ALGEBRA II

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Lecture # 1

For Understanding:

If (G,+) is Abelian group.

If that G is

(i) (G , .) closed and
(ii) (G, .) associative

then (G, +, .) is called a Ring

And If (G, .) contain "e"

$\Rightarrow$ (G, +, .) called Identity Ring. Or Ring with unity.

If (G, .) contain inverse

$\Rightarrow$ (G, +, .) called Division Ring

If (G , .) holds commutativity

$\Rightarrow$ (G, +, .) called Abelian Ring

If (G, +, .) holds distributive laws (left and right distributive law) then

(G, +, .) is called a Field.

(G, +, .) become (F, +, .)

e.g. set of real number is a field and set of rational number is a field.

Vector Space:

Let (V,+) be an abelian group and ($\mathbb{F}$, +, .) be a field define a scalar multiplication

"." : $\mathbb{F} \times V \rightarrow V$ since (. is function)

Such that $\forall\ \alpha \in F$, $v \in V$, $\alpha.v \in V$

Then V is said to be a Vector space over F if the following axioms are true

(i) $\alpha(u+v) = \alpha u + \alpha v$
(ii) $(\alpha+\beta)\, u = \alpha u+\beta u$
(iii) $\alpha(\beta u) = (\alpha\beta)u$ (iv) $1.u = u$ $\forall\ \alpha,\beta \in \mathbb{F}$, $u.v \in V$

Example:

Let F be a field consider the set V ={(α,β) : α , β ∈ F} then V is vector space.

Solution:

Define Addition and scalar multiplication in V as

Let (α_1, β_1) , $(\alpha_2, \beta_2) = (\alpha_1 + \alpha_2 , \beta_1 + \beta_2)$

Let α ∈ F and $(\alpha_1, \beta_1) \in V$ then $\alpha.(\alpha_1, \beta_1) = (\alpha\alpha_1, \alpha\beta_1)$

Then V form a vector space over $\mathbb{F}$

Now we make (V, +) is abelian

(i) Let (α_1, β_1) , $(\alpha_2, \beta_2) \in V$
$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2 , \beta_1 + \beta_2)$
Closure law is hold

(ii) Associating is trivial

(iii) Let O = (0,0) ∈ V
Where O ∈ F
(α,β) + (0,0) = (α+0 , β+0) = (α,β)
Identity law is hold

(iv) Since α ∈ F ⇒ -α ∈ F
Also β ∈ F , ⇒ -β ∈ F
Now (α,β) ∈ F ⇒ (-α, -β) ∈ F

And (α,β) + (-α,-β) = (α-α , β-β) = (0,0) ∈ V inverse exist

(v) $(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2 , \beta_1 + \beta_2)$
$= (\alpha_2 + \alpha_1 , \beta_2 + \beta_1)$
$= (\alpha_2, \beta_2) + (\alpha_1, \beta_1)$
Commutative law hold.

Hence (V, +) is abelian group. Now we prove V is vector space by following axioms.

(i) Let α ∈ F and $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in V$

then $\alpha\,[(\alpha_1, \beta_1) + (\alpha_2, \beta_2)] = \alpha\,[(\alpha_1 + \alpha_2 , \beta_1 + \beta_2)]$

$= (\alpha\,[\alpha_1 + \alpha_2] , \alpha\,[\beta_1 + \beta_2])$

$= (\alpha\alpha_1 + \alpha\alpha_2 , \alpha\beta_1 + \alpha\beta_2)$

$= (\alpha\alpha_1 , \alpha\beta_1) + (\alpha\alpha_2 , \alpha\beta_2)$

$= \alpha(\alpha_1, \beta_1) + \alpha(\alpha_2, \beta_2)$

(ii) $[\alpha+\beta]\ (\alpha_1, \beta_1) = ([\alpha+\beta]\alpha_1 , [\alpha+\beta]\beta_1)$
$= (\alpha\alpha_1 + \beta\alpha_1,\ \alpha\beta_1 + \beta\beta_1)$
$= (\alpha\alpha_1 + \alpha\beta_1) + (\beta\alpha_1,\ \beta\beta_1)$
$= \alpha(\alpha_1, \beta_1) + \beta(\alpha_1, \beta_1)$

(iii) $\alpha[\beta(\alpha_1, \beta_1)] = \alpha(\beta\alpha_1,\ \beta\beta_1)$

$= (\alpha\beta\alpha_1,\ \