightarrow\begin{pmatrix} I_{C11} & I_{C12} & I_{C13}\\ I_{C12} & I_{C22} & I_{C23}\\ I_{C13} & I_{C23} & I_{C33}\end{pmatrix}=\begin{pmatrix} I_{O11} & I_{O12} & I_{O13}\\ I_{O12} & I_{O22} & I_{O23}\\ I_{O13} & I_{O23} & I_{O33}\end{pmatrix}-M\begin{pmatrix}\mathbf{r}_c^2 & 0 & 0\\ 0 & \mathbf{r}_c^2 & 0\\ 0 & 0 & \mathbf{r}_c^2\end{pmatrix}+M\begin{pmatrix} x_{c,1}x_{c,1} & x_{c,1}x_{c,2} & x_{c,1}x_{c,3}\\ x_{c,1}x_{c,2} & x_{c,2}x_{c,2} & x_{c,2}x_{c,3}\\ x_{c,1}x_{c,3} & x_{c,2}x_{c,3} & x_{c,3}x_{c,3}\end{pmatrix},$$

where, $\mathbf{r}_c=(x_{c,1},\ x_{c,2},\ x_{c,3})=\left(0,\ 0,\ \frac{3}{8}a\right)$ is the position vector of centre of mass C with respect to coordinate system $Ox'y'z'$.

$$\Rightarrow\begin{pmatrix} I_{C11} & I_{C12} & I_{C13}\\ I_{C12} & I_{C22} & I_{C23}\\ I_{C13} & I_{C23} & I_{C33}\end{pmatrix}=\begin{pmatrix}\frac{2}{5}Ma^2 & 0 & 0\\ 0 & \frac{2}{5}Ma^2 & 0\\ 0 & 0 & \frac{2}{5}Ma^2\end{pmatrix}-M\begin{pmatrix}\frac{9}{64}a^2 & 0 & 0\\ 0 & \frac{9}{64}a^2 & 0\\ 0 & 0 & \frac{9}{64}a^2\end{pmatrix}+M\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \frac{9}{64}a^2\end{pmatrix}$$

$$\Rightarrow\quad[\mathbf{I}_C]=\begin{pmatrix} I_{C11} & I_{C12} & I_{C13}\\ I_{C12} & I_{C22} & I_{C23}\\ I_{C13} & I_{C23} & I_{C33}\end{pmatrix}=\begin{pmatrix}\frac{2}{5}Ma^2-\frac{9}{64}Ma^2 & 0 & 0\\ 0 & \frac{2}{5}Ma^2-\frac{9}{64}Ma^2 & 0\\ 0 & 0 & \frac{2}{5}Ma^2-\frac{9}{64}Ma^2+\frac{9}{64}Ma^2\end{pmatrix}=\begin{pmatrix}\frac{83}{320}Ma^2 & 0 & 0\\ 0 & \frac{83}{320}Ma^2 & 0\\ 0 & 0 & \frac{2}{5}Ma^2\end{pmatrix}.$$

Now, we apply parallel axis theorem in tensor notation to find inertia tensor $[\mathbf{I}_A]$ with respect to coordinate system $Axyz$, as follows

$$I_{Aij}=I_{Cij}+M\mathbf{r}_c'^2\delta_{ij}-Mx'_{c,i}x'_{c,j}$$

$$\Rightarrow\begin{pmatrix} I_{A11} & I_{A12} & I_{A13}\\ I_{A12} & I_{A22} & I_{A23}\\ I_{A13} & I_{A23} & I_{A33}\end{pmatrix}=\begin{pmatrix} I_{C11} & I_{C12} & I_{C13}\\ I_{C12} & I_{C22} & I_{C23}\\ I_{C13} & I_{C23} & I_{C33}\end{pmatrix}+M\begin{pmatrix}\mathbf{r}_c'^2 & 0 & 0\\ 0 & \mathbf{r}_c'^2 & 0\\ 0 & 0 & \mathbf{r}_c'^2\end{pmatrix}-M\begin{pmatrix} x'_{c,1}x'_{c,1} & x'_{c,1}x'_{c,2} & x'_{c,1}x'_{c,3}\\ x'_{c,1}x'_{c,2} & x'_{c,2}x'_{c,2} & x'_{c,2}x'_{c,3}\\ x'_{c,1}x'_{c,3} & x'_{c,2}x'_{c,3} & x'_{c,3}x'_{c,3}\end{pmatrix},$$

where, $\mathbf{r}'_c=(x'_{c,1},\ x'_{c,1},\ x'_{c,1})=\left(0,\ a,\ \frac{3}{8}a\right)$ is the position vector of centre of mass C with respect to coordinate system $Axyz$.

$$\Rightarrow\begin{pmatrix} I_{A11} & I_{A12} & I_{A13}\\ I_{A12} & I_{A22} & I_{A23}\\ I_{A13} & I_{A23} & I_{A33}\end{pmatrix}=\begin{pmatrix}\frac{83}{320}Ma^2 & 0 & 0\\ 0 & \frac{83}{320}Ma^2 & 0\\ 0 & 0 & \frac{2}{5}Ma^2\end{pmatrix}+M\begin{pmatrix}\frac{73}{64}a^2 & 0 & 0\\ 0 & \frac{73}{64}a^2 & 0\\ 0 & 0 & \frac{73}{64}a^2\end{pmatrix}-M\begin{pmatrix}0 & 0 & 0\\ 0 & a^2 & \frac{3}{8}a^2\\ 0 & \frac{3}{8}a^2 & \frac{9}{64}a^2\end{pmatrix}$$

$$[\mathbf{I}_A]=\begin{pmatrix} I_{A11} & I_{A12} & I_{A13}\\ I_{A12} & I_{A22} & I_{A23}\\ I_{A13} & I_{A23} & I_{A33}\end{pmatrix}=\begin{pmatrix}\frac{83}{320}Ma^2+\frac{73}{64}Ma^2 & 0 & 0\\ 0 & \frac{83}{320}Ma^2+\frac{73}{64}Ma^2-Ma^2 & -\frac{3}{8}Ma^2\\ 0 & -\frac{3}{8}a^2 & \frac{2}{5}Ma^2+\frac{73}{64}Ma^2-\frac{9}{64}Ma^2\end{pmatrix}$$

$$[\mathbf{I}_A]=\begin{pmatrix}\frac{7}{5}Ma^2 & 0 & 0\\ 0 & \frac{2}{5}Ma^2 & -\frac{3}{8}Ma^2\\ 0 & -\frac{3}{8}Ma^2 & \frac{7}{5}Ma^2\end{pmatrix}=\begin{pmatrix}56\alpha & 0 & 0\\ 0 & 16\alpha & -15\alpha\\ 0 & -15\alpha & 56\alpha\end{pmatrix},\quad\text{where,}\quad\alpha=\frac{1}{40}Ma^2$$

To find the eigenvalues, we solve characteristic equation $\det([I_A]-\lambda[I_3])=0$, where $[I_3]$ is unit matrix of order 3.

$$\det([\mathbf{I}_A]-\lambda[I_3])=0\quad\Rightarrow\quad\begin{vmatrix}56\alpha-\lambda & 0 & 0\\ 0 & 16\alpha-\lambda & -15\alpha\\ 0 & -15\alpha & 56\alpha-\lambda\end{vmatrix}=0$$

On expanding by first row, we get,

$(56\alpha-\lambda)[(16\alpha-\lambda)(56\alpha-\lambda)-(-15\alpha)^2]=0\ \Rightarrow\ (56\alpha-\lambda)[896\alpha^2-16\alpha\lambda-56\alpha\lambda+\lambda^2-225\alpha^2]=0$

$\Rightarrow(56\alpha-\lambda)[\lambda^2-72\alpha\lambda+671\alpha^2]=0$

Either $56\alpha-\lambda=0\ \Rightarrow\ \lambda=56\alpha$	or, $\lambda^2-72\alpha\lambda+671\alpha^2=0\ \Rightarrow\ \lambda=\frac{72\alpha\pm\sqrt{(72\alpha)^2-4(1)(671\alpha^2)}}{2(1)}$ $\Rightarrow\ \lambda=\frac{72\alpha\pm\sqrt{5184\alpha^2-2684\alpha^2}}{2}=\frac{72\alpha\pm\sqrt{2500\alpha^2}}{2}=\frac{72\alpha\pm50\alpha}{2}$ $\Rightarrow\ \lambda=\frac{72\alpha+50\alpha}{2},\ \frac{72\alpha-50\alpha}{2}=\frac{122\alpha}{2},\ \frac{22\alpha}{2}=61\alpha,\ 11\alpha$

Thus, $\lambda_1 = 56\alpha$, $\lambda_2 = 61\alpha$, and $\lambda_3 = 11\alpha$.

These eigenvalues gives principal moments of inertia at point A. To find the direction of corresponding principal axes, we find eigenvectors corresponding to each eigenvalue.

