hmo	bod	Prepared	by: Dr	. Amir Mah	mood	Prepar	ed by:	Dr. A	mir Ma	ahmood	Pre	par
hmo	ood Mecł	Prepared nanics Made Easy	by: Dr	. Amir Mah	mood	l Prepar	ed by:	Dr. A	mir Ma	Moment of In	Prep	par
hm	pod	Prepared	by Mc	oment of	Ine	d Prepar	red by:	Dr. A	mir M	ahmood	Pre	par
hmo	Defi	nition: <u>Moment</u>	of inert	tia of a particle	of mass	<i>m</i> about a	line (calle	d axis	of	Ĩ	e	bar
hm	rotat	tion) is defined as	by: Dr	Amir Mah	2000	l Prepar	ed by:	Dr. A	m 📝		e	par
hm	when	re, r is the perpe	endicular	distance of partic	le from	line. repar	ed by:	Dr. A	mi		e	par
hm	abou	it a line (called as	xis of rota	ation) is defined as		d Prepai	red by:	Dr. A	m	-	е	bar
hm	when	re, r_i is the perp	endicular	$I = \sum_{i} m_{i}$ r distance of <i>i</i> -th	r_i^2 , particle of	of mass m_i fr	om line.	Dr. A	m	253	e	bar
hmo	Defi	nition: <u>Moment</u>	of iner	tia of a continu	ous dis	tribution of	mass, suc	h as th	ieni	dm	e	par
hmo	line i	is defined as	by: Dr	A figure), having h		Prepar	ed by:	Dr. A	mi		e	bar
hm	whe	re. r is the perpe	endicular	$I = \int_{M} r^2 dm = \rho$ distance of point	$\int_M r^2 d$ mass ele	V, ement <i>dm</i> of :	the body a	d dV i	s its eleme	ntary volume	Pre	bar
hmo	Mom	ents of inertia wi	ith respec	ct to Cartesian coo	rdinate a	axes are defin	ed in the fo	ollowing	table:	ahmood	Pre	par
hm	Dod	loment of inertia	Momen particle	nt of inertia of a e with respect to	Mome	ent of inertia les with res	of a set of pect to 3-	Mome contin	ent of i Nuous rigi	nertia of d body wit	a hPrei	par
hm	bod	Prepared	3-dime	nsioal Cartesian	dimer coord	nsioal inate system	Cartesian	respe Cartes	ct to sian coordi	3-dimensioa nate system	al Prei	bar
hm	bod	About x-axis $I_{xx} = I_{11}$	by: Dr	$m(y^2 + z^2)$ ah	mood	$\sum m_i(y_i^2 +$	z_i^2) by:	Dr. A	$\int (y^2 +$	z^2)dmood	Pre	par
hm	bod	About <i>v</i> -axis	by: Dr	Amir Mah	mood	Prega	ed by:	Dr. A	mar M	ahmood	Pre	bar
hmo	bod	$P^{I_{yy}} = I_{22}$	by: Dr	$m(x^2 + z^2)$ and	mood	$\sum_{i} m_i (x_i^2 +$	z_i^2) ed by:	Dr. A	$\int (x^2 +$	z²)dm	Prep	pare
nmo	bod	About <i>z</i> -axis $I_{zz} = I_{33}$	oy: Dr	$m(x^2 + y^2)$ ahi	no	$\sum_{i} m_i (x_i^2 +$	y_i^2) by:	Dr. A	$\int (x^2 + x)^2$	y ²)dmood	Prep	are
hmo	Prod	ucts of inertia wi	th respec	t to Cartesian cooi	dinate a	xes are defin	ed in the fo	llowing	table: Ma	ahmood	Pre	par
nmo	ode	roduct of inertia	by: DP	roduct of inertia particle with respe	ofa P ct to p	roduct of ine articles with	ertia of a se respect to	t of P	roduct of ontinuous	inertia of rigid bod	a rep	pare
hmo	ood	Prepared I	by: Di	dimensioal Carte	sian d	imensioal oordinate sys	Carte	sian w	rith resj imensioal	pect to S	Pre	pare
nmo	od	Prepared I	oy: Dr.	Annr Mahi	nooc	Prepar	ed by: I	Dr. A	oordinate	system	Prep	pare
hm	bod	$I_{xy} = I_{yx} = I_{12} =$	∍ I ₂₁ Dr	. Am <i>ir mxy</i> ah	mood	l Prep <u>p</u> i	$n_i x_i y_i$	Dr. A	mir ₩	xy dmood	Pre	par
hmo	bod	Prepared I	by: Dr.	. Amir Mahi	nood	Prepar	$m_{i}v_{i}z_{i}$	Dr. A	mir M	ahmood	Prep	pare
nmo	bod	$I_{yz} = I_{zy} = I_{23} =$	= I ₃₂ Dr.	. Amir Wahi	nooc	Prepar	ed by: I	Dr. A	mir M	ahmood	Prep	pare
nmo	bod	$I_{xz} = I_{zx} = I_{13} =$	oγ ₃₁ Dr.	Ami <i>r mxz</i> hi	nooc	Prep	$m_i x_i z_i$	Dr. A	mir №l	xz dm od	Prep	are
hmo	Defi	nition: <u>Radius o</u>	f gyratio	on k of a rigid body	of mas	s <i>M</i> with resp	pect to a lin	e <i>l</i> is de	efined as	ahmood	Pre	par
nmo	where	Prepared	nt of iner	tia of the body wi	k =	$\sqrt{I/M}$,	ed by:	Dr. A	mir Ma	ahmood	Prep	bare
hmo	Prob	lem: Prove in ma	itrix notal	$\frac{\text{tion that}}{\text{tion that}} [L] = [I]$][ω], <u>w</u>	here, all the r	notations u	sed hav	e their usi	ual meanings.	Prep	pare
nmo	a fixe	f: The angular m ed point, is given	by Dr	. Amir Mahi	n the fol	rm of a set of Prepar	ed by:	bout a Dr. A	n instanta mir Ma	ahmood	Prer	pare
hm	bod	Prepared	by: Dr	$= \sum \mathbf{r}_i \times (m_i \mathbf{v}_i)$	$=\sum m$	$\mathbf{r}_i(\mathbf{r}_i \times \mathbf{v}_i) = \mathbf{v}_i$	$\sum m_i(\mathbf{r}_i \times$	$(\boldsymbol{\omega} \times \mathbf{r}_i)$)) hir Mi	$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$	Pre	par
hmo	bod	Prepared I	by: Dr.	Amir Mah	$\frac{\overline{i}}{1000}$	Prepar	ed by:	Dr. A	mir Ma	ahmood	Prer	pare
1000		IN CONTRACTOR IN		$= m_i(\mathbf{r}_i \times (\boldsymbol{\omega}))$	(\mathbf{r}_i) =	$m_i [(\mathbf{r}_i \cdot \mathbf{r}_i)]$	$(\mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \mathbf{r}_i)$	$\omega \mathbf{r}_i$			-	
nmo	bod	Prepared	by: Dr	<u>. An</u> ir Mah	mood	Prepar	ed by:	Dr. A	mir Ma	ahmood	Pre	Jan

nood Mechanics Made Easy $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$ and Let, $L = L_1 i + L_2 i$ $A\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$ $x_i^2 + y_i^2 + z_i^2$ and $\mathbf{r}_i \cdot \boldsymbol{\omega} = x_i \boldsymbol{\omega}_1 + y_i \boldsymbol{\omega}_2 + z_i \boldsymbol{\omega}_3$ Prepared by: Dr. $m_i [(x_i^2 + y_i^2 + z_i^2)(\omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_2 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_2 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_2 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_2 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_2 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + y_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3)(x_i \mathbf{i} + \omega_3 \mathbf{k}) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3 + z_i \omega_3) - (x_i \omega_1 + z_i \omega_3) - (x_i \omega_$ $L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k} =$ Comparing corresponding components on both sides of above vector equation, we get $-(x_i\omega_1+y_i\omega_2+z_i\omega_3)x_i]$ $+ z_i^2 \omega_1$ ood Prepared by: $+ z_i^2 \omega_2 - (x_i \omega_1 + y_i \omega_2 + z_i \omega_3) y_i = - - -$ $m_i(x_i)$ Prepared by: $+ z_i^2 \omega_3$ $-(x_i\omega_1 + y_i\omega_2 + z_i\omega_3)z_i$ $m_i | (x_i)$ pared $L_{1} = \sum m_{i} \left[x_{i}^{2} \omega_{1} + (y_{i}^{2} + z_{i}^{2}) \omega_{1} - x_{i}^{2} \omega_{1} - x_{i} y_{i} \omega_{2} \right]$ From Eq. (1), we get, $m_i x_i z$ $m_i x_i y_i$ $-\omega_2$ ood Prepared b $= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3$ Similarly, from (2) and (3), we get $L_2 = I_{12}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3$ and $L_3 = I_{13}\omega_1 + I_{23}\omega_2 + I_{33}\omega_3 -$ Mahmood/L1 $\langle \omega_1 \rangle$ od Prepared by: Dr. Amir $(I_{11} \ I_{12} \ I_{13})$ Writing Eqs. (4), (5) and (6) in matrix form, we get, L_2 I_{23} ω_2 122 121 Hence proved. Ιω L = **Problem:** Prove that $T = \frac{1}{2}M\mathbf{v}^2 + \frac{1}{2}\boldsymbol{\omega}$. L, where all the notations used have their (or) prove that $T = T_{tr} + T_{rot}^2$ where, $T_{tr} = \frac{1}{2}M\mathbf{v}^2 = \text{total translational kinetic energy of the body}$ and $T_{rot} = \frac{1}{2}\omega$. L = total rotational kinetic energy of the body. **Proof**: Consider a rigid body, in the form of a set of particles, which is in general state of motion (i.e., having both translation and rotation) with respect to a fixed (inertial) frame of reference Oxyz. Ivianmood Prepared by Let, M = total mass of the body \mathbf{r}_i = position vector of *i*-th particle of mass m_i with respect to origin "O" \mathbf{r}'_i = position vector of *i*-th particle of mass m_i with respect to centre of mass "C" $\mathbf{r} = \text{position vector of centre of mass "}C"$ with respect to origin "O" \mathbf{v}_i = velocity of *i*-th particle of mass m_i with respect to origin "O" \mathbf{v}_i' = velocity of *i*-th particle of mass m_i with respect to centre of mass "C **v** = velocity of centre of mass "*C*" with respect to origin "*O*" ω = instantaneous angular velocity of body about instantaneous axis through centre of mass From figure, $\mathbf{r}_i = \mathbf{r} + \mathbf{r}'_i$ Dr. Amir Differentiating both sides with respect to time "t", we get Prepared by: Dr. Amir Mahmo $\dot{\mathbf{r}}_i = \dot{\mathbf{r}} + \dot{\mathbf{r}}'_i$ $\Rightarrow \mathbf{v}_i = \mathbf{v} + \mathbf{v}_i = \mathbf{v} + \mathbf{o}_i \times \mathbf{r}'_i$ $\mathbf{v}_i = \mathbf{v} + \mathbf{v}'_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}'_i$ $= \omega \times \mathbf{r}_i'$ Kinetic energy of the *i*-th particle is $T_i = \frac{1}{2}m_i \mathbf{v}_i^2$ Kinetic energy of the whole body is $m_i \{ (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}'_i) \cdot (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}'_i) \}$ Prepared by: Dr. Amir Mahmood

hmo	ood	1 Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir	Mahmood	Prepare
hmo	ood Mec	Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir chanics Made Easy	Mahmood Moment of I	Prepare
hm	ood	$T = \frac{1}{2} \sum m_i \{ \mathbf{v}^2 + 2\mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}'_i) + \boldsymbol{\omega} \cdot \mathbf{r}'_i \times (\boldsymbol{\omega} \times \mathbf{r}'_i) \}$	Mahmood	Prepare
hmo	ood	$\frac{1}{2}$ Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir $-\frac{1}{2}\left(\sum_{m}\right)y^{2} + \sum_{m}y_{n}\left(y_{n} \times r'\right) + \frac{1}{2}\sum_{m}\left(y_{n} \times r'\right)^{2}$	Mahmood	Prepare
hm	ood	$d \operatorname{Prepare}^{2}(\frac{2}{4}\gamma) \operatorname{Dr}^{2} \frac{2}{4} \operatorname{min}^{2} \operatorname{Mahm}^{2} \frac{2}{4} \operatorname{d}^{2} \operatorname{Prepared}^{2} \operatorname{by}: \operatorname{Dr}. \operatorname{Amin}^{2}$	Mahmood	Prepare
hmo	ood	$ Prepa = \frac{1}{2}M\mathbf{v}^2 + \mathbf{v} \cdot \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}'_i\right) + \frac{1}{2}\boldsymbol{\omega} \cdot \sum_i m_i \mathbf{r}'_i \times (\boldsymbol{\omega} \times \mathbf{r}'_i), \text{ where, } M = \sum_i m_i = \frac{1}{2}m_i \mathbf{r}'_i + \frac{1}{2}\mathbf{v} \cdot \sum_i m_i \mathbf{r}'_i \times (\mathbf{v} \times \mathbf{r}'_i), \text{ where, } M = \sum_i m_i = \frac{1}{2}m_i \mathbf{r}'_i + \frac{1}{2}\mathbf{v} \cdot \sum_i m_i \mathbf{r}'_i \times (\mathbf{v} \times \mathbf{r}'_i), \text{ where, } M = \sum_i m_i \mathbf{r}'_i + \frac{1}{2}\mathbf{v} \cdot \sum_i m_i \mathbf{r}'_i \times (\mathbf{v} \times \mathbf{r}'_i), \text{ where, } M = \sum_i m_i \mathbf{r}'_i + \frac{1}{2}\mathbf{v} \cdot \sum_i m_i \mathbf{r}'_i \times (\mathbf{v} \times \mathbf{r}'_i), \text{ where, } M = \sum_i m_i \mathbf{r}'_i + \frac{1}{2}\mathbf{v} \cdot \sum_i m_i \mathbf{r}'_i + $	total mass of the b	odyepare
hm	ood	Also, $\sum m_i \mathbf{r}'_i = 0$, as \mathbf{r}'_i is the position vector of <i>i</i> th particle of mass m_i with respect to	centre of mass "C	"Prepare
hm	ood	d Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir $\Rightarrow T = -M\mathbf{y}^2 + -\boldsymbol{\omega} \cdot \sum m_i \mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}') =(1)$	Mahmood	Prepare
hmo	ood	Prepared by: Dr. Amir Mainfood Prepared by: Dr. Amir	Mahmood	Prepare
hmo	But,	t, angular momentum L of the body with respect to centre of mass " \mathcal{L} " is given by	Mahmood	Prepare
hm	bod	$\mathbf{L} = \sum_{i} \mathbf{r}'_{i} \times (m_{i} \mathbf{v}'_{i}) = \sum_{i} \mathbf{r}'_{i} \times \{m_{i} (\boldsymbol{\omega} \times \mathbf{r}'_{i})\} = \sum_{i} m_{i} \mathbf{r}'_{i} \times (\boldsymbol{\omega} \times \mathbf{r}'_{i}) $	(2) Mahmood	Prepare
hm	Usin	ng (2) in (1), we get,	Mahmood	Propara
hm	bod	Prepared by Dr. Amir Mahmand Propared by Dr. Amir	Mahmood	Propart
	000	Prepared by: Dr. Amir Mahr $T = T_{tr} + T_{rot}$ Propared by Dr. Amir	Mahmood	Proport
nme	000	where, $T_{tr} = \frac{1}{2}M\mathbf{v}^2 = \text{total translational kinetic energy of the body}$	Manmood	Frepare
hm	ood	and $T_{rot} = \frac{1}{2}\omega$. L = total rotational kinetic energy of the body	Mahmood	Prepare
hm	Pro	blem: Find moment of inertia of a rigid body about a given line Z I	Given lin	pare
hmo	pass	ssing through the origin and having direction cosines are (λ, μ, ν) .		pare
hmo	give	en line as z-axis, as shown in the figure.	<i>u</i> ₁ m .	pare
hme	Let,	, $M = \text{total mass of the body}$	•	pare
hmo	$d_i = d_i = \theta_i = \theta_i$	= perpendicular distance of <i>i</i> -th particle of mass m_i from given line l = angle between position vector \mathbf{r}_i and given line l	/	pare
hmo	e =	: unit vector in the direction of given line l		vbare
hmo	The	e required moment of inertia I_l is given by		nare
hm	hod	$\frac{X}{2}$	Mahmood	Prepare
hm	and	$I_l = \sum_i m_i a_i^z = \sum_i m_i (\mathbf{r}_i \sin \theta_i)^z = \sum_i m_i (\mathbf{e} \times \mathbf{r}_i)^z - \dots \rightarrow (1) \forall \ \sin \theta_i = \frac{ \mathbf{r}_i }{ \mathbf{r}_i } \text{ and } \mathbf{r}_i $	$\sin \theta_i = \mathbf{e} \times \mathbf{r}_i \leq 1$	Droppare
hm	Now	w, $\mathbf{e} \times \mathbf{r}_i = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda & \mu & \nu \end{vmatrix} = (\mu z_i - \nu y_i)\mathbf{i} + (\nu x_i - \lambda z_i)\mathbf{j} + (\lambda y_i - \mu x_i)\mathbf{k}$	Mahmood	Prepare
LINE	JUG	$\Rightarrow (\mathbf{e} \times \mathbf{r}_i)^2 = (\mu z_i - \nu y_i)^2 + (\nu x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2 - \dots - \dots \rightarrow \dots$	(2)	Droppire
nmo	Usin	ng (2) in (1), we get	vianmood	Frepare
hmo	ood	$Prepare O_{i} = \sum_{i} m_{i} [(\mu z_{i} - \nu y_{i})^{2} + (\nu x_{i} - \lambda z_{i})^{2} + (\lambda y_{i} - \mu x_{i})^{2}] = 0 \text{ by: Dr. Amir}$	Mahmood	
hmo	bod			riepait
	JUU	Prepared b = $\sum_{i} m_i [(\mu^2 z_i^2 + \nu^2 y_i^2 - 2\mu v y_i z_i) + (\nu^2 x_i^2 + \lambda^2 z_i^2 - 2\lambda v x_i z_i) + (\lambda^2 y_i^2 + \mu^2 z_i^2 - 2\lambda v x_i z_i) + (\lambda^2 y_i^2 + \mu^2 z_i^2 - 2\lambda v x_i z_i) + (\lambda^2 y_i^2 + \mu^2 z_i^2 - 2\lambda v x_i z_i) + (\lambda^2 y_i^2 - 2\mu v y_i z_i) + (\lambda^2 y_i^2 - 2\lambda v x_i z_i) + (\lambda^2 y_i^2 - 2\lambda v x_i) + (\lambda^2 y_i^2 - 2\lambda v x_i$	$2x_i^2 - 2\lambda\mu x_i y_i$	Prepare
hmo	pod	Prepared b = $\sum_{i} m_{i} [(\mu^{2} z_{i}^{2} + \nu^{2} y_{i}^{2} - 2\mu \nu y_{i} z_{i}) + (\nu^{2} x_{i}^{2} + \lambda^{2} z_{i}^{2} - 2\lambda \nu x_{i} z_{i}) + (\lambda^{2} y_{i}^{2} + \mu)^{2}$ = $\lambda^{2} \sum_{i} m_{i} (y_{i}^{2} + z_{i}^{2}) + \mu^{2} \sum_{i} m_{i} (x_{i}^{2} + z_{i}^{2}) + \nu^{2} \sum_{i} m_{i} (x_{i}^{2} + y_{i}^{2}) + 2\lambda \mu \Big(- \frac{1}{2} \sum_{i} m_{i} (y_{i}^{2} + z_{i}^{2}) + \mu^{2} \sum_{i} m_{i} (x_{i}^{2} + z_{i}^{2}) + \nu^{2} \sum_{i} m_{i} (x_{i}^{2} + y_{i}^{2}) + 2\lambda \mu \Big(- \frac{1}{2} \sum_{i} m_{i} (y_{i}^{2} + z_{i}^{2}) + \mu^{2} \sum_{i} m_{i} (y_{i}^{2} + z_{i}^{2}) + \mu^$	$\sum_{i=1}^{2} m_i x_i y_i $	Prepare Prepare
hmo hmo	ood ood	Prepared b = $\sum_{i} m_{i} [(\mu^{2} z_{i}^{2} + \nu^{2} y_{i}^{2} - 2\mu \nu y_{i} z_{i}) + (\nu^{2} x_{i}^{2} + \lambda^{2} z_{i}^{2} - 2\lambda \nu x_{i} z_{i}) + (\lambda^{2} y_{i}^{2} + \mu^{2} z_{i}^{2} + \lambda^{2} z_{i}^{2} - 2\lambda \nu x_{i} z_{i}) + (\lambda^{2} y_{i}^{2} + \mu^{2} z_{i}^{2} + \mu^{2} z_{i}^{2} + \lambda^{2} z_{i}^{2} + \lambda^{2} z_{i}^{2} - 2\lambda \nu x_{i} z_{i}) + (\lambda^{2} y_{i}^{2} + \mu^{2} z_{i}^{2} + \mu^{2} z_{i}^{2} + \lambda^{2} z$	$\sum_{i}^{2} m_{i} x_{i} y_{i} $	Prepare Prepare Prepare
hma hma hma	ood ood ood	$\begin{aligned} & = \sum_{i} m_{i} [(\mu^{2} z_{i}^{2} + \nu^{2} y_{i}^{2} - 2\mu \nu y_{i} z_{i}) + (\nu^{2} x_{i}^{2} + \lambda^{2} z_{i}^{2} - 2\lambda \nu x_{i} z_{i}) + (\lambda^{2} y_{i}^{2} + \mu z_{i}^{2}) \\ & = \lambda^{2} \sum_{i} m_{i} (y_{i}^{2} + z_{i}^{2}) + \mu^{2} \sum_{i} m_{i} (x_{i}^{2} + z_{i}^{2}) + \nu^{2} \sum_{i} m_{i} (x_{i}^{2} + y_{i}^{2}) + 2\lambda \mu \left(-\sum_{i} m_{i} y_{i} z_{i} \right) \\ & + 2\mu \nu \left(-\sum_{i} m_{i} y_{i} z_{i} \right) + 2\lambda \nu \left(-\sum_{i} m_{i} x_{i} z_{i} \right) \end{aligned}$	$\sum_{i=1}^{2} m_i x_i y_i $	Prepare Prepare Prepare Prepare Prepare
hma hma hma hma		$ = \sum_{i} m_{i} [(\mu^{2} z_{i}^{2} + \nu^{2} y_{i}^{2} - 2\mu \nu y_{i} z_{i}) + (\nu^{2} x_{i}^{2} + \lambda^{2} z_{i}^{2} - 2\lambda \nu x_{i} z_{i}) + (\lambda^{2} y_{i}^{2} + \mu) $ $ = \lambda^{2} \sum_{i} m_{i} (y_{i}^{2} + z_{i}^{2}) + \mu^{2} \sum_{i} m_{i} (x_{i}^{2} + z_{i}^{2}) + \nu^{2} \sum_{i} m_{i} (x_{i}^{2} + y_{i}^{2}) + 2\lambda \mu \left(-\frac{1}{2} \sum_{i} m_{i} y_{i} z_{i} \right) + 2\lambda \nu \left(-\sum_{i} m_{i} x_{i} z_{i} \right) $ $ = \lambda^{2} I_{l} = \lambda^{2} I_{xx} + \mu^{2} I_{yy} + \nu^{2} I_{zz} + 2\lambda \mu I_{xy} + 2\mu \nu I_{yz} + 2\lambda \nu I_{xz} $	$\sum_{i=1}^{2} x_i^2 - 2\lambda \mu x_i y_i $	Prepare Prepare Prepare Prepare Prepare
hma hma hma hma hma	ood ood ood This	$= \sum_{i} m_{i} [(\mu^{2} z_{i}^{2} + \nu^{2} y_{i}^{2} - 2\mu \nu y_{i} z_{i}) + (\nu^{2} x_{i}^{2} + \lambda^{2} z_{i}^{2} - 2\lambda \nu x_{i} z_{i}) + (\lambda^{2} y_{i}^{2} + \mu) \\ = \lambda^{2} \sum_{i} m_{i} (y_{i}^{2} + z_{i}^{2}) + \mu^{2} \sum_{i} m_{i} (x_{i}^{2} + z_{i}^{2}) + \nu^{2} \sum_{i} m_{i} (x_{i}^{2} + y_{i}^{2}) + 2\lambda \mu \left(- \frac{1}{2} + 2\mu \nu \left(-\sum_{i} m_{i} y_{i} z_{i} \right) + 2\lambda \nu \left(-\sum_{i} m_{i} x_{i} z_{i} \right) \right) \\ \Rightarrow I_{l} = \lambda^{2} I_{xx} + \mu^{2} I_{yy} + \nu^{2} I_{zz} + 2\lambda \mu I_{xy} + 2\mu \nu I_{yz} + 2\lambda \nu I_{xz} \end{bmatrix}$ s is required moment of inertia.	$\sum_{i=1}^{2} \lambda \mu x_i y_i $	Prepare Prepare Prepare Prepare Prepare Prepare

hmc	ood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood	Prepare
hmc	Mechanics Made Easy Moment of International Mechanics Mechan	Prepare
hmo	Problem: Find the equation of "ellipsoid of inertia" or "momental ellipsoid" of a rigid body.	Prepare
hme	Solution : As we know that moment of inertia of a rigid body about a given line l having direction cosines (λ with respect to a coordinate system $Oxyz$, whose origin " O " lies on the line l , is given by	,μ,ν) Prepare
hmo	$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz} + 2\lambda \mu I_{xy} + 2\mu \nu I_{yz} + 2\lambda \nu I_{xz} \rightarrow (1)$ On the line <i>l</i> , choose a point <i>P</i> such that $ \overrightarrow{OP} = 1/\sqrt{I_l}$. If coordinates of <i>P</i> are (<i>x</i> , <i>y</i> , <i>z</i>), then	Prepare
hmo	bod Prepared by: Dr. Amir $\frac{x_1}{ \overline{OP} }$ by: Dr. Amir Mahmood	Prepare
hmo	bod Prepared by: Dr. Amir Jahmodz, Prepared by: Dr. Amir Mahmood Elimination 1 word from (1) and (2) and (2)	Prepare
hmo	Eliminating λ, μ and ν from (1) and (2), we get $I_{l} = I_{l} (I_{xx}x^{2} + I_{yy}y^{2} + I_{zz}z^{2} + 2I_{xy}xy + 2I_{yz}yz + 2I_{xz}xz)$	Prepare
hmo	Since $I = I$ and I are all positive therefore above equation represents an ellipsoid called "ellipsoid of inert	Prepare
hme	"momental ellipsoid" of the rigid body.	Brabare
hmo	Note:	Prepare
hmc	(<i>i</i>) The momental ellipsoid of a rigid body contains information about moments and product of inertia of the	arepare
hmo	(<i>ii</i>) The centre of momental ellipsoid lies at the origin of the coordinate system.	Prepare
hmo	(<i>iii</i>) If P is any point on momental ellipsoid, then $1 \rightarrow 1 \rightarrow 1 = 1$	Prepare
hmo	od Prepared by: Dr. Amir Mahm $\sqrt{t_l}$ d Pre $\overline{ OP ^2}$ by: Dr. Amir Mahmood	Prepare
hmo	showing that moment of inertia about line \overrightarrow{OP} is equal to the reciprocal of square of distance of point <i>P</i> from origin <i>O</i> .	o ^m repare
hmo	Problem: State and prove perpendicular axis theorem for a discrete mass distribution.	Prepare
hmo	Statement : The moment of inertia of a plane rigid body in the form of discrete mass distribution (i.e., a set of particles) about a given axis perpendicular to the plane of the	appare
hmo	body is equal to the sum of moments of inertia about two mutually perpendicular axes lying in the plane of the body and meeting at a common point on the given axis. m_i	epare
hmo	Proof : We choose Cartesian coordinate system $Oxyz$ such that xy -plane lies in the plane of the body, while <i>z</i> -axis lies perpendicular to it, which is assumed to the given axis.	epare
hmo	Let, $\mathbf{r}_i = x_i \mathbf{i} + y_i \mathbf{j}$ be the position vector of <i>i</i> -th particle of mass m_i w.r.t. origin "0".	
hmo	ood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amz	epare
hmo	$I_{zz} = \sum_{i} m_{i} \mathbf{r}_{i} ^{2} = \sum_{i} m_{i} (x_{i}^{2} + y_{i}^{2}) = \sum_{i} m_{i} x_{i}^{2} + \sum_{i} m_{i} y_{i}^{2} = I_{xx} + I_{yy}$	Prepare
hmo	pod, Prepared by: Dr. Artic $I_{zz} = I_{xx} + I_{yy}$ Prepared by: Dr. Amir Mahmood	Prepare
hmo	Statement : The moment of inertia of a plane rigid body in the form of continuous mass distribution about a	givenpare
hmo	axis perpendicular to the plane of the body is equal to the sum of moments of inertia of same body abou mutually perpendicular axes lying in the plane of body and meeting at a common point	t two Prepare
hmo	on the given axis. Proof: We choose Cartesian coordinate system $0xyz$ such that xy -plane lies in the plane	pare
hmo	of the body having mass <i>M</i> , while <i>z</i> -axis lies perpendicular to it, which is assumed to the given axis	dm pare
hmc	Let, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector of elementary particle of body of mass dm w.r.t.	pare
hmo	Then moment of inertia of the body about z-axis is ood Prepared by: Dr. Am	pare
hmo	$I_{zz} = \int \mathbf{r} ^2 dm = \int (x^2 + y^2) dm = \int x^2 dm + \int y^2 dm = I_{xx} + I_{yy} \text{ red by: Dr. Am}$	
hmo	bod Prepared $\Rightarrow: I_{zz} = I_{xx} + I_{yy}$ hence proved. red by: Dr. Am ^z	pare
hme	Problem: State and prove parallel axis theorem for the case of moment of inertia for a discrete	mass pare
hmo	Prepared by: Dr. Amir Mahmood min Mahmood Prepared by: Dr. Amir Mahmood P	age 4 pare