For $\lambda_1 = 56\alpha$: Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_1 = 56\alpha$, then

$$([\mathbf{I}_A] - \lambda_1[I_3])X = \mathbf{0} \Rightarrow \begin{pmatrix} 56\alpha - 56\alpha & 0 & 0 \\ 0 & 16\alpha - 56\alpha & -15\alpha \\ 0 & -15\alpha & 56\alpha - 56\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -40\alpha & -15\alpha \\ 0 & -15\alpha & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -40\alpha x_2 - 15\alpha x_3 = 0 \\ -15\alpha x_2 = 0 \end{cases} \Rightarrow \begin{cases} 8x_2 + 3x_3 = 0 ------(1) \\ x_2 = 0 ------(2) \end{cases}$$

Thus we have, $x_2 = x_3 = 0$ and $x_1 = r$, where, $r \in \mathbb{R}$, $r \neq 0$

Thus, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$, $\Rightarrow$ For $r = 1$, we get, $X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{i}$

For $\lambda_2 = 61\alpha$: Let $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_2 = 61\alpha$, then

$$([\mathbf{I}_A] - \lambda_2[I_3])Y = \mathbf{0} \Rightarrow \begin{pmatrix} 56\alpha - 61\alpha & 0 & 0 \\ 0 & 16\alpha - 61\alpha & -15\alpha \\ 0 & -15\alpha & 56\alpha - 61\alpha \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -5\alpha & 0 & 0 \\ 0 & -45\alpha & -15\alpha \\ 0 & -15\alpha & -5\alpha \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -5\alpha y_1 = 0 \\ -45\alpha y_2 - 15\alpha y_3 = 0 \\ -15\alpha y_2 - 5\alpha y_3 = 0 \end{cases} \Rightarrow \begin{cases} y_1 = 0 \\ 3y_2 + y_3 = 0 \\ 3y_2 + y_3 = 0 \end{cases} \Rightarrow \begin{cases} y_1 = 0 \\ 3y_2 + y_3 = 0 \end{cases}$$

Let, $y_2 = s$, where, $s \in \mathbb{R}$, $s \neq 0$ $\Rightarrow$ $y_3 = -3s$

Thus, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ -3s \end{pmatrix}$ $\Rightarrow$ For $s = 1$, we get, $Y = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0\mathbf{i} + \mathbf{j} - 3\mathbf{k} = \mathbf{j} - 3\mathbf{k}$

For $\lambda_3 = 11\alpha$: Let $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ be the required eigenvector corresponding to eigenvalue $\lambda_3 = 11\alpha$, then

$$([\mathbf{I}_A] - \lambda_3[I_3])Z = \mathbf{0} \Rightarrow \begin{pmatrix} 56\alpha - 11\alpha & 0 & 0 \\ 0 & 16\alpha - 11\alpha & -15\alpha \\ 0 & -15\alpha & 56\alpha - 11\alpha \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 45\alpha & 0 & 0 \\ 0 & 5\alpha & -15\alpha \\ 0 & -15\alpha & 45\alpha \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -45\alpha z_1 = 0 \\ 5\alpha z_2 - 15\alpha z_3 = 0 \\ -15\alpha z_2 + 45\alpha z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = 0 \\ z_2 - 3z_3 = 0 \\ z_2 - 3z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = 0 \\ z_2 - 3z_3 = 0 \end{cases}$$

Let, $z_3 = t$, where, $t \in \mathbb{R}$, $t \neq 0$ $\Rightarrow$ $z_2 = 3t$

Thus, $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3t \\ t \end{pmatrix}$ $\Rightarrow$ For $t = 1$, we get, $Z = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = 0\mathbf{i} + 3\mathbf{j} + \mathbf{k} = 3\mathbf{j} + \mathbf{k}$

Principal moment of inertia	Principal axis	Normalized principal axis
$\lambda_1 = 56\alpha$	$X = \mathbf{i}$	$\hat{X} = \mathbf{i}$
$\lambda_2 = 61\alpha$	$Y = \mathbf{j} - 3\mathbf{k}$	$\hat{Y} = (1/\sqrt{10})(\mathbf{j} - 3\mathbf{k})$
$\lambda_3 = 11\alpha$	$Z = 3\mathbf{j} + \mathbf{k}$	$\hat{Z} = (1/\sqrt{10})(3\mathbf{j} + \mathbf{k})$

Definition: Two distributions of matter are said to be "equimomental" if they have the same moment of inertia about any line in spase.

Theorem: Two systems S_1 and S_2 are equimomental if and only if the following three conditions are satisfied,

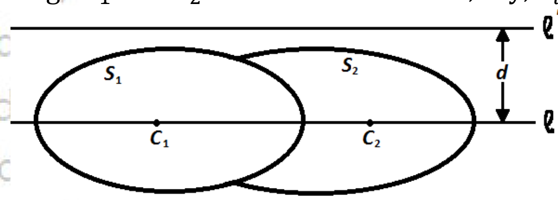
(i) they have same mass,

(ii) they have same centre of mass, and

(iii) they have same principal axes and principal moments of inertia at centre of mass.

Proof: Suppose that two systems S_1 and S_2 are equimomental. We will show that conditions (i), (ii) and (iii) are satisfied.

(i) Let M_1 and M_2, respectively, be the masses of the systems S_1 and S_2 and C_1 and C_2, respectively, be their centres of mass. Since the systems are supposed to be equimomental, therefore their moments of inertia about any line should be same. In particular, their moments of inertia about line l through C_1 and C_2 should also be same, say, I_l. Let l' be any line parallel to l and d be the perpendicular distance between l and l'. Further suppose that $I_{l'}$ be the common moment of inertia of both systems about line l'.



By parallel axis theorem, we have,

$$I_{l'} = I_l + M_1 d^2 \quad \text{(for system } S_1) \longrightarrow (1)$$
$$I_{l'} = I_l + M_2 d^2 \quad \text{(for system } S_2) \longrightarrow (2)$$

From equations (1) and (2), we have,

$$I_l + M_1 d^2 = I_l + M_2 d^2 \quad \Rightarrow \quad M_1 = M_2 = M \text{ (say)}$$

$\Rightarrow$ masses of both systems are same $\Rightarrow$ condition (i) is satisfied.

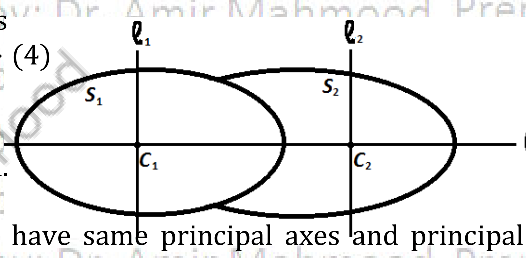
(ii) Now, let l_1 and l_2, respectively, be the lines through C_1 and C_2 and perpendicular to line l. Let common moment of inertia of each system about line l_1 be I_{l_1} and about line l_2 be I_{l_2}.

By parallel axis theorem, moment of inertia of system S_1 about l_2 is

$$I_{l_2} = I_{l_1} + M|C_1C_2|^2 \longrightarrow (3)$$

Again, by parallel axis theorem, moment of inertia of system S_2 about l_2 is

$$I_{l_2} = I_{l_1} - M|C_1C_2|^2 \longrightarrow (4)$$



From equations (3) and (4), we get

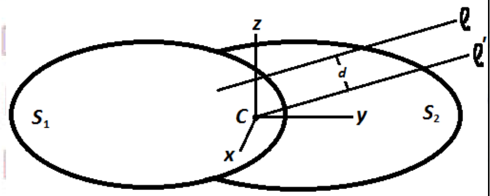
$I_{l_1} + M|C_1C_2|^2 = I_{l_1} - M|C_1C_2|^2 \;\Rightarrow\; |C_1C_2| = 0 \;\Rightarrow\; C_1 \equiv C_2 \equiv C$ (say)

$\Rightarrow$ centres of mass of both systems are same $\Rightarrow$ condition (ii) is satisfied.

(iii) Since both system have same centre of mass C and same mass M, Therefore, they both have same momental ellipsoid at C. Hence, they have same principal axes and principal moments of inertia at centre of mass C. $\Rightarrow$ condition (iii) is satisfied.

Conversely, suppose that for two systems S_1 and S_2, conditions (i), (ii) and (iii) are satisfied. We will show that both systems are equimomental.

Let C and M, respectively, be the common centre of mass and common mass of both systems. Further let that I_1, I_2 and I_3 be the common principal moments of inertia about common principal axes at centre of mass C. In figure, common principal axes at C are shown by Cartesian coordinate system $Cxyz$.



Let l be an arbitrary line in space. Draw a line l' through C parallel to l. Then the moment of inertia of each system about l' is given by

$$I_{l'} = I_1\lambda^2 + I_2\mu^2 + I_3\nu^2,$$

where, λ, μ and ν are direction cosines of line l'. Now, by using parallel axis theorem, the moment of inertia of each system about line l is given by

$$I_l = I_{l'} + Md^2 = I_1\lambda^2 + I_2\mu^2 + I_3\nu^2 + Md^2,$$

where, d is the perpendicular distance between lines l and l'. Since the moment of inertia of both system about an arbitrary line l in space is same. This shows that both systems S_1 and S_2 are equimomental.

Problem: Show that a hoop of mass m and radius $a/\sqrt{2}$ is equimomental with a circular plate of mass m radius a.

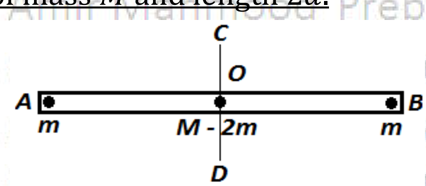
Proof: The moment of inertia of a circular hoop (or ring) of mass m and radius $a/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $\quad I_1 = m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{1}{2}ma^2.$

The moment of inertia of a circular plate (or disc) of mass m and radius a about an axis through its centre and perpendicular to its plane is $\quad I_2 = \frac{1}{2}ma^2.$

Since, both moments of inertia are same. Therefore both systems are equimomental.

Problem: Find the equimomental system of particles for a uniform rod AB of mass M and length $2a$.

Solution: Let O be the centre of mass of the rod AB having mass M. If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass m, are placed at end points A and B of the rod and third particle of mass $M - 2m$ is placed at its centre of mass O, then it is clear that,



(i) mass of both systems is equal to M,

(ii) centre of mass of both systems is same (i.e., point O),
(iii) symmetry axes (and hence, principal axes) of both systems are also same at centre of mass O.
Moment of inertia of the rod about an axis CD through O and perpendicular to the rod is given by

$$I_1 = \frac{1}{12}M(2a)^2 = \frac{1}{3}Ma^2$$

Moment of inertia of the system of particles about axis CD is given by

$$I_2 = ma^2 + 0 + ma^2 = 2ma^2$$

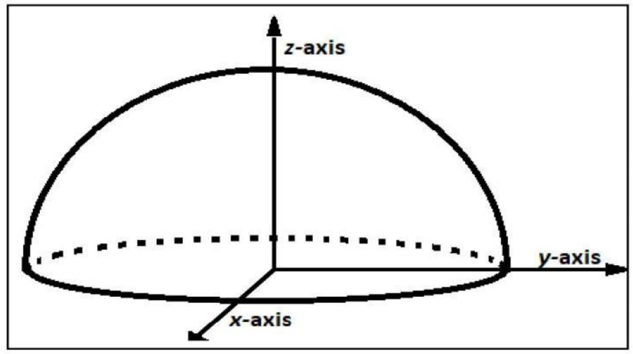
The two systems will equimomental if

$$I_1 = I_2 \quad \Rightarrow \quad \frac{1}{3}Ma^2 = 2ma^2 \quad \Rightarrow \quad m = \frac{1}{6}M$$

Hence, equimomental system of particles is given by first particle of mass M at A, second particle of mass $M - 2m$ at O and third particle of mass M at B.