hmo	od	Prep	ared	by:[Dr. A	mir M	lahmoo	d Pre	pared	by: D	r. Ami	r Mah	mood	Prepar	6
hmo	od	Prepa	ared	by:[Dr. A	mir M	lahmoo	d Pre	pared	by: D	r. Ami	r Mah	mood	Prepar	e
hmo	distri	ibution.	ared	by: I	Dr. A	mir IV	lahmoo	d Pre	pared	by: D	r. Ami	r Mah	mood	Prepar	1
hmo	States about	nent : Th a giver	e mon axis is	nent of equal	inertia to the	of a rigi sum of r	d body in t noment of i	he form o nertia of	of discrete same boo	e mass o dy about	listributio t a paralle	n (i.e., a l axis (to	set of part the given	icles) axis)	Ę
hmo	throu centr	gh the o e of mas	entre o s, abou	of mass t given	s of the axis.	e body an	id the mom	ent of ine	ertia due	to the to	otal mass	of the bo	dy placed	atitspar	e
hmo	Proof and p	: Consid assing t	er a rig prough	id body centre	y, in the of mas	e form of s of the h	a set of par ody. Let. <i>M</i>	ticles. Let = total n	<i>l</i> be the grass of th	given an e bodv	d <i>l'</i> be an	axis whic	h is parall	eltolpar	E
hmo	$r_0 = 1$	osition	vector	of <i>i</i> -th	particle	e of mass	m_i with res	pect to or	rigin " <i>O</i> "	by: C	Dr. Ami	r Mah	mood	Prepai	1
hmo	$\mathbf{r}_{c} =$	position	vector	of cen	tre of n	nass "C"	with respec	t to origin	1"0" ed	bv: D	r. Ami	r Mah	mood	Prepar	e
hmo	$\theta_i = a$ $d_i = 1$	ingle be perpend	ween p icular c	oositior listance	n vector e of <i>i-</i> th	r r_i and g 1 particle	iven line <i>l</i> of mass <i>m_i</i>	from give	en axis <i>l</i>		^ • /		l <mark>i</mark> •	bar	e
hmo	$d'_i = p$ $d_c = 1$	erpendi perpend	cular d icular c	istance listance	e of <i>i-</i> th e of cer	particle	of mass m_i from g	from para tiven axis	illel axis <i>l</i> 1	′•	┶╞		- m	bar	e
hmo		perpend	icular o	distanc direct	e betw	een <i>l</i> and	láhmoo	d Pre	pared	(Give		1/	•	par	1
hmo	From	figure,	sin θ	$\frac{d_i}{d_i} = \frac{d_i}{ \mathbf{r}_i }$	$D \Rightarrow d$	$i = \mathbf{r}_i \text{ si}$	$n \theta_i = \mathbf{e} \times$	rd Pre	pared	axis	ı le	/ ri	•	par	E
hmo	Simila Mom	arly, d'_i	= e × ertia of	\mathbf{r}'_i and the bo	$d d_c =$ dv about	= e × r _c ut given a	lahmoo	d Pre	pared	- 0 ⁰	o Vá	C−r.	•	bar	E
hmo	R_{l}	$\sum m_i d_i^2$	$\pm \sum_{i}$	$m_i(\mathbf{e})$	$(\mathbf{r}_i)^2$	$=\sum m_i($	$(\mathbf{e} \times \mathbf{r}_i) \cdot (\mathbf{e})$	$(\mathbf{r}_i)^{\text{pre}}$	pared	nor	17			— y par	e
hmo	bod	Prep		by:	Dr. A		ahmoox (r + r)	d Pre	parec	X /	igure)	•		pai	1
hmo	od	Prep	area	by: [Dr. A	mir N	lahmoo	d Pre	pared	by: D	r. Ami	r Mah	mood	Prepar	1
hmo	od	Prepa	$ar\bar{e}\sum_{i}$	$m_i(\mathbf{e})$	\mathbf{r}_{c}	$\mathbf{e} \times \mathbf{r}'_i \cdot \mathbf{e}$	$(\mathbf{e} \times \mathbf{r}_c + \mathbf{e})$	r i)	pared	by: D	r. Ami	Mah	mood	Prepar	e
nmo	od	Prepa	ire∑	<i>m_i</i> [(e	$\times \mathbf{r}_{c})$	$(\mathbf{e} \times \mathbf{r}_c)$	$+ 2(\mathbf{e} \times \mathbf{r}_c)$	$\cdot (\mathbf{e} \times \mathbf{r}'_i)$	+ (e × r	$(e \times i) \cdot (e \times i)$	r í)]Amii	Mah	mood	Prepar	e
hmo	od	Prepa	arest	$m_i[(e$	$\mathbf{e} \times \mathbf{r}_c$)	$e^{2} + 2(e = 1)$	$(\mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}_c)$	r_{i}^{\prime}) + (e	$(\mathbf{r}_i')^2$	by: D	r. Ami	r Mah	mood	Prepar	e
hmo	od	Prepa	ared	by: [Dr. A	mir M	aomoo	d Pre	pared	by: D	r. Ami	r Mah	mood	Prepar	e
hmo	od	Prepa	ared	$\left(\frac{m_i}{i}\right)$	(e × r	$\frac{1}{c})^2 + 2($	$\mathbf{e} \times \mathbf{r}_c \cdot \sum_i$	$m_i (\mathbf{e} \times \mathbf{i})$	$\sum_{i} + \sum_{i}$	$m_i(\mathbf{e} \times \mathbf{e})$	r¦)² Ami	r Mah	mood	Prepar	e
hmo	od	Prepa	n≘ M	$d_c^2 + 2$	$2(e \times r_c)$.) · (e ×	$\sum m_i \mathbf{r}'_i +$	$\sum m_i d'_i$	2 arewh	ere, M	$=\sum m_i$	= total m	ass of the l	bodypar	e
hmo	od	Prepa	ared	by: [Dr. A	mir M	làhmoo	d ⁱ Pre	pared	by: D	r. Ami	r Mah	mood	Prepar	e
hmo	Als	$o, \sum r$	$n_i \mathbf{r}'_i \equiv$	0 , as r	i' is the	position	vector of <i>i</i> t	h particle	of mass r	m _i with 1	respect to	centre of	mass <i>"C</i> " a	andepar	e
hmo	od	Prepa	ared	by: [$\operatorname{Dr.}_{I_{I'}}$	$=\sum m_i d$	$d_i'^2 = mome$	ent of iner	tia of the	body ax	r "Ami	Mah	mood	Prepar	e
nmo	od	Prepa	ared	by: [Dr. A	m <mark>r M</mark>	$\frac{1}{h_{1}} = L_{1} + N$	$\frac{1}{d^2}$	oared	by: D	r. Ami	Mah	mood	Prepar	e
hmo	Prob	lem: Sta	ite and	prove	e para	llel axis	theorem for	or the ca	se of mo	oment c	of inertia	for a co	ntinuous	<u>mass</u> par	e
hmo	distri State	ibution. nent: Tł	ie mom	ent of	inertia	of a rigid	l body in the	form of	a continu	ious mas	s distribu	tion abou	it a given a	Prepar axis is	e
hmo	equal mass	to the sof the b	um of ody an	momer d the r	nt of in momen	ertia of s t of iner	ame body a tia due to th	bout a pa ne total m	arallel axi nass of th	is (to the ie body j	e given ax placed at :	s) throu ts centre	gh the center of mass, a	tre of about	e
hmo	given Proof	axis. : Consid	er a rig	id body	v. in the	e form of	a h m o o	d Pre	oared istributio	by: D	e the give	n and l' h	mood be an axis y	Prepar	e
hmo	is par	allel to l	and pa	issing t	hrough	centre o	of mass of th	e body.	pared	by: D	r. Ami	r Mah	mood	Prepar	e
hmo	$\mathbf{r} = \mathbf{p}$	osition	vector	of elem	entary	mass dm	with respe	ct to origi	n <i>"0"</i> ed	by: D	r. Ami	r Mah	mood	Prepar	e
hmo	S₫	osition	vector	of elem	ientary	mass dr	n with respe	ct to cent	re of mas	s C D	r. Ami	r Mah	mood	Prepar	E
	Prena	red hv	Dr Am	ir Mahr	mood	main NA	abmaa	d Dee	a sea of	Law D	- A	0.4 - 1-	P P	age 5	G.

hmoPrepared by: Dr. Amir Mahmood mir Mahmood Prepared by: Dr. Amir Mahmood Page 5 pare

hmood	Prepared by: Dr	. Amir Mahmood	Prepared	by: Dr. Ami	r Mahmood	Prepare
hmood Mec	Prepared by: Dr hanics Made Easy	. Amir Mahmood	Prepared	by: Dr. Ami	r Mahmood Moment of In	Prepare
hmor _c =	position vector of centre of	of mass "C" with respect to	origin " <i>O</i> " red	by: Dr. Ami	r Mahmood	Prepare
$hm \phi d = d' = d' = d'$	perpendicular distance of	elementary mass dm from g elementary mass dm from g	given axis <i>l</i>	ZŁ	Ľ	Dipare
hmoad	perpendicular distance o	f centre of mass C from give	n axis l	- I /	/_d′ _	pare
hmo <u>ođ</u>	 perpendicular distance b unit vector in the directior 	etween l and l' and of given line l	Prepar		d dm	pare
hm Fror	n figure, $\sin \theta = \mathbf{e} \times \mathbf{r}' $ and σ	$d/ \mathbf{r} \Rightarrow d = \mathbf{r} \sin\theta =$	e × r bai	(11)		pare
hmoMon	nent of inertia of the body	about given axis <i>l</i> is given b	yPrepai G	iven 🖌 e	/ \^	pare
hmobਰ	$\int_{M} d^2 \mathrm{d}m = \int_{M} (\mathbf{e} \times \mathbf{r})^2 \mathrm{d}m =$	$\int_{M} (\mathbf{e} \times \mathbf{r}) \cdot (\mathbf{e} \times \mathbf{r}) dm$	Prepar a	xis l 🖌	r. Ac	pare
hmood	Prepare $\int [e \times (r_c + r)]$	$[\mathbf{e} \times (\mathbf{r}_c + \mathbf{r}')] \mathrm{d}m $: \mathbf{r}	$= \mathbf{r}_c + \mathbf{r}^a \mathbf{r}$	6	×P	pare
hmood	Prepared by: Dr	. Amir Mahmood	Prepai		/	ypare
hmood	$= \int_{M} [(\mathbf{e} \times \mathbf{r}_{c}) \cdot (\mathbf{e} \times \mathbf{r}_{c})] d\mathbf{r}_{M}$	$\mathbf{e} \times \mathbf{r}_c + 2(\mathbf{e} \times \mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}') +$	(e × r') (e × r') Prepar	d_c	\square	pare
hmood	$Prepa= \int_{M} [(\mathbf{e} \times \mathbf{r}_{c})^{2}]$	$+2(\mathbf{e} \times \mathbf{r}_c) \cdot (\mathbf{e} \times \mathbf{r}') + (\mathbf{e} \times \mathbf{r}')$	$[']^2]dm_{area}$	by or. Ami	r ivianmood	Prepare
hmood	Prepare $\left(\int dm\right) (le \times$	\mathbf{r}_{c}]) ² + 2($\mathbf{e} \times \mathbf{r}_{c}$) · $\int (\mathbf{e} \times \mathbf{r}') d\mathbf{r}_{c}$	$m + \int (\mathbf{e} \times \mathbf{r}')^2 d\mathbf{r}$	dm Dr. Ami	r Mahmood	Prepare
hmood	Prepared by: Dr	. Amir Mahmood	Prepared	by: Dr. Ami	r Mahmood	Prepare
hmood	$= Md_c^2 + 2(\mathbf{e} \times$	\mathbf{r}_c) $\cdot \left(\mathbf{e} \times \int_M \mathbf{r}' dm \right) + \int_M d'^2 dr$	n, where, $M =$	$= \int \mathrm{d}m = \mathrm{total} \mathrm{mas}$	ss of the body	Prepare
hmoAlso	, $\int_M \mathbf{r}' dm = 0$, as \mathbf{r}' is the	position vector of mass elen	nent dm with res	spect to centre of	mass "C", mood	Prepare
and	$I_{l'} = \int_M d^{\prime 2} dm = \text{momen}$ Prepared by Dr	t of inertia of the body axis $I \rightarrow I_{1} - I_{2} + Md^{2}$	l' Hence prove	y: Dr. Ami	r Mahmood	Prepare
hmo Pro	blem: Prove in matrix no	$\frac{1}{\text{tation that}} \begin{bmatrix} \mathbf{\dot{L}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega} \times \mathbf{L} \end{bmatrix}$	+ [I][ώ], <u>whe</u>	re, all the notati	ons used have the	iPrepare
hmo Proc	<u>al meanings.</u> o f : As we know that the an	gular momentum of a syster	n of particles is g	given byr. Ami	r Mahmood	Prepare
hmood	Prepared by: Dr	$Amir \mathbf{L} = \sum \mathbf{r}_i \times (m_i \mathbf{v})$	$(r_i) = \sum m_i \mathbf{r}_i \times \mathbf{v}_i$	by: Dr. Ami	r Mahmood	Prepare
hmo Diffe	erentiating both sides with	respect to time "t", we get	Prepared	by: Dr. Ami	r Mahmood	Prepare
hmood	$Pre_{i}\mathbf{\dot{L}}=\sum m_{i}\dot{\mathbf{r}}_{i}\times\mathbf{v}_{i}+$	$-\sum m_i \mathbf{r}_i imes \dot{\mathbf{v}}_i = \sum m_i \mathbf{v}_i imes$	$\mathbf{v}_i + \sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_i$	by: Dr. Ami	r Mahmood	Prepare
hmood	Prepared by: Dr	$\sum_{m=1}^{i} \frac{d}{d} \sum_{m=1}^{i} \frac{d}{d} $	Prepared	and $\mathbf{v}_i - \frac{d\mathbf{v}_i}{d\mathbf{v}_i}$	d Mahmood	Prepare
hmood	Prepared by: Dr		Prepared	by: Dr. Ami	Mahmood	Prepare
hmood	Prepared by: Dr	$\sum_{i} m_i \mathbf{r}_i \times [(\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) + (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)]$	$[\mathbf{r}_i)] = \sum_i m_i \mathbf{r}_i \times$	$(\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) + \sum_i m$	$_{i}\mathbf{r}_{i} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_{i})$	Prepare
hmoWrit	ing in matrix form, we get	Amir Mahmood	Prepared	by: Dr. Ami	r Mahmood	Prepare
hmood	Prepared by:	$= \left \sum m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) \right + \left \sum m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) \right $	$m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)$		r Mahmood	Prepare
hmood	As we know that V	Amir Mahmoo	Prepared	by: Dr. Ami	Mannood	Prepare
hmood	Prepared by: Dr		Prepared I	bv: Dr. Ami	r Mahmood	Prepare
hmood	Prepared by: Dr	. Amir Mahn $\sum m_i \mathbf{r}$	$\mathbf{v}_i \times \mathbf{v}_i = [\mathbf{I}][\boldsymbol{\omega}]$	by: Dr. Ami	r Mahmood	Prepare
hmood	Prepared by: Dr	Amir Mannood	Preparado	by: Dr. Ami	r Mahmood	Prepare
hmood	Prepared by: Dr	$ = \prod_{i=1}^{n} m_i \mathbf{r}_i \times (\mathbf{\omega}) $	$r_i = [1][\omega]$	by: Dr. Ami	r Mahmood	Prepare

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rebared by Mechanics Made Easy Replace ω by $\dot{\omega}$ on both sides, we get $\mathbf{r}_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) = [\mathbf{I}] [\dot{\boldsymbol{\omega}}]$ Prebared Prepared by: Now consider, rec vlanmood $m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i) = m_i \mathbf{r}_i \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = m_i \mathbf{r}_i \times [(\boldsymbol{\omega} \times \mathbf{v}_i) + \mathbf{v}_i)]$ $(\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{r}_i$ $\mathbf{r}_i)\boldsymbol{\omega}$ $\sum m_i [(\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_i \times \mathbf{r}_i)] = \sum m_i (\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega}) - - -$ ebared Further consider that $\boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{v}_i) = \boldsymbol{\omega} \times [\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = \boldsymbol{\omega} \times [(\mathbf{r}_i \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i] = (\mathbf{r}_i \cdot \mathbf{r}_i)(\boldsymbol{\omega} \times \boldsymbol{\omega}) - (\mathbf{r}_i \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{r}_i)$ $= -(\mathbf{r}_i \cdot \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{r}_i) = (\boldsymbol{\omega} \cdot \mathbf{r}_i)(\mathbf{r}_i \times \boldsymbol{\omega})$ ÷ω× Using (4) in (3), we get DL AMI $\sum m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \sum m_i \boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{v}_i) = \boldsymbol{\omega} \times \sum \mathbf{r}_i \times (m_i \mathbf{v}_i) =$ $\therefore \mathbf{L} = \mathbf{r}_i \times (m_i \mathbf{v})$ $\omega \times L$ Writing above equation in matrix form, we get, $m_i \mathbf{r}_i$ $\times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i)$ $= |\omega \times L|$ od Prepared by:)r. Amir Mahm Using (2) and (5) in (1), we get bod Prepared by: Dr. $A[\mathbf{L}] = [\boldsymbol{\omega} \times \mathbf{L}] + [\mathbf{L}][\boldsymbol{\omega}]$ Prephence proved. Problem: Show that inertia matrix [I] is a Cartesian tensor of rank 2. Proof: As we know that the angular momentum of a system of particles is given by $\mathbf{r}_{\alpha} \times (m_{\alpha} \mathbf{v}_{\alpha}) = \sum m_{\alpha} (\mathbf{r}_{\alpha} \times \mathbf{v}_{\alpha}) = \sum m_{\alpha} (\mathbf{r}_{\alpha} \times (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}))$ $= \omega \times r_{\alpha}$ epared by: Dr. A α $\sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_{\alpha}) \mathbf{r}_{\alpha}]$ $m_{\alpha}[(\mathbf{r}_{\alpha}\cdot\mathbf{r}_{\alpha})\boldsymbol{\omega}-(\mathbf{r}_{\alpha}\cdot\boldsymbol{\omega})\mathbf{r}_{\alpha}]=$ and Let, $\boldsymbol{\omega}=(\omega_1,\,\omega_2,\,\omega_3)$ L3), $\mathbf{r}_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ Then, pre $\omega_1 x_{\alpha,1} + \omega_2 x_{\alpha,2} + \omega_3 x_{\alpha,3} =$ $\omega_j x_{\alpha,j}$ od Prepared by: $\sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha}^{2}(\omega_{1}, \omega_{2}, \omega_{3}) - \mathbf{r}_{\alpha}^{2}(\omega_{1}, \omega_{3}, \omega_{3}) - \mathbf{r}_{\alpha}^{2}(\omega_{$ So (1) can be written as: $(L_1, L_2, L_3) =$ $\omega_i x_{i,\alpha}$ $x_{\alpha,2}, x_{\alpha,3}$ $\omega_j x_{\alpha,j}$ $x_{\alpha,i}$ $\mathbf{r}_{\alpha}^{2}\omega_{i}$ m_{lpha} α $\sum \omega_j x_{\alpha,j} | x_{\alpha,i}$ $\langle \omega_i \delta_{ii} \rangle$ m_{α} m_{α} ω_i $x_{\alpha,i}|\omega_i =$ $m_{\alpha} \mathbf{r}_{\alpha}^2 \delta_i$ = ij'th component of ine where Since, both the angular velocity $\boldsymbol{\omega} = (\omega_i)$ and the angular momentum $\mathbf{L} = (L_i)$ are known to be vectors (i.e., Cartesian tensors of rank 1), it follows from equation (2) and quotient theorem that the inertia tensor $[\mathbf{I}] = (I_{ii})$ is a Cartesian tensor of rank 2. Problem: Express angular momentum in tensor notation. **Solution:** As we know that the angular momentum of a system of particles Prepared by: Dr. Amir Mahmood

r. Amir Mahmood Prepared Moment of Inertia Mechanics Made Easy $\sum m_{\alpha}(\mathbf{r}_{\alpha} \times \mathbf{v}_{\alpha}) = \sum m_{\alpha}(\mathbf{r}_{\alpha} \times (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}))$ $\mathbf{r}_{\alpha} \times (m_{\alpha} \mathbf{v}_{\alpha}) = \gamma$ \mathbf{v}_{α} $= \omega \times \mathbf{r}_{\alpha}$ Mahmood Prepared by: Dr. Amir Mahmood α $\sum m_{\alpha}[(\mathbf{r}_{\alpha}\cdot\mathbf{r}_{\alpha})\boldsymbol{\omega}]$ $-(\mathbf{r}_{\alpha}\cdot\boldsymbol{\omega})\mathbf{r}_{\alpha}] = \sum m_{\alpha}[\mathbf{r}_{\alpha}^{2}\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r}_{\alpha})\mathbf{r}_{\alpha}]$ and $\mathbf{r}_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ $\mathbf{L} = (L_1, L_2, L_3), \qquad \mathbf{\omega} = (\omega_1, \omega_2, \omega_3)$ Let, Then, Prepa $\mathbf{\omega} \cdot \mathbf{r}_{\alpha} = \omega_1 x_{\alpha,1} + \omega_2 x_{\alpha,2} + \omega_3 x_{\alpha,3} = \sum \omega_j x_{\alpha,j}$ pared by Dr Amir repare Prepared by: Dr. Amir Mahmoo So, (1) can be written as $m_{\alpha} | \mathbf{r}_{\alpha}^2(\omega_1, \omega_2,$ $\sum \omega_j x_{j,\alpha}$ ur Vlahm mα Mahmood $\overline{j=1}$ $\Rightarrow L_i = \sum m_{\alpha} \left| \mathbf{r}_{\alpha}^2 \omega_i - \left(\sum \omega_j x_{\alpha,j} \right) x_{\alpha,i} \right|, \qquad i = 1, 2, 3$ repare repare $\sum m_{\alpha} \left| \mathbf{r}_{\alpha}^{2} \sum_{i=1}^{n} \omega_{j} \delta_{ij} - \left(\sum_{i=1}^{n} \omega_{j} x_{\alpha,i} \right) x_{\alpha,i} \right| = \mathsf{par}$ $\omega_i = \lambda$ $\omega_i \delta_{ii}$ Prepare Prepared by: Dr. 003 3 Prepare $\sum_{\alpha} m_{\alpha} \sum_{j=1} \left[\mathbf{r}_{\alpha}^{2} \delta_{ij} - x_{\alpha,j} x_{\alpha,i} \right] \omega_{j} = \sum_{j=1} \omega_{j} \sum_{\alpha} m_{\alpha} \left[\mathbf{r}_{\alpha}^{2} \delta_{ij} \right]$ $x_{\alpha,i}x_{\alpha,j} =$ repared where, repart $I_{ij} = \sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}] = ij$ th component of inertia tensor. A min Mahm repare Equation (2) is required tensor form of angular momentum. Problem: Express rotational kinetic energy in tensor notation. Prepare Solution: As we know that the rotational kinetic energy of a system is given by $\frac{1}{2}\omega$ epared Prepared by: Dr. Amir Mal repare od by: Dr. A Let, $\mathbf{P} \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3), \quad \mathbf{L} = (L_1, L_2, L_3)$ ood Prepared Prepare by: Dr. An $\Rightarrow T_{rot} = \frac{1}{2}(\omega_1 L_1 + \omega_2 L_2 + \omega_3 L_3) = \frac{1}{2}$ > wiLiA $\sum_{i=1}^{3} \omega_j \sum_{\alpha} m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}]$ epared $\overline{2}\sum_{i=1}^{\omega_i}\omega_i$ $\omega_j \sum m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij}]$ $r_{\sim i} = \frac{1}{2} \sum_{i=1}^{n} \omega_i \omega_j I$ $=\frac{1}{2}\sum_{i}\omega_{i}$ $\sum_{\alpha=1}^{3} \omega_j \left(\sum_{\alpha} m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij} - x_{\alpha} \right) \right)$ $-x_{\alpha,i}x_{\alpha,j}$ $[x_{\alpha,j}] = \frac{1}{2} \sum_{i,j=1} \omega_i \omega_j$ i=1 j=1 $\sum_{\alpha} m_{\alpha} [\mathbf{r}_{\alpha}^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j}] = ij$ th component of inertia tensor where, Equation (1) is required tensor form of rotational kinetic energy. Problem: Express parallel axis theorem in tensor notation for discrete mass distribution. Solution: Consider a rigid body in the form of discrete mass distribution (i.e., a set of particles). Let, C be the centre of mass of the body. We consider two parallel coordinate systems Oxyz and Cx'y'z', as shown in the figure. Let, M = total mass of the body min Mahmood Prepared b \mathbf{r}_{α} = position vector of α -th particle of mass m_{α} with respect to origin "O" \mathbf{r}'_{α} = position vector of α -th particle of mass m_{α} with respect to centre of mass "C" $\mathbf{r}_c = \text{position vector of centre of mass "}\mathcal{C}$ " with respect to origin " \mathcal{O} " From figure, a re $\mathbf{r}_{\alpha} = \mathbf{r}_{c} + \mathbf{r}_{\alpha}' = - - - -$

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epared by: repared by: D Prepared Mechanics Made Easy Let, $\mathbf{r}_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}), \quad \mathbf{r}_{c} = (x_{c,1}, x_{c,2}, x_{c,3}) \text{ and } \mathbf{r}_{\alpha}' = (x_{\alpha,1}', x_{\alpha,2}', x_{\alpha,3}')$ Equation (1) becomes $(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$ ood Prepar $x_{\alpha,i} = x_{c,i} + x'_{\alpha,i}$ i = 1, 2, 3 for a prepare (2) As we know that $\begin{bmatrix} p_{\alpha,i} x_{\alpha,i} \\ x_{\alpha,i} \\ x_{\alpha,i} \end{bmatrix}$ $\mathbf{x}_{\alpha,i}\mathbf{x}_{\alpha,j}] = \sum m_{\alpha} [(\mathbf{r}_{\alpha} \cdot \mathbf{r}_{\alpha})\delta_{ij}]$ Mahamood Prepared by: Dr. Amir $m_{\alpha}\left[\left((\mathbf{r}_{c}+\mathbf{r}_{\alpha}')\cdot(\mathbf{r}_{c}+\mathbf{r}_{\alpha}')\right)\delta_{ij}-\left(x_{c,i}+x_{\alpha,i}'\right)\left(x_{c,j}+x_{\alpha,j}'\right)\right]$ (by using (1) and (2)) $\sum m_{\alpha} [(\mathbf{r}_{c} \cdot \mathbf{r}_{c} + 2 \mathbf{r}_{c} \cdot \mathbf{r}_{\alpha}' + \mathbf{r}_{\alpha}' \cdot \mathbf{r}_{\alpha}') \delta_{ij} - x_{c,i} x_{c,j} - x_{c,i} x_{\alpha,j}']$ $-x_{c,j}x'_{\alpha,i}$ $-x'_{\alpha,i}x'_{\alpha,i}$ Mahmood Prepared by: Dr. Amir $m_{\alpha} [(\mathbf{r}_{c}^{2} + 2 \mathbf{r}_{c} \cdot \mathbf{r}_{\alpha}' + \mathbf{r}_{\alpha}'^{2}) \delta_{ij}]$ $-x_{c,i}x_{c,j}$ $-x_{c,i}x'_{\alpha,j}$ – $x_{c,j}x_{\alpha,i}' - x_{\alpha,i}'x_{\alpha,j}'$ $Dare = \sum m_{\alpha} [\mathbf{r}_{\alpha}^{\prime 2} \delta_{ij} - x_{\alpha,i}^{\prime} x_{\alpha,j}^{\prime}] + 2 \mathbf{r}_{c} \cdot (\sum m_{\alpha} \mathbf{r}_{\alpha}^{\prime}) \delta_{ij} + (\sum m_{\alpha}) \mathbf{r}_{c}^{2} \delta_{ij} - \mathbf{r}_{c}$ m_{α} $x_{c,i}$ Now, $\sum m_{\alpha} [\mathbf{r}_{\alpha}'^2 \delta_{ij} - x'_{\alpha,i} x'_{\alpha,j}] = I'_{ij} = ij$ th component of inertia tensor with respect to Cx'y'z' system Also, $\sum m_{\alpha} \mathbf{r}'_{\alpha} = \mathbf{0}$, (:: \mathbf{r}'_{α} is the position vector of α -th particle of mass m_{α} with respect to centre of mass "C") d by: Dr. Amir Mahmood Repared by: Dr. Amir Mahmood Prepar $\sum_{\alpha} m_{\alpha} (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3}) = (0, 0, 0) \implies \sum_{\alpha} m_{\alpha} x'_{\alpha,i} = 0, \quad i = 1, 2, 3$ \therefore $\mathbf{r}'_{\alpha} = (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$ And, $m_{\alpha} = M = \text{total mass of the body}$ Prepared by Prepared by: Dr. A So equation (3) becomes $I_{ij} = I'_{ij} + M\mathbf{r}_c^2 \delta_{ij} - Mx_{c,i} x_{c,j}$ This is required tensor form of parallel axis theorem for discrete mass distribution **Problem:** Express parallel axis theorem in tensor notation for a continuous mass distri Solution: Consider a rigid body in the form of continuous mass distribution. Let, C be the centre of mass of We consider two parallel coordinate systems Oxyz and Cx'y'z', as shown in the figure. Let, M = total mass of the body \mathbf{r} = position vector of elementary mass dm with respect to origin "0" = 0 $\mathbf{r}' =$ position vector of elementary mass dm with respect to centre of mass "C" \mathbf{r}_{c} = position vector of centre of mass "*C*" with respect to origin "*O*" From figure, $\mathbf{r}_{c} = (x_{c,1}, x_{c,2}, x_{c,3})$ and $\mathbf{r}' = (x'_{1}, x'_{2}, x'_{3})$ Let, $\mathbf{r} = (x_1, x_2, x_3)$, So, equation (1) becomes $(x_1, x_2, x_3) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_1, x'_2, x'_3)$ i = 1, 2, 3 $x_i = x_{c,i} + x_i',$ are ir iylai $\int_{M} \left[\mathbf{r}^{2} \delta_{ij} - x_{\alpha,i} x_{\alpha,j} \right] \mathrm{d}m = \int_{M} \left[(\mathbf{r} \cdot \mathbf{r}) \delta_{ij} - x_{\alpha,i} x_{\alpha,j} \right] \mathrm{d}m$ $= \left[\left[\left((\mathbf{r}_{c} + \mathbf{r}') \cdot (\mathbf{r}_{c} + \mathbf{r}') \right) \delta_{ij} - (x_{c,i} + x_{i}') (x_{c,j} + x_{j}') \right] dm \right]$ (by using (1) and (2)) lahmood Prepared by: Dr. Amir N $[(\mathbf{r}_c \cdot \mathbf{r}_c + 2 \mathbf{r}_c \cdot$ $x'_i x'_i | dm$ δ_{ii} $-x_{c,i}x_i'$ $-x_{c,i}x_i'$ $-x_{c,i}x_{c,i}$ Prepared by: Dr. Amir Mahmood

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$I_{ii} = \int [(\mathbf{r}_{c}^{2} + 2 \mathbf{r}_{c} \cdot \mathbf{r}' + \mathbf{r}'^{2}) \delta_{ii} - x_{ci} x_{ci} - x_{ci} x'_{i} - x_{ci} x'_{i} - x'_{i} x'_{i}] dm$	epare
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hmod Prepare $\int [\mathbf{r}'^2 \delta_{ij} - x'_i x'_j] dm + 2\mathbf{r}_c \cdot \left(\int \mathbf{r}' dm\right) \delta_{ij} + \left(\int dm\right) \mathbf{r}_c^2 \delta_{ij} - \left(\int dm\right) x_{c,i} x_{c,j} - \left(\int x'_j dm\right) x_{c,i}$	epare
hmood Prepared by: D) Amir Mahmood Prepared by: Dr. Amir Mahmood Pr	epare
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Now, $\Pr[\mathbf{r}'^2 \delta_{ij} - x'_i x'_j] dm = I'_{ij} = ij$ th component of inertia tensor with respect to $Cx'y'z'$ system OOO	epare
hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Pl Also $\int \mathbf{r}' d\mathbf{m} = 0$ (:: \mathbf{r}' is the position vector of mass element $d\mathbf{m}$ with respect to centre of mass "C")	epare
hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Pr	epare
hmod Prepa $\Rightarrow \int (x'_1, x'_2, x'_3) dm = (0, 0, 0) \Rightarrow 0 \int x'_i dm = 0, ri = 1, 2, 3 Dr. A: r' = (x'_1, x'_2, x'_3) dP$	repare
hmood Prepared by: Dr. Amir Mahmood	epare
hmood Prepared by: Dr. Amir Mahmood Prepared by Or. Amir Mahmood Pr	epare
hm od Prepared by: Dr. Amir $I_{ij} = I'_{ij} + Mr_c^2 \delta_{ij} - Mx_{c,i}x_{c,j}$ Dr. Amir Mahmood Pr	epare
This is required tensor form of parallel axis theorem for continuous mass distribution. Problem: State and prove parallel axis theorem for the case of products of inertia for a discrete ma	repare
hmodistribution ared by: Dr. Amir Mahmood Prepared by: Dr. Amir Ma	epare
centre of mass of the body. If $Oxyz$ and $Cx'y'z'$ be two parallel coordinate systems as shown in figure, then we have	epare
$I_{ij} = I'_{ij} - Mx_{c,i}x_{c,j}, i \neq j, i,j \in \{1,2,3\}$ $I_{ij} = \text{product of inertia with respect to } Oxyz\text{-system}$	bare
hm l'_{ij} = product of inertia with respect to $Cx'y'z'$ -system d Prepared by: Dr	bare
$(x_{c,1}, x_{c,2}, x_{c,3}) =$ position vector of centre of mass "C" with respect to origin "O" M = total mass of the body	_y ^{pare}
Proof : Consider a rigid body, in the form of a set of particles. Let, \mathbf{r}_{α} = position vector of α -th particle of mass m_{α} with respect to origin " O "	pare
\mathbf{r}'_{α} = position vector of α -th particle of mass m_{α} with respect to centre of mass "C" \mathbf{r}_{α} = position vector of centre of mass "C" with respect to origin "O"	bare
From figure, ared by: $Dr r_{\alpha} = r_{c} + r_{\alpha}' = 1 \mod -1 \implies (1) \operatorname{red} by: Dr$	pare
So, equation (1) becomes $(x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) = (x_{c,1}, x_{c,2}, x_{c,3}) + (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3})$	pare
hmood Prepare xai = xci + x'ai mi + 1,2,3mood Prepare (2) by: Dr. Amir Mahmood Pr	epare
how consider for $i \neq j$, $I_{ij} = -\sum_{\alpha} m_{\alpha} x_{\alpha,i} x_{\alpha,j} = -\sum_{\alpha} m_{\alpha} (x_{c,i} + x'_{\alpha,i}) (x_{c,j} + x'_{\alpha,j})$ min Mahmood Pr	epare
hmod Prepared = $-\left(\sum m_{\alpha}\right)x_{c,i}x_{c,j} - \left(\sum m_{\alpha}x'_{\alpha,j}\right)x_{c,i} - \left(\sum m_{\alpha}x'_{\alpha,i}\right)x_{c,j} - \sum m_{\alpha}x'_{\alpha,i}x'_{\alpha,j} - \cdots \rightarrow (3)$	epare
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hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Pr	epare
hand Also, $\Pr_{\Box} \sum_{i} m_{\alpha} x'_{\alpha,i} x'_{\alpha,j} = I'_{ij} = \text{product of inertia with respect to } Cx'y'z' - system A mir Mahmood Pr$	epare
hand $\operatorname{Prep}'_{\lambda} m_{\alpha} \mathbf{r}'_{\alpha} = 0 \Rightarrow \sum m_{\alpha} (x'_{\alpha,1}, x'_{\alpha,2}, x'_{\alpha,3}) = (0, 0, 0) \Rightarrow \sum m_{\alpha} x'_{\alpha,i} = 0, i = 1, 2, 3$	epare
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hmo	distr State	<u>ibution.</u> ment: Consi	der a rig	gid boc	ly, in the	e form of a	a continu	ious mass dis	tributio	n. Let,	C be th	ne centre c	f mass o	Prepare
hme	body	. If Oxyz and	Cx'y'z	I' be tw $I_{ij} = I'_{ij}$	vo parall – Mx _{c,i}	el coordir x _{c,j} , i ₹	iate syste ≤ j, i,j	ems as shown ∈ {1, 2, 3}	in the f	igure,	then we	have	lood	Prepare
hmo	$I_{ij} = I'_{ii} =$	product of i product of i	nertia w nertia w	vith res vith res	spect to (Oxyz-syst Cx'v'z'-sy	em od	Prepare	d b	i	Z		Ĭ	dm pare
hmc	$(x_{c,1})$	$(x_{c,2}, x_{c,3}) =$	= posit	ion veo	tor of ce	entre of m	ass "C " v	with respect t	o origin	"0"		r 🌙	<th>, pare</th>	, pare
hmo	M =	f: Consider a	rigid bo	ody ody, in	the form	of a set o	of particle	Prepare es.	d b			r	λ_{c}	-{y'pare
hmo	r = p	position vect	or of ele tor of ele	menta ementa	ry mass ary mass	d <i>m</i> with 1 s d <i>m</i> with	respect to respect t	o origin " <i>O"</i> to centre of m	ass "C"			~		
hmo	$\mathbf{r}_c =$ From	position vec 1 figure,	tor of ce	entre o	f mass " $r = r_c +$	<i>C</i> " with re r' – – –	espect to	origin " O " \rightarrow (1)	db	0			<u> </u>	pare
hme	Let,	$\mathbf{r} = (x_1, x_2, x_3, x_4)$	$(x_3), y_1$	$\mathbf{r}_c = (z)$	$x_{c,1}, x_{c,2},$	$x_{c,3}$) and $x_{c,3}$	$\mathbf{d} \mathbf{r}' =$	(x'_1, x'_2, x'_3)						pare
hmo	30, 20		$x_i = x_{c,i}$	$x_i + x_i'$	i = 1	$1, 2, 3 - (x_{c,1})$	$ \overset{\lambda_{C,2}}{-} \overset{\lambda_{C,2}}{-}$	$(x_1, x_2, \dots, x_n) \rightarrow (2)$))	Dr.	Amir	Ivianm	1000	Prepare
nme	Now	consider for	i≠j,	$I_{ij} =$	$-\int x_i x$	$c_j \mathrm{d}m = -$	$\int (x_{c,i} +$	$(x_{c,j} + x_j')(x_{c,j} + x_j')$)d <i>m</i>	3 ⁰ r.	Amir	Iviann	1000	Prepare
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hme	bod	Prepar		$m \int x_{c}$	$i^{x_{c,j}}$	$\int_{M} x_j dm$	$x_{c,i} \supset \left(\prod_{i} \right)$	$\int_{M} x_{i} dm \int x_{c,j}$	$-\int_{M} x_i x$	c _j dm	Amir	Iviann	1000	Prepare
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nmo	AISO,		$x_j um =$		product	ormertia	with res	pect to cx y.	z -syster	Dr.	Amir	Mahm	boo	Propare
nmo	And	$\int \mathbf{r}' \mathrm{d}m = 0$	$0 \Rightarrow \int (\mathbf{z})$	$x'_{1}, x'_{2},$	x'_3)dm	$= \left(\int x'_1 d \right)$	$m, \int x_2'$	dm , $\int x'_2 dm$) = (0,0),0) ⇒	$\int x'_i \mathrm{d}n$	ı = 0, i =	= 1, 2, 3	Propare
nmo	So ec	\tilde{M} [uation (3) g	й tives	Dr.	Amir	M	TO MIC	Prepare	a by:	Dr.	M	Mahm	000	Drepare
nmo	od	Prepare	ea by	$I_{ij} =$	$I_{ij}' - M$	$x_{c,i}x_{c,j},$	<i>i</i> ≠ <i>j</i> ,	$i, j \in \{1, 2, 3\}$	Hen	ce pro	ved.	Iviann	000	Prepare
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$$\Rightarrow I_{xx} = \frac{1}{2}Ma^2$$

Example 4: Find the moment of inertia of a (uniform) circular disc of Mass M and radius a about (i) an axis passing through its centre and perpendicular to its plane, (*ii*) its diameter. Solution: (i) Moment of inertia about central axis: Let M, a and σ , respectively, be the mass, radius and surface (areal) mass density of the disc. Choose coordinate axes as shown in figure. We divide disc into large number of concentric circular rings of infinitesimal widths. One typical elementary ring

of mass dm, radius r, width dr and area dA is shown in the figure.

Moment of inertia of typical elementary ring about z-axis is given by

$$\mathrm{d}I_{zz} = r^2 \mathrm{d}m$$

Thus, moment of inertia of disc about z-axis is

$$I_{zz} = \int_{\text{Disc}} r^2 dm$$

$$= 2\pi\sigma \int_{\text{Disc}} r^3 dr$$

$$= \frac{2M}{a^2} \int_{r=0}^a r^3 dr = \frac{2M}{a^2} \left(\frac{a^4}{4}\right) = \frac{1}{2}Ma^2$$

$$\therefore \sigma = \frac{dm}{dA} = \frac{dm}{(2\pi r)dr} = \text{constant}$$

$$\therefore \sigma = \frac{M}{\pi a^2} \text{ (for disc)}$$

(ii) Moment of inertia about diameter: By perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx},$$

$$\Rightarrow I_{xx} = \frac{1}{4}Ma^{2}$$

Example 5: Find the moment of inertia of a (uniform) elliptical plate with semi-major axis and semi minor axis a and b, respectively **about** (i) major axis,

(*ii*) minor axis,

(iii) an axis passing through centre of plate and perpendicular to its plane.

Solution: Consider an elliptical plate in xy-plane whose boundary curve is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b$$

Let M and σ , respectively, be the mass and surface (areal) mass density of the elliptical plate. To find moment of inertia about major axis (x-axis), we proceed as follows. We divide plate into large number of

elementary rectangular pieces of infinitesimal areas with sides parallel to x and y axis. One typical area element, located at point (x, y), having mass dm, area dS, length dx and width dy is shown in the figure. Moment of inertia of typical area element about x-axis is given by

$$\mathrm{d}I_{xx} = y^2 \mathrm{d}m$$

Thus, moment of inertia of elliptical plate about x-axis is

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Put $x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta, \ x = 0 \Rightarrow \theta = 0, \ x = a \Rightarrow \theta = \pi/2$

$$I_{xx} = \frac{4Mb^2}{3\pi a^4} \int_{x=0}^{\pi/2} a^4 \cos^4\theta d\theta$$

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Using Wallis cosine formula, we get,

$$I_{xx} = \frac{4Mb^2}{3\pi} \left(\frac{3}{4}\right) \left(\frac{1}{2}\right) \left(\frac{\pi}{2}\right) = \frac{1}{4}Mb^2 \tag{3}$$

Similarly, moment of inertia about minor axis is $I_{yy} = \frac{1}{4}Ma^2$ By perpendicular axis theorem, the moment of inertia about the

By perpendicular axis theorem, the moment of inertia about the axis passing through centre of the elliptical plate and perpendicular to its plane, is

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{4}M(a^2 + b^2)$$
(4)

Corollary: The moment of inertia of a (uniform) circular disc of radius a about (i) its diameter and (ii) an axis passing through its centre and perpendicular to its plane can be obtained by putting b = a in (3) and (4), to give (respectively)

$$I_{xx} = \frac{1}{4}Ma^2 \tag{5}$$

and

$$I_{zz} = \frac{1}{2}Ma^2 \tag{6}$$

Note that, the results obtained in (5) and (6) are in accordance (as they should be) with the results, obtained in (2) and (1), respectively.

Example 6: Find the moment of inertia of a (uniform) triangular lamina (i.e., two dimensional triangular plate) of mass M about one of its sides. **Solution:** Let M and σ , respectively, be the mass and surface (areal) mass density of the triangular lamina lying in xy-plane. Choose x-axis and y-axis as shown in figure. We divide lamina into large number of strips of infinitesimal widths parallel to the base AB of lamina. One typical elementary strip DE of mass dm, width dy and area dS is shown in the figure.

Moment of inertia of typical elementary strip about side AB (x-axis) is given by

$$\mathrm{d}I_{xx} = y^2 \mathrm{d}m$$

Thus, moment of inertia of triangular lamina about x-axis is

$$I_{xx} = \int_{\text{Triangular lamina}} y^2 dm = \sigma \int_{\text{Triangular lamina}} y^2 |DE| dy$$

$$\therefore \sigma = \frac{dm}{dS} = \frac{dm}{|DE|dy} = \text{constant}$$

$$= \frac{2M}{h} \int_{\text{Triangular lamina}} y^2 \frac{|DE|}{|AB|} dy$$

$$\therefore \sigma = \frac{M}{\frac{1}{2}|AB|h} \text{ (for tiangular lamina)}$$

From equivalent triangles ABC and DEC, we have

$$\frac{|DE|}{|AB|} = \frac{\text{height of triangle } DEC}{\text{height of triangle } ABC} = \frac{h-y}{h}$$

$$\Rightarrow \qquad I_{xx} = \frac{2M}{h} \int_{\text{Triangular lamina}} y^2 \left(\frac{h-y}{h}\right) dy$$

$$= \frac{2M}{h^2} \int_{y=0}^h y^2 (h-y) dy = \frac{2M}{h^2} \left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{1}{6} Mh^2$$

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hmood Example 7: Calculate the inertia matrix of a (uniform solid) rectangular box (rectangular paralhmood lelopiped or cuboid) of mass M at one of its corners, by taking coordinate axes along its edges.

Solution: Let M and ρ , respectively, be the mass and volume mass density of the rectangular box. Let the lengths of adjacent edges be a, b and c. Choose coordinate axis along the edges of box, as shown in inmood figure. We divide box into large number of elementary rectangular boxes of infinitesimal volumes. One hmood typical elementary volume element of mass dm, volume dV and dimensions dx, dy and dz, is shown in hmood the figure.

> Moment of inertia of typical elementary volume element (or elementary box) about x-axis is given by

$$\mathrm{d}I_{xx} = (y^2 + z^2)\mathrm{d}n$$

Thus, moment of inertia of rectangular box about x-axis is

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$$I_{yy} = \frac{M}{3} (a^2 + c^2)$$
 and $I_{zz} = \frac{M}{3} (a^2 + b^2)$

For product of inertia

$$I_{xy} = \int_{\text{Rectangular box}} x \, y \, \mathrm{d}m = -\frac{M}{abc} \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} x \, y \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = -\frac{M}{abc} \left(\frac{a^2}{2}\right) \left(\frac{b^2}{2}\right) c = -\frac{1}{4} Mab$$

hmood Similarly,

$$I_{yz} = -\frac{1}{4}Mbc$$
 and $I_{xz} = -\frac{1}{4}Mac$

The required inertia matrix is given by

$$[I_O] = \begin{bmatrix} (1/3)M(b^2 + c^2) & -(1/4)Mab & -(1/4)Mac \\ -(1/4)Mab & (1/3)M(a^2 + c^2) & -(1/4)Mbc \\ -(1/4)Mac & -(1/4)Mbc & (1/3)M(a^2 + b^2) \end{bmatrix} = \frac{1}{12}M \begin{bmatrix} 4(b^2 + c^2) & -3ab & -3ac \\ -3ab & 4(a^2 + c^2) & -3bc \\ -3ac & -3bc & 4(a^2 + b^2) \end{bmatrix}$$

Example 8: Calculate the inertia matrix of a (uniform solid) cube of mass M at one of its corners, by taking coordinate axes along its edges.

Solution: Repeat example 7 for a = b = c and get

$$[I_O] = \frac{1}{12} M a^2 \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$



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hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare d Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare Mechanics Made Easy Moment of Inertia Example 9: Find the moment of inertia of a (uniform solid) hemisphere of mass M about hmood (i) its axis of symmetry (*ii*) an axis perpendicular to the axis of symmetry and passing hmood through the centre of the base. Solution: (i) Moment of inertia about axis of symmetry: hmood Let M, a and ρ , respectively, be the mass, base radius and volume mass density of the hemisphere. Choose coordinate axes as shown in figure. ahmood Moment of inertia of typical volume element of hemisphere, having mass dm and volume dV, about z-axis is given by hmood $\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$ hmood Thus, moment of inertia of hemisphere about z-axis is $I_{zz} = \int_{\text{Hemisphere}} (x^2 + y^2) dm = \rho \int_{\text{Hemisphere}} (x^2 + y^2) dV$ hmood $\therefore \rho = \frac{\mathrm{d}m}{\mathrm{d}V} = \mathrm{constant}$ hmood $\therefore \rho = \frac{M}{(2/3)\pi a^3}$ $= \frac{3M}{2\pi a^3} \int_{\text{Homisphere}} (x^2 + y^2) \,\mathrm{d}V$ (for hemisphere) hmood To make the computation simpler, we transform the problem from Cartesian coordinates (x, y, z) to spherical hmood coordinates (r, θ, ϕ) by using $z = r\cos\theta$ $x = r \sin \theta \cos \phi, \qquad y = r \sin \theta \sin \phi,$ hmood where, volume element in spherical coordinates is given by ahmood $dV = dr (r d\theta) (r \sin \theta d\phi) = r^2 \sin \theta dr d\theta d\phi$ hmood $\Rightarrow \quad x^2 + y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\sin^2\theta(\sin^2\phi) = r^2\theta(\sin^2\phi) = r$ hmood $0 \le r \le a, \qquad 0 \le \theta \le \pi/2, \qquad 0 \le \phi < 2\pi$ hmood For hemisphere, $\Rightarrow I_{zz} = \frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin^3\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4 \, \mathrm{d}r \int_{\theta=0}^{\pi/2} \sin^3\theta \, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi$ hmood (7)hmood $\int_{\theta=0}^{\pi/2} \sin^3 \theta \,\mathrm{d}\theta = \frac{1}{4} \int_{\theta=0}^{\pi/2} (3\sin\theta - \sin 3\theta)$ $\therefore \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$ hmood where, hmood $= \frac{1}{4} \left(-3\cos\theta + \frac{1}{3}\cos 3\theta \right) \left| \frac{\pi/2}{\theta = 0} \right|^{\pi/2} = \frac{1}{4} \left(3 - \frac{1}{3} \right) = \frac{2}{3}$ hmood Using (8) in (7), we get hmood $I_{zz} = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5}\right) \left(\frac{2}{3}\right) (2\pi) = \frac{2}{5}Ma^2$ hmood (ii) Moment of inertia about a diameter of the base: hmood $I_{xx} = \int_{\text{Hemisphere}} (y^2 + z^2) \mathrm{d}m = \frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} (y^2 + z^2) \, \mathrm{d}V$ hmood Transforming problem in spherical coordinates (r, θ, ϕ) , we get hmood $I_{xx} = \frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 (\sin^3\theta \sin^2\phi + \cos^2\theta \sin\theta) \mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\phi$ hmood $=\frac{3M}{2\pi a^3}\int_{r=0}^a r^4 \,\mathrm{d}r \left(\int_{\theta=0}^{\pi/2} \sin^3\theta \,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin^2\phi \,\mathrm{d}\phi + \int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta \,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \,\mathrm{d}\phi\right),$ (9)hmood where, hmood

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 $\int_{\phi=0}^{2\pi} \sin^2 \phi \, \mathrm{d}\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1 - \cos 2\phi) \, \mathrm{d}\phi = \frac{1}{2} \left(\phi - \frac{1}{2} \sin 2\phi \right) \left| \begin{array}{c} 2\pi \\ \phi = 0 \end{array} \right|_{\phi=0}^{2\pi} = \frac{1}{2} (2\pi) = \pi$ (10)

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 $\int_{\theta=0}^{\pi/2} \cos^2 \theta \sin \theta \, \mathrm{d}\theta = -\frac{1}{3} \cos^3 \theta \Big|_{\theta=0}^{\pi/2} = \frac{1}{3}$ (11)

Using (8), (10) and (11) in (9), we get

$$I_{xx} = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5}\right) \left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5}\right) \left(\frac{4\pi}{3}\right) = \frac{2}{5}Ma^2$$

Example 10: Find three products of inertia of a (uniform) solid hemisphere of mass M with respect to coordinate axes as in figure of example 9. Solution:

Now,

$$I_{xz} = -\int_{\text{Hemisphere}} x \, z \, \mathrm{d}m = -\frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} x \, z \, \mathrm{d}V = -\frac{3M}{2\pi a^3} \int_{r=0}^a \int_{\theta=0}^{a} \int_{\phi=0}^{2\pi} r^4 \sin^2 \theta \cos \theta \, \mathrm{d}\phi \, \mathrm{d}\phi$$

$$I_{xy} = I_{xz} = I_{yz} = 0,$$
 $\therefore I_{xz} = I_{yz} \text{ (by symmetry)}$

Example 11: Find the moments and products of inertia of a (uniform solid) sphere of mass M and radius a with respect to its axes of symmetry.

Solution: (i) Moment of inertia about axis of symmetry:

Let M, a and $\overline{\rho}$, respectively, be the mass, radius and volume mass density of the sphere. Choose coordinate axes as shown in figure. Moment of inertia of typical volume element of sphere, having mass dm and volume dV, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}n$$

Thus, moment of inertia of sphere about z-axis is

$$I_{zz} = \int_{\text{Sphere}} (x^2 + y^2) dm$$

= $\rho \int_{\text{Sphere}} (x^2 + y^2) dV$ $\therefore \rho = \frac{dm}{dV} = \text{constant}$
= $\frac{3M}{4\pi a^3} \int_{\text{Sphere}} (x^2 + y^2) dV$ $\therefore \rho = \frac{M}{(4/3)\pi a^3}$ (for sphere)

To make the computation simpler, we transform the problem from Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) by using

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

where, volume element in spherical coordinates is given by

$$dV = dr (r d\theta) (r \sin \theta d\phi) = r^2 \sin \theta dr d\theta d\phi$$

$$\Rightarrow \quad x^2 + y^2 = r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\sin^2\theta$$

For sphere, $0 \le r \le a$, $0 \leq \theta \leq \pi$, $0 \le \phi < 2\pi$

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$$dI_{yy} = dI_o + (dm)z^2 = \frac{1}{4}r^2dm + (dm)z^2 = \left(\frac{1}{4}r^2 + z^2\right)dm = \frac{3M}{a^2h}\left(\frac{1}{4}r^4 + r^2z^2\right)dz, \quad \boxed{\because dm = \rho \, dV = \frac{3M}{\pi a^2h}(\pi r^2 \, dz)} \quad epare = \frac{3M}{2}\left(\frac{1}{4}r^4 + r^2z^2\right)dz$$

centre of the base O) is given by

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 $\int_{\theta=0}^{\pi/2} \sin^3 \theta \,\mathrm{d}\theta = \frac{1}{4} \int_{\theta=0}^{\pi/2} (3\sin\theta - \sin 3\theta)$ $\therefore \quad \sin 3\theta = 3\sin \theta - 4\sin^3 \theta$ $= \frac{1}{4} \left(-3\cos\theta + \frac{1}{3}\cos 3\theta \right) \Big|_{\theta=0}^{\pi/2} = \frac{1}{4} \left(3 - \frac{1}{3} \right) = \frac{2}{3}$

Moment of Inertia

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$$I_{zz} = \frac{Ma^2}{2\pi} \left(\frac{2}{3}\right) (2\pi) = \frac{2}{3}Ma^2$$

(ii) Moment of inertia about a diameter of the base:

$$I_{xx} = \int_{S} (y^2 + z^2) dm = \sigma \int_{S} (y^2 + z^2) dS = \frac{M}{2\pi a^2} \int_{S} (y^2 + z^2) dS, \quad S: \text{ hemispherical shell}$$

Using parametric equations of hemispherical shell, we get

$$\int_{\phi=0}^{2\pi} \sin^2 \phi \, \mathrm{d}\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1 - \cos 2\phi) \, \mathrm{d}\phi = \frac{1}{2} \left(\phi - \frac{1}{2} \sin 2\phi \right) \, \left| \begin{array}{c} 2\pi \\ \phi = 0 \end{array} \right|_{\phi=0}^{2\pi} = \frac{1}{2} (2\pi) = \pi \tag{15}$$

$$\int_{\theta=0}^{\pi/2} \cos^2 \theta \sin \theta \, \mathrm{d}\theta = -\frac{1}{3} \cos^3 \theta \left| \frac{\pi/2}{\theta=0} \right|^2 = \frac{1}{3}$$
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Using (13), (15) and (16) in (14), we get

$$I_{xx} = \frac{Ma^2}{2\pi} \left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{Ma^2}{2\pi} \left(\frac{4\pi}{3}\right) = \frac{2}{3}Ma^2$$

Example 16: Find three products of inertia of a (uniform) hemispherical shell of mass M with respect to coordinate axes as in figure of example 15.

$$I_{xy} = -\int_S x \, y \, \mathrm{d}m = -\sigma \int_S x \, y \, \mathrm{d}S = -\frac{M}{2\pi a^2} \int_S x \, y \, \mathrm{d}S, \quad S: \text{ hemispherical shell}$$

Using parametric equations of hemispherical shell, we get

$$I_{xy} = -\frac{M}{2\pi a^2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} a^4 \sin^3 \theta \sin \phi \cos \phi \, \mathrm{d}\theta \, \mathrm{d}\phi = -\frac{Ma^2}{2\pi} \int_{\theta=0}^{\pi/2} \sin^3 \theta \, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, \mathrm{d}\phi$$

$$\int_{\phi=0}^{2\pi} \sin\phi \cos\phi \,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi \Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

$$I_{xz} = -\int_{S} x \, z \, \mathrm{d}m = -\frac{M}{2\pi a^2} \int_{S} x \, z \, \mathrm{d}S = -\frac{M}{2\pi a^2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} a^4 \sin^2 \theta \cos \theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$
$$= -\frac{Ma^2}{2\pi} \int_{\theta=0}^{\pi/2} \sin^2 \theta \cos \theta \, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \cos \phi \, \mathrm{d}\phi$$

$$\int_{\phi=0}^{2\pi} \cos\phi \,\mathrm{d}\phi = \sin\phi \Big|_{\phi=0}^{2\pi} = 0 \implies I_{xz} = 0 = I_{yz}, \qquad \qquad \boxed{\because I_{xz} = I_{yz} \text{ (by symmetry)}}$$

$$I_{xy} = I_{xz} = I_{yz} = 0$$

Prepare Example 17: Find the moments and products of inertia of a (uniform) spherical shell of mass M and ra-Prepare with dius respect its axes of symmetry.