Problem: ***Find the (direction of) principal axes and principal moments of inertia*** of a (uniform) solid hemisphere of mass M at centre of its base.

Solution: **Moment of inertia about axis of symmetry:**



Let M, a and ρ, respectively, be the mass, base radius and volume mass density of the hemisphere. Choose coordinate axes as shown in figure.
Moment of inertia of typical volume element of hemisphere, having mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of hemisphere about z-axis is

$$I_{zz} = \int_{\text{Hemisphere}} (x^2 + y^2)\mathrm{d}m = \rho \int_{\text{Hemisphere}} (x^2 + y^2)\,\mathrm{d}V \qquad \boxed{\because \;\; \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \text{constant}}$$

$$= \frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} (x^2 + y^2)\,\mathrm{d}V \qquad \boxed{\because \;\; \rho = \tfrac{M}{(2/3)\pi a^3} \quad \text{(for hemisphere)}}$$

To make the computation simpler, we transform the problem from Cartesian coordinates $(x,\, y,\, z)$ to spherical coordinates $(r,\, \theta,\, \phi)$ by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

where, volume element in spherical coordinates is given by

$$\mathrm{d}V = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\, d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$\Rightarrow \quad x^2 + y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\sin^2\theta$$

For hemisphere, $\qquad 0 \le r \le a, \qquad 0 \le \theta \le \pi/2, \qquad 0 \le \phi < 2\pi$

$$\Rightarrow \quad I_{zz} = \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r \int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta \int_{\phi=0}^{2\pi}\mathrm{d}\phi \tag{1}$$

where, $\qquad \displaystyle\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2}(3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi/2} = \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{2}{3} \tag{2}$$

Using (2) in (1), we get

$$I_{zz} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

Moment of inertia about diameter of the base:

$$I_{xx} = \int_{\text{Hemisphere}} (y^2 + z^2)\mathrm{d}m = \frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} (y^2 + z^2)\,\mathrm{d}V$$

Transforming problem in spherical coordinates $(r,\, \theta,\, \phi)$, we get

$$I_{xx} = \frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4(\sin^3\theta\sin^2\phi + \cos^2\theta\sin\theta)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\left(\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi + \int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi\right), \tag{3}$$

where,

$$\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1 - \cos 2\phi)\,\mathrm{d}\phi = \frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi \tag{4}$$

and

$$\int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta \,\mathrm{d}\theta = -\frac{1}{3}\cos^3\theta\Big|_{\theta=0}^{\pi/2} = \frac{1}{3} \tag{5}$$

Using (2), (4) and (5) in (3), we get

$$I_{xx} = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right) = \frac{3M}{2\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4\pi}{3}\right) = \frac{2}{5}Ma^2$$

Products of inertia:

$$I_{xy} = -\int_{\text{Hemisphere}} x\,y\,\mathrm{d}m = -\frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} x\,y\,\mathrm{d}V = -\frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = 0 \qquad \boxed{\because\ \int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \tfrac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0}$$

Now,

$$I_{xz} = -\int_{\text{Hemisphere}} x\,z\,\mathrm{d}m = -\frac{3M}{2\pi a^3}\int_{\text{Hemisphere}} x\,z\,\mathrm{d}V = -\frac{3M}{2\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\sin^2\theta\cos\theta\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{3M}{2\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi/2}\sin^2\theta\cos\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi = 0 \qquad \boxed{\because\ \int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi = \sin\phi\Big|_{\phi=0}^{2\pi} = 0}$$

Thus,

$$I_{xy} = I_{xz} = I_{yz} = 0, \qquad \boxed{\because\ I_{xz} = I_{yz}\ \text{(by symmetry)}}$$

The inertia matrix with respect to coordinate system $Oxyz$ is given by

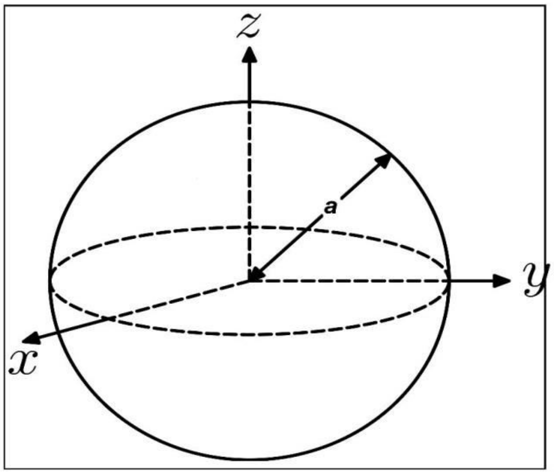
$$[I_O] = \begin{pmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix}$$

Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}Ma^2$ are principal moments of inertia.

Problem: ***Find the (direction of) principal axes and principal moments of inertia*** of a (uniform) solid sphere of mass M at its centre.

Solution: (i) **Moment of inertia about axis of symmetry:**

Let M, a and ρ, respectively, be the mass, radius and volume mass density of the sphere. Choose coordinate axes as shown in figure. Moment of inertia of typical volume element of sphere, having mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by



$$\mathrm{d}I_{zz} = (x^2+y^2)\mathrm{d}m$$

Thus, moment of inertia of sphere about z-axis is

$$I_{zz} = \int_{\text{Sphere}} (x^2+y^2)\mathrm{d}m$$

$$= \rho\int_{\text{Sphere}} (x^2+y^2)\,\mathrm{d}V \qquad \boxed{\because\ \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \text{constant}}$$

$$= \frac{3M}{4\pi a^3}\int_{\text{Sphere}} (x^2+y^2)\,\mathrm{d}V \qquad \boxed{\because\ \rho = \tfrac{M}{(4/3)\pi a^3}\quad \text{(for sphere)}}$$

To make the computation simpler, we transform the problem from Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) by using

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

where, volume element in spherical coordinates is given by

$$\mathrm{d}V = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$\Rightarrow \quad x^2+y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = r^2\sin^2\theta$$

For sphere, $\quad 0 \le r \le a, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi$

$$\Rightarrow \quad I_{zz} = \frac{3M}{4\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3M}{4\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi,$$

where,

$$\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3}$$

Thus,

$$I_{zz} = \frac{3M}{4\pi a^3}\left(\frac{a^5}{5}\right)\left(\frac{4}{3}\right)(2\pi) = \frac{2}{5}Ma^2$$

Similarly,

$$I_{xx} = I_{yy} = \frac{2}{5}Ma^2 \qquad \boxed{\because \quad I_{xx} = I_{yy} = I_{zz} \ \text{(by symmetry)}}$$

(ii) <u>Products of inertia with respect to axes of symmetry:</u>

$$I_{xy} = -\int_{\text{Sphere}} x\,y\,\mathrm{d}m = -\frac{3M}{4\pi a^3}\int_{\text{Sphere}} x\,y\,\mathrm{d}V = -\frac{3M}{4\pi a^3}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\sin\phi\cos\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{3M}{4\pi a^3}\int_{r=0}^{a} r^4\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = 0 \qquad \boxed{\because \quad \int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \tfrac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0}$$

Similarly,

$$I_{yz} = I_{xz} = 0 \qquad \boxed{\because \quad I_{xy} = I_{yz} = I_{xz} \ \text{(by symmetry)}}$$

The inertia matrix with respect to coordinate system $Oxyz$ is given by

$$[I_O] = \begin{pmatrix} \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix}$$

Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}Ma^2$ are principal moments of inertia.

Problem: ***Find the (direction of) principal axes and principal moments of inertia*** of a (uniform) solid ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ of mass M at its centre.

Solution: (i) <u>Moments of inertia about axis of symmetry:</u>

Let M and ρ, respectively, be the mass and volume mass density of the ellipsoid. Choose coordinate axes as shown in figure.

Moment of inertia of typical volume element of ellipsoid, with mass $\mathrm{d}m$ and volume $\mathrm{d}V$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2+y^2)\mathrm{d}m$$

Thus, moment of inertia of ellipsoid about z-axis is

$$I_{zz} = \int_{\text{Ellipsoid}}(x^2+y^2)\mathrm{d}m$$

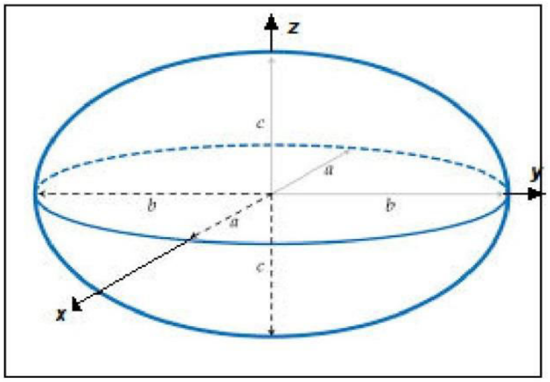
$$= \rho\int_{\text{Ellipsoid}}(x^2+y^2)\,\mathrm{d}V \qquad \boxed{\because \quad \rho = \tfrac{\mathrm{d}m}{\mathrm{d}V} = \text{constant}}$$

$$= \frac{3M}{4\pi a\,b\,c}\int_{\text{Ellipsoid}}(x^2+y^2)\,\mathrm{d}V \qquad \boxed{\because \quad \rho = \tfrac{M}{(4/3)\pi a\,b\,c} \quad \text{(for ellipsoid)}}$$

Let us substitute

$$x/a = x', \quad y/b = y', \quad z/c = z'$$

$$\Rightarrow \quad \mathrm{d}x/a = \mathrm{d}x', \quad \mathrm{d}y/b = \mathrm{d}y', \quad \mathrm{d}z/c = \mathrm{d}z', \qquad \mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = a\,b\,c\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$



Under the above transformation, the given ellipsoid is transformed into the unit sphere

$$S : x'^2 + y'^2 + z'^2 = 1.$$

$$\Rightarrow \quad I_{zz} = \frac{3M}{4\pi a\,b\,c}\int_S (a^2x'^2 + b^2y'^2)(a\,b\,c\,dx'\,dy'\,dz') = \frac{3M}{4\pi}\int_S (a^2x'^2 + b^2y'^2)dV', \quad \text{where, } dV' = dx'\,dy'\,dz'$$

$$= \frac{3M(a^2+b^2)}{4\pi}\int_S x'^2 dV' \qquad \boxed{\because \int_S x'^2 dV' = \int_S y'^2 dV' \quad \text{(by symmetry)}}$$