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hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare **Mechanics Made Easy** Moment of Inertia epare

hmoo hmoo hmoo ahmoo hmoo hmoo hmood hmoo hmoo hmoo hmoo ahmoo hmoo hmood hmood hmood hmood hmood hmoo hmood hmood hmood hmood hmood hmood hmood hmoo hmood

Solution: (i) Moment of inertia about axis of symmetry: Let M, a and σ , respectively, be the mass, radius of base and areal mass density of the spherical shell. Choose coordinate axes as shown in figure.

Moment of inertia of typical area element of spherical shell, with mass dm and area dS, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of spherical shell about z-axis is

 $I_{zz} = \int_{S} (x^2 + y^2) \mathrm{d}m, \qquad S: \text{ spherical shell}$

To make the computation simpler, we use the parametric equations of spherical shell as follows

 $x = a\sin\theta\cos\phi,$ $y = a\sin\theta\sin\phi,$ $z = a \cos \theta$

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For spherical shell: $0 \le \theta \le \pi,$ $dS = (a d\theta) (a \sin \theta d\phi) = a^2 \sin \theta d\theta d\phi$

$$x^{2} + y^{2} = a^{2}(\sin^{2}\theta\cos^{2}\phi + \sin^{2}\theta\sin^{2}\phi) = a^{2}\sin^{2}\theta(\cos^{2}\phi + \sin^{2}\phi) = a^{2}\sin^{2}\theta$$

$$M = \ell^{\pi} - \ell^{2\pi}$$

$$M = \ell^{\pi} - \ell^{2\pi} - \ell^{2\pi}$$

$$\Rightarrow I_{zz} = \frac{M}{4\pi a^2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} a^4 \sin^3\theta \,\mathrm{d}\theta \,\mathrm{d}\phi = \frac{Ma^2}{4\pi} \int_{\theta=0}^{\pi} \sin^3\theta \,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi, \tag{17}$$

where,

$$= \frac{1}{4} \left(-3\cos\theta + \frac{1}{3}\cos 3\theta \right) \left| \begin{matrix} \pi \\ \theta = 0 \end{matrix} = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3}$$
(18)

Using (18) in (17), we get

$$I_{zz} = \frac{Ma^2}{4\pi} \left(\frac{4}{3}\right) (2\pi) = \frac{2}{3}Ma^2$$

(ii) Products of inertia with respect to axes of symmetry:

$$I_{xy} = -\int_{S} x \, y \, \mathrm{d}m = -\sigma \int_{S} x \, y \, \mathrm{d}S = -\frac{M}{4\pi a^2} \int_{S} x \, y \, \mathrm{d}S, \quad S: \text{ spherical shell}$$

Using parametric equations of spherical shell, we get

$$I_{xy} = -\frac{M}{4\pi a^2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} a^4 \sin^3\theta \sin\phi \cos\phi \,\mathrm{d}\theta \,\mathrm{d}\phi$$

$$= -\frac{Ma^2}{4\pi} \int_{\theta=0}^{\pi} \sin^3\theta \,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin\phi \cos\phi \,\mathrm{d}\phi$$

But

$$\int_{\phi=0}^{2\pi} \sin\phi \cos\phi \,\mathrm{d}\phi = \frac{1}{2}\sin^2\phi \left| \begin{array}{c} 2\pi\\ \phi=0 \end{array} \right| \implies I_{xy} = 0$$

Similarly,

$$I_{yz} = I_{xz} = 0$$
 $\therefore I_{xy} = I_{yz} = I_{xz}$ (by symmetry)

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hmoo	od Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood P Mechanics Made Easy	repare
hmod	Definition: A set of three mutually perpendicular axes having origin <i>O</i> which are fixed in the rigid body	andpar
hmod	axes of inertia" or simply "principal axes" of body at point 0.	repare
hmod	Definition: An axis is called " <u>principal axis of inertia</u> " or simply " <u>principal axis</u> " of a rigid body if direction	s of
	Theorem: Above two definitions of principal axes are Conversely , suppose that for a rigid body we	have
nmoo	equivalent.	ually
hmod	Proof: Suppose that for a rigid body we have three perpendicular axes for which second definition in mutually concurrent and mutually perpendicular axes for Choosing these axes as Cartesian coordinate axes	, and
hmod	which first definition holds. Choosing these axes as assuming that body rotates about $x - axis$, we have a summary that body rotates about $x - axis$, we have a summary that body rotates about $x - axis$.	have, ar
hmod	to this coordinate system is given by	repare
hmod	Prepared $\begin{bmatrix} I_{11} & 0 & 0\\ 0 & I_{22} & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & 0 & 0\\ 0 & I_{22} & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & 0\\ 0 & \lambda_2 & 0 \end{bmatrix}$ above Prepared $\begin{bmatrix} \lambda_1 & \lambda_2 & $	repare
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hmod	velocity has the form $\boldsymbol{\omega}_x = \begin{pmatrix} \omega_{x1} \\ 0 \end{pmatrix}$ As we know that $[\mathbf{L}_x] = [\mathbf{I}][\boldsymbol{\omega}_x]$	repare
hmod	As we know that $[\mathbf{L}_x] = [\mathbf{I}][\boldsymbol{\omega}_x]$ As we know that $[\mathbf{L}_x] = [\mathbf{I}][\boldsymbol{\omega}_x]$ As we know that $[\mathbf{L}_x] = \begin{bmatrix} I_{1} \\ I_{2} \end{bmatrix} = \begin{bmatrix} I_{12} \\ I_{22} \end{bmatrix} \begin{bmatrix} I_{12} \\ I_{22} \end{bmatrix} \begin{bmatrix} u_{23} \\ 0 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{12} \\ u_{23} \end{bmatrix} \rightarrow \mathbf{I}_{12}$	(2)
hmod	$ = \begin{pmatrix} L_{x1} \\ L_{x2} \end{pmatrix} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \end{pmatrix} \begin{pmatrix} \omega_{x1} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{11}\omega_{x1} \\ 0 \end{pmatrix} = \begin{pmatrix} L_{x3} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} I_{13}\omega_{x1} \\ I_{13}\omega_{x1} \end{pmatrix} $ From (1) and (2), we have	repare
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hmod	$d^{11}\left(r_{0}^{0}\right)$ ared by: Dr. Amir Mahmood Prevared by: $r_{0}A$ $hi(I_{13}\omega_{x1})$ mood P	repar
nmoo	$\Rightarrow \mathbf{L}_{x} = I_{11}\boldsymbol{\omega}_{x}$ This shows that angular momentum is parallel to angular Similarly, assuming the rotation of body about	repare
mag	velocity. Similarly, we can show that when body rotates axis ($\mathbf{L}_y = \lambda_2 \boldsymbol{\omega}_y$), we get, $I_{12} = I_{23} = 0$.	renard
11100	\rightarrow All product of inertia and gave Hence first define	ition
hmod	about y or z axis then angular momentum is parallel to angular velocity. Hence second definition also holds for also holds for given axes. (Note: $\lambda_i = I_{ii}$, $i = 1,2,3$)	nition 3)
hmod	about y or z axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes. (Note: $\lambda_i = I_{ii}$, $i = 1,2,3$)	nition 3)e pare
hmoc	 about <i>y</i> or <i>z</i> axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes. (Note: λ_i = l_{ii}, i = 1,2,3) Definition: The moment of inertia with respect to a principal axis is called "principal moment of inertia". 	nition 3) pare
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hmod hmod hmod hmod hmod hmod hmod	about y or z axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes. Definition: The moment of inertia with respect to a principal axis is called " <u>principal moment of inertia</u> ". Theorem: Prove that for a rigid body a set of three mutually perpendicular principal axes exists at given point. Proof: As we know from the definition of principal axis that if a rigid body rotates bout principal axes, pass through a point <i>O</i> , then the angular momentum L and the angular velocity $\boldsymbol{\omega}$ of the body are in same direction we can write, $\mathbf{L} = \lambda \boldsymbol{\omega}$, where, λ is constant Let, $\mathbf{L} = L_1 \mathbf{i} + L_2 \mathbf{j} + L_3 \mathbf{k}$, $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$ Then, $L_1 \mathbf{i} + L_2 \mathbf{j} + L_3 \mathbf{k} = \lambda(\omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k})$ Comparing corresponding components on both sides of above vector equation, we get $L_1 = \lambda \omega_1$, $L_2 = \lambda \omega_2$, $L_3 = \lambda \omega_3 \rightarrow (1)$ As we know that, $\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{12} & l_{22} & l_{23} \\ l_{13} & l_{23} & l_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \rightarrow \rightarrow (2)$ From (1) and (2), we get, $l_1 \omega_1 + l_{12} \omega_2 + l_{13} \omega_3 = \lambda \omega_1$	nition 3) e pare repare sing are repare repare repare repare repare repare
hmoc hmoc hmoc hmoc hmoc hmoc hmoc hmoc	about y or z axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes. Definition: The moment of inertia with respect to a principal axis is called "principal moment of inertia". Theorem: Prove that for a rigid body a set of three mutually perpendicular principal axes exists at giv point. Proof: As we know from the definition of principal axis that if a rigid body rotates bout principal axes, pass through a point 0, then the angular momentum L and the angular velocity $\boldsymbol{\omega}$ of the body are in same direction we can write, $\mathbf{L} = \lambda \omega$, where, λ is constant Let, $\mathbf{L} = L_1 \mathbf{i} + L_2 \mathbf{j} + L_3 \mathbf{k}$, $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$ Then, $L_1 \mathbf{i} + L_2 \mathbf{j} + L_3 \mathbf{k} = \lambda (\omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k})$ Comparing corresponding components on both sides of above vector equation, we get $L_1 = \lambda \omega_1$, $L_2 = \lambda \omega_2$, $L_3 = \lambda \omega_3 \rightarrow (1)$ As we know that, $\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{12} & l_{23} & l_{33} \end{bmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \rightarrow (2)$ From (1) and (2), we get, $I_1 + U_2 \mathbf{j} + U_3 \mathbf{k} = I_{10} \omega_1$ $I_1 = (\mathbf{k} + U_1 + U_2 - U_1 + U_2 - U_1 + U_2 - $	nition 3) = p are repare sing are repare repare repare repare repare repare repare repare
hmod hmod hmod hmod hmod hmod hmod hmod	about <i>y</i> or <i>z</i> axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes. Definition: The moment of inertia with respect to a principal axis is called "principal moment of inertia". Theorem: Prove that for a rigid body a set of three mutually perpendicular principal axes exists at given point. Proof: As we know from the definition of principal axis that if a rigid body rotates bout principal axes, pass through a point <i>O</i> , then the angular momentum L and the angular velocity $\boldsymbol{\omega}$ of the body are in same direction we can write, $\mathbf{L} = \lambda \omega$, where, λ is constant Let, $\mathbf{L} = L_1 \mathbf{i} + L_2 \mathbf{j} + L_3 \mathbf{k}$, $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$ Then, $L_1 \mathbf{i} + L_2 \mathbf{j} + L_3 \mathbf{k}$, $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$ Comparing corresponding components on both sides of above vector equation, we get $L_1 = \lambda \omega_1$, $L_2 = \lambda \omega_2$, $L_3 = \lambda \omega_3 \rightarrow (1)$ As we know that, $\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{12} \\ I_{12} \\ I_{23} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \rightarrow (2)$ From (1) and (2), we get, $I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = \lambda \omega_1$ $I_{12}\omega_1 + I_{22}\omega_2 + I_{13}\omega_3 = \lambda \omega_3$ This system can be written as, $(L = 2)\omega_1 + L_2(\omega_1 + L_2)\omega_2 + I_{23}\omega_3 = \lambda \omega_3$	nition 3) e pare repare sing are repare repare repare repare repare repare repare repare repare
hmod hmod hmod hmod hmod hmod hmod hmod	about <i>y</i> or <i>z</i> axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes. Definition: The moment of inertia with respect to a principal axis is called "principal moment of inertia". Theorem: Prove that for a rigid body a set of three mutually perpendicular principal axes exists at given point. Proof: As we know from the definition of principal axis that if a rigid body rotates bout principal axes, pass through a point <i>O</i> , then the angular momentum L and the angular velocity $\boldsymbol{\omega}$ of the body are in same direction we can write, $\mathbf{L} = \lambda \boldsymbol{\omega}$, where, λ is constant Let, $\mathbf{L} = L_1 \mathbf{i} + L_2 \mathbf{j} + L_3 \mathbf{k}$, $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$ Then, $L_1 = \lambda \omega_1$, $L_2 = \lambda \omega_2$, $L_3 = \lambda \omega_3 \rightarrow (1)$ As we know that, $\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_1 \\ I_1 \\ I_2 \\ I_1 \end{bmatrix} \begin{bmatrix} U_1 \\ I_2 \\ I_1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_2 \end{bmatrix} \begin{bmatrix} U_1 \\ I_1 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \rightarrow (2)$ From (1) and (2), we get, $I_1 + L_2 \mathbf{y} + L_1 \mathbf{w} + L_1 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} = L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} = L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} = L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} = L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} = L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} = L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} = L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_1 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} + L_2 \mathbf{w} + L_1 \mathbf{w} + L_2 \mathbf{w} +$	nition a)epare arepare singpare repare repare repare repare repare repare repare repare repare repare repare
hmod hmod hmod hmod hmod hmod hmod hmod	about y or z axis then angular momentum is parallel to angular velocity. Hence second definition also holds for given axes. Definition: The moment of inertia with respect to a principal axis is called "principal moment of inertia". Theorem: Prove that for a rigid body a set of three mutually perpendicular principal axes exists at given point. Proof: As we know from the definition of principal axis that if a rigid body rotates bout principal axes, pass through a point 0, then the angular momentum L and the angular velocity ω of the body are in same direction we can write, $\mathbf{L} = \lambda \omega$, where, λ is constant Let, $\mathbf{L} = L_1 \mathbf{i} + L_2 \mathbf{j} + L_3 \mathbf{k}$, $\omega = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$ Theorem: corresponding components on both sides of above vector equation, we get $L_1 = \lambda \omega_1$, $L_2 = \lambda \omega_2$, $L_3 = \lambda \omega_3 \rightarrow (1)$ As we know that, $\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{12} & l_{23} & l_{33} \\ l_{13} & l_{23} & l_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \rightarrow (2)$ From (1) and (2), we get, This system can be written as, $\begin{pmatrix} (l_{11} - \lambda)\omega_1 + l_{12}\omega_2 + l_{13}\omega_3 = \lambda \omega_1 \\ l_{12}\omega_1 + (l_{22} - \lambda)\omega_2 + l_{23}\omega_3 = \lambda \omega_3 \\ l_{13}\omega_1 + l_{23}\omega_2 + l_{23}\omega_3 = \lambda \omega_3 \end{pmatrix}$ This is homogeneous system of three equations in three unknowns ω_1 , ω_2 and ω_3 . This system will have	nition a) pare repare ven are pare repare repare repare repare repare repare repare repare repare repare repare repare

hmoPrepared by: Dr. Amir Mahmood, mir Mahmood, Prepared by: Dr. Amir Mahmoo Page 24 pare

hmod	od Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepa	an
hmoo	A Prepared by: Dr. Amir Mahmood Prep	ar
hmod	trivial solution if an only if r. Amir Mahmood Prepared by: Dr. Amir Mahmood Prep	ar
hmod	d Prepared by: Dr. Amir Mahməədi، Prepared by: Dr. Amir Mahmood Prep	ar
hmod	pd Prepared by: Dr. Amir Mah I_{12}^{12} $I_{22}^{-\lambda}$ I_{23}^{23} d by: Dr. Amir Mahmood Prepared	ar
hmod	This is cubic equation in <i>I</i> which is called characteristic equation of inertia matrix [I]. It has three roots, say, λ_1 , λ_2 and λ_3 , which are, in fact, principal moments of inertia. By substituting $\lambda = \lambda_1$ in system (3), we can obtain the	are
hmo	ratios $\omega_1: \omega_2: \omega_3$, which give direction of principal axes relative to which moment of inertia is λ_1 . Similarly, we	ar
hmor	can find direction of other two principal axes corresponding to moments of inertia λ_2 and λ_3 . We can always find three mutually perpendicular principal axes because [1] is symmetric. This shows that there exists three mutually	ari
h	perpendicular principal axes passing through given point <i>O</i> .	90
nmod	Problem: A triangular plate is made of uniform material and has sides of lengths a, 2a and $\sqrt{3}a$.	an
hmoo	Determine the (direction of) principal axes and corresponding principal moments of inertia at 30° corper (or vertex)	are
hmod	Solution: Let M and σ , respectively, be the mass and surface (areal) mass density of triangular plate OAB lying in	ar
hmar	<i>xy</i> -plane, as shown in the figure, with $ OA = \sqrt{3}a$, $ AB = a$ and $ OB = 2a$.	ar
nmod	Clearly, $ OB ^2 = (2a)^2 = (\sqrt{3}a)^2 + a^2 = OA ^2 + AB ^2$.	
hmod	This shows that OAB is right angled triangle with right angle at A.	ar
hmod	Also, $\tan(m \angle AOB) = \frac{1}{ OA } = \frac{1}{\sqrt{3}a} \implies m \angle AOB = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^{\circ}$.	are
hmo	moments of inertia at vertex 0. The moment of inertia of triangular	ar
hmo	plate about side OA (x-axis) is given by $2a$	ar
	The moment of inertia of triangular plate about side AB is given by C	are
nmoc	$I_{AB} = \frac{1}{2}M OA ^2 = \frac{1}{2}M(\sqrt{3}a)^2 = \frac{1}{2}Ma^2$ 30	210
nmod	Let C be the centre of mass of the plate and take D on OA and E on D D A	are
hmod	OB such that DE is passing through C and parallel to AB . Z	are
hmod	Then moment of inertia of plate about <i>DE</i> is given by (using parallel axis theorem) $I_{DE} = I_{AB} - M AD ^2 = \frac{1}{2}Ma^2 - M AD ^2 \rightarrow (1)$	are
	From figure. $ AD = OA - OD = \sqrt{3}a - (x \text{-coordinate of centre of mass C}) = \sqrt{3}a - \frac{1}{2}(x_0 + x_1 + x_2)$	
nmoc	$= \sqrt{3}a - \frac{1}{2}(0 + \sqrt{3}a + \sqrt{3}a) = \sqrt{3}a - \frac{2\sqrt{3}a}{\sqrt{3}a - 2\sqrt{3}a} = \frac{\sqrt{3}a}{\sqrt{3}a - 2\sqrt{3}a} = \frac{a}{\sqrt{3}a} = - \rightarrow (2)$	
hmod	$a^{-\sqrt{3}a} = \sqrt{3}a^{-\sqrt{3}a} = \sqrt{3}a^{-$	are
hmod	Using (2) in (1), we get, $I_{DE} = \frac{1}{2}Ma^2 - M(\frac{\pi}{\sqrt{3}}) = \frac{1}{2}Ma^2 - \frac{1}{3}Ma^2 = \frac{1}{6}Ma^2 = \frac{1}{6}Ma^2$	are
hmor	Then moment of inertia of plate about y-axis is given by (using parallel axis theorem), as follows, $1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} + \frac{\sqrt{2}$	are
	$I_{yy} = I_{22} = I_{DE} + M OD ^2 = \frac{1}{6}Ma^2 + M(x \text{-coordinate of centre of mass } C)^2 = \frac{1}{6}Ma^2 + M\left(\frac{0+\sqrt{3}a+\sqrt{3}a}{3}\right)$	
nmoc	$1 - (2\sqrt{3}a)^2 + 4 - Ma^2 + 8Ma^2 + 3$	210
nmod	d Prepared = $\frac{1}{6}Ma^2 + M\left(\frac{1}{3}\right) = \frac{1}{6}Ma^2 + \frac{1}{3}Ma^2 = \frac{1}{6}Ma^2 = \frac{1}{6}Ma^2 = \frac{1}{2}Ma^2 = $	are
hmod		100
	Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows, or Prepa	an
nmoc	Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows, $I_{zz} = I_{33} = I_{xx} + I_{yy} = \frac{1}{6}Ma^2 + \frac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$	are
hmod	Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows, $I_{zz} = I_{33} = I_{xx} + I_{yy} = \frac{1}{6}Ma^2 + \frac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$ $I_{xy} = I_{12} = -\int xy dm = -\sigma \int xy dx dy = -\sigma \int_{x=0}^{\sqrt{3}a} \left(\int_{y=0}^{\frac{x}{\sqrt{3}}} xy dy\right) dx = -\sigma \int_{x=0}^{\sqrt{3}a} \left(x \left(\frac{y^2}{2}\right)\Big _{y=0}^{\frac{x}{\sqrt{3}}}\right) dx \because dm = \sigma dx dy$	are are
hmoc hmoc	Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows, $I_{zz} = I_{33} = I_{xx} + I_{yy} = \frac{1}{6}Ma^2 + \frac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$ $I_{xy} = I_{12} = -\int xy dm = -\sigma \int xy dx dy = -\sigma \int_{x=0}^{\sqrt{3}a} \left(\int_{y=0}^{\frac{x}{\sqrt{3}}} xy dy\right) dx = -\sigma \int_{x=0}^{\sqrt{3}a} \left(x \left(\frac{y^2}{2}\right)\Big _{y=0}^{\frac{x}{\sqrt{3}}}\right) dx \because dm = \sigma dx dy$ $= -\frac{\sigma}{6} \int_{x=0}^{\sqrt{3}a} x^3 dx = -\frac{1}{6} \left(\frac{2M}{\sqrt{3}a^2}\right) \left(\frac{x^4}{4}\right)\Big _{x=0}^{\sqrt{3}a} = -\frac{1}{6} \left(\frac{2M}{\sqrt{3}a^2}\right) \left(\frac{9a^4}{4}\right) = -\frac{\sqrt{3}}{4}Ma^2 \because \sigma = \frac{M}{\frac{1}{2} OA AB } = \frac{M}{\frac{1}{2}(\sqrt{3}a)(a)} = \frac{2M}{\sqrt{3}a^2}$	are are are
hmoc hmoc hmoc hmoc	Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows, $I_{zz} = I_{33} = I_{xx} + I_{yy} = \frac{1}{6}Ma^2 + \frac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$ $I_{xy} = I_{12} = -\int xy dm = -\sigma \int xy dxdy = -\sigma \int_{x=0}^{\sqrt{3}a} \left(\int_{y=0}^{\frac{x}{\sqrt{3}}} xy dy\right) dx = -\sigma \int_{x=0}^{\sqrt{3}a} \left(x \left(\frac{y^2}{2}\right)\Big _{y=0}^{\frac{x}{\sqrt{3}}}\right) dx \because dm = \sigma dxdy$ $= -\frac{\sigma}{6} \int_{x=0}^{\sqrt{3}a} x^3 dx = -\frac{1}{6} \left(\frac{2M}{\sqrt{3}a^2}\right) \left(\frac{x^4}{4}\right)\Big _{x=0}^{\sqrt{3}a} = -\frac{1}{6} \left(\frac{2M}{\sqrt{3}a^2}\right) \left(\frac{9a^4}{4}\right) = -\frac{\sqrt{3}}{4}Ma^2 \because \sigma = \frac{M}{\frac{1}{2} OA AB } = \frac{M}{\frac{1}{2}(\sqrt{3}a)(a)} = \frac{2M}{\sqrt{3}a^2}$ As $z = 0$ in xy -plane, therefore, $I_{xz} = I_{13} = -\int xz dm = 0$ and $I_{yz} = I_{23} = -\int yz dm = 0$	are are are are
hmoc hmoc hmoc hmoc	Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows, $I_{zz} = I_{33} = I_{xx} + I_{yy} = \frac{1}{6}Ma^2 + \frac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$ $I_{xy} = I_{12} = -\int xy dm = -\sigma \int xy dxdy = -\sigma \int_{x=0}^{\sqrt{3}a} \left(\int_{y=0}^{\frac{x}{\sqrt{3}}} xy dy\right) dx = -\sigma \int_{x=0}^{\sqrt{3}a} \left(x\left(\frac{y^2}{2}\right)\Big _{y=0}^{\frac{x}{\sqrt{3}}}\right) dx \because dm = \sigma dxdy$ $= -\frac{\sigma}{6}\int_{x=0}^{\sqrt{3}a} x^3 dx = -\frac{1}{6}\left(\frac{2M}{\sqrt{3}a^2}\right)\left(\frac{x^4}{4}\right)\Big _{x=0}^{\sqrt{3}a} = -\frac{1}{6}\left(\frac{2M}{\sqrt{3}a^2}\right)\left(\frac{9a^4}{4}\right) = -\frac{\sqrt{3}}{4}Ma^2 \because \sigma = \frac{M}{\frac{1}{2} OA AB } = \frac{M}{\frac{1}{2}(\sqrt{3}a)(a)} = \frac{2M}{\sqrt{3}a^2}$ As $z = 0$ in xy -plane, therefore, $I_{xz} = I_{13} = -\int xz dm = 0$ and $I_{yz} = I_{23} = -\int yz dm = 0$ The inertia matrix at point 0 , with respect to coordinate system $Oxyz$, is given by	are are are are
hmoc hmoc hmoc hmoc	Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows, $I_{zz} = I_{33} = I_{xx} + I_{yy} = \frac{1}{6}Ma^2 + \frac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$ $I_{xy} = I_{12} = -\int xy dm = -\sigma \int xy dxdy = -\sigma \int_{x=0}^{\sqrt{3}a} \left(\int_{y=0}^{x} xy dy\right) dx = -\sigma \int_{x=0}^{\sqrt{3}a} \left(x \left(\frac{y^2}{2}\right)\Big _{y=0}^{x}\right) dx \because dm = \sigma dxdy$ $= -\frac{\sigma}{6} \int_{x=0}^{\sqrt{3}a} x^3 dx = -\frac{1}{6} \left(\frac{2M}{\sqrt{3}a^2}\right) \left(\frac{x^4}{4}\right)\Big _{x=0}^{\sqrt{3}a} = -\frac{1}{6} \left(\frac{2M}{\sqrt{3}a^2}\right) \left(\frac{9a^4}{4}\right) = -\frac{\sqrt{3}}{4}Ma^2 \because \sigma = \frac{M}{\frac{1}{2} OA AB } = \frac{M}{\frac{1}{2}(\sqrt{3}a)(a)} = \frac{2M}{\sqrt{3}a^2}$ As $z = 0$ in xy -plane, therefore, $I_{xz} = I_{13} = -\int xz dm = 0$ and $I_{yz} = I_{23} = -\int yz dm = 0$ The inertia matrix at point O , with respect to coordinate system $Oxyz$, is given by	are are are are are
hmoc hmoc hmoc hmoc	Then moment of inertia of plate about z-axis is given by (using perpendicular axis theorem), as follows, $I_{zz} = I_{33} = I_{xx} + I_{yy} = \frac{1}{6}Ma^2 + \frac{3}{2}Ma^2 = \frac{Ma^2 + 9Ma^2}{6} = \frac{10}{6}Ma^2 = \frac{5}{3}Ma^2$ $I_{xy} = I_{12} = -\int xy dm = -\sigma \int xy dxdy = -\sigma \int_{x=0}^{\sqrt{3}a} \left(\int_{y=0}^{x} xy dy\right) dx = -\sigma \int_{x=0}^{\sqrt{3}a} \left(x\left(\frac{y^2}{2}\right)\right) \Big _{y=0}^{x}\right) dx \because dm = \sigma dxdy$ $= -\frac{\sigma}{6} \int_{x=0}^{\sqrt{3}a} x^3 dx = -\frac{1}{6} \left(\frac{2M}{\sqrt{3}a^2}\right) \left(\frac{x^4}{4}\right) \Big _{x=0}^{\sqrt{3}a} = -\frac{1}{6} \left(\frac{2M}{\sqrt{3}a^2}\right) \left(\frac{9a^4}{4}\right) = -\frac{\sqrt{3}}{4}Ma^2 \because \sigma = \frac{M}{\frac{1}{2} OA AB } = \frac{M}{\frac{1}{2}(\sqrt{3}a)(a)} = \frac{2M}{\sqrt{3}a^2}$ As $z = 0$ in xy -plane, therefore, $I_{xz} = I_{13} = -\int xz dm = 0$ and $I_{yz} = I_{23} = -\int yz dm = 0$ The inertia matrix at point O , with respect to coordinate system $Oxyz$, is given by	are are are are are

Mechanics Made Eas $o] = \begin{pmatrix} I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{4}Ma^2 & \frac{3}{2}Ma^2 & 0 \\ -\frac{\sqrt{3}}{4}Ma^2 & \frac{3}{2}Ma^2 & 0 \end{pmatrix} = \begin{pmatrix} 2\alpha & -3\sqrt{3}\alpha & 0 \\ -3\sqrt{3}\alpha & 18\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$ I_{12} I_{13} To find the eigenvalues, we have the characteristic equation $det([I_0] - \lambda[I_3]) = 0$, where $[I_3]$ is unit matrix order 3. Prepared by: [-λ Prepared by: $det([\mathbf{I}_0])$ $\Rightarrow -3\sqrt{3}\alpha \quad 18\alpha - \lambda \qquad 0$ repared by: D On expanding by third row, we get, $\left[(20\alpha - \lambda) \left[(2\alpha - \lambda)(18\alpha - \lambda) - (-3\sqrt{3}\alpha)^2 \right] = 0 \right]$ $\Rightarrow (20\alpha - \lambda)[36\alpha^2 - 2\alpha\lambda - 18\alpha\lambda + \lambda^2 - 27\alpha^2]$ $\Rightarrow (20\alpha - \lambda)[\lambda^2]$ $-20\alpha\lambda + 9\alpha^2] = 0$ Either $20\alpha - \lambda$ $= 0 \Rightarrow \lambda = 20\alpha$ $20\alpha \pm \sqrt{(20\alpha)^2 - 4(1)(9\alpha^2)}$ $\lambda^2 - 20\alpha\lambda + 9\alpha^2 = 0 \Rightarrow$ or. epared by: Dr. Amir $\Rightarrow \lambda = \frac{20\alpha \pm \sqrt{400\alpha^2 - 36\alpha^2}}{20\alpha \pm \sqrt{364\alpha^2}} = \frac{20\alpha \pm \sqrt{364\alpha^2}}{20\alpha \pm \sqrt{364\alpha^2}}$ $20\alpha \pm 2\sqrt{91}\alpha$ $=(10\pm\sqrt{91})\alpha^{2}$ $\lambda_3 = (10 - \sqrt{91})\alpha$ Thus, $\lambda_1 = 20\alpha$, $\lambda_2 = (10 + \sqrt{91})\alpha$ These eigenvalues gives principal moments of inertia at point 0. To find the direction of corres principal axes, we find eigenvectors corresponding to each eigenvalue. **For** $\lambda_1 = 20\alpha$: Let $X = \{x_2\}$ be the required eigenvector corresponding to eigenvalue $\lambda_1 = 20\alpha$, then d Prepared D $(2\alpha - 20\alpha - 3\sqrt{3}\alpha - 0)$ $(x_1) = (0) - (-18\alpha - 3\sqrt{3}\alpha - 0)$ $(x_1) = (0)$ $([\mathbf{I}_0] - \lambda_1[I_3])X = \mathbf{0} \Rightarrow \Big|$ $-3\sqrt{3}\alpha \quad 18\alpha - 20\alpha \quad 0 \\ 0 \quad 20\alpha - 20\alpha \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3\sqrt{3}\alpha & -2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0$ Prepared $(-3\sqrt{3}\alpha x_1 - 2\alpha x_2 = 0)$ $(3\sqrt{3}x_1 + 2x_2 = 0)$ From Eq. (3), we have $x_1 = -\frac{\sqrt{3}}{6}x_2$ and putting it in (4), we get, $3\sqrt{3}\left(\frac{\sqrt{3}}{6}x_2\right) - 2x_2 = 0 \Rightarrow \frac{3}{2}x_2$ Put $x_2 = 0$ in (3), we get, $x_1 = 0$ Amir Mał Thus, $X = \{x_2\} = \{0\}$, where, $r \in \mathbb{R}$, $r \neq 0$ get, X = [0] = 0iFor $\lambda_2 = (10 + \sqrt{91})\alpha$: Let $Y = (y_2)$ be the required eigenvector corresponding to eigenvalue Prepared $\sqrt{91}\alpha$, then $Ma_{1} = (8 + \sqrt{91})\alpha = 2\sqrt{3}\alpha$ by: $Dr_0 Am_1$ epared by: Dr. Am $\mathsf{Ma}(\mathsf{nm}-3\sqrt{3}\alpha \operatorname{Pre}(8-\sqrt{91})\alpha \vee \operatorname{Dro}\mathsf{Ami})(\mathcal{Y}_2) = (0)$ $\partial \left([\mathbf{I}_0] - \lambda_2 [I_3] \right) Y = \mathbf{0}$ $1a 0 0 Prepared by (10 - \sqrt{91}) \alpha$ $-(8+\sqrt{91})\alpha y_1 - 3\sqrt{3}\alpha y_2 = 0$ $\begin{cases} -(8+\sqrt{91})\alpha y_1 - 3\sqrt{3}\alpha y_2 = 0\\ -3\sqrt{3}\alpha y_1 + (8-\sqrt{91})\alpha y_2 = 0 \end{cases} \Rightarrow \begin{cases} (8+\sqrt{91})y_1 + 3\sqrt{3}y_2 = 0\\ 3\sqrt{3}y_1 - (8-\sqrt{91})y_2 = 0 \end{cases}$ by: Dr. $(10 - \sqrt{91})\alpha y_3 = 0$ epared by: $Dy_3 = 0$ From Eq. (5), we have $\frac{y_1}{y_2} = \frac{-3\sqrt{3}}{8+\sqrt{91}}$ and from Eq. (6), we have $\frac{y_1}{y_1}$ $3\sqrt{3}(8+\sqrt{91})$ Thus, Eq. (5) and Eq. (6) are mutually identical, therefore, last system of equati $(8 + \sqrt{91})y_1 + 3\sqrt{3}y_2 = 0$ repy3=€0 Let, $v_2 = s$, where, $s \in \mathbb{R}$, $s \neq 0$ Prepared by: Dr. Amir Mahmood

hmod	bd	Prepa	red	by: I	Dr.	Amir	Mar	mood	Prepar	ed b	y: Dr	. Ami	rivia	nmod	bd	Frei	Jan
hmog	od Mecha	Prepa anics Mad	red le Easy	by: l	Dr.	Amir	Mah	mood	Prepar	ed b	y: Dr	, Ami	r Ma	hmoc Moment	of In	Prep ertia	pare
hmod	pd	Prepa	repy1	py:/	/3√	3-5	Mah	nmood	Prepar	ed b	y:/Dr	3√3	nMa	hmod	bd	Pre	pare
hmod	The	erefore, Y	$= \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}$),≣	$8 + \sqrt{5}$	An)ir	For s	$= -(8 + 1)^{-1}$	$\sqrt{91}$), we g	et, Y		$8 + \sqrt{91}$		√3i – (8	d 4	91)j	are
hmo	For	Prepa	(10 -	√91)	<u>α</u> : 1	Amir Let Z =	$=\begin{pmatrix} z_1\\z_2 \end{pmatrix}$	be the	required	ed b eigen	vector	corres	pondin	hmog	eigen	value	are
hmod	d da	Prepa = (10 -)	$\sqrt{91}\alpha$	by: then	Dr.	Amir	$\langle z_3 \rangle$	mood	Prepar	ed b	y: Dr	. Ami	r Ma	hmod	bd	Prep	are
hmo	od	Prepa	red	by:	Dr.	Amir	Ma	-(8-√9	$\overline{1})\alpha$ α $-3\sqrt{2}$	<u>3</u> α b	y: Dr	o Ami	$\left(\frac{z_1}{z_1} \right)$	hmod	bd	Prep	pare
hmod	bd	Prepa	([I ₀] -	$-\lambda_2[I_3]$	3]) <i>Z</i> =	Amir	Mah	- <u>3√3</u> a	Pre 8+1	91)α	y: Dr	Ami	$\left \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} \right $	$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} $	bd	Prep	are
hmod	bd	Prepa	red	-(8 -	√ 91	$Amir_{\alpha z_1} - 3$	$\sqrt{3\alpha z_2}$		Prepar	$\frac{1}{91}$	$(10 + 3\sqrt{3}z)$	$\sqrt{91}\alpha$	r Ma	$\frac{hmoc}{-(7)}$	bd	Prep	are
hmod	bd	Prepa	red	-3√3c	$xz_1 +$	(8 + √9	$\overline{(1)}\alpha z_2$	=nood	$3\sqrt{3}z_1$	- (8 +	$+\sqrt{91}z$		r_Ma	hm(8)	d	Prep	are
hmod	pd	Prepa	red	by:	Dr.	$(10 + \sqrt{3})^{-3}$	$91)\alpha z_3$	T rood	Prepar	ed b	y: DZ	$x_3 = 0$	r Ma	hmod	bd	Prep	oar
hmod	Fro	m Eq. (7)	, we ha	ave $\frac{z_1}{z_2}$	8	$\frac{5\sqrt{5}}{-\sqrt{91}}$ an	d from	Eq. (8), we	e have $\frac{z_1}{z_2} =$	$3\sqrt{3}$	$=\frac{3\sqrt{3}}{3\sqrt{3}}$	$\frac{1}{8} - \sqrt{9}$	$\frac{1}{91} = \frac{1}{3\sqrt{3}}$	$\frac{27}{(8-\sqrt{91})}$	8 -	- √91⊖ (are
innor	Thu	us, Eq. (7)	and E	q. (8)	are r	nutually	identio	cal, therefo	ore, last syste	em of e	equation	is can be	e writte	n as			
hmod	od	Prepa	red	by: I	Dr.	Amır	Mat	(8 - √91)	$z_1 + 3\sqrt{3}z_2$	≘0 D =0	your	. Ami	r IVIa	hmod	bd	Prep	are
hmod	od Let	$\Pr_{z_2 = t,}$	red wher	by: e, t∈	Dr. R,	Amir t≠0	Mah	$z_1 = \frac{1}{2}$	³ Prepáří	ed o	M. Dr	. Ami	r Ma	hmod	bd	Prep	are
hmo	od	Prepa	red	by:	2_3√3	$\frac{3}{21}t$	Mah	mood	Prepar	ed p	Dr:Dr	3√3	ir Ma	hmod	bd	Pre	par
hmod	The	erefore, Z	$= \begin{bmatrix} z_2 \\ z_3 \end{bmatrix}$	Þ₹(Anh	For t	T (8-1	√91), we ge	t, dZ	₽ E(8	$3 - \sqrt{91}$) = 31	√3i – (8	σdV	91)j	are
nmod	d	Princip	al mor	nent o	ofine	rtia	Mah	Princir	alaxis	ed b	y: Dr	Normali	zed prir	ncipal axi	s	Prep	are
			$\lambda_1 =$	20α	36	Amir	A dia la	X =	= k		Die	A	$\hat{X} = \mathbf{k}$	0	1	Dise	are
		Uro no												and the second second second	100		
nmoc	d	λ_2	= (10	+ √9	1)α	- All HILL	Y	$= 3\sqrt{3}i -$	$(8 + \sqrt{91})j$	su p	$\gamma = 0$	Ami	[3 ₁	$\frac{1}{2i} - (8)$			are
nmoo hmoo	od od	Prepa λ ₂ Prepa	= (10 red	+√9 by: [<u>1</u>)α	Amir	Mah Mah	$= 3\sqrt{3}i$	$(8+\sqrt{91})\mathbf{j}$	ed b	$\hat{Y} = \frac{1}{\sqrt{18}}$	$\frac{1}{32+16}$	<u></u> [3v √91	∕ <mark>∃i – (</mark> 8 ·	+ √9	91)j]	bare
nmoc hmoc nmoc	d d	$\frac{\operatorname{Prepa}}{\lambda_2}$ Prepa $\operatorname{Prep}^{\lambda_3}$	=(10) =(10)	$+\sqrt{9}$ $-\sqrt{9}$	1)α 1)α	Amir Amir	Mah Mah Ma ^z	$= 3\sqrt{3}\mathbf{i} - \frac{1}{3}\mathbf{i} - $	$\frac{(8+\sqrt{91})\mathbf{j}}{(8-\sqrt{91})\mathbf{j}}$	ed b ed b	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = \frac{1}{\sqrt{18}}$	$\frac{1}{32 + 16}$ $\frac{32 + 16}{32 + 16}$	[3v √91 [3v √91	∕3i – (8 · ∕3i – (8 ·	+ √9 - √9	91)j] 91)j]	pare
nmoc hmoc nmoc hmoc	Pro	Prepa Prepa Prep $^{\lambda_3}$ Oblem: De	= (10) $= (10)$ $= (10)$ eterm	$+\sqrt{9}$ $-\sqrt{9}$ ine th	1)α 1)α 1e (di	Amir Amir rection	Y Z of) pr	$= 3\sqrt{3}\mathbf{i} - $ $= 3\sqrt{3}\mathbf{i} - $ incipal ax	$(8 + \sqrt{91})\mathbf{j}$ $(8 - \sqrt{91})\mathbf{j}$ es and corr	ed b ed b espon	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = \frac{1}{\sqrt{18}}$ $\frac{1}{\sqrt{18}}$	$\frac{1}{32 + 16^{\circ}}$ $\frac{32 + 16^{\circ}}{1}$	$\frac{1}{\sqrt{91}} \begin{bmatrix} 3v \\ \sqrt{91} \end{bmatrix}$	√3i — (8 - √3i — (8 - ents of in	$+\sqrt{9}$ $-\sqrt{9}$ nerti	91)j] 91)j] a of a	bare bare bare
nmoc hmoc nmoc hmoc nmoc	Pro uni	λ_2 Prepa Prep λ_3 blem: Do form sol	= (10) $= (10)$ $= (10)$ $= (10)$ $= (10)$	$+\sqrt{9}$ $-\sqrt{9}$ <u>ine th</u> <u>nisph</u> and <i>o</i>	$\overline{1}$) α $\overline{1}$) α <u>le (di</u> <u>ere a</u> resp	Amir Amir rection	Y Z of) pr nt on it be the	$= 3\sqrt{3}i -$ $= 3\sqrt{3}i -$ $\frac{1}{3}i - \frac{1}{3}i - \frac{1}{3}$	$(8 + \sqrt{91})\mathbf{j}$ $(8 - \sqrt{91})\mathbf{j}$ es and corr	í z espon	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = \frac{1}{\sqrt{18}}$	$\frac{1}{32 + 16}$ $\frac{32 + 16}{1}$ $\frac{32 + 16}{1}$ cincipal	$\frac{3\sqrt{91}}{\sqrt{91}}$	$\sqrt{3}i - (8 - \sqrt{3}i - (8 - \sqrt{3}i - $	$+\sqrt{9}$ $-\sqrt{9}$ <u>nerti</u>	91)j] 91)j] <u>a of a</u>	bare bare bare bare
nmoc hmoc hmoc hmoc hmoc	Pro uni Solu	λ_2 Prep ^{λ_3} blem: Do form sol ution: Let nisphere.	= (10) $= (10)$ $= (1$	$+\sqrt{9}$ $-\sqrt{9}$ ine th nisph and ρ , t, O an	$1)\alpha$ $1)\alpha$ $1 = (di)\alpha$ resp d C,	Amir Amir rection at a poin ectively respection	of) pr of) pr of on it be the vely, b	$= 3\sqrt{3}i -$ $= 3\sqrt{3}i -$ incipal ax s rim. mass, rad e point o	$(8 + \sqrt{91})\mathbf{j}$ $(8 - \sqrt{91})\mathbf{j}$ es and corr lius of the bann the rim, o	ý ź espon ise and rentre	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = \frac{1}{\sqrt{18}}$ $\frac{1}{\sqrt{18}}$	$\frac{1}{32 + 16}$ $\frac{32 + 16}{1}$ $\frac{32 + 16}{1}$ rincipal e mass base an	$\frac{3}{\sqrt{91}} \begin{bmatrix} 3\sqrt{91} \\ \sqrt{91} \end{bmatrix}$ mome density nd cent	√3i – (8 · √3i – (8 · •nts of in •of a unit re of ma	$+\sqrt{9}$ $-\sqrt{9}$ <u>nerti</u> form	(1)j] (1)j] a of a a of a a solid of the	bare bare bare bare bare
hmoc hmoc hmoc hmoc hmoc hmoc	Pro uni Solu her <i>Cx</i> '	λ_2 λ_3 blem: Defined form solution: Let nisphere. nisphere. 'y''z'' as	= (10) $= (10)$ $= (10)$ $= (10)$ $M, a = a$ $Let A$ $Choos$ $showr$	$+\sqrt{9}$ $-\sqrt{9}$ ine th nisph and ρ , ρ an e three μ in th	$1)\alpha$ $1)\alpha$ resp d C, recome fign	Amir Amir rection at a poin ectively respection ordinate ure.	of) pr nt on it , be the vely, b axes A	$= 3\sqrt{3}i -$ $= 3\sqrt{3}i -$ incipal ax s rim. mass, rad e point of xyz, Ox'y	$(8 + \sqrt{91})\mathbf{j}$ $(8 - \sqrt{91})\mathbf{j}$ es and corr lius of the ban n the rim, o y'z' and	se and entre	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = \frac{1}{\sqrt{18}}$ $\frac{1}{\sqrt{18}}$ $$	$\frac{1}{32 + 16^{4}}$ $\frac{32 + 16^{4}}{1}$ $\frac{32 + 16^{4}}{1}$ $\frac{32 + 16^{4}}{1}$ $\frac{32 + 16^{4}}{1}$	$\frac{1}{\sqrt{91}} \begin{bmatrix} 3\sqrt{91} \\ \sqrt{91} \end{bmatrix}$ $\frac{1}{\sqrt{91}} \begin{bmatrix} 3\sqrt{91} \\ 0 \end{bmatrix}$ $\frac{1}{\sqrt{91}} \begin{bmatrix} 3\sqrt{91} \\ 0 \end{bmatrix}$ $\frac{1}{\sqrt{91}} \begin{bmatrix} 1\sqrt{91} \\ 0 \end{bmatrix}$	$\sqrt{3}i - (8 + 7)$	$+\sqrt{9}$ $-\sqrt{9}$ nerti form ass o	(1)j] (1)j] a of a a solid of the	bare bare bare bare bare bare bare
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hmoc hmoc hmoc hmoc hmoc hmoc hmoc	Prouni Solution herr Cx' As res Io1: The sys	λ_2 λ_3 oblem: Do form sol ution: Let nisphere. "y"z" as we know pect to $\mu = I_{022} =$ erefore, to tem $Ox'y$	= (10) $= (10)$	$+\sqrt{9}$ $-\sqrt{9}$ <u>ine th</u> <u>nisph</u> and ρ , 0 an e three 1 in th the n dinate $=\frac{2}{5}M$ ertia given t	1) α 1) α 1	Amir Amir arection at a poin ectively respection ordinate are. ents and stem (and rix with	$\begin{array}{c c} Y \\ \hline \\$	$= 3\sqrt{3}\mathbf{i} -$ $= 3\sqrt{3}\sqrt{3}\mathbf{i} -$ $= 3\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}$ $= 3\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{3}3$	$(8 + \sqrt{91})\mathbf{j}$ $(8 - \sqrt{91})\mathbf{j}$ es and corr lius of the ban n the rim, of z' and tia with ven by $z_{013} = 0$. ordinate	espon espon entre	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = $	$\frac{1}{32 + 16}$ $\frac{32 + 16}{1}$ $\frac{32 + 16}{1}$ incipal e mass base an	$\frac{\sqrt{91}}{\sqrt{91}} \begin{bmatrix} 3\sqrt{91} \\ \sqrt{91} \\ mome \\ density \\ d cent \\ z' \\ z'' \\ c \\ y \\ c \\ y \\ z'' \\ c \\ y \\ y \\ z'' \\ c \\ y \\ y$	$\sqrt{3}i - (8 - \sqrt{3}i - (8 - \sqrt{3}i - (8 - \sqrt{3}i - \sqrt{3}i$	$+\sqrt{9}$ - $\sqrt{9}$ nerti form ass o	1)j] 1)j] a of a solid of the	are bare bare bare are are are
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hmod hmod hmod hmod hmod hmod hmod hmod	Pro uni Solu her <i>Cx'</i> As res <i>I</i> ₀₁ . The sys	λ_2 λ_3 oblem: Definition Definition Solution: Let nisphere. nisphere. "y"z" as we know pect to $\mu = I_{022} =$ erefore, for tem $Ox'y$ $= (I_{0ij}) =$	$= (10)$ $= (10)$ $= (10)$ $= (10)$ $= (10)$ $= I_{033}$ $= I_{033}$ $= I_{033}$ $= I_{033}$ $= I_{033}$ $= I_{033}$	$+\sqrt{9}$ $-\sqrt{9}$ ine th nisph and ρ , and ρ , b, O an e three the the n dinate $=\frac{2}{5}M$ ertia given th I_{012} I_{022} I_{023}	1) α 1) α	Amir Amir Amir arection ectively respection ordinate are. ents and stem (and fix with $\frac{2}{5}$	$\begin{array}{c c} Y \\ \hline \\$	$= 3\sqrt{3}i -$ $= 3\sqrt{3}i -$ $incipal ax$ $s rim.$ mass, rad e point o xyz, $Ox'yct of inerare giv= I_{023} = I ect to coo0 \qquad 0 Ma^2 \qquad 0$	$(8 + \sqrt{91})j$ $(8 - \sqrt{91})j$ es and corr lius of the ban n the rim, of z'z' and tia with ven by $z_{013} = 0.$ ordinate	espon ese and centre	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = $	$\frac{1}{32 + 16}$ $\frac{32 + 16}{1}$ $\frac{32 + 16}{1}$ $\frac{1}{2}$ $\frac{1}{2$	$\frac{3\sqrt{91}}{\sqrt{91}}$	73i – (8 - 73i – (8 - nts of in of a unit re of ma	$+\sqrt{9}$ $-\sqrt{9}$ form ass of	(1)j] (1)j] a of a solid of the	are bare bare bare bare are are are are
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hmod hmod hmod hmod hmod hmod hmod hmod	Pro uni Solu her her Cx' As res Io1: The sys [Io] Nez sys	λ_2 λ_3 oblem: Definition λ_3 oblem: Definit λ_3 oblem: Definition λ_3 oblem:	$= (10)$ $= (10)$ $= (10)$ $= (10)$ $= (10)$ $= (10)$ $= I_{033}$	$+\sqrt{9}$ $=\sqrt{9}$ ine th nisph and ρ , ρ an e three in th the n dinate $=\frac{2}{5}M$ ertia given th I_{012} I_{022} I_{023} rallel is follo	1) α 1) α	Amir Amir Amir arection ectively respection ordinate are. ents and stem (and stem (and and and and and and and and	$\begin{array}{c c} Y \\ \hline Z \\ \hline content \\ con$	$= 3\sqrt{3}i -$ $= 3\sqrt{3}i -$ $incipal ax$ s rim. mass, rade e point of xyz, Ox'y ct of iner are giv = $I_{023} = I$ ect to cou 0 0 Ma^2 0 0 $\frac{2}{5}Ma$ msor notat $i = I_{Cij} + I$ $cij = I_{0ij} -$	$(8 + \sqrt{91})\mathbf{j}$ $(8 - \sqrt{91}$	inertia $c_{c,i}x_{c,j}x_{c}$	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = $	$\frac{1}{32 + 16}$ $\frac{32 + 16}{1}$ $\frac{32 + 16}{1$	$\frac{3\sqrt{91}}{\sqrt{91}}$	3i - (8 3i - (8 mts of in of a unit re of ma ," ," pect to a nmod	$+\sqrt{9}$ $-\sqrt{9}$ herti form ass o	(1)j] (1)j] <u>a of a</u> i solid of the of the y dinate	are bare bare bare bare are are are bare b
hmod hmod hmod hmod hmod hmod hmod hmod	Pro uni Solu her her Cx' As res Io1: The sys [Io] Nez sys	λ_2 blem: Definition: Let nisphere. <i>i'y'' z''</i> as we know pect to $\mu = I_{022} =$ erefore, for tem $Ox'y$ $P = (I_{0ij}) =$ kt, we ap tem $Cx''y$	$= (10)$ $= (10)$ $= (10)$ $= (10)$ $= (10)$ $= (10)$ $= I_{033}$	$+\sqrt{9}$ $=\sqrt{9}$ ine th nisph and ρ , 0 an e three in th the n dinate $=\frac{2}{5}M$ ertia given th I_{012} I_{022} I_{023} rallel as follo	1) α 1) α	Amir Amir Amir Amir ection ectively respection ordinate and ix with $\binom{2}{5}$ theorem Amir Amir	$\begin{array}{c c} Y \\ \hline \\$	$= 3\sqrt{3}\mathbf{i} -$ $= 3\sqrt{3}\mathbf{i} -$ $\frac{\mathbf{incipal ax}}{\mathbf{s rim.}}$ mass, rad e point of <i>xyz</i> , <i>Ox'y</i> ct of iner are giv = $I_{023} = I$ ect to counce $0 \qquad 0$ $Ma^2 \qquad 0$ $0 \qquad \frac{2}{5}Ma$ msor notat $x = I_{Cij} + I$	$(8 + \sqrt{91})\mathbf{j}$ $(8 - \sqrt{91}$	inertia $c_{c,i}x_{c,j}x_{c}$	$\hat{Y} = \frac{1}{\sqrt{18}}$ $\hat{Z} = $	$\frac{1}{32 + 16}$ $\frac{32 + 16}{1}$ $\frac{32 + 16}{1$	$\frac{\sqrt{91}}{\sqrt{91}} \begin{bmatrix} 3\sqrt{91} \\ \sqrt{91} \end{bmatrix}$	3i - (8 3i - (8 mts of in of a unit re of ma "" "" pect to a hmod hmod	$+\sqrt{9}$ $-\sqrt{9}$ herti form ass o	(1)j] (1)j] a of a solid of the of the y dinate Prep Prep	are bare bare bare bare are are are bare b