To make the computation simpler, we transform the problem from Cartesian coordinates (x', y', z') to spherical coordinates (r, θ, ϕ) by using

$$x' = r\sin\theta\cos\phi, \qquad y' = r\sin\theta\sin\phi, \qquad z' = r\cos\theta,$$

where, volume element in spherical coordinates is given by, $\quad dV' = dr\,(r\,d\theta)\,(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$

For unit sphere, $\quad 0 \le r \le 1, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\cos^2\phi\,dr\,d\theta\,d\phi = \frac{3M(a^2+b^2)}{4\pi}\int_{r=0}^{1} r^4\,dr\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta\int_{\phi=0}^{2\pi}\cos^2\phi\,d\phi,$$

where, $\displaystyle\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) = \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3}$

and $\displaystyle\int_{\phi=0}^{2\pi}\cos^2\phi\,d\phi = \frac{1}{2}\int_{\phi=0}^{2\pi}(1+\cos 2\phi)\,d\phi = \frac{1}{2}\left(\phi + \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi$

$$\Rightarrow \quad I_{zz} = \frac{3M(a^2+b^2)}{4\pi}\left(\frac{1}{5}\right)\left(\frac{4}{3}\right)(\pi) = \frac{1}{5}M(a^2+b^2)$$

Similarly, $\quad I_{xx} = \frac{1}{5}M(b^2+c^2) \quad$ and $\quad I_{yy} = \frac{1}{5}M(a^2+c^2)$

(*ii*) <u>Products of inertia with respect to axes of symmetry:</u>

$$I_{xy} = -\int_{\text{Ellipsoid}} x\,y\,dm = -\frac{3M}{4\pi a\,b\,c}\int_{\text{Ellipsoid}} x\,y\,dV = -\frac{3M}{4\pi a\,b\,c}\int_{S}(a\,b\,x'\,y')(a\,b\,c\,dx'\,dy'\,dz')$$

$$= -\frac{3a\,b\,M}{4\pi}\int_S x'\,y'dV' = -\frac{3\,a\,b\,M}{4\pi}\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin^3\theta\sin\phi\cos\phi\,dr\,d\theta\,d\phi$$

$$= -\frac{3\,a\,b\,M}{4\pi}\int_{r=0}^{1} r^4\,dr\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,d\phi = 0 \qquad \boxed{\because \int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,d\phi = \tfrac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0}$$

Similarly, it is not difficult to show that

$$I_{yz} = I_{xz} = 0$$

The inertia matrix with respect to coordinate system $Oxyz$ is given by

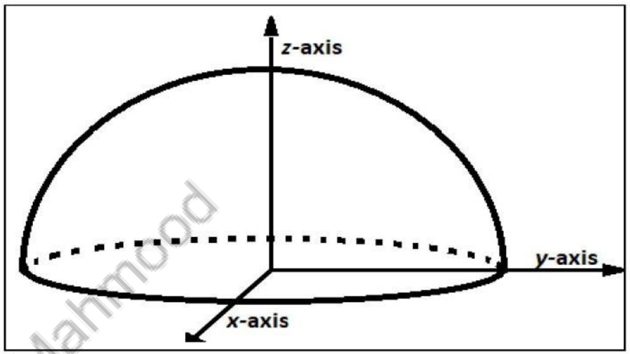
$$[I_O] = \begin{pmatrix} \frac{1}{5}M(b^2+c^2) & 0 & 0 \\ 0 & \frac{1}{5}M(a^2+c^2) & 0 \\ 0 & 0 & \frac{1}{5}M(a^2+b^2) \end{pmatrix}$$

Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia I_{xx}, I_{yy} and I_{zz} are principal moments of inertia.

Problem: ***Find the (direction of) principal axes and principal moments of inertia*** of a (uniform) hemispherical shell (hollow hemisphere) of mass M at centre of its base.

Solution: (i) **Moment of inertia about axis of symmetry:**

Let M, a and σ, respectively, be the mass, radius of base and areal mass density of the hemispherical shell. Choose coordinate axes as shown in figure.



Moment of inertia of typical area element of hemispherical shell, with mass dm and area dS, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2+y^2)\mathrm{d}m$$

Thus, moment of inertia of hemisphere about z-axis is

$$\begin{aligned} I_{zz} &= \int_S (x^2+y^2)\mathrm{d}m, \qquad S:\ \text{hemispherical shell} \\ &= \sigma \int_S (x^2+y^2)\,\mathrm{d}S \qquad \boxed{\because\ \ \sigma = \tfrac{\mathrm{d}m}{\mathrm{d}S} = \text{constant}} \\ &= \frac{M}{2\pi a^2}\int_S (x^2+y^2)\,\mathrm{d}S \qquad \boxed{\because\ \ \sigma = \tfrac{M}{2\pi a^2}\ \ \text{(for hemispherical shell)}} \end{aligned}$$

To make the computation simpler, we use the parametric equations of hemispherical shell as follows

$$x = a\sin\theta\cos\phi, \qquad y = a\sin\theta\sin\phi, \qquad z = a\cos\theta$$

$$\text{For hemispherical shell}: \quad 0 \le \theta \le \pi/2, \qquad 0 \le \phi < 2\pi$$

$$\mathrm{d}S = (a\,\mathrm{d}\theta)\,(a\sin\theta\,d\phi) = a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2+y^2 = a^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = a^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = a^2\sin^2\theta$$

$$\Rightarrow \quad I_{zz} = \frac{M}{2\pi a^2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^4\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{Ma^2}{2\pi}\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi, \tag{6}$$

where,

$$\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi/2}(3\sin\theta - \sin 3\theta) \qquad \boxed{\because\ \ \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi/2} = \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{2}{3} \tag{7}$$

Using (7) in (6), we get

$$I_{zz} = \frac{Ma^2}{2\pi}\left(\frac{2}{3}\right)(2\pi) = \frac{2}{3}Ma^2$$

(ii) **Moment of inertia about a diameter of the base:**

$$I_{xx} = \int_S (y^2+z^2)\mathrm{d}m = \sigma\int_S (y^2+z^2)\mathrm{d}S = \frac{M}{2\pi a^2}\int_S (y^2+z^2)\,\mathrm{d}S, \qquad S:\ \text{hemispherical shell}$$

Using parametric equations of hemispherical shell, we get

$$\begin{aligned} I_{xx} &= \frac{M}{2\pi a^2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^4(\sin^3\theta\sin^2\phi + \cos^2\theta\sin\theta)\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{Ma^2}{2\pi}\left(\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin^2\phi\,\mathrm{d}\phi + \int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi\right) \end{aligned} \tag{8}$$

where,

$$\int_{\phi=0}^{2\pi} \sin^2\phi \, \mathrm{d}\phi = \frac{1}{2}\int_{\phi=0}^{2\pi} (1-\cos 2\phi)\, \mathrm{d}\phi = \frac{1}{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_{\phi=0}^{2\pi} = \frac{1}{2}(2\pi) = \pi \tag{9}$$

and

$$\int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta \, \mathrm{d}\theta = -\frac{1}{3}\cos^3\theta\Big|_{\theta=0}^{\pi/2} = \frac{1}{3} \tag{10}$$

Using (7), (9) and (10) in (8), we get

$$I_{xx} = \frac{Ma^2}{2\pi}\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{Ma^2}{2\pi}\left(\frac{4\pi}{3}\right) = \frac{2}{3}Ma^2$$

(*iii*) **Products of inertia:**

$$I_{xy} = -\int_S x\,y\,\mathrm{d}m = -\sigma\int_S x\,y\,\mathrm{d}S = -\frac{M}{2\pi a^2}\int_S x\,y\,\mathrm{d}S, \quad S: \text{ hemispherical shell}$$

Using parametric equations of hemispherical shell, we get

$$I_{xy} = -\frac{M}{2\pi a^2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^4\sin^3\theta\sin\phi\cos\phi\,\mathrm{d}\theta\,\mathrm{d}\phi = -\frac{Ma^2}{2\pi}\int_{\theta=0}^{\pi/2}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \quad \Longrightarrow \quad I_{xy} = 0$$

Now,

$$\begin{aligned} I_{xz} &= -\int_S x\,z\,\mathrm{d}m = -\frac{M}{2\pi a^2}\int_S x\,z\,\mathrm{d}S = -\frac{M}{2\pi a^2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^4\sin^2\theta\cos\theta\cos\phi\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -\frac{Ma^2}{2\pi}\int_{\theta=0}^{\pi/2}\sin^2\theta\cos\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi \end{aligned}$$

But

$$\int_{\phi=0}^{2\pi}\cos\phi\,\mathrm{d}\phi = \sin\phi\Big|_{\phi=0}^{2\pi} = 0 \quad \Longrightarrow \quad I_{xz} = 0 = I_{yz}, \qquad \boxed{\because \; I_{xz} = I_{yz} \text{ (by symmetry)}}$$

Thus,

$$I_{xy} = I_{xz} = I_{yz} = 0$$

The inertia matrix with respect to coordinate system $Oxyz$ is given by

$$[I_O] = \begin{pmatrix} \frac{2}{3}Ma^2 & 0 & 0 \\ 0 & \frac{2}{3}Ma^2 & 0 \\ 0 & 0 & \frac{2}{3}Ma^2 \end{pmatrix}$$

Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx} = I_{yy} = I_{zz} = \frac{2}{3}Ma^2$ are principal moments of inertia.

Problem: ***Find the (direction of) principal axes and principal moments of inertia*** of a (uniform) spherical shell (hollow sphere) of mass M at its centre.

Solution: (*i*) **Moment of inertia about axis of symmetry:**

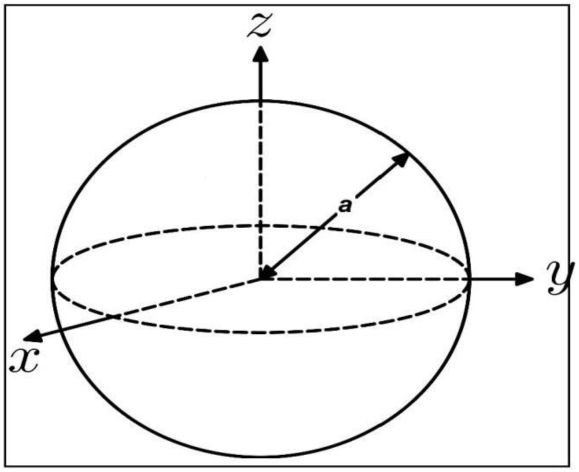
Let M, a and σ, respectively, be the mass, radius of base and areal mass density of the spherical shell. Choose coordinate axes as shown in figure.