hmod	od Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepar
hmoo	d Prepared by: Dr. Amir Mahmood Prep
hmod	$\frac{1}{1} \left(I_{c11} - I_{c12} - I_{c13} \right) - \left(I_{011} - I_{013} \right) - \left(r_c^2 - 0 - 0 \right) - \left(x_{c,1} x_{c,1} - x_{c,1} x_{c,2} - x_{c,1} x_{c,3} \right) = 0$
hmod	$\Rightarrow \begin{pmatrix} I_{c12} & I_{c22} & I_{c23} \\ I_{c13} & I_{c23} & I_{c33} \end{pmatrix} = \begin{pmatrix} I_{012} & I_{022} & I_{023} \\ I_{013} & I_{023} & I_{033} \end{pmatrix} - M \begin{pmatrix} 0 & \mathbf{r}_c^2 & 0 \\ 0 & 0 & \mathbf{r}_c^2 \end{pmatrix} + M \begin{pmatrix} x_{c,1}x_{c,2} & x_{c,2}x_{c,2} & x_{c,2}x_{c,3} \\ x_{c,1}x_{c,3} & x_{c,2}x_{c,3} & x_{c,3}x_{c,3} \end{pmatrix},$
hmod	where, $\mathbf{r}_c = (x_{c,1}, x_{c,2}, x_{c,3}) = (0, 0, \frac{3}{8}a)$ is the position vector of centre of mass C with respect to coordinate
hmod	system $0x'y'z'$.
hmo	pd $\operatorname{Pre}_{I_{c11}} = I_{c12} \vee I_{c13}$. $\left\{ 5 \text{ in } M_2 \text{ in mood } \right\}$ $\operatorname{Pre}_{164} = 4 \text{ by : Dr. Amir } \left\{ 0 \text{ bound}_{0} = 0 \text{ constants} \right\}$
hmod	$\Rightarrow \begin{pmatrix} I_{c12} & I_{c22} & I_{c23} \\ I_{c13} & I_{c23} & I_{c33} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{5}Ma^2 & 0 \\ 0 & -\frac{1}{5}Ma^2 & 0 \\ 0 & -\frac{1}{64}a^2 & 0 \\ 0 & -\frac{1}{64}a^2 & 0 \\ 0 & -\frac{1}{64}a^2 \end{pmatrix} \text{Preparative}$
hmod	od Prepared by: Dr. Am 0 Ma 0 m $\frac{5}{5}Ma^2/repared by 0$ D $\frac{64}{64}a^2/repared by 0$ D $\frac{1}{64}a^2/repared by 0$
hmoo	$\Rightarrow [\mathbf{I}_{C}] = \begin{pmatrix} I_{C11} & I_{C12} & I_{C13} \\ I_{C12} & I_{C22} & I_{C23} \end{pmatrix} = \begin{pmatrix} \frac{2}{5}Ma^{2} - \frac{9}{64}Ma^{2} & 0 & 0 \\ 0 & \frac{2}{5}Ma^{2} - \frac{9}{64}Ma^{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{83}{320}Ma^{2} & 0 & 0 \\ 0 & \frac{83}{320}Ma^{2} & 0 \\ 0 & \frac{83}{320}Ma^{2} & 0 \end{pmatrix}.$
hmod	$\int dr Pre V_{c13} V_{c23} V_{c33} D \left(A_0 i r M_0^3 M_0^3 dr V_{\frac{2}{5}Ma^2} - \frac{9}{64}Ma^2 + \frac{9}{64}Ma^2 \right) A_0 i r V_0^3 h N_{\frac{2}{5}Ma^2} + Pre Par$
hmod	Now, we apply parallel axis theorem in tensor notation to find inertia tensor $[I_A]$ with respect to coordinate system Axyz, as follows
hmod	od Prepared by: Dr. Amir $MI_{Aij} = I_{cij} + Mr_c^{\prime 2} \delta_{ij} - Mx_{c,i}^{\prime} x_{c,j}^{\prime}$ Or. Amir Mahmood Prepar
hmod	$\Rightarrow \begin{pmatrix} I_{A11} & I_{A12} & I_{A13} \\ I_{A12} & I_{A22} & I_{A23} \end{pmatrix} = \begin{pmatrix} I_{C11} & I_{C12} & I_{C13} \\ I_{C12} & I_{C22} & I_{C23} \end{pmatrix} + M \begin{pmatrix} \mathbf{r}_c^{r_2} & 0 & 0 \\ 0 & \mathbf{r}_c^{r_2} & 0 \end{pmatrix} - M \begin{pmatrix} x_{c,1}x_{c,1} & x_{c,1}x_{c,2} & x_{c,1}x_{c,3} \\ x_{c,1}x_{c,2} & x_{c,2}x_{c,2}' & x_{c,2}' \\ x_{c,2}r_{c,2} & x_{c,2}' & x_{c,3}' \end{pmatrix},$
hmod	$V_{A13} I_{A23} I_{A33} / I_{C13} I_{C23} I_{C33} / 0 0 \mathbf{r}_{c}^{\prime 2} / \langle x_{c,1}^{\prime} x_{c,3}^{\prime} x_{c,3}^{\prime} x_{c,3}^{\prime} x_{c,3}^{\prime} \rangle$
hmod	where, $\mathbf{r}_c = (x_{c,1}, x_{c,1}, x_{c,1}) = \begin{pmatrix} 0, a, \frac{1}{8}a \end{pmatrix}$ is the position vector of centre of mass <i>C</i> with respect to coordinate system <i>Axyz</i> . $\begin{pmatrix} \frac{83}{8}Ma^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{73}{8}a^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$
hmoo	$\begin{pmatrix} I_{A11} & I_{A12} & I_{A13} \end{pmatrix} \begin{pmatrix} 320 & 1 & 1 & 1 \\ & & & & & & \\ & & & & & &$
hmod	$\Rightarrow \begin{pmatrix} I_{A12} & I_{A22} & I_{A23} \\ I_{A13} & I_{A23} & I_{A33} \end{pmatrix} = \begin{pmatrix} 0 & \overline{320} & Ma^2 & 0 \\ 0 & \overline{320} & Ma^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} M & 0 & \overline{64} & a^2 & 0 \\ 0 & \overline{64} & \overline{64} & 0 \\ $
hmoc	d Prepared by: Dr(Ar0ir Malon $\sqrt{5}Ma^2$ /epa(e0 by: 0). $\frac{1}{64}a^2$ /r Mahn ⁸ ood ⁶⁴ Prépar
hmoc	d Prepared by: Dr. A $\begin{pmatrix} \frac{33}{320}Ma^2 + \frac{73}{64}Ma^2 d \\ \frac{83}{320}Ma^2 + \frac{73}{64}Ma^2 d \\ \frac{83}{73} & \frac{73}{73} \end{pmatrix}$ Dr. Amir Ma@mood P epar
hmoc	$ = \begin{pmatrix} I_{A12} & I_{A22} & I_{A23} \\ I_{A12} & I_{A23} & I_{A23} \end{pmatrix} = \min \left\{ 0 \mod \frac{33}{320} Ma^2 + \frac{13}{64} Ma^2 - Ma^2 \right\} = \min \left\{ -\frac{3}{8} Ma^2 \log A + \frac{1}{8} Ma^2 \log A $
hmoc	d Prepared by Dr. $\frac{3}{8}a^2$ by Dr. $\frac{2}{5}Ma^2 + \frac{73}{64}Ma^2 - \frac{9}{64}Ma^2$ parameters of the prepared by Dr. $\frac{2}{5}Ma^2 + \frac{73}{64}Ma^2 - \frac{9}{64}Ma^2$ parameters of the prepared by Dr. $\frac{2}{5}Ma^2 + \frac{73}{64}Ma^2 - \frac{9}{64}Ma^2$ parameters of the prepared by Dr. $\frac{2}{5}Ma^2 + \frac{73}{64}Ma^2 - \frac{9}{64}Ma^2$ parameters of the prepared by Dr. $\frac{2}{5}Ma^2 + \frac{73}{64}Ma^2 - \frac{9}{64}Ma^2$
hmod	od Prepared b/7-Ma². Amir Mahmood Prepared by: Dr. Amir Mahmood Prepar
hmod	$ \mathbf{I}_{A} = \begin{pmatrix} 5 & 0 & -\frac{2}{5}Ma^{2} & -\frac{3}{8}Ma^{2} \\ -\frac{3}{8}Ma^{2} & -\frac{3}{8}Ma^{2} \end{pmatrix} = \begin{pmatrix} 56\alpha & 0 & 0 \\ 0 & 16\alpha & -15\alpha \end{pmatrix}, \text{ where, } \alpha = \frac{1}{40}Ma^{2} \text{ or } Prepare$
hmoo	d Prepared by: Dr. Again Magnopol 0 repared 56 α /Dr. Amin Manmood Prepared 0
hmoo	To find the eigenvalues, we solve characteristic equation det($[I_A] - \lambda[I_3]$) = 0, where $[I_3]$ is unit matrix of order 3.
hmod	det($[I_A] - \lambda [I_3]$) = 0 \Rightarrow $\begin{pmatrix} 56\alpha - \lambda \\ 0 \\ 16\alpha - \lambda \\ -15\alpha \end{pmatrix}$ = 0 Mahmood Prepar
hmoo	On expanding by first row, we get, $\Omega_{1} = 0$ Prepared $\Sigma_{2} = 0$ Pre
hmod	$(56\alpha - \lambda)[(16\alpha - \lambda)(56\alpha - \lambda) - (-15\alpha)^2] = 0 \implies (56\alpha - \lambda)[896\alpha^2 - 16\alpha\lambda - 56\alpha\lambda + \lambda^2 - 225\alpha^2] = 0$ $\implies (56\alpha - \lambda)[\lambda^2 - 72\alpha\lambda + 671\alpha^2] = 0$
hmod	d Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepar
hmod	Either $56\alpha - \lambda = 0 \Rightarrow \lambda = 56\alpha$ or, $\lambda^2 - 72\alpha\lambda + 671\alpha^2 = 0 \Rightarrow \lambda = \frac{72\alpha \pm \sqrt{(72\alpha)^2 - 4(1)(671\alpha^2)}}{2(1)}$
hmod	d Prepared by: Dr. Amir Mahmood $\lambda = \frac{724 \pm 93044}{724 \pm 506} = \frac{724 \pm 9204}{2}$
hmod	Prepared by: Dr. Amir Mahmob $\Rightarrow \lambda = \frac{72\alpha+30\alpha}{2}, \frac{72\alpha-30\alpha}{2} = \frac{122\alpha}{2}, \frac{22\alpha}{2} = 61\alpha, 11\alpha$ Prepared
D	reneved by: Dr. Amin Mahmaad

hmoPrepared by: Dr. Amir Mahmood mir Mahmood Prepared by: Dr. Amir Mahmoo Page 28 pare

hmoc	d Prepared by: Dr. Amir	Mahmood Prepared by	: Dr. Amir Mahmood	Prepare			
hmoo	d Prepared by: Dr. Amir lechanics Made Easy	Mahmood Prepared by	: Dr. Amir Mahmood Moment of J	Prepare			
hmod	Thus, $\lambda_1 = 56\alpha$, $\lambda_2 = 61\alpha$,	and $\lambda_3 = 11\alpha$. repared by	: Dr. Amir Mahmood	Prepare			
hmod	These eigenvalues gives principal mo axes, we find eigenvectors correspond	ments of inertia at point <i>A</i> . To find t ling to each eigenvalue.	he direction of corresponding pi	Prepare			
hmoo	For $\lambda_1 = 56\alpha$: Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_1 \end{pmatrix}$ be the red	quired eigenvector corresponding to	eigenvalue $\lambda_1 = 56\alpha$, then 000	Prepare			
hmod	d Prepared by: $\int 56\alpha - 56\alpha$	Mahonood Poeparxi by	xېرopr. Amir Mahroopx	¹ Prepare			
hmod	$([\mathbf{I}_{A}] - \lambda_{1}[I_{3}])X = 0 \Rightarrow \begin{pmatrix} 0 \\ r & A_{0} \end{pmatrix}$	$\begin{array}{ccc} 16\alpha - 56\alpha & -15\alpha \\ -15\alpha & 56\alpha - 56\alpha \end{array} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} =$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -40\alpha & -15\alpha \\ 0 & -15\alpha & 0 \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix}$	² ² ² ² ² ² ² ² ² ²			
hmod	Prepared by: Dr. Amir	Mahmood Prepared by	: Dr. Amir Mahmood	Prepare			
hmod	0 Prepared by: Dr _{40ax₂}	$-15\alpha x_3 = 0 \qquad \Rightarrow \qquad 8x_2 + 3x_3 = 0$	Dr. Amin Mahmood	Prepare			
hmod	Thus we have, $x_2 = x_3 = 0$ and $x_1 =$	$-15\alpha x_2 = 0 \qquad (\qquad x_2 = 0)$ r, where, $r \in \mathbb{R}$, $r \neq 0$: Dr. Amī(²)/Jahmood	Prepare			
hmod	Thus, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} T \\ 0 \end{pmatrix}, \Rightarrow Fo$	r $r = 1$, we get, $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{i} + 0$: Dr. Amir Mahmood +0k=i	Prepare			
hmod	d Prepa ^x ³ d b ⁰ , y ₁ , Amir	Mahmood Prepared by	: Dr. Amir Mahmood	Prepare			
hmod	For $\lambda_2 = 61\alpha$: Let $Y = \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}$ be the re	quired eigenvector corresponding to	eigenvalue $\lambda_2 = 61\alpha$, then	Prepare			
hmod	$([\mathbf{I}_{4}] - \lambda_{2}[I_{2}])Y = 0 \Rightarrow \begin{pmatrix} 56\alpha - 61\alpha \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 16\alpha - 61\alpha & -15\alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\Rightarrow \begin{pmatrix} -5\alpha & 0 & 0 \\ 0 & -45\alpha & -15\alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} =$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix} e p a re$			
hmod	od Prepared by: Dr. Amisa	$\begin{array}{ccc} 15\alpha & 51\alpha & 15\alpha \\ -15\alpha & 56\alpha - 61\alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_3 \end{pmatrix} \begin{pmatrix} 0 \\ y_1 \end{pmatrix} = 0 \end{array}$	$\begin{bmatrix} 0 & 15\alpha \\ 0 & -15\alpha \\ 0 & -5\alpha \end{bmatrix} \begin{bmatrix} 5\alpha \\ 3 \end{bmatrix}$	Prepare			
hmod	d Prepared by $\left\{ \begin{array}{c} -45\alpha y_2 - 15\alpha y_2 - 5\alpha $	$\begin{array}{l} xy_3 = 0 \\ xy_2 = 0 \end{array} \Rightarrow \begin{cases} 3y_2 + y_3 = 0 \\ 3y_2 + y_2 = 0 \end{cases}$	$\Rightarrow \begin{cases} y_1 = 0 \\ 3y_2 + y_3 = 0 \end{cases}$	Prepare			
hmoo	Let, $y_2 = s$, where, $s \in \mathbb{R}$, $s \neq 0$	$V \Rightarrow h y_3 = -3s$ Bepared by	: Dr. Amir Mahmood	Prepare			
hmoo	Thus, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ -3s \end{pmatrix}$, $A \Rightarrow ir$	For $s = 1$, we get, $Y = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} =$	$0\mathbf{i} + \mathbf{j} - 3\mathbf{k} = \mathbf{j} - 3\mathbf{k}$	Prepare			
hmoo	For $\lambda_3 = 11\alpha$: Let $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ be the re	equired eigenvector corresponding to	eigenvalue $\lambda_3 = 11\alpha$, then	Prepare			
hmoo	d Prepared by: (z_3) Amir $(56\alpha - 11\alpha)$	Machino Prepared by $\binom{1}{2}$: D_{r} Amir Mahmood	Prepare			
hmod	$([\mathbf{I}_{A}] - \lambda_{3}[I_{3}])Z = 0 \Rightarrow \left(Dr_{0}^{0} Ami^{10} \right)$	$5\alpha - 11\alpha \qquad -15\alpha \\ -15\alpha \qquad 56\alpha - 11\alpha \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$	$ \begin{pmatrix} 0 & 5\alpha & -15\alpha \\ 0 & -15\alpha & 45\alpha \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_2 \\ $	0)repare			
hmoo	d Prepared by: $\int r_{5\alpha z_2} - 15$	$\begin{array}{l} \alpha z_1 = 0 \\ \alpha z_3 = 0 \end{array} \Rightarrow \begin{array}{l} z_1 = 0 \\ z_2 - 3z_3 = 0 \end{array}$	Dr. Amirz1/#0hmood	Prepare			
hmod	Let, $z_3 = t$, where, $t \in \mathbb{R}$, $t \neq 0$	$\begin{array}{l} \alpha z_3 = 0 \\ \Rightarrow z_2 = 3t \end{array} \qquad \begin{pmatrix} z_2 - 3z_3 = 0 \\ z_2 - 3z_3 = 0 \end{pmatrix}$	$(z_2 - 3z_3 = 0)$	Prepare			
hmod	Thus $T = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2t \end{pmatrix}$	Mahmood Prepare $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	Dr. Amir Mahmood	Prepare			
hmoo	$\frac{1}{z_3} = \frac{1}{z_3} = \frac{3}{t} = \frac{3}{t}$	Mahmood Prepared 1	<u>: Dr. Amir Mahmood</u>	Prepare			
amoo	Principal moment of inertia	Principal axis	Normalized principal axis	Prepare			
11100	$\lambda_1 = 56\alpha$ $\lambda_2 = 61\alpha$	$Y = \mathbf{i} - 3\mathbf{k}$	$\hat{Y} = (1/\sqrt{10})(\mathbf{i} - 3\mathbf{k})$	Descare			
hmod	$\lambda_3 = 11\alpha$	$Z = 3\mathbf{j} + \mathbf{k}$	$\hat{Z} = (1/\sqrt{10})(3\mathbf{i} + \mathbf{k})$	Prepare			
hmoo	Definition: Two distributions of matte	er are said to be "equimomental" if	they have the same moment of	inertia are			
hmod	about any line in spase. Theorem: Two systems S_1 and S_2	are equimomental if and only if	the following three conditio	ns arepare			
hmoo	satisfied, area by Dr. Amir	Mahmood Prepared by	: Dr. Amir Mahmood	Prepare			
hmod	(<i>ii</i>) they have same centre of mass,	and hmood Prepared by	: Dr. Amir Mahmood	Prepare			
hmod	(<i>iii</i>) they have same principal axes and principal moments of inertia at centre of mass. Proof: Suppose that two systems S ₁ and S ₂ are equimomental. We will show that conditions (i) (ii) and (iii) are						
hmod	satisfied. ared by Dr. Amir	Mahmood Prenared by	Dr. Amir Mahmood	Prepare			
innou	C	mannessa rieparea by					

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Note that is Made Easy Mathematical and the sense of the system sense of the system set of the system sense of the system se	hmod	d Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare
(f) Let M_{1} and M_{2} respectively, be the masses of the systems S_{1} and S_{2} should also be same, say, T_{1} Let L be any line particular, their moments of merita about line T through C_{2} and C_{2} should also be same, say, T_{1} Let L be any line particular, their moments of merita about line L and T further suppose that L/r be the common moment T function D to any line particular, their moments of L/r be the expenditular distance between L and L' . Further suppose that L/r be the common moment T function D to T system S_{1}) in moment C_{2} . The expectively is the same same L is L/r and L/r be the expectively. The the common moment T is expectively. The the line L is the origin L/r and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through C_{2} and L/r and L/r respectively. But he lines through L/r and L/r and L/r respectively. But he lines through L/r and L/r and L/r and L/r and L/r respectively. But L/r and L	hmoo	d Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare lechanics Made Easy Moment of Inertia
of mass. Since the systems are supposed to be equimomental, therefore their moments of interia about any, line between it and <i>P</i> . Where <i>i</i> and	hmod	(i) Let M_1 and M_2 , respectively, be the masses of the systems S_1 and S_2 and C_1 and C_2 , respectively, be their centres
Let l' be any line parallel to 1 and d be the perpendicular distance between 1 and l' . Whither suppose that l_i be the common moment of inertia of both systems are bound in l' . By parallel axis theorem, we have, $l_i = l_i + M_i d^2$ (for system S_1) $ (2)$ From equations (1) and (2), we have, $l_i + M_i d^2 = l_i + M_i d^2 = M_i = M_i = M_i = M_i (agy)$ \leq masses of both systems are same \leq condition (1) is satisfied. (1) Now, let l_i and l_i respectively, be the lines through C_i and C_i and perpendicular to line. Let common moment of inertia of system S_i about l_i is $l_i = l_i - M(i, C_i)^2 = $	hmod	of mass. Since the systems are supposed to be equimomental, therefore their moments of inertia about any line should be same. In particular, their moments of inertia about line l through C_1 and C_2 should also be same, say, I_l .
of inertia of both systems about line l'. Ity parallel axis theorem, we have, $l_1 = l_1 + M_1 d^2$ (for system S ₂) $$ (2) From equations (1) and (2), we have, $l_1 = l_1 + M_1 d^2$ (low system S ₂) $$ (3) From equations (1) and (2), we have, $l_1 = l_1 + M_1 d^2$ (low system S ₁) $$ (3) From equations (1) and (2), we have, $l_1 = l_1 + M_1 d^2$ (low system shows that both system about line l ₁ be l ₁ and about line l ₂ be l ₁ By parallel axis theorem, moment of inertia of system S ₂ about l ₂ is $l_1 = l_1 + M_1 (l_1 C_2)^2(3)$ Again, by parallel axis theorem, moment of inertia of system S ₂ about l ₂ is $l_1 = l_1 + M_1 (l_1 C_2)^2(4)$ Again, by parallel axis theorem, moment of inertia of system S ₂ about l ₂ is $l_1 = M_1 (l_1 - M_1 (l_2 - l_1^2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_2 - l_1^2 - l_2 - l_1 - M_1 (l_1 - M_1 (l_1 - M_1 - M_1 (l_1 - M_1 - M_1 (l_1 - M_1 - M_1 - M_1 (l_1 - M_1 - M_1 - M_1 (l_1 - M_1 - M_1 - M_1 - M_1 (l_1 - M_1 - M_1 - M_1 - M_1 - M_1 (l_1 - M_1 (l_1 - M_1 -$	hmod	Let l' be any line parallel to l and d be the perpendicular distance between l and l' . Further suppose that $I_{l'}$ be the common moment
$\begin{aligned} P_{i} = l_{i} + M_{i}d^{2} (\text{tor system } S_{i}) = (2) \\ \text{From equations (1) and (2), we have,} \\ l_{i} = l_{i} + M_{i}d^{2} (l_{i} + M_{i}d^{2} = l_{i} + M_{i}d^{2} = M_{i} = M_{i} = M_{i} = M_{i} = M_{i} \\ (1) Now, let l_{i} and l_{i} - respectively, be the lines through C_{i} and C_{i} and prependicular to line l. Let common moment of inertia of each system about line l_{i} be l_{i}, and about line l_{i} be l_{i} \\ By parallel axis theorem, moment of inertia of system S_{i} about l_{i} lis \\ l_{i} = l_{i} + M_{i}C_{i}^{2} = l_{i} - \dots = (3) \\ l_{i} = l_{i} + M_{i}C_{i}^{2} = l_{i} - \dots = (3) \\ l_{i} = l_{i} + M_{i}C_{i}C_{i}^{2} = (3) \\ l_{i} = l_{i} + M_{i}C_{i}C_{i}^{2} = (3) \\ l_{i} = l_{i} - M_{i}(C_{i}C_{i}^{2})^{2} = (3) \\ l_{i} = l_{i} - M_{i}(C_{i}C_{i}^{2})^{2} = (3) \\ l_{i} = l_{i} - M_{i}(C_{i}C_{i}^{2})^{2} = l_{i} - \dots = (1) \\ l_{i} = l_{i} - M_{i}(C_{i}C_{i}^{2})^{2} = (3) \\ l_{i} = l_{i} - M_{i}(C_{i}C_{i}^{2})^{2} = l_{i} - \dots = (1) \\ l_{i} = l_{i} - M_{i}(C_{i}C_{i}^{2})^{2} = (3) \\ l_{i} = l_{i} - M_{i}(C_{i}C_{i}^{2})^{2} = l_{i} - M_{i}(C_{i}C_{i}^{2})^{2} = l_{i} - \dots = (1) \\ l_{i} = M_{i}(C_{i}^{2})^{2} = l_{i} - M_{i}(C_{i}^{2})^{2} = l_{i} - M_{i}(C_{i}^{2})^{2} = l_{i} - M_{i}(C_{i}^{2})^{2} = l_{i} - M_{i} - M_{i$	hmod	of inertia of both systems about line l' . By parallel axis theorem, we have, c_1 c_2 c_2 c_2 c_3
From equations (1) and (2), we have, $h_1 + M_4 d^2 = \mu_1 + M_6 d^2 = \mu_1 + M_1 = M_2 = M_1 (say)$ The masses of both systems are same \Rightarrow condition (1) is satisfied. (1) Now, let A_1 and A_2 , respectively, be the lines through C_2 and C_2 and perpendicular to line 1. Let common moment of inertia of each system about line 1, be h_1 , and about line 1, be h_1 . By parallel axis theorem, moment of inertia of systems S_1 about L_2 is $h_2 = h_1 + M(C_2)^2 =(3)$. Again, by parallel axis theorem, moment of inertia of system S_2 about L_2 is $h_1 = h_1 - M(C_2)^2 =(3)$. Again, by parallel axis theorem, moment of inertia of system S_2 about L_2 is $h_1 = h_1 - M(C_2)^2 =(3)$. From equations (3) and (4), we get $h_1 = h_1 - M(C_2)^2 =(4)$. From equations (3) and (4), we get $h_2 = h_1 + M(C_2)^2 =$	hmo	$PreI_{l'} = I_l + M_1 d^2 (\text{for system } S_1) = \rightarrow (1) \text{ parec}$ $I_{l'} = I_l + M_2 d^2 (\text{for system } S_2) = \rightarrow (2)$
$ \begin{array}{l} \begin{array}{l} \begin{array}{l} \label{eq:constraint} \label{eq:constraint} \begin{array}{l} \label{eq:constraint} \label{eq:constraint} \begin{array}{l} \label{eq:constraint} \label{eq:constraint} \begin{array}{l} \label{eq:constraint} \label{eq:constraint} \label{eq:constraint} \label{eq:constraint} \begin{array}{l} \label{eq:constraint} eq:con$	hmod	From equations (1) and (2), we have, Mahmood Prepared by: Dr. Amir Mahmood Prepare
(if) Now, let l_i and l_p respectively, be the lines through f_i and f_2 and perpendicular to line l_i be l_i , and about line l_i be l_i , and about line l_i be l_i . By parallel axis theorem, moment of inertia of system 3s, about l_2 is $l_{i_p} = l_i + M[C_i C_2]^2 = (4)$ Again, by parallel axis theorem, moment of inertia of system S, about l_2 is $l_{i_p} = l_{i_1} - M[C_i C_2]^2 = (4)$ From equations (3) and (4), we get $l_i + M[C_i C_2]^2 = l_p - M[C_i C_2]^2 = 0 \Rightarrow C_i = C_2 \equiv C$ (say) $e \text{ centres of mass of both systems are same \Rightarrow condition (ii) is satisfied. (iii) Since both system have same contre of mass C and same mass M. Therefore, they both have same momental ellipsoid at C. Hence, they have same principal axes and principal moments of inertia at centre of mass C and same mass M. Therefore, they both have same momental ellipsoid at C. Hence, they have same principal axes and principal moments of inertia at centre of mass C \rightarrow condition (ii) is satisfied. Conversely, suppose that for two systems S_1 and S_2, conditions (i), (ii) and (iii) are satisfied. We will show that both systems. Further let that l_1, l_2 and l_2 be the common principal moments of inertia at centre of mass and common mass of both systems. Further let that l_1, l_2 and l_2 be the common principal moments of inertia of each system about l_1 is given by l_1 = l_1 l_2 + l_2 l_2 + l_2 l_3^2 where, l_1 and v_1 are direction costnes of line l'. Now, by using parallel axis theorem, the moment of inertia of each system about line l is given by l_1 = l_1 l_2 + l_2 l_2 + l_3 l_3^2. The moment of inertia of a circular hoop (or ring) of mass m and radius a /\sqrt{2} about an axis through its centre and perpendicular to its plane is l_1 = \frac{1}{2} ma^2. The moment of inertia of a circular hoop (or ring) of mass m and radius a /\sqrt{2} about an axis through its centre and perpendicular to its plane is l_1 = \frac{1}{2} ma^2. The moment of inertia a fact are show. Therefore both system$	hmod	\Rightarrow masses of both systems are same \Rightarrow condition (i) is satisfied.
By parallel axis theorem, moment of inertia of system S ₁ about l_2 is $l_1 \leftarrow H(C_1C_2)^2 (3)$ Again, by parallel axis theorem, moment of inertia of system S ₂ about l_2 is $l_2 \leftarrow H_1 - M(C_1C_2)^2 - H_2 - M(C_1C_2)^2 = (3)$ $l_1 \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_2(C_2) = 0 \Rightarrow C_1 = C_2 = C$ (say) $\downarrow \leftarrow H(C_1C_2)^2 = l_1 - M(C_1C_2)^2 = l_1(C_1C_2)^2 = l$	hmod	(<i>ii</i>) Now, let l_1 and l_2 , respectively, be the lines through C_1 and C_2 and perpendicular to line l . Let common moment of inertia of each system about line l_1 be l_{l_1} and about line l_2 be l_{l_2} .
Again, by parallel axis theorem, moment of inertia of system S ₂ about l_2 is $l_1 = l_1 - M[C_1C_2]^2 + (l_1 - l_2)^2 + (l_1C_2)^2 - l_2 - M[C_1C_2]^2 + (l_1 - l_2)^2 + (l_1C_2)^2 - l_1 - M[C_1C_2]^2 - (l_1C_2)^2 - ($	hmoo	By parallel axis theorem, moment of inertia of system S_1 about l_2 is d by: Dr. Amir Mahmood Prepare
From equations (3) and (4), we get $l_1 = l_1 - M[C_1C_2]^2 (4)$ $r_1 = M[C_1C_2]^2 = l_1 - M[C_1C_2]^2 = 0 \Rightarrow C_1 \equiv C_2 \equiv C(say)$ r_2 centres of mass of both systems are same $r \Rightarrow$ condition (<i>ii</i>) is satisfied (<i>iii</i>) Since both system have same centre of mass C and same mass M. Therefore, they both have same momental ellipsoid at C. Hence, they have same principal axes and principal moments of inertia at centre of mass C. $r \Rightarrow$ condition (<i>iii</i>) is satisfied. Conversely, suppose that for two systems S ₁ and S ₂ , conditions (<i>i</i>). (<i>ii</i>) and (<i>iii</i>) are satisfied. We will show that both systems are equimomental. Let C and M, respectively, be the common centre of mass and common mass of both systems. Further let hat l_1 , l_2 and l_1 be the common principal moments of inertia about common principal axes at centre of mass C. In figure, common principal axes at C are shown by Cartesian coordinate system <i>Caryz</i> . Let <i>l</i> be an arbitrary line in space. Draw a line <i>l'</i> through C parallel to <i>l</i> . Then the moment of inertia of each system about <i>l'</i> is given by $l_1 = l_1^2 + l_2 \mu^2 + l_3 \nu^2$, where, d_1 is the perpendicular distance between lines l and l' . Since the moment of inertia of both system about an arbitrary line <i>l</i> in space is same. This shows that both systems S ₁ and S ₂ are equimomental. Problem: Show that a hoog of mass m and radius $a/\sqrt{2}$ is equimomental. Problem: Show that a hoog of mass m and radius a $d/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $l_1 = m(\frac{\pi}{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of a circular plate (or disc) of mass m and radius $a dott an axis through its centre and perpendicular to its plane is l_2 = \frac{1}{2}ma^2.Since, both moments of inertia are same. Therefore both systems are equimomental.Problem: Show that a hoog of mass M and length 2a.Solution: Let O be the centre of mass O, then it is clear that;(i) mass of both systems is equal to M.$	hmod	Again, by parallel axis theorem, moment of inertia of system S_2 about l_2 is ℓ_1 .
$l_{L} + M[C_{1}C_{2}]^{2} = l_{L} - M[C_{1}C_{2}]^{2} \Rightarrow C_{2}C_{2} = 0 \Rightarrow C_{1} \equiv C_{2} \equiv C_{3}$ $\Rightarrow centres of mass of both systems are same \Rightarrow condition (ii) is satisfied. (ii) Since both system have same momental ellipsoid at C. Hence, they have same principal axes and principal moments of mertia at centre of mass C. \Rightarrow condition (ii) is satisfied. (Goversely, suppose that for two systems S1 and S2, conditions (i), (i) and (iii) are satisfied. We will show that both systems are equimomental. Let G and M, respectively, be the common centre of mass and common mass of both systems. Further let that l_{1}, l_{2} and l_{3} be the common principal moments of inertia about common principal axes at c are shown by Cartesian coordinate system Cxyz. Let l be an arbitrary line in space. Draw a line l' through C parallel to l. Then the moment of inertia of each system soft and l' is work by using parallel axis theorem, the moment of inertia of each system share both they stems S1 and S2 are equimomental. Problem: Show that a hoop of mass m and radius a/\sqrt{2} is equimomental. Problem: Show that a hoop of mass m and radius a/\sqrt{2} is equimomental. Problem: Show that a hoop of mass m and radius a/\sqrt{2} is equimomental. Problem: Show that a hoop of the cycle of mass m and radius a and radius a and s for the system and the system S and s are equimomental. Problem: Find the equimomental system of particles for a sufficient of inertia of a circular hoop (or ring) of mass m and radius a A/\2 about an axis through its centre and perpendicular to its plane is I_1 = I_1 m^2 - \frac{1}{2} ma^2. Since, both moments of inertia as as shown in the figure, such that two particles, each having mass M. Hwe replace the refer of mass M and be the cond and third particles of mass M and length 2a. Solution: Let O be the centre of mass O, then it is clear that, I_1 = I_2 ma^2. Since, both moments of inertia are same. Therefore both systems same ar$	hmo	From equations (3) and (4), we get $I_{l_2} = I_{l_1} - M C_1 C_2 ^2 \rightarrow (4)$
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Therefore, they both have same momental ellipsoid at <i>C</i> . Hence, they have same principal axes and principal moments of merita at centre of mass S_1 and S_2 , conditions (ii) (ii) and (iii) are satisfied. We will show that both systems are equimomental. Let <i>C</i> and <i>M</i> , respectively, be the common centre of mass and common mass of both systems. Further let that I_1, I_2 and I_3 be the common principal moments of inertia about common principal axes at centre of mass <i>C</i> . In figure, common principal axes at <i>C</i> are shown by Cartesian coordinate system $Cxyz$. Let <i>I</i> be an arbitrary line in space. Draw a line <i>I'</i> through <i>C</i> parallel to I. Then the moment of inertia of each system about <i>I'</i> is given by $I_1 = I_1\lambda^2 + I_2\mu^2 + I_3v^2$, where, λ, μ and <i>v</i> are direction cosines of line <i>I'</i> . Now, by using parallel axis theorem, the moment of inertia of each system by $I_1 = I_1\lambda^2 + I_2\mu^2 + I_3v^2 + Md^2$, where, <i>d</i> is the perpendicular distance between lines <i>l</i> and <i>l'</i> . Since the moment of inertia of both system about an arbitrary line <i>I</i> in space is same. This shows that both systems S_1 and S_2 are equimomental. Problem: Show that a hoop of mass <i>m</i> and radius $a/\sqrt{2}$ is equimomental. Problem: Show that a hoop of mass <i>m</i> and radius $a/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $I_2 = \frac{1}{2}ma^2$. Since, both moments of inertia are same. Therefore both systems are and radius <i>a</i> about an axis through its centre and perpendicular to its plane is $I_2 = \frac{1}{2}ma^2$. Since, both moments of merita are same. Therefore both systems are equinomental. Problem: Find the equinomental <i>A</i> and <i>B</i> of the rod and third particles of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass <i>O</i> , then it is clear that, (i) mass of both systems is equal to <i>M</i> .	ahmoo	(<i>iii</i>) Since both system have same centre of mass <i>C</i> and same mass <i>M</i> ,
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system $Cxyz$. Let l be an arbitrary line in space. Draw a line l' through C parallel to l . Then the moment of inertia of each system about l' is given by $l_{l'} = l_1\lambda^2 + l_2\mu^2 + l_3v^2$, where, λ, μ and v are direction cosines of line l' . Now, by using parallel axis theorem, the moment of inertia of each system about line l is given by $l_l = l_{l'} + Md^2 = l_1\lambda^2 + l_2\mu^2 + l_3v^2 + Md^2$, where, d is the perpendicular distance between lines l and l' . Since the moment of inertia of both system about an arbitrary line l in space is same. This shows that both systems S_1 and S_2 are equimomental. Problem: Show that a hoop of mass m and radius $a/\sqrt{2}$ is equimomental with a circular plate of mass m radius a . Proof: The moment of inertia of a circular hoop (or ring) of mass m and radius $a/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $l_1 = m\left(\frac{\alpha}{\sqrt{2}}\right)^2 = \frac{1}{2}ma^2$. The moment of inertia of a circular plate (or disc) of mass m and radius a about an axis through its centre and perpendicular to its plane is $l_2 = \frac{1}{2}ma^2$. Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod AB of mass M and length $2a$. Solution: Let O be the centre of mass O the rod AB having mass M . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass m , are placed at its centre of mass O , then it is clear that, (i) mass of both systems is equal to M .	hmod	moments of inertia about common principal axes at centre of mass C. In figure, common principal axes at C are shown by Cartesian coordinate $\begin{pmatrix} s_1 & c & f_2 \\ c & f_3 \end{pmatrix}$ are
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$I_{l'} = I_1 \lambda^2 + I_2 \mu^2 + I_3 \nu^5,$ where, λ , μ and ν are direction cosines of line l' . Now, by using parallel axis theorem, the moment of inertia of each system about line l is given by $I_l = I_{l'} + Md^2 = I_1 \lambda^2 + I_2 \mu^2 + I_3 \nu^2 + Md^2,$ where, d is the perpendicular distance between lines l and l' . Since the moment of inertia of both system about an arbitrary line l in space is same. This shows that both systems S_1 and S_2 are equimomental. Problem: Show that a hoop of mass m and radius $a/\sqrt{2}$ is equimomental with a circular plate of mass m radius a . Proof: The moment of inertia of a circular hoop (or ring) of mass m and radius $a/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $I_1 = m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{1}{2}ma^2.$ The moment of inertia of a circular plate (or disc) of mass m and radius a about an axis through its centre and perpendicular to its plane is $I_2 = \frac{1}{2}ma^2.$ Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod AB of mass M and length $2a$. Solution: Let O be the centre of mass of the rod AB having mass M . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass m , are placed at end points A and B of the rod and third particle of mass $M - 2m$ is placed at its centre of mass O , then it is clear that, (i) mass of both systems is equal to M , b	hmod	Let l be an arbitrary line in space. Draw a line l' through C parallel to l . Then the moment of inertia of each system about l' is given by a red by Dr. Amir Mahmood Prepare
where, <i>d</i> is the perpendicular distance between lines <i>l</i> and <i>l'</i> . Since the moment of inertia of both system about an arbitrary line <i>l</i> in space is same. This shows that both systems S_1 and S_2 are equimomental. Problem: Show that a hoop of mass <i>m</i> and radius $a/\sqrt{2}$ is equimomental with a circular plate of mass <i>m</i> radius <i>a</i> . Proof: The moment of inertia of a circular hoop (or ring) of mass <i>m</i> and radius $a/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $l_1 = m(\frac{a}{\sqrt{2}})^2 = \frac{1}{2}ma^2$. The moment of inertia of a circular plate (or disc) of mass <i>m</i> and radius <i>a</i> about an axis through its centre and perpendicular to its plane is $l_2 = \frac{1}{2}ma^2$. Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod <i>AB</i> of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass of the rod <i>AB</i> having mass <i>M</i> . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass <i>m</i> , are placed at end points <i>A</i> and <i>B</i> of the rod and third particle of mass <i>M</i> and length 2 <i>a</i> . (i) mass of both systems is equal to <i>M</i> ,	hmod	$I_{l'} = I_1 \lambda^2 + I_2 \mu^2 + I_3 \nu^2$, where $\lambda_{l'}$ and $\nu_{l'}$ are direction cosines of line l' . Now, by using parallel axis theorem, the moment of inertia of each
where, <i>d</i> is the perpendicular distance between lines <i>l</i> and <i>l'</i> . Since the moment of inertia of both system about an arbitrary line <i>l</i> in space is same. This shows that both systems S_1 and S_2 are equimomental. Problem: Show that a hoop of mass <i>m</i> and radius $a/\sqrt{2}$ is equimomental with a circular plate of mass <i>m</i> radius <i>a</i> . Proof: The moment of inertia of a circular hoop (or ring) of mass <i>m</i> and radius $a/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $I_1 = m(\frac{a}{\sqrt{2}})^2 = \frac{1}{2}ma^2$. The moment of inertia of a circular plate (or disc) of mass <i>m</i> and radius <i>a</i> about an axis through its centre and perpendicular to its plane is $I_2 = \frac{1}{2}ma^2$. Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod <i>AB</i> of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass of the rod <i>AB</i> having mass <i>M</i> . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass <i>m</i> , are placed at end points <i>A</i> and <i>B</i> of the rod and third particle of $mass M - 2m$ is placed at its centre of mass <i>O</i> , then it is clear that, (i) mass of both systems is equal to <i>M</i> , D	hmod	system about line <i>l</i> is given by $I_{l} = I_{l} + Md^{2} = I_{1}\lambda^{2} + I_{2}\mu^{2} + I_{2}\nu^{2} + Md^{2}.$
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Proof: The moment of inertia of a circular hoop (or ring) of mass <i>m</i> and radius $a/\sqrt{2}$ about an axis through its centre and perpendicular to its plane is $I_1 = m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{1}{2}ma^2$. The moment of inertia of a circular plate (or disc) of mass <i>m</i> and radius <i>a</i> about an axis through its centre and perpendicular to its plane is $I_2 = \frac{1}{2}ma^2$. Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod <i>AB</i> of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass of the rod <i>AB</i> having mass <i>M</i> . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass <i>m</i> , are placed at end points <i>A</i> and <i>B</i> of the rod and third particle of $mass M - 2m$ is placed at its centre of mass <i>O</i> , then it is clear that, (<i>i</i>) mass of both systems is equal to <i>M</i> , <i>D</i>	hmod	Problem: Show that a hoop of mass m and radius $a/\sqrt{2}$ is equimomental with a circular plate of mass m
centre and perpendicular to its plane is $I_1 = m \left(\frac{a}{\sqrt{2}}\right)^2 = \frac{1}{2}ma^2$. The moment of inertia of a circular plate (or disc) of mass <i>m</i> and radius <i>a</i> about an axis through its centre and perpendicular to its plane is $I_2 = \frac{1}{2}ma^2$. Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod <i>AB</i> of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass of the rod <i>AB</i> having mass <i>M</i> . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass <i>m</i> , are placed at end points <i>A</i> and <i>B</i> of the rod and third particle of mass of both systems is equal to <i>M</i> , i mass of both systems is equal to <i>M</i> ,	hmoc	<u>radius a.</u> Proof: The moment of inertia of a circular hoop (or ring) of mass m and radius $a/\sqrt{2}$ about an axis through its
The moment of inertia of a circular plate (or disc) of mass <i>m</i> and radius <i>a</i> about an axis through its centre and perpendicular to its plane is $I_2 = \frac{1}{2}ma^2$. Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod <i>AB</i> of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass of the rod <i>AB</i> having mass <i>M</i> . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass <i>m</i> , are placed at end points <i>A</i> and <i>B</i> of the rod and third particle of $M - 2m$ is placed at its centre of mass <i>O</i> , then it is clear that, (<i>i</i>) mass of both systems is equal to <i>M</i> , D	hmod	centre and perpendicular to its plane is $l_1 = m \left(\frac{a}{c}\right)^2 = \frac{1}{2}ma^2$. A mir Mahmood Prepare
perpendicular to its plane is $I_2 = \frac{1}{2}ma^2$. Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod <i>AB</i> of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass of the rod <i>AB</i> having mass <i>M</i> . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass <i>m</i> , are placed at end points <i>A</i> and <i>B</i> of the rod and third particle of mass $M - 2m$ is placed at its centre of mass <i>O</i> , then it is clear that, (<i>i</i>) mass of both systems is equal to <i>M</i> , <i>D</i>	hmod	The moment of inertia of a circular plate (or disc) of mass m and radius a about an axis through its centre and repare
Since, both moments of inertia are same. Therefore both systems are equimomental. Problem: Find the equimomental system of particles for a uniform rod <i>AB</i> of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass of the rod <i>AB</i> having mass <i>M</i> . If we replace the rod by three particles, as shown in the figure, such that two particles, each having mass <i>m</i> , are placed at end points <i>A</i> and <i>B</i> of the rod and third particle of mass $M - 2m$ is placed at its centre of mass <i>O</i> , then it is clear that, (<i>i</i>) mass of both systems is equal to <i>M</i> , <i>D</i>	Innoc	perpendicular to its plane is $I_2 = \frac{1}{2}ma^2$.
Problem: Find the equimomental system of particles for a uniform rod AB of mass M and length 2a.Solution: Let O be the centre of mass of the rod AB having mass M. If we replacethe rod by three particles, as shown in the figure, such that two particles, eachhaving mass m, are placed at end points A and B of the rod and third particle ofmass $M - 2m$ is placed at its centre of mass O, then it is clear that,(i) mass of both systems is equal to M,	hmod	Since, both moments of inertia are same. Therefore both systems are equimomental.
the rod by three particles, as shown in the figure, such that two particles, each having mass m , are placed at end points A and B of the rod and third particle of mass $M - 2m$ is placed at its centre of mass O , then it is clear that, (i) mass of both systems is equal to M , D	hmoc	Problem: Find the equimomental system of particles for a uniform rod <i>AB</i> of mass <i>M</i> and length 2 <i>a</i> . Solution: Let <i>O</i> be the centre of mass of the rod <i>AB</i> having mass <i>M</i> . If we replace <i>c</i>
maxing mass <i>m</i> , are placed at end points <i>A</i> and <i>D</i> of the rot and time particle of $A = \frac{1}{m} + \frac{1}{m} + \frac{1}{m} = \frac{1}{m} + \frac{1}$	hmod	the rod by three particles, as shown in the figure, such that two particles, each o
hmo (i) mass of both systems is equal to M, Annood Prepared by Dr. D	hmod	mass $M - 2m$ is placed at its centre of mass O , then it is clear that, $C = 0$ $M - \frac{1}{2m}$ $M = \frac{1}{m}$ $M = \frac{1}{2m}$
	hmod	(i) mass of both systems is equal to M, Prepared by Dr. D

Prepared by: Dr. Amir Mahmood, mir Mahmood, Prepared by: Dr. Amir Mahmoo Page 30 pare hmo

hmood	Prepared	by: Dr.	Amir	Mahmood	Prepared	by: Di	, Amir	Mahmood	Prepare
hmood Mecl	Prepared nanics Made Eas	by: Dr.	Amir	Mahmood	Prepared	by: Dr	. Amir	Mahmood Moment of I	Prepare nertia
hmoo(ii) centre of mass	of both sy	stems is	same (i.e., point 0),Prepared	by: Di	r. Amir	Mahmood	Prepare
hmoom	<i>i</i>) symmetry axe oment of inertia	es (and her of the rod	ice, princ about an	axis <i>CD</i> through	systems are als 0 and perpendi	o same at cular to tl	centre of ne rod is g	mass 0. iven by mood	Prepare
hmood	Prepared	by: Dr.	Amir	$I_1 = \frac{1}{12}M($	$(2a)^2 = -Ma^2$	by: Di	r. Amir	Mahmood	Prepare
hmood	Prepared	by: Dr.	Amir	$\int_{f} I_2 = ma^2 + 0$	$(1) + ma^2 = 2ma$	²by: Di	r. Amir	Mahmood	Prepare
hmood	Prepared	by: Dr.	Amir	$= I_2 \implies \frac{1}{2}Ma^2$	$=2ma^2$	$m = \frac{1}{\epsilon}M$	r. Amir	Mahmood	Prepare
hmood	ence, equimome	ental system	n of part	icles is given by fin	rst particle of m	ass M at A	4, second p	particle of mass M	-P2mpare
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전문 가슴을 걸 방송했는 것



$$= \frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4 \, \mathrm{d}r \left(\int_{\theta=0}^{\pi/2} \sin^3\theta \, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin^2\phi \, \mathrm{d}\phi + \int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta \, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \, \mathrm{d}\phi \right), \tag{3}$$

where

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$$\int_{\phi=0}^{2\pi} \sin^2 \phi \, d\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1 - \cos 2\phi) \, d\phi = \frac{1}{2} \left(\phi - \frac{1}{2} \sin 2\phi \right) \Big|_{\phi=0}^{2\pi} = \frac{1}{2} (2\pi) = \pi$$
 (4)

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hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare Mechanics Made Easy Moment of Inertia hmo and pare

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$$\int_{\theta=0}^{\pi/2} \cos^2 \theta \sin \theta \, \mathrm{d}\theta = -\frac{1}{3} \cos^3 \theta \left| \frac{\pi/2}{\theta=0} = \frac{1}{3} \right|$$
(5)

Using (2), (4) and (5) in (3), we get hmo

$$I_{xx} = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5}\right) \left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{3M}{2\pi a^3} \left(\frac{a^5}{5}\right) \left(\frac{4\pi}{3}\right) = \frac{2}{5}Ma^2$$
a:

hmo <u>Products of inertia</u>:

$$I_{xy} = -\int_{\text{Hemisphere}} x \, y \, dm = -\frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} x \, y \, dV = -\frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin^3 \theta \sin \phi \cos \phi \, dr \, d\theta \, d\phi$$

$$= -\frac{3M}{2\pi a^3} \int_{r=0}^{a} r^4 \, dr \int_{\theta=0}^{\pi/2} \sin^3 \theta \, d\theta \int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, d\phi = 0$$

$$\therefore \int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, d\phi = \frac{1}{2} \sin^2 \phi \Big|_{\phi=0}^{2\pi} = 0$$

Now, hmo

$$I_{xz} = -\int_{\text{Hemisphere}} x \, z \, \mathrm{d}m = -\frac{3M}{2\pi a^3} \int_{\text{Hemisphere}} x \, z \, \mathrm{d}V = -\frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin^2 \theta \cos \theta \cos \phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$M_{xz} = -\frac{3M}{2\pi a^3} \int_{\mathrm{Hemisphere}} x \, z \, \mathrm{d}V = -\frac{3M}{2\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^4 \sin^2 \theta \cos \phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$$

Thus,

$$I_{xy} = I_{xz} = I_{yz} = 0,$$
 $\therefore I_{xz} = I_{yz}$ (by symmetry)

The inertia matrix with respect to coordinate system Oxyz is given by hmo

$$[I_O] = \begin{pmatrix} \frac{2}{5}Ma^2 & 0 & 0\\ 0 & \frac{2}{5}Ma^2 & 0\\ 0 & 0 & \frac{2}{5}Ma^2 \end{pmatrix}$$

hmo Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}Ma^2$ are principal moments of inertia. hmo

Problem: Find the (direction of) principal axes and principal moments of inertia of a (uniform) hmo solid sphere of mass Mat its Solution: (i) Moment of inertia about axis of symmetry:

hmo Let M, a and ρ , respectively, be the mass, radius and volume mass density of the sphere. Choose coordinate axes as shown in figure. hmo Moment of inertia of typical volume element of sphere, having mass dm and volume dV, about z-axis is given by hmo

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

Thus, moment of inertia of sphere about z-axis is

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$$I_{zz} = \int_{\text{Sphere}} (x^2 + y^2) dm$$

$$= \rho \int_{\text{Sphere}} (x^2 + y^2) dV$$

$$\therefore \rho = \frac{dm}{dV} = \text{constant}$$

$$= \frac{3M}{4\pi a^3} \int_{\text{Sphere}} (x^2 + y^2) dV$$

$$\therefore \rho = \frac{M}{(4/3)\pi a^3} \text{ (for sphere)}$$

To make the computation simpler, we transform the problem from Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) by using hmo

$$x = r \sin \theta \cos \phi, \qquad y = r \sin \theta \sin \phi, \qquad z = r \cos \theta$$

where, volume element in spherical coordinates is given by

$$dV = dr (r d\theta) (r \sin \theta d\phi) = r^2 \sin \theta dr d\theta d\phi$$



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Page 33 pare Prepared by: Dr. Amir Mahmood mir Mahmood Prepared by: Dr. Amir Mahmoo hmo

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hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare hmood Prepared by: Dr. Amir Mahmood Prepared by: Dr. Amir Mahmood Prepare Mechanics Made Easy Moment of Inertia hmo + z hmo Dare hmo hmo ahmo pare hmo Dare hmo Dare hmo Dare pare hmo Under the above transformation, the given ellipsoid is transformed into the unit sphere $S: x'^2 + y'^2 + z'^2 = 1.$ hmo Dare hmo $\Rightarrow I_{zz} = \frac{3M}{4\pi a \, b \, c} \int_{S} (a^{2} x^{\prime 2} + b^{2} y^{\prime 2}) (a \, b \, c \, \mathrm{d}x^{\prime} \, \mathrm{d}y^{\prime} \, \mathrm{d}z^{\prime}) = \frac{3M}{4\pi} \int_{S} (a^{2} x^{\prime 2} + b^{2} y^{\prime 2}) \mathrm{d}V^{\prime}, \quad \text{where, } \mathrm{d}V^{\prime} = \mathrm{d}x^{\prime} \, \mathrm{d}y^{\prime} \, \mathrm{d}z^{\prime}$ hmo pare ahmo pare To make the computation simpler, we transform the problem from Cartesian coordinates (x', y', z') to spherical hmo coordinates (r, θ, ϕ) by using $x' = r \sin \theta \cos \phi, \quad y' = r \sin \theta \sin \phi, \quad z' = r \cos \theta,$ hmo hmo $dV' = dr (r d\theta) (r \sin \theta d\phi) = r^2 \sin \theta dr d\theta d\phi$ where, volume element in spherical coordinates is given by, hmo pare $0 \le r \le 1, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi < 2\pi$ For unit sphere, hmo $\Rightarrow I_{zz} = \frac{3M(a^2 + b^2)}{4\pi} \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^4 \sin^3 \theta \cos^2 \phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{3M(a^2 + b^2)}{4\pi} \int_{r=0}^{1} r^4 \, \mathrm{d}r \int_{\theta=0}^{\pi} \sin^3 \theta \, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \cos^2 \phi \, \mathrm{d}\phi,$ hmo pare hmo are where, $\int_{\theta=0}^{\pi} \sin^3 \theta \, \mathrm{d}\theta = \frac{1}{4} \int_{\theta=0}^{\pi} (3\sin\theta - \sin 3\theta) = \frac{1}{4} \left(-3\cos\theta + \frac{1}{3}\cos 3\theta \right) \left| \frac{\pi}{\theta=0} = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3} \int_{\theta=0}^{\pi} (-3\theta) \, \mathrm{d}\theta = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3} \int_{\theta=0}^{\pi} (-3\theta) \, \mathrm{d}\theta = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3} \int_{\theta=0}^{\pi} (-3\theta) \, \mathrm{d}\theta = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3} \int_{\theta=0}^{\pi} (-3\theta) \, \mathrm{d}\theta = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3} \int_{\theta=0}^{\pi} (-3\theta) \, \mathrm{d}\theta = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3} \int_{\theta=0}^{\pi} (-3\theta) \, \mathrm{d}\theta = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3} \int_{\theta=0}^{\pi} (-3\theta) \, \mathrm{d}\theta = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right] = \frac{4}{3} \int_{\theta=0}^{\pi} (-3\theta) \, \mathrm{d}\theta = \frac{1}{4} \left[\left(3 - \frac{1}{3} \right) - \left(-3 + \frac{1}{3} \right) \right]$ hmo Dare hmo Dare and $\int_{\phi=0}^{2\pi} \cos^2 \phi \, d\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1 + \cos 2\phi) \, d\phi = \frac{1}{2} \left(\phi + \frac{1}{2} \sin 2\phi \right) \Big|_{\phi=0}^{2\pi} = \frac{1}{2} (2\pi) = \pi$ hmo $\Rightarrow I_{zz} = \frac{3M(a^2 + b^2)}{4\pi} \left(\frac{1}{5}\right) \left(\frac{4}{3}\right) (\pi) = \frac{1}{5}M(a^2 + b^2)$ hmo Similarly, $I_{xx} = \frac{1}{5}M(b^2 + c^2)$ and $I_{yy} = \frac{1}{5}M(a^2 + c^2)$ hmo Dare hmo (ii) Products of inertia with respect to axes of symmetry: hmo $I_{xy} = -\int_{\text{Ellipsoid}} x \, y \, \mathrm{d}m = -\frac{3M}{4\pi a \, b \, c} \int_{\text{Ellipsoid}} x \, y \, \mathrm{d}V = -\frac{3M}{4\pi a \, b \, c} \int_{\mathrm{S}} (a \, b \, x' \, y') (a \, b \, c \, \mathrm{d}x' \, \mathrm{d}y' \, \mathrm{d}z')$ hmo are $= -\frac{3abM}{4\pi} \int_{S} x' y' dV' = -\frac{3abM}{4\pi} \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^{4} \sin^{3}\theta \sin\phi \cos\phi \, dr \, d\theta \, d\phi$ hmo $= -\frac{3 a b M}{4\pi} \int_{r=0}^{1} r^4 dr \int_{\theta=0}^{\pi} \sin^3 \theta d\theta \int_{\phi=0}^{2\pi} \sin \phi \cos \phi d\phi = 0 \qquad \boxed{\because \quad \int_{\phi=0}^{2\pi} \sin \phi \cos \phi d\phi = \frac{1}{2} \sin^2 \phi \Big|_{\phi=0}^{2\pi} = 0}$ hmo

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$$\begin{aligned} & \text{Homod} \text{Prepared by: Dr. Amir Mahmood Prepared by: Dr. Dr. Amir Mahmood Prepared by: Dr. Dr. Dr. Dr. Dr. Dr. Dr. Dr.$$

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$$\int_{\phi=0}^{2\pi} \sin^2 \phi \, d\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1 - \cos 2\phi) \, d\phi = \frac{1}{2} \left(\phi - \frac{1}{2} \sin 2\phi \right) \Big|_{\phi=0}^{2\pi} = \frac{1}{2} (2\pi) = \pi$$
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and
 $\int_{\phi=0}^{\pi/2} - 2 \cos 2\phi \, d\phi = \frac{1}{2} \int_{\phi=0}^{\pi/2} - 1 \cos 2\phi \, d\phi = \frac{1}{2} \int_{\phi=0}^{\pi/2} - 1 \cos 2\phi \, d\phi = \frac{1}{2} \int_{\phi=0}^{\pi/2} - 2 \cos 2\phi \, d\phi = \frac{1}{2}$

$$\int_{\theta=0}^{\pi/2} \cos^2 \theta \sin \theta \, \mathrm{d}\theta = -\frac{1}{3} \cos^3 \theta \left| \frac{\pi/2}{\theta=0} = \frac{1}{3} \right| \tag{10}$$

Using (7), (9) and (10) in (8), we get

$$I_{xx} = \frac{Ma^2}{2\pi} \left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \frac{Ma^2}{2\pi} \left(\frac{4\pi}{3}\right) = \frac{2}{3}Ma^2$$

(iii) Products of inertia:

$$I_{xy} = -\int_{S} x \, y \, \mathrm{d}m = -\sigma \int_{S} x \, y \, \mathrm{d}S = -\frac{M}{2\pi a^2} \int_{S} x \, y \, \mathrm{d}S, \quad S: \text{ hemispherical shell}$$

Using parametric equations of hemispherical shell, we get

$$I_{xy} = -\frac{M}{2\pi a^2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} a^4 \sin^3 \theta \sin \phi \cos \phi \, \mathrm{d}\theta \, \mathrm{d}\phi = -\frac{Ma^2}{2\pi} \int_{\theta=0}^{\pi/2} \sin^3 \theta \, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \sin \phi \cos \phi \, \mathrm{d}\phi$$

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$$\int_{\phi=0}^{2\pi} \sin\phi \cos\phi \,\mathrm{d}\phi = \frac{1}{2} \sin^2\phi \Big|_{\phi=0}^{2\pi} = 0 \qquad \Longrightarrow \qquad I_{xy} = 0$$

hmo Now,

$$I_{xz} = -\int_{S} x \, z \, \mathrm{d}m = -\frac{M}{2\pi a^2} \int_{S} x \, z \, \mathrm{d}S = -\frac{M}{2\pi a^2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} a^4 \sin^2 \theta \cos \theta \cos \phi \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$= -\frac{Ma^2}{2\pi} \int_{\theta=0}^{\pi/2} \sin^2\theta \cos\theta \,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \cos\phi \,\mathrm{d}\phi$$

hmo But

$$\int_{\phi=0}^{2\pi} \cos\phi \,\mathrm{d}\phi = \sin\phi \left| \begin{array}{c} 2\pi \\ \phi=0 \end{array} \right| \implies I_{xz} = 0 = I_{yz}, \qquad \qquad \boxed{ \because I_{xz} = I_{yz} \text{ (by symmetry)} } \qquad \qquad \text{pare}$$

hmo _{Thus},

$$I_{xy} = I_{xz} = I_{yz} = 0$$

The inertia matrix with respect to coordinate system Oxyz is given by

$$[I_O] = \begin{pmatrix} \frac{2}{3}Ma^2 & 0 & 0\\ 0 & \frac{2}{3}Ma^2 & 0\\ 0 & 0 & \frac{2}{3}Ma^2 \end{pmatrix}$$

Since, all products of inertia are zero, therefore coordinate axes shown in the figure are required principle axes and corresponding moments of inertia $I_{xx} = I_{yy} = I_{zz} = \frac{2}{3}Ma^2$ are principal moments of inertia.

hmo Problem: Find the (direction of) principal axes and principal moments of inertia of a (uniform) principal spherical shell (hollow sphere) of mass M at its centre.