Moment of inertia of typical area element of spherical shell, with mass $\mathrm{d}m$ and area $\mathrm{d}S$, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of spherical shell about z-axis is

$$\begin{aligned} I_{zz} &= \int_S (x^2+y^2)\mathrm{d}m, \qquad S: \text{ spherical shell} \\ &= \sigma\int_S (x^2+y^2)\,\mathrm{d}S \qquad \boxed{\because \; \sigma = \tfrac{\mathrm{d}m}{\mathrm{d}S} = \text{constant}} \\ &= \frac{M}{4\pi a^2}\int_S (x^2+y^2)\,\mathrm{d}S \qquad \boxed{\because \; \sigma = \tfrac{M}{4\pi a^2} \text{ (for spherical shell)}} \end{aligned}$$



To make the computation simpler, we use the parametric equations of spherical shell as follows

$$x = a\sin\theta\cos\phi, \qquad y = a\sin\theta\sin\phi, \qquad z = a\cos\theta$$

$$\text{For spherical shell}: \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi$$

$$\mathrm{d}S = (a\,\mathrm{d}\theta)\,(a\sin\theta\, d\phi) = a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$x^2+y^2 = a^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = a^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = a^2\sin^2\theta$$

$$\Rightarrow \quad I_{zz} = \frac{M}{4\pi a^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a^4\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{Ma^2}{4\pi}\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi, \tag{11}$$

where,

$$\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\theta=0}^{\pi}(3\sin\theta - \sin 3\theta) \qquad \boxed{\because \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

$$= \frac{1}{4}\left(-3\cos\theta + \frac{1}{3}\cos 3\theta\right)\Big|_{\theta=0}^{\pi} = \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3} \tag{12}$$

Using (12) in (11), we get

$$I_{zz} = \frac{Ma^2}{4\pi}\left(\frac{4}{3}\right)(2\pi) = \frac{2}{3}Ma^2$$

(ii) **Products of inertia with respect to axes of symmetry:**

$$I_{xy} = -\int_S x\,y\,\mathrm{d}m = -\sigma\int_S x\,y\,\mathrm{d}S = -\frac{M}{4\pi a^2}\int_S x\,y\,\mathrm{d}S, \qquad S: \text{ spherical shell}$$

Using parametric equations of spherical shell, we get

$$I_{xy} = -\frac{M}{4\pi a^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a^4\sin^3\theta\sin\phi\cos\phi\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -\frac{Ma^2}{4\pi}\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi}\sin\phi\cos\phi\,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

Similarly,

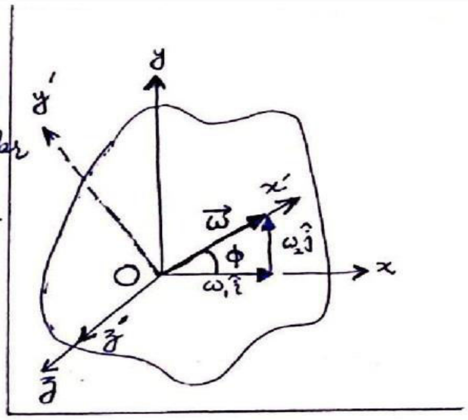
$$I_{yz} = I_{xz} = 0 \qquad \boxed{\because \ I_{xy} = I_{yz} = I_{xz} \ \text{(by symmetry)}}$$

The inertia matrix with respect to coordinate system $Oxyz$ is given by

$$[I_O] = \begin{pmatrix} \frac{2}{3}Ma^2 & 0 & 0 \\ 0 & \frac{2}{3}Ma^2 & 0 \\ 0 & 0 & \frac{2}{3}Ma^2 \end{pmatrix}$$

Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx} = I_{yy} = I_{zz} = \frac{2}{3}Ma^2$ are principal moments of inertia.

Problem: Determine principal axes of a plane rigid body (lamina) at a point in its plane, whose one principal axis perpendicular to its plane is known.



Solution: Consider a plane rigid body in xy-plane, with z-axis as known principal axis. The other two principal axes will lie in plane of body.

As body is in xy-plane, therefore $I_{13} = I_{23} = 0$. Furthermore, $I_{12} \neq 0$.

As we know that if I is principal moment of inertia, then

$$\begin{cases}(I_{11}-I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0\\ I_{12}\omega_1 + (I_{22}-I)\omega_2 + I_{23}\omega_3 = 0\\ I_{13}\omega_1 + I_{23}\omega_2 + (I_{33}-I)\omega_3 = 0\end{cases}$$

where, $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$ is the angular velocity in the direction of principle axis. Put $I_{13} = I_{23} = 0$ in above system, we get

$$\begin{cases}(I_{11}-I)\omega_1 + I_{12}\omega_2 = 0\\ I_{12}\omega_1 + (I_{22}-I)\omega_2 = 0\\ (I_{33}-I)\omega_3 = 0\end{cases}$$

let Ox' and Oy' be the principal axes in xy-plane at "O". let ϕ be the angle between Ox and Ox'. If we take $I = I_1$ as principal moment of inertia w.r.t principal axis Ox', then above system gives

$$\begin{cases}(I_{11}-I_1)\omega_1 + I_{12}\omega_2 = 0 \quad \text{——} (1)\\ I_{12}\omega_1 + (I_{22}-I_1)\omega_2 = 0 \quad \text{——} (2)\\ \omega_3 = 0 \quad \because I_{33} - I_1 \neq 0 \rightarrow\end{cases}$$

$\left(\because I'_{33} = I'_{11} + I'_{22} \text{ (for plane body)} \Rightarrow I_{33} = I_1 + I_2, \text{ where } I_1 \neq 0, I_2 \neq 0\right)$

Thus, $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j}$ is the angular velocity in the direction of Ox' (i.e; x'-axis), therefore we have

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\omega_2}{\omega_1} \Rightarrow \frac{\omega_1}{\cos\phi} = \frac{\omega_2}{\sin\phi} = k \text{ (say)}$$

$$\Rightarrow \omega_1 = k\cos\phi \quad \text{——} (3) \quad \text{and} \quad \omega_2 = k\sin\phi \quad \text{——} (4)$$

Using (3) and (4) in (1) and (2), we get

$$\begin{cases}(I_{11}-I_1)\cos\phi + I_{12}\sin\phi = 0 \quad \text{——} (5)\\ I_{12}\cos\phi + (I_{22}-I_1)\sin\phi = 0 \quad \text{——} (6)\end{cases}$$

From equations ⑤ and ⑥, we get

$$I_{11} - I_1 = -I_{12}\frac{\sin\phi}{\cos\phi} \quad \text{—— ⑦}$$

$$I_{22} - I_1 = -I_{12}\frac{\cos\phi}{\sin\phi} \quad \text{—— ⑧}$$

Subtracting ⑦ and ⑧, we get

$$I_{11} - I_{22} = I_{12}\left(-\frac{\sin\phi}{\cos\phi} + \frac{\cos\phi}{\sin\phi}\right)$$

$$= I_{12}\left(\frac{-\sin^2\phi + \cos^2\phi}{\sin\phi\cos\phi}\right) = +2I_{12}\left(\frac{\cos^2\phi - \sin^2\phi}{2\sin\phi\cos\phi}\right)$$

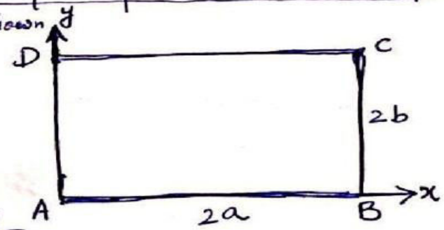
$$= +2I_{12}\frac{\cos 2\phi}{\sin 2\phi} = 2I_{12}\cot 2\phi$$

$$\Rightarrow \cot 2\phi = \frac{I_{11} - I_{22}}{2I_{12}} \Rightarrow \boxed{\tan 2\phi = \frac{2I_{12}}{I_{11} - I_{22}}} \quad \text{—— ⑨}$$

Equation ⑨ gives two values of ϕ between 0° and 180°, which are the directions of principal axes.

Problem: A uniform rectangular lamina (or thin plate) ABCD with sides of lengths $2a, 2b$, $(b > a)$. Find the direction of principal axes at corner A.

Solution: We take x-axis and y-axis as shown in the figure.



Moment of inertia about side AB is

$$I_{11} = I_{AB} = \int y^2 dm = \sigma\int_{x=0}^{2a}\int_{y=0}^{2b} y^2\, dy\, dx$$

$$= \sigma\int_{x=0}^{2a} dx \int_{y=0}^{2b} y^2 dy = \frac{M}{(2a)(2b)}(2a)\frac{(2b)^3}{3} \qquad \because \sigma = \frac{M}{(2a)(2b)}$$

$$= \frac{4}{3}Mb^2$$

Similarly $I_{22} = I_{AD} = \frac{4}{3}Ma^2$

And $I_{12} = -\int xy\, dm = -\sigma\int_{x=0}^{2a}\int_{y=0}^{2b} xy\, dy\, dx = -\frac{M}{(2a)(2b)}\int_{x=0}^{2a} x\,dx \int_{y=0}^{2b} y\,dy$

$$= -\frac{M}{4ab}\cdot\frac{(2a)^2}{2}\cdot\frac{(2b)^2}{2} = -Mab$$

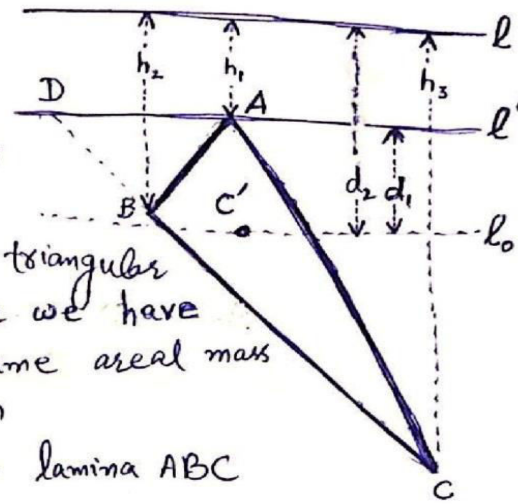
Thus, the direction of principal axis at vertex A is given by

$$\tan 2\phi = \frac{2I_{12}}{I_{11} - I_{22}} = \frac{-2Mab}{\frac{4}{3}Mb^2 - \frac{4}{3}Ma^2} = \frac{3ab}{2(a^2 - b^2)}$$

Problem: Determine an equimomental system for a uniform triangular lamina of mass M.

Solution: Let ABC be a triangular lamina. Let h_1, h_2 and h_3 respectively be the distances of the vertices A, B and C from a given line "ℓ", such that $h_1 < h_2 < h_3$. Draw a line ℓ' parallel to ℓ

and passing through vertex A. Suppose that the side **CB**, when produced, meets line l' at point D. The distances of A, B and C from l' are 0, $h_2 - h_1$ and $h_3 - h_1$, respectively.



If M_1 and M_2 denote the masses of triangular laminas ACD and ABD, respectively (where we have assumed that laminas ACD and ABD have same areal mass density (σ) as that of lamina ABC), then

$$M = \text{mass of triangular lamina ABC}$$
$$= M_1 - M_2$$

But
$$\frac{M_1}{M_2} = \frac{(\text{Areal density})(\text{Area of } \Delta ACD)}{(\text{Areal density})(\text{Area of } \Delta ABD)} = \frac{\cancel{\sigma}\left(\cancel{\frac{1}{2}}\times\cancel{|AD|}\times(h_3-h_1)\right)}{\cancel{\sigma}\left(\cancel{\frac{1}{2}}\times\cancel{|AD|}\times(h_2-h_1)\right)}$$

$$= \frac{h_3-h_1}{h_2-h_1}$$

Therefore,
$$\frac{M_1}{M_2} - 1 = \frac{h_3-h_1}{h_2-h_1} - 1$$

$$\Rightarrow \frac{M_1-M_2}{M_2} = \frac{M}{M_2} = \frac{h_3-h_2}{h_2-h_1}$$

$$\Rightarrow M_2 = \frac{M(h_2-h_1)}{h_3-h_2} \quad \text{and} \quad M_1 = M + M_2 = M + \frac{M(h_2-h_1)}{h_3-h_2} = \frac{M(h_3-h_1)}{h_3-h_2}$$

Now, moment of inertia ($I_{l'}$) of lamina ABC about the line l' is given by

$$I_{l'} = \begin{pmatrix}\text{Moment of inertia of lamina ACD} \\ \text{about line } l'\end{pmatrix} - \begin{pmatrix}\text{Moment of inertia of lamina ABD} \\ \text{about line } l'\end{pmatrix}$$

$$= \frac{1}{6}M_1(h_3-h_1)^2 - \frac{1}{6}M_2(h_2-h_1)^2$$

$$= \frac{1}{6}M\left(\frac{h_3-h_1}{h_3-h_2}\right)(h_3-h_1)^2 - \frac{1}{6}M\left(\frac{h_2-h_1}{h_3-h_2}\right)(h_2-h_1)^2$$

$$= \frac{M}{6(h_3-h_2)}\left[(h_3-h_1)^3 - (h_2-h_1)^3\right]$$

$$= \frac{M}{6\cancel{(h_3-h_2)}}\cancel{\left[(h_3-h_1)-(h_2-h_1)\right]}\left\{(h_3-h_1)^2 + (h_3-h_1)(h_2-h_1) + (h_2-h_1)^2\right\} \quad \because a^3-b^3 = (a-b)(a^2+ab+b^2)$$

$$= \frac{M}{6}\left(h_3^2 + h_1^2 - 2h_1h_3 + h_2h_3 - h_1h_3 - h_1h_2 + h_1^2 + h_2^2 + h_1^2 - 2h_1h_2\right)$$

$$= \frac{M}{6}\left(3h_1^2 + h_2^2 + h_3^2 - 3h_1h_2 + h_2h_3 - 3h_1h_3\right) \quad \text{—— (1)}$$

Let l_0 be the line parallel to l' (and l) and passing through centre of mass C' of lamina ABC. Then, (by parallel axis theorem) moment of inertia I_0 of lamina ABC about line l_0 is given by

$$I_0 = I_{l'} - Md_1^2 \quad (2)$$

where, d_1 is the distance of centre of mass C' from line l'

Here $d_1 = \frac{1}{3}\left[\left(\begin{matrix}\text{distance of vertex A}\\ \text{from line } l'\end{matrix}\right)+\left(\begin{matrix}\text{distance of vertex B}\\ \text{from line } l'\end{matrix}\right)+\left(\begin{matrix}\text{distance of vertex C}\\ \text{from line } l'\end{matrix}\right)\right]$

$$= \frac{1}{3}\left[0+(h_2-h_1)+(h_3-h_1)\right] = \frac{1}{3}(h_2+h_3-2h_1) \quad (3)$$

Using (1) and (3) in (2), we get

$$I_0 = \frac{M}{6}(3h_1^2+h_2^2+h_3^2-3h_1h_2+h_2h_3-3h_1h_3) - \frac{M}{9}(h_2+h_3-2h_1)^2$$

$$I_0 = \frac{M}{6}(3h_1^2+h_2^2+h_3^2-3h_1h_2+h_2h_3-3h_1h_3) - \frac{M}{9}(h_2^2+h_3^2+4h_1^2+2h_2h_3-4h_1h_3-4h_1h_2) \quad (4)$$

Again, by parallel axis theorem, the moment of inertia (I_l) of lamina ABC about line l is given by

$$I_l = I_0 + Md_2^2 \quad (5)$$

where, d_2 is the distance of centre of mass C' from line l.

Here $d_2 = \frac{1}{3}(h_1+h_2+h_3) \quad (6)$

Using (4) and (6) in (5), we get

$$I_l = \frac{M}{6}(3h_1^2+h_2^2+h_3^2-3h_1h_2+h_2h_3-3h_1h_3) - \frac{M}{9}(h_2^2+h_3^2+4h_1^2+2h_2h_3-4h_1h_3-4h_1h_2) + \frac{M}{9}(h_1^2+h_2^2+h_3^2+2h_1h_2+2h_2h_3+2h_1h_3)$$

$$= \frac{M}{18}\left[9h_1^2+3h_2^2+3h_3^2-9h_1h_2+3h_2h_3-9h_1h_3-\cancel{2h_2^2}-\cancel{2h_3^2}-8h_1^2-\cancel{4h_2h_3}+8h_1h_3+8h_1h_2+2h_1^2+\cancel{2h_2^2}+\cancel{2h_3^2}+4h_1h_2+\cancel{4h_2h_3}+4h_1h_3\right]$$

$$= \frac{M}{18}\left[3h_1^2+3h_2^2+3h_3^2+3h_1h_2+3h_2h_3+3h_1h_3\right]$$

$$= \frac{M}{6}\left[h_1^2+h_2^2+h_3^2+h_1h_2+h_2h_3+h_1h_3\right] = \frac{M}{12}\left[2h_1^2+2h_2^2+2h_3^2+2h_1h_2+2h_2h_3+2h_1h_3\right]$$

$$= \frac{M}{12}\left[(h_1^2+h_2^2+2h_1h_2)+(h_2^2+h_3^2+2h_2h_3)+(h_1^2+h_3^2+2h_1h_3)\right] = \frac{M}{12}\left[(h_1+h_2)^2+(h_2+h_3)^2+(h_1+h_3)^2\right]$$

$$= \frac{M}{3}\left[\left(\frac{h_1+h_2}{2}\right)^2+\left(\frac{h_2+h_3}{2}\right)^2+\left(\frac{h_1+h_3}{2}\right)^2\right],$$

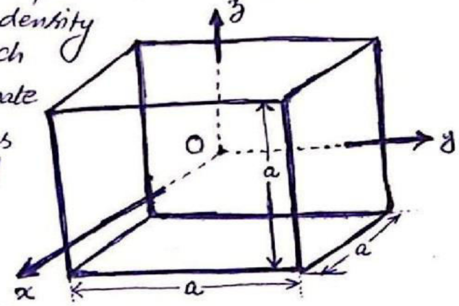
where, the terms on R.H.S can be interpreted as moment of inertia of three particles each of mass $M/3$ at distances $\frac{h_1+h_2}{2}$, $\frac{h_2+h_3}{2}$ and $\frac{h_1+h_3}{2}$ from the line "l". This shows that equimomental system of triangular lamina consists of three particles, each of mass $M/3$, placed at the mid-points of the sides of the triangular lamina.

What is Cube? Ans: Cube = Rectangular box with same dimensions.

Problem: Show that a uniform solid cube of mass M is equimomental with **(a)** masses $M/24$ at midpoints of its edges and $M/2$ at its centre. **(b)** masses $M/24$ at its corners and $2M/3$ at it centre. respectively

Note: This problem is for cube, (not for cuboid)

Solution: Let M and ρ be the mass and density of cube. Choose coordinate system $Oxyz$ such that O lies at centre of mass and coordinate axes are along edges of the cube. ∴ Thus the coordinate axes are symmetry axes (and hence are principal axes) of cube at O.