Solution: (i) Moment of inertia about axis of symmetry:

hmo Let M, a and σ , respectively, be the mass, radius of base and areal mass density of the spherical shell. Choose coordinate axes as shown in figure. Moment of inertia of typical area element of spherical shell, with mass dm and area dS, about z-axis is given by

$$\mathrm{d}I_{zz} = (x^2 + y^2)\mathrm{d}m$$

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hmo Thus, moment of inertia of spherical shell about z-axis is

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hmo Problem: Determine principal axes of a hmo plane rigid body (lamina) at a point in its 8 thmo plane, whose one poincipal axis perpendicular hmo to its plane is known. Solution: Consider a plane highed body in ahmo epare xy-plane, with z-axis as Rna un proincipal hmo axis. The other two poincipal axes will hmo epare lie in plane of body. As body is in xy-plane, therefore I13 = I23 = 0. Furthermose, hmo epare hmo epare Liz = 0. As we know that if I is poincipal moment of inertia, then hmo $((I_{11} - I)\omega_1 + I_{12}\omega_2 + \tilde{I}_{13}\omega_3 = 0$ $I_{12}\omega_{1} + (I_{22} - I)\omega_{2} + I_{23}\omega_{3} = 0$ hmo epare $I_{13}\omega_1 + I_{23}\omega_2 + (I_{33} - I)\omega_3 = 0$ ahmo where, $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ is the angular velocity in the direction epare hmo principle aris. Put I13 = I23 =0 in above system, we get epare hmo $(I_{11} - I)\omega_1 + I_{12}\omega_2 = 0$ pare $I_{12} \omega_1 + (I_{22} - \hat{I}) \omega_2 = 0$ epare $(I_{33}-I)\omega_3=0$ let Ox and Oy' be the principal axies in xy-plane at "O". let epare angle between Ox' and Ox'. 96 as poincipal moment of inectia wirt principal axis Ox, then above system epare of be the epare hmo $\int (I_{1} - I_{1})^{U} \omega_{1} + I_{12} \omega_{2} = 0$ hmo epare $I_{12} \omega_1 + (I_{22} - I_1) \omega_2 = 0$ $: I_{33} - I_1 \neq 0 \rightarrow$ hmo epare $\overline{\omega} = \omega_1 i + \omega_2 j$ is the angular velocity in the direction hmo Ox'(ie; x'-axis), therefores we have pare $\Rightarrow \frac{\omega_1}{\cos \phi} = \frac{\omega_2}{4\sin \phi} = k \left(\frac{\delta \alpha_1}{\delta \alpha_2} \right)$ Thus, hmo epare and $\omega_2 = k \sin \phi - \phi$ $\tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{\omega_2}{\omega_1}$ epare hmo - 3 @ in () and @, we get WI = Kcus¢ ([I, - I,) cos q + I,2 kin q = 0 ______ =7 hmo 3 and $I_{12}\cos\phi + (I_{22} - \hat{I}_{1})\sin\phi = 0$ - 0 Using

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hmo lamina Let ABC be a triangular lamina let h, h in a given pare Solution: Let ABC be a triangular lamina let h, h in a given pare hmo tively be the distances of the vertices A, i and C from a given pare tively be the distances of the vertices A, i and C from a given pare hmo line "l", such that Dh, < h_2 < h_3. Draw a line l' parallel to l pare

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and passing through vertex A. Suppose that the side CB, when produced, meets hmo pare A-h hmo line l'at point D. The distances D hmo A, B and C from l'are 0, hz-h, pare ahmo and ha-hi, respectively. pare dr d, c 9) Mi and M2 denote the masses of triangular -·lo pare assumed that laminas ACD and ABD have same areal mass pare pare denkity as that of lamina ABC), then pare M = mass of toringular lamina ABC pare = M, - M2 pare $\frac{M_1}{M_2} = \frac{(Akeal denkity)(Akea] \Delta ACD)}{(Akeal denkity)(Akea] \Delta ABD)} = \frac{(d)(\frac{1}{2} \times |AD| \times (h_3 - h_1))}{(d)(\frac{1}{2} \times |AD| \times (h_2 - h_1))}$ But pare pare $=\frac{h_3-h_1}{h_3-h_1}$ Thesefore, $\frac{M_1}{M_2} - 1 = \frac{h_3 - h_1}{h_2 - h_1}$ $\Rightarrow \frac{M_1 - M_2}{M_2} = \frac{M}{M_2} = \frac{h_3 - h_2}{h_2 - h_1}$ $\Rightarrow M_1 = \frac{M(h_2 - h_1)}{h_3 - h_2} \quad and \quad M_1 = M + M_2 = M + \frac{M(h_2 - h_1)}{h_3 - h_2} = \frac{M(h_3 - h_1)}{h_3 - h_2}$ Now, moment of inertia for lamina ABC about the line l' is given by pare 2 pare pare Ie' = (Moment q'inestia of lamina ACD) - (Moment q inestia q lamina ABD) about line l') - (Moment line l') pare $= \frac{1}{6} M_1 (h_3 - h_1)^2 - \frac{1}{6} M_2 (h_2 - h_1)^2$ $=\frac{1}{6}M\left(\frac{h_{3}-h_{1}}{h_{3}-h_{2}}\right)\left(h_{3}-h_{1}\right)^{2}-\frac{1}{6}M\left(\frac{h_{2}-h_{1}}{h_{3}-h_{2}}\right)\left(h_{2}-h_{1}\right)^{2}$ $=\frac{M}{6(h_{3}-h_{1})}\left[\left(h_{3}-h_{1}\right)^{3}-\left(h_{2}-h_{1}\right)^{3}\right]$ pare $= \frac{M}{6(h_3-h_1)} \left[(h_3-h_1) - (h_2-h_1) \right] \left[(h_3-h_1)^2 + (h_3-h_1)(h_2-h_1) + (h_2-h_1)^2 \right]$ •: a³-b³ = (a-b)(a+ab+b) oare pare $= \frac{M}{6} \left(h_3^2 + h_1^2 - \lambda h_1 h_3 + h_2 h_3 - h_1 h_3 - h_1 h_2 + h_1^2 + h_1^2 + h_1^2 - \lambda h_1 h_2 \right)$ pare $= \frac{M}{6} \left(3h_1^2 + h_2^2 + h_3^2 - 3h_1h_2 + h_2h_3 - 3h_1h_3 \right) - \frac{M}{6} \left(3h_1^2 + h_2^2 + h_3^2 - 3h_1h_2 + h_2h_3 - 3h_1h_3 \right)$ ____ (i)

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ared by: Dr. Amir Mah Let lo be the line parallel to l'(and l) and passing that centre q mass C'q lamina ABC. Then, moment q inertia Io lamina ABC about line lo is given by hmo where, d, is the distance q centre q mass C' from line l're $d_1 = \frac{1}{3} \left[\begin{pmatrix} distance \ q \end{pmatrix} \text{ vertex } A \\ from \ line \ l' \end{pmatrix} + \begin{pmatrix} distance \ q \end{pmatrix} \text{ vertex } B \\ from \ line \ l' \end{pmatrix} + \begin{pmatrix} distance \ q \end{pmatrix} \text{ vertex } B \\ from \ line \ l' \end{pmatrix} + \begin{pmatrix} distance \ q \end{pmatrix} \text{ vertex } B \\ from \ line \ l' \end{pmatrix}$ pare pare $= \frac{1}{3} \left[0 + (h_2 - h_1) + (h_3 - h_1) \right] = \frac{1}{3} \left(h_2 + h_3 - \frac{1}{3} h_1 \right)$ pare Using D and 3 in D, we get $I_{0} = \frac{M}{6} \left(3h_{1}^{2} + h_{2}^{2} + h_{3}^{2} - 3h_{1}h_{2} + h_{2}h_{3} - 3h_{1}h_{3} \right) - \frac{M}{9} \left(h_{1} + h_{3} - 2h_{1} \right)^{2}$ pare
$$\begin{split} & I_0 = \frac{M}{6} \left(3h_1^2 + h_2^2 - 3h_1h_2 + h_2h_3 - 3h_1h_3 \right) - \frac{M}{9} \left(h_2^2 + h_3^2 + 4h_1^2 + 4h_1^2 + 4h_1h_3 - 4h_1h_3 - 4h_1h_2 \right) - 9 \\ & Again, by parallel axis theorem, the moment 9 inestrial 9 kamina \\ & ABC about line & is given by \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_0 + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + Md_2 - 6 \\ & I_e = I_e + I_e +$$
where, d_2 is the distance of centre of mess C' from line l. Here $d_2 = \frac{1}{3}(h_1 + h_2 + h_3)$ (b), we get Using (c) and (c) in (c), we get pare $I_{\ell} = \frac{N}{6} \left(3h_{1}^{2} + h_{2}^{2} + h_{3}^{2} - 3h_{1}h_{2} + h_{2}h_{3} - 3h_{1}h_{3} \right) - \frac{N}{9} \left(h_{2}^{2} + h_{3}^{2} + 4h_{1}^{2} + 2h_{2}h_{3} - 4h_{1}h_{3} - 4h_{1}h_{2} \right)$ $= \frac{M}{18} \left[\frac{4h_1^2 + h_2^2 + h_3^2 + 2h_1h_2 + 2h_2h_3 + 2h_1h_3}{4 - 4h_1h_3} - \frac{2h_1^2 - 2h_2^2 - 8h_1^2 - 4h_2h_3 + 8h_1h_3 + 8h_1h_3 + 4h_1h_3}{4 - 4h_1^2 + 3h_2^2 - 9h_1h_2 + 3h_2h_3 - 9h_1h_3 - 2h_1^2 - 2h_3^2 - 8h_1^2 - 4h_2h_3 + 4h_1h_2 + 4h_1h_3}{4 - 4h_1^2 + 4h_1h_2 + 4h_1h_3} \right]$ nmo $= \frac{M}{18} \left[3h_1^2 + 3h_2^2 + 3h_3^2 + 3h_1h_2 + 3h_2h_3 + 3h_1h_3 \right]$ $= \frac{M}{6} \left[h_1^2 + h_2^2 + h_3^2 + h_1 h_2 + h_2 h_3 + h_1 h_3 \right] = \frac{M}{12} \left[2h_1^2 + 2h_2^2 + 2h_3^2 + 2h_1 h_2^2 + 3h_1 h_2^2 + 3h_1 h_3^2 \right]^{2}$ $= \frac{M}{12} \left[\frac{(h_1^2 + h_2^2 + a_{h_1h_2}) + (h_2^2 + h_3^2 + a_{h_2h_3}) + (h_1^2 + h_3^2 + a_{h_1h_3})}{12} \right] = \frac{M}{12} \left[\frac{(h_1^2 + h_2^2 + a_{h_1h_2}) + (h_2 + h_3^2 + a_{h_2h_3})}{12} \right]$ $= \frac{M}{3} \left[\left(\frac{h_1 + h_2}{2} \right)^2 + \left(\frac{h_2 + h_3}{2} \right)^2 + \left(\frac{h_1 + h_3}{2} \right)^2 \right],$ = $\frac{M}{3} \left(\frac{h_1 + h_2}{2} \right)^+ \left(\frac{h_1 + h_2}{2}$

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Brothem: Show that a uniform bolid cube of mass M is equimomental with (a) masses M/24 at midpoints of its edges and M/2 at its centre. (b) masses M/24 at its corners and 2M/3 at it centre. Solution: Let M and Py be the mass and density 3 I cube. Choose coordinate system Oxy & such that O lies at centre of mass and coordinate axes are along edges of the cube: Thus the coordinate axes are symmetry axes (and 0; hence principal axes) of cube at Moment of linestia of b cube rabout x-axis is 2 $I_{11} = \int (y^2 + 3^2) dm = P \int \int \int (y^2 + 3^2) dx dy dg$ pare problem 3= 2 8: 9 1= -92 $= \frac{M}{a^3} \int_{x=a}^{a} dx \int_{y=a}^{a} \int_{y=a}^{a} (y^2 + y^2) dy dy = \frac{M}{a^3} (a) \int_{y=a}^{a} (y^3 + y^3) dy dy = \frac{M}{a^3} \int_{x=a}^{a} (a) \int_{y=a}^{a} ($ This $= \frac{M}{a^{2}} \left(\frac{a^{3}}{a^{3}} \frac{3}{4} + \frac{a}{3} \frac{3^{3}}{3} \right) \Big|_{3=-9_{2}}^{9_{2}} = \frac{M}{a^{2}} \Big[3 \left(\frac{a^{4}}{a^{4}} + \frac{a^{4}}{a^{4}} \right) \Big] = \frac{M}{a^{2}} \left(\frac{a^{4}}{12} + \frac{a^{4}}{a^{4}} \right) \Big]$ By symmetry, $I_{33} = I_{22} = I_{11} = Ma^2$ (a) There are twelve (12) edges 9 a cube. When masses 1/24 are placed at the midpoints of these edges and a mass 1/2 is placed at centre, the total mass of the system of thirteen (13) particles is cube 18×(1/24)+サ = サ+サ = Thus is messes of both systems (cube and system of particles) are same, (ii) centres of mass of both bystems are same (at point 0) and (iii) symmetry axes (and hence poincipal axis) of both systems are same at 0. Cultorid, The two systems will be equimomental if pystem 9 pasticles also has moments 9 inestia about principal axes (coordinate axes), equal to particles also 3 Mat $I_{11} = I_{12} = I_{33} = I_{33}$ white Moment of inectia of system of particles about 3-axis is $I_{33} = 8 \times \left(\frac{M}{24} \times \left(\frac{a}{2} \right)^{2} \right)^{2} + 4 \times \left[\frac{M}{24} \times \left(\frac{a}{\sqrt{2}} \right)^{2} \right] = \frac{Ma^{2}}{12} + \frac{Ma^{2}}{12} = \frac{Ma^{2}}{6} = I_{11} = I_{22}$ Helloyin $I_{11} = I_{11}'$, $I_{22} = I_{22}$ and $I_{33} = I_{33}'$ i.e; corresponding poincipal moment q inestia q both systems same. Thus both systems lase dequimomental. ase also -2 よ (b) Total mass of the system of nine (9) particles = 8× 1/24 + 2M3 = M Thus above conditions (i), (ii) and (iii) are satisfied. Moment of inertia particles about J-axis is given $\begin{array}{cccc} 0 & I & I_{33}'' = 8 \times \left[(M_{14}) \times (\frac{a}{12})^2 \right] &= \frac{Ma^2}{6} = I_{11}'' = I_{22}'' \\ I_{11} &= I_{11}'' , & I_{22} = I_{22}'' , & I_{33} = I_{33}'' = \end{array}$ both systems are equimement. g system g

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Fixed and Rotating Frames of Reference pare G: Let Oxyz-coordinate system be ratating with respect to OXXZ-coordinate system fixed in space. Desive expression for the time desirative nate system fixed in space. Desive expression for the time desirative nector A(t) for the observers in two coordinate systems. pare pare Q: State and prove rotating axis theorem. IP: Statements = 40 a time dependent vector Function A: A(t) is pare pare sepsescuted by A, and An in fixed and hotating coordinate system, then pare $\left(\frac{dA}{dt}\right)_{f} = \left(\frac{dA}{dt}\right)_{h} + \vec{\omega} \times \vec{A}$ pare where $\omega = angulars velocity ? notating coordinate bystem wirt$ fixed coordinate bystempare and it is assumed that origins of two coordinates system pare are at some point (ie; origins love coincident.) Prof: Let OXYZ be fined co-ordinate pystem with if; it's as base (unit) vectors and OxyZ be notating coordinate system with i, j, k as its base (unit) vectors. pare pare We note that $A = A_{f} = A_{h}$ $\vec{A}_{f} = A_{i}\hat{i}' + A_{j}\hat{j}' + A_{j}\hat{k}'$ and $\vec{A}_{k} = A_{i}\hat{i} + A_{j}\hat{j} + A_{3}\hat{k}$ pare ket $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$ Thesefore, Also, (dA) dA; dA; dA; i't dA; i't dA; R (in fixed coordinate) system OXYZ) and $(\frac{dA}{dt}) \frac{dA_{h}}{dt} = \frac{dA_{l}}{dt}\hat{i} + \frac{dA_{2}}{dt}\hat{j} + \frac{dA_{3}}{dt}\hat{k} (in hotating Coordinate)$ pare As, $A_{i} = A_{k} = A_{i}\hat{i} + A_{i}\hat{j} + A_{3}\hat{k}$ Diggesentiating both sides with time "t" (viewing above vectors equation in fixed coordinate /system OXYZ), we get $\begin{pmatrix} d \\ dt \end{pmatrix}\hat{k} = \begin{pmatrix} d \\ dt \end{pmatrix}_{f} \begin{pmatrix} A_{i}\hat{i} + A_{3}\hat{j} + A_{3}\hat{k} \end{pmatrix}$ $\Rightarrow \begin{pmatrix} dA_{i} \\ dt \end{pmatrix}_{f} = \begin{pmatrix} dA \\ dt \end{pmatrix}_{f} = \frac{dA_{i}\hat{i}}{dt} + \frac{dA_{2}\hat{j}}{dt} + \frac{dA_{3}\hat{k}}{dt} + A_{1}\frac{d\hat{i}}{dt} + A_{2}\frac{d\hat{j}}{dt} + A_{3}\frac{d\hat{k}}{dt} \begin{pmatrix} we have duped \\ windex f flow \\ windex f flow \\ convinience \end{pmatrix}$ pare pare equation in pare $\Rightarrow \left(\frac{dA}{dt}\right)_{f} = \left(\frac{dA}{dt}\right)_{h} + A_{i}\frac{di}{dt} + A_{2}\frac{di}{dt} + A_{3}\frac{dk}{dt} - 0 \approx \left(\frac{dA}{dt}\right)_{h} = \frac{dA_{i}\hat{i} + dA_{h}\hat{j} + \frac{dA_{3}}{dt}\hat{k}$ pare To find di, we proceed as follows: We note that as $\hat{i} = \overline{op}$ notates about the axis of notation, is displaced to point Q in time interval &t point P He

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hmo hmo hmo ahmo hmo hmo hmo hmo hmo hmc hmo ahmo hmo buch that it distances of P and G from notation semain same, as shown in the figure that perpens anis pare From Figure; /PG1 = 18i1 = 1MP180 : in DOMP ê pare $= \beta \sin \phi \, \delta \phi \quad \Rightarrow \ |\overline{oP}| = |\widehat{u}| = 1 |\overline{OP}|$ = (10P/bing) 60 pare pare St becomes so small such that St >0, then pare $\left|\frac{di}{dt}\right| = \dot{\theta}/\sin\phi = \omega/\sin\phi$, where $\dot{\theta} = \frac{d\theta}{dt} = \omega = \left(\frac{magnitude}{angular} + velocity\right)$ pare 4) pare $= |(\hat{\theta}\hat{e}) \times \hat{i}| = |\vec{\omega} \times \hat{i}|, \quad \text{where } \hat{e} \text{ is unit vector} \\ = |(\hat{\theta}\hat{e}) \times \hat{i}| = |\vec{\omega} \times \hat{i}|, \quad \text{where } \hat{e} \text{ is unit vector} \\ = \hat{\theta}\hat{e}$ pare $\Rightarrow \frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i} \qquad \therefore \frac{d\hat{i}}{dt} \text{ and } \vec{\omega} \times \hat{i} \text{ have same magnitudes and bare where, the direction of <math>\vec{\omega}$ is related to that $\int \hat{i}$ and $\frac{d\hat{i}}{dt}$ by right hand hule. pare pare Similarly, $\frac{dj}{dt} = \vec{\omega} \times j$ and $\frac{d\hat{\kappa}}{dt} = \vec{\omega} \times \hat{k}$ Thus, equation D becomes pare Thus, equation $\begin{pmatrix} d\tilde{A} \\ dt \end{pmatrix}_{f} = \begin{pmatrix} d\tilde{A} \\ dt \end{pmatrix}_{\lambda} + A_{\lambda}\tilde{\omega}\tilde{x}\hat{i} + A_{\lambda}\tilde{\omega}\tilde{x}\hat{j} + A_{3}\tilde{\omega}\tilde{x}\hat{k}$ pare $\begin{pmatrix} d\vec{A} \\ dt \end{pmatrix}_{f} = \begin{pmatrix} d\vec{A} \\ dt \end{pmatrix}_{h} + \vec{\omega} \times (A_{i}\hat{i} + A_{j}\hat{j} + A_{k}\hat{k})$ pare $\frac{\partial (d\vec{A})}{\partial (dt)_{f}} = (\vec{O}(\vec{A})_{f} + \vec{\omega} \times \vec{A})$ (Hence proved.)pare pare Remarn: Above result can also be written as pare as follows $D_f = D_h + \overline{\omega} x$ pare pare Where L.H.S Khowld be viewed in fined coordinate bystem, hile R.H.S in notating coordinate system. pare while

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a hotating Coordinate System. with hespect motion Q: Find the equation expressions of a sotating coordinate 6: Find velocity and acceleration with respect to - fixed coordinate Sol: Case I: When the origins g fixed and System hotating coordinate systems are coincident: pare let x be the position vector of a moving particle P, as shown in the figure, then $\overline{h} = \overline{h}_f = \overline{h}_h$ Sotating coordinate pare and h shows the where subscripts f pare quantities viewed in fixed and hotating peopeetively. coordinate systems, last will proceed in the same manner as À and, ultimately, we will seach at: Dear students, we question by taking i in place of $\begin{pmatrix} d\tilde{u} \\ dt \end{pmatrix}_{f} = \begin{pmatrix} dt \\ dt \end{pmatrix}_{h} + \tilde{\omega} \times \tilde{h}_{h}$ pare $\vec{v}_f = \vec{v}_h + \vec{\omega} \times \vec{h} \otimes \vec{v}_h \otimes \vec{v}_h$ we can drop subscript h from vector \vec{h} , as it is same in both coordinate systems. equivilently, $\vec{v}_{k} = \vec{v}_{f} - \vec{\omega} \times \vec{k}$ with hespect to Frame of heference. hotating This is required velocity operator equation as get (Follows we $\begin{pmatrix} d \\ dt \end{pmatrix}_{f} = \begin{pmatrix} d \\ dt \end{pmatrix}_{h} + \tilde{w}\chi$ (Remembring that L.H.S is viewed in fixed coordinate system, while R:H:S is viewed in rotating one.) From O, fined coordinate @ Www.r.t both kides 9 OXYZ Differentiating $\begin{pmatrix} d \\ dt \end{pmatrix}_{f}^{\vec{v}_{f}} = \begin{pmatrix} d \\ dt \end{pmatrix}_{f}^{\vec{v}_{h}} \begin{pmatrix} \vec{v}_{h} + \vec{\omega} \times \vec{h} \end{pmatrix}$ both $\left(\frac{dv_f}{dt}\right)_f = \left(\left(\frac{d}{dt}\right)_n + \vec{\omega} \times\right) \left(\vec{v}_n + \vec{\omega} \times \vec{\lambda}\right)$ 3 $\begin{pmatrix} dv_{f} \\ dt \end{pmatrix}_{f} = \begin{pmatrix} d\vec{v}_{h} \\ dt \end{pmatrix}_{h} + \begin{pmatrix} d(\vec{\omega} \times \vec{h}) \\ dt \end{pmatrix}_{h} + \vec{\omega} \times \vec{v}_{h} + \vec{\omega} \times (\vec{\omega} \times \vec{h}) \end{pmatrix}$ bame 2. ヨ $\vec{a}_{f} = \vec{a}_{h} + \vec{\omega} \times (\vec{a}_{h}) + \vec{\omega} \times \vec{h} + \vec{\omega} \times \vec{v}_{h} + \vec{\omega} \times (\vec{\omega} \times \vec{h})$ 13 $= \vec{a}_{f} = \vec{a}_{x} + \vec{a}\vec{\omega}\times\vec{v}_{y} + \vec{\omega}\times\vec{i} + \vec{\omega}\times(\vec{\omega}\times\vec{i}) -$ Note $\left\{\vec{a}_{\lambda} = \vec{a}_{\mu} - 2\vec{\omega} \times \vec{v}_{\mu} - \vec{\omega} \times \hbar - \vec{\omega} \times (\vec{\omega} \times \vec{k})\right\}$ required acceleration with respect to rotating coordinate bystem. This is the

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= acceleration viewed in fixed coordinate System. epare Hose $\vec{a}_{f} = \left(\frac{d\vec{v}_{f}}{dt}\right)_{f}$ $\vec{a}_{k} = \begin{pmatrix} d\vec{v}_{k} \\ dt \end{pmatrix}_{k}^{T} = \stackrel{acceleration}{=} \vec{v} = \vec{v} \neq \vec{v} \neq \vec{v} = centripetal (acceleration.)$ $\vec{a}_{cor} = -2\vec{\omega} \neq \vec{v}_{k} = Coriclib acceleration.$ pare pare $\tilde{a}_{tr} = -\tilde{\omega} \times \tilde{n} = thans verse or linear acceleration.$ pare epare $\vec{\alpha}_{cf} = -\vec{\omega} \times (\vec{\omega} \times \vec{n}) = \text{centrifugal}$ acceleration. pare The relation 3 is referred to as "Coriolis theorem". The equations of motion in fixed and moving coordinate bystems are given by \vec{r} pare pare epare $\vec{F}_f = m \vec{a}_f$ and $\vec{F}_h = m \vec{a}_h$, respectively. pare Where, Fr and Fr are total forces in fixed and hotating condinate systems, respectively and m is mass of moving pasticle. pare pare epare Multiplying equation @ by "m" on both kides, we get pare $m\vec{a}_{k} = m\vec{a}_{f} - a_{m}\vec{\omega} \times \vec{v}_{k} - m\vec{\omega} \times \vec{k} - m\vec{\omega} \times (\vec{\omega} \times \vec{k})$ $\Rightarrow [\vec{F}_{\lambda} = \vec{F}_{f} + \vec{F}_{ex} + \vec{F}_{th} + \vec{F}_{ef}]$ pare pare For = - 2m wx Vr = Coriolis force. pare where, For = - mäxi = transverse force. pare pare For = - m wix (wixh) = contrifugal force. pare These forces are called fictitions or apparent or inertial forces. Equation & is required equation of motion in retating name I betweence. pare pare Observations: i) As $\vec{v}_h \cdot \vec{F}_{ar} = -2m \vec{v}_h \cdot \vec{\omega} \times \vec{v}_h = 0$ This knows that the Coriole's force has a direction which be at right angle to the direction of motion defined by Vr.9 in the Vrotating coordinate Departem. (ii) The centrigal force F_{cf} is always directed away from the axis of rotation. pare pare pare pare pare pare

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Prepared by: Mechanics Made Eas (11) The transverse force For depends on angular acceleration is and will disappear when angular velocity is constant. (iii) hmo pare (iv) The Cosioli's force is a special interest. It is present only if the particle is moving in notating coordinate system (i.e.; Vin = i) hmc pare hmo pare hmc pare Case II : When the origin of hotating coordinate system is moving theely wirt fixed coordinate system. hmo pare Let OXYZ be fixed (space) condinate Zi bystem with baser vectors i'j', i' hmo in the show pare hmo along coordinate axes and Ozyz pare hmo hotating coordinate system be sotating (body) coordinate system with base (unit) vectors i, j, k wong of its coordinate axes. pare hmo pare Fixed coordinate system hmo let, pare Ry=R = Position vector of moving X positicle P wirt fixed coordinate system OXYZ. hmo pare hmo hy = h = Position vector of moving pasticle P wint notating coordinate system Oxyz. pare hmo ho = Position vector of O (origin of notating frame Oxyz) w. It fixed frame Oxyz. pare hmo From tig: $\overline{k_{g}} = \overline{k_{o}} + \overline{k_{s}} = \overline{k_{o}} + \chi i + g j + g \hat{\kappa}$, where $\overline{h} = h = \chi i + g \bar{s} + g \hat{\kappa}$ Differentiate both brides of above equation court time t (wirt fined frame OXYZ) From Fig: hmo hmo $\Rightarrow \left(\frac{d}{dt}\right) \vec{h} = \left(\frac{d}{dt}\right) \left(\vec{h}_0 + x\hat{i} + y\hat{j} + g\hat{k}\right)$ hmo $= \left(\frac{d\vec{k}_{f}}{dt}\right)_{f} = \frac{d\vec{k}_{o}}{dt} + \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dy}{dt}\hat{k}\right) + \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{k} + \frac{dy}{dt}\hat{k}\right) + \left(\frac{dx}{dt}\hat{k} + \frac$ hmc pare But $\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dy}{dt}\hat{k} = (\frac{dx}{dt}) = \vec{v}_k$ and $\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i}$, $\frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j}$, $\frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$ pare hmc where Vo= dr hmc pare Theorefore, $\vec{Y}_{f} = \vec{V}_{0} + \vec{V}_{k} + x(\vec{\omega} \times \hat{i}) + y(\vec{\omega} \times \hat{j}) + \hat{j}(\vec{\omega} \times \hat{k})$, and $\vec{V}_{f} = (\vec{d}\hat{i})$ hmo Note: From this eq. we can find \Rightarrow $\vec{V}_f = \vec{V}_0 + \vec{V}_k + \vec{\omega} \times (x_1^2 + y_1^2 + 3\hat{\kappa}) = \vec{V}_0 + \vec{V}_k + \vec{\omega} \times \vec{\lambda}$ $\vec{v}_f = \vec{k}_f = x_1^2 + y_1^2 + 3\hat{\mu}$ pare hmo pare Differentiating both kides with time t (with fixed Frame OXYZ) $\begin{pmatrix} d \\ dt \end{pmatrix}_{f}^{\vec{v}_{f}} = \begin{pmatrix} d \\ dt \end{pmatrix}_{f}^{\vec{v}_{e} + \vec{v}_{h} + \vec{\omega} \times \vec{h}}) = \begin{pmatrix} d \\ dt \end{pmatrix}_{f}^{\vec{v}_{e} + (\frac{d}{dt})_{f}} (\vec{v}_{h} + \vec{\omega} \times \vec{h})$ pare hmo Note: Farm here: we can find => $\{\hat{a}_{\mu} = \hat{a}_{\nu} + \hat{a}_{\mu} + \hat{\omega} \times \hat{v}_{\mu} + \hat{\omega} \times \hat{v}_{\mu}$ $\frac{\overline{a}_{k}}{F_{som}} \underbrace{\textcircled{(})}_{k=\sqrt{k}-\sqrt{\omega}\times\overline{k}} \xrightarrow{\text{(})}_{k=\sqrt{k}-\sqrt{\omega}\times\overline{k}} \xrightarrow{\text{(})$ pare pare $= \overline{V_{f} - V_{o}} - \overline{\omega} \times \overline{h}$ and $(\underline{u}, \underline{v})$ both bides, we get eq. 9 metion in moving frame as Multiplying eq. (a) by m, on both bides, we get eq. 9 metion in moving frame as $\overline{F_{k}} = \overline{F_{f}} - \overline{F_{o}} + \overline{F_{cos}} + \overline{F_{th}} + \overline{F_{ef}}$; (where $\overline{F_{s}} = m\overline{a_{o}}$) and $(\overline{F_{s,s}} + \overline{F_{th}} - \overline{F_{ef}})$ are $\overline{F_{k}} = \overline{F_{f}} - \overline{F_{o}} + \overline{F_{cos}} + \overline{F_{th}} + \overline{F_{ef}}$; (where $\overline{F_{s}} = m\overline{a_{o}}$) and ($\overline{F_{s,s}} + \overline{F_{th}} - \overline{F_{ef}})$ (before) pare

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