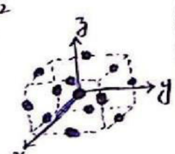


Moment of inertia of cube about x-axis is

$$I_{11} = \int_{cube} (y^2+z^2)\,dm = \rho \int_{z=-\frac{a}{2}}^{a/2}\int_{y=-\frac{a}{2}}^{a/2}\int_{x=-a/2}^{a/2} (y^2+z^2)\,dx\,dy\,dz$$

$$= \frac{M}{a^3}\int_{x=-\frac{a}{2}}^{a/2} dx \int_{z=-\frac{a}{2}}^{a/2}\int_{y=-\frac{a}{2}}^{\frac{a}{2}} (y^2+z^2)\,dy\,dz = \frac{M}{a^3}(a)\int_{z=-\frac{a}{2}}^{\frac{a}{2}} \left(\frac{y^3}{3}+yz^2\right)\Big|_{y=-\frac{a}{2}}^{a/2} dz = \frac{M}{a^2}\int_{z=-\frac{a}{2}}^{\frac{a}{2}} 2\left(\frac{a^3}{24}+\frac{a}{2}z^2\right)dz$$

$$= \frac{M}{a^2}\left(\frac{a^3}{12}z + \frac{a}{3}z^3\right)\Big|_{z=-a/2}^{a/2} = \frac{M}{a^2}\left[2\left(\frac{a^4}{24}+\frac{a^4}{24}\right)\right] = \frac{M}{a^2}\left(\frac{a^4}{12}+\frac{a^4}{12}\right) = \frac{Ma^2}{6}$$



By symmetry, $I_{33} = I_{22} = I_{11} = \frac{Ma^2}{6}$

(a) There are twelve (12) edges of a cube. When masses $M/24$ are placed at the midpoints of these edges and a mass $M/2$ is placed at centre, the total mass of the system of thirteen (13) particles is

$12\times(M/24) + \frac{M}{2} = \frac{M}{2}+\frac{M}{2} = M$

Thus (i) masses of both systems (cube and system of particles) are same, (ii) centres of mass of both systems are same (at point O) and (iii) symmetry axes (and hence principal axis) of both systems are same at O. The two systems will be equimomental if system of particles also has (same) moments of inertia about principal axes (coordinate axes), equal to $I_{11} = I_{22} = I_{33} = \frac{Ma^2}{6}$

Moment of inertia of system of particles about z-axis is

$$I'_{33} = 8\times\left[\frac{M}{24}\times\left(\frac{a}{2}\right)^2\right] + 4\times\left[\frac{M}{24}\times\left(\frac{a}{\sqrt{2}}\right)^2\right] = \frac{Ma^2}{12}+\frac{Ma^2}{12} = \frac{Ma^2}{6} = I'_{11} = I'_{22}$$

Thus $I_{11} = I'_{11}$, $I_{22} = I'_{22}$ and $I_{33} = I'_{33}$

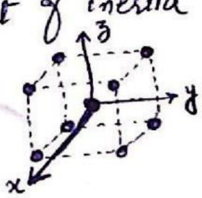
i.e; corresponding principal moment of inertia of both systems are also same. Thus both systems are equimomental.

In book, it is wrongly written as cuboid, instead of cube

(b) Total mass of the system of nine (9) particles $= 8\times M/24 + 2M/3 = M$

Thus above conditions (i), (ii) and (iii) are satisfied. Moment of inertia of system of particles about z-axis is given by

$$I''_{33} = 8\times\left[(M/24)\times\left(\frac{a}{\sqrt{2}}\right)^2\right] = \frac{Ma^2}{6} = I''_{11} = I''_{22}$$



$\Rightarrow I_{11} = I''_{11}$, $I_{22} = I''_{22}$, $I_{33} = I''_{33}$ $\Rightarrow$ both systems are equimomental.

Fixed and Rotating Frames of Reference

Q: Let Oxyz-coordinate system be rotating with respect to OXYZ-coordinate system fixed in space. Derive expression for the time derivative of vector $\vec{A}(t)$ for the observers in two coordinate systems.

Q: State and prove rotating axis theorem.

Statement: If a time dependent vector function $\vec{A}=\vec{A}(t)$ is represented by $\vec{A}_f$ and $\vec{A}_r$ in fixed and rotating coordinate system, then

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega}\times\vec{A}$$

where $\vec{\omega}$ = angular velocity of rotating coordinate system w.r.t fixed coordinate system

and it is assumed that origins of two coordinate system are at same point (i.e; origins are coincident.)

Proof: Let OXYZ be fixed co-ordinate system with $\hat{i}', \hat{j}', \hat{k}'$ as base (unit) vectors and Oxyz be rotating coordinate system with $\hat{i}, \hat{j}, \hat{k}$ as its base (unit) vectors.

We note that $\vec{A} = \vec{A}_f = \vec{A}_r$

Let $\vec{A}_f = A_1'\hat{i}' + A_2'\hat{j}' + A_3'\hat{k}'$ and $\vec{A}_r = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

Therefore, $\vec{A} = A_1'\hat{i}' + A_2'\hat{j}' + A_3'\hat{k}' = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

Also, $\left(\frac{d\vec{A}}{dt}\right)_f = \frac{d\vec{A}_f}{dt} = \frac{dA_1'}{dt}\hat{i}' + \frac{dA_2'}{dt}\hat{j}' + \frac{dA_3'}{dt}\hat{k}'$ (in fixed coordinate system OXYZ)

and $\left(\frac{d\vec{A}}{dt}\right)_r = \frac{d\vec{A}_r}{dt} = \frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}$ (in rotating system Oxyz)

As, $\vec{A}_f = \vec{A}_r = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ (viewing above vector w.r.t fixed coordinate system OXYZ)

Differentiating both sides w.r.t time "t" in fixed coordinate system OXYZ), we get equation

$$\left(\frac{d}{dt}\right)_f\vec{A} = \left(\frac{d}{dt}\right)_f(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$\Rightarrow \left(\frac{d\vec{A}_f}{dt}\right) = \left(\frac{d\vec{A}}{dt}\right)_f = \frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k} + A_1\frac{d\hat{i}}{dt} + A_2\frac{d\hat{j}}{dt} + A_3\frac{d\hat{k}}{dt}$$

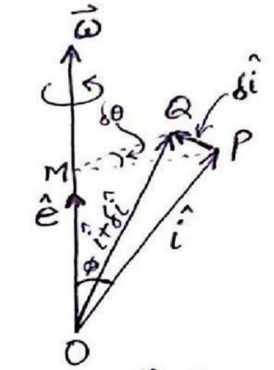
(we have dropped index f from R.H.S for convinience)

$$\Rightarrow \left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + A_1\frac{d\hat{i}}{dt} + A_2\frac{d\hat{j}}{dt} + A_3\frac{d\hat{k}}{dt} \quad \text{--- (1)} \quad \because \left(\frac{d\vec{A}}{dt}\right)_r = \frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k}$$

we proceed as follows:

To find $\frac{d\hat{i}}{dt}$, we note that as $\hat{i} = \vec{OP}$ rotates about the axis of rotation, the point P is displaced to point Q in time interval δt

such that (perpendicular) distances of P and Q from axis of rotation remain same, as shown in the figure



From figure;

$$|\vec{PQ}| = |\delta\hat{i}| = |\vec{MP}|\,\delta\theta$$

$$= (|\vec{OP}|\sin\phi)\,\delta\theta \qquad \because \text{in } \Delta OMP \quad \sin\phi = \frac{|\vec{MP}|}{|\vec{OP}|}$$

$$= \sin\phi\,\delta\theta \qquad \because |\vec{OP}| = |\hat{i}| = 1$$

$$\left|\frac{\delta\hat{i}}{\delta t}\right| = \sin\phi\,\frac{\delta\theta}{\delta t}$$

If δt becomes so small such that $\delta t \to 0$, then

where $\dot{\theta} = \frac{d\theta}{dt} = \omega = $ (magnitude of angular velocity)

$$\left|\frac{d\hat{i}}{dt}\right| = \dot{\theta}\sin\phi = \omega\sin\phi,$$ where $\hat{e}$ is unit vector in the direction of $\vec{\omega} = \dot{\theta}\hat{e}$

$$= |(\dot{\theta}\hat{e})\times\hat{i}| = |\vec{\omega}\times\hat{i}|$$

$$\Rightarrow \frac{d\hat{i}}{dt} = \vec{\omega}\times\hat{i} \qquad \because \frac{d\hat{i}}{dt} \text{ and } \vec{\omega}\times\hat{i} \text{ have same magnitudes and same directions.}$$

where, the direction of $\vec{\omega}$ is related to that of $\hat{i}$ and $\frac{d\hat{i}}{dt}$ by right hand rule.

Similarly, $\frac{d\hat{j}}{dt} = \vec{\omega}\times\hat{j}$ and $\frac{d\hat{k}}{dt} = \vec{\omega}\times\hat{k}$

Thus, equation (1) becomes

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + A_1\vec{\omega}\times\hat{i} + A_2\vec{\omega}\times\hat{j} + A_3\vec{\omega}\times\hat{k}$$

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega}\times(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$\boxed{\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega}\times\vec{A}} \qquad \because \vec{A} = \vec{A}_r = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

(Hence proved.)

Remark: Above result can also be written as

$$D_f\vec{A} = D_r\vec{A} + \vec{\omega}\times\vec{A}$$

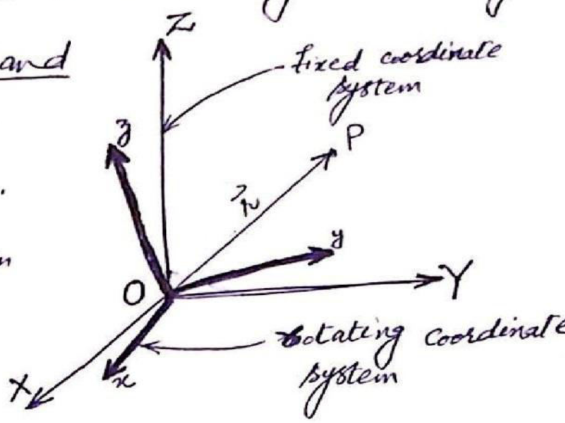
From which we get formula in differential operator form as follows $\boxed{D_f = D_r + \vec{\omega}\times}$

where L.H.S should be viewed in fixed coordinate system, while R.H.S in rotating coordinate system.

Q: Find the equation of motion with respect to a rotating coordinate system.

Q: Find expressions of velocity and acceleration with respect to a rotating coordinate system.

Sol: **Case I: When the origins of fixed and rotating coordinate systems are coincident:**



Let $\vec{r}$ be the position vector of a moving particle P, as shown in the figure, then

$$\vec{r} = \vec{r}_f = \vec{r}_r$$

where subscripts f and r shows quantities viewed in fixed and rotating coordinate systems, respectively.

> Dear students, we will proceed in the same manner as in last question by taking $\vec{r}$ in place of $\vec{A}$ and, ultimately, we will reach at:

$$\left(\frac{d\vec{r}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times\vec{r}_r \quad \text{——— (1)}$$

or equivilently, $\vec{V}_f = \vec{V}_r + \vec{\omega}\times\vec{r}$ ——— (2) (we can drop subscript r from vector $\vec{r}$, as it is same in both coordinate systems.)

$$\Rightarrow \boxed{\vec{V}_r = \vec{V}_f - \vec{\omega}\times\vec{r}}$$

This is required velocity with respect to rotating frame of reference.

From (1), we get operator equation as follows

$$\left(\frac{d}{dt}\right)_f = \left(\frac{d}{dt}\right)_r + \vec{\omega}\times$$

(Remembring that L.H.S is viewed in fixed coordinate system, while R.H.S is viewed in rotating one.)

Differentiating both sides of (2) w.r.t time t (w.r.t fixed coordinate system, OXYZ)

$$\left(\frac{d}{dt}\right)_f \vec{V}_f = \left(\frac{d}{dt}\right)_f (\vec{V}_r + \vec{\omega}\times\vec{r})$$

$$\Rightarrow \left(\frac{d\vec{V}_f}{dt}\right)_f = \left(\left(\frac{d}{dt}\right)_r + \vec{\omega}\times\right)(\vec{V}_r + \vec{\omega}\times\vec{r})$$

$$\Rightarrow \left(\frac{dV_f}{dt}\right)_f = \left(\frac{d\vec{V}_r}{dt}\right)_r + \left(\frac{d(\vec{\omega}\times\vec{r})}{dt}\right)_r + \vec{\omega}\times\vec{V}_r + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\Rightarrow \vec{a}_f = \vec{a}_r + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_r + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\vec{V}_r + \vec{\omega}\times(\vec{\omega}\times\vec{r}) \quad \text{——— (3)}$$

$$\Rightarrow \vec{a}_f = \vec{a}_r + 2\vec{\omega}\times\vec{V}_r + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) \quad \text{——— (4)}$$

$$\Rightarrow \boxed{\vec{a}_r = \vec{a}_f - 2\vec{\omega}\times\vec{V}_r - \dot{\vec{\omega}}\times\vec{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r})}$$

This is the required acceleration with respect to rotating coordinate system.

Note: $\vec{\omega}$ is same in both coordinate systems.

Here $\vec{a}_f = \left(\frac{d\vec{v}_f}{dt}\right)_f$ = acceleration viewed in fixed coordinate system.

$\vec{a}_r = \left(\frac{d\vec{v}_r}{dt}\right)_r$ = acceleration viewed in rotating coordinate system.

$\vec{a}_{cp} = \vec{\omega}\times(\vec{\omega}\times\vec{r})$ = centripetal acceleration.

$\vec{a}_{cor} = -2\vec{\omega}\times\vec{v}_r$ = Coriolis acceleration.

$\vec{a}_{tr} = -\dot{\vec{\omega}}\times\vec{r}$ = transverse or linear acceleration.

$\vec{a}_{cf} = -\vec{\omega}\times(\vec{\omega}\times\vec{r})$ = centrifugal acceleration.

The relation ③ is referred to as "<u>Coriolis theorem</u>".

The equations of motion in fixed and moving coordinate systems are given by

$$\vec{F}_f = m\vec{a}_f \quad \text{and} \quad \vec{F}_r = m\vec{a}_r, \text{ respectively.}$$

Where, $\vec{F}_f$ and $\vec{F}_r$ are total forces in fixed and rotating coordinate systems, respectively and m is mass of moving particle.

Multiplying equation ④ by "m" on both sides, we get

$$m\vec{a}_r = m\vec{a}_f - 2m\vec{\omega}\times\vec{v}_r - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) \quad \text{—— ⑤}$$

$$\Rightarrow \boxed{\vec{F}_r = \vec{F}_f + \vec{F}_{cor} + \vec{F}_{tr} + \vec{F}_{cf}}$$

where,

$\vec{F}_{cor} = -2m\vec{\omega}\times\vec{v}_r$ = Coriolis force.

$\vec{F}_{tr} = -m\dot{\vec{\omega}}\times\vec{r}$ = transverse force.

$\vec{F}_{cf} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ = centrifugal force.

These three forces are called fictitious or apparent or inertial forces. Equation ⑤ is required equation of motion in rotating frame of reference.

<u>Observations</u>: (i) As $\vec{v}_r\cdot\vec{F}_{cor} = -2m\,\vec{v}_r\cdot\vec{\omega}\times\vec{v}_r = 0$

This shows that the Coriolis force has a direction which is at right angle to the direction of motion defined by $\vec{v}_r$, in the rotating coordinate system.

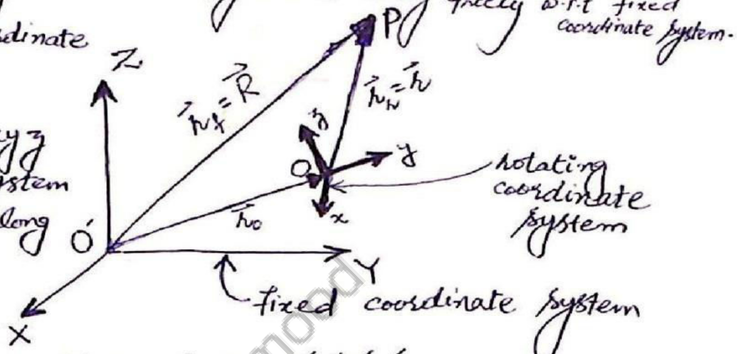
(ii) The centrifugal force $\vec{F}_{cf}$ is always directed away from the axis of rotation.

(iii) The transverse force $\vec{F}_{tr}$ depends on angular acceleration $\dot{\vec{\omega}}$ and will disappear when angular velocity $\vec{\omega}$ is constant.

(iv) The Coriolis force is of special interest. It is present only if the particle is moving in rotating coordinate system (i.e; $\vec{v}_r \neq 0$)

Case II : When the origin of rotating coordinate system is moving freely w.r.t fixed coordinate system.

Let $O'XYZ$ be fixed (space) coordinate system with base (unit) vectors $\hat{i}', \hat{j}', \hat{k}'$ along its coordinate axes and $Oxyz$ be rotating (body) coordinate system with base (unit) vectors $\hat{i}, \hat{j}, \hat{k}$ along its coordinate axes.



Let,

$\vec{r}_f = \vec{R}$ = Position vector of moving particle P w.r.t fixed coordinate system $O'X'Y'Z'$.

$\vec{r}_r = \vec{r}$ = Position vector of moving particle P w.r.t rotating coordinate system $Oxyz$.

$\vec{r}_o$ = Position vector of O (origin of rotating frame $Oxyz$) w.r.t fixed frame $Oxyz$.

From fig:
$$\vec{r}_f = \vec{r}_o + \vec{r}_r = \vec{r}_o + x\hat{i} + y\hat{j} + z\hat{k}, \quad \text{where } \vec{r}_r = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Differentiate both sides of above equation w.r.t time t (w.r.t fixed frame OXYZ)

$$\Rightarrow \left(\frac{d}{dt}\right)_f \vec{r}_f = \left(\frac{d}{dt}\right)_f \left(\vec{r}_o + x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\Rightarrow \left(\frac{d\vec{r}_f}{dt}\right)_f = \frac{d\vec{r}_o}{dt} + \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) + \left(x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}\right)$$
(We have dropped index f from R.H.S for convinience)

But $\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = \left(\frac{d\vec{r}_r}{dt}\right)_r = \vec{v}_r$ and $\frac{d\hat{i}}{dt} = \vec{\omega}\times\hat{i}$, $\frac{d\hat{j}}{dt} = \vec{\omega}\times\hat{j}$, $\frac{d\hat{k}}{dt} = \vec{\omega}\times\hat{k}$ where $\vec{v}_o = \frac{d\vec{r}_o}{dt}$

Therefore, $\vec{v}_f = \vec{v}_o + \vec{v}_r + x(\vec{\omega}\times\hat{i}) + y(\vec{\omega}\times\hat{j}) + z(\vec{\omega}\times\hat{k})$, and $\vec{v}_f = \left(\frac{d\vec{r}}{dt}\right)$

Note: From this eq. we can find $\vec{v}_r$

$$\Rightarrow \boxed{\vec{v}_f = \vec{v}_o + \vec{v}_r + \vec{\omega}\times(x\hat{i} + y\hat{j} + z\hat{k}) = \vec{v}_o + \vec{v}_r + \vec{\omega}\times\vec{r}} \quad (6) \qquad \because \vec{r} = \vec{r}_r = x\hat{i} + y\hat{j} + z\hat{k}$$

Differentiating both sides w.r.t time t (w.r.t fixed frame OXYZ)

$$\left(\frac{d}{dt}\right)_f \vec{v}_f = \left(\frac{d}{dt}\right)_f \left(\vec{v}_o + \vec{v}_r + \vec{\omega}\times\vec{r}\right) = \left(\frac{d}{dt}\right)_f \vec{v}_o + \left(\frac{d}{dt}\right)_f \left(\vec{v}_r + \vec{\omega}\times\vec{r}\right)$$

$$= \vec{a}_o + \vec{a}_r + 2\vec{\omega}\times\vec{v}_r + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

where $\left(\frac{d\vec{v}_o}{dt}\right)_f = \vec{a}_o$

Note: From here we can find $\vec{a}_r$

$$\Rightarrow \boxed{\vec{a}_f = \vec{a}_o + \vec{a}_r + 2\vec{\omega}\times\vec{v}_r + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})} \quad (7)$$

From (6) and (7), velocity $\boxed{\vec{v}_r = \vec{v}_f - \vec{v}_o - \vec{\omega}\times\vec{r}}$ and acceleration in rotating frame are given by

and $\boxed{\vec{a}_r = \vec{a}_f - \vec{a}_o - 2\vec{\omega}\times\vec{v}_r - \dot{\vec{\omega}}\times\vec{r} - \omega\times(\vec{\omega}\times\vec{r})}$ (8), respectively.

Multiplying eq. (8) by m, on both sides, we get eq. of motion in moving frame as

$$\boxed{\vec{F}_r = \vec{F}_f - \vec{F}_o + \vec{F}_{cor} + \vec{F}_{tr} + \vec{F}_{cf}}$$

(where $\vec{F}_o = m\vec{a}_o$, $\vec{F}_r = m\vec{a}_r$, $\vec{F}_f = m\vec{a}_f$) and ($\vec{F}_{cor}$, $\vec{F}_{tr}$, $\vec{F}_{cf}$ are same as given before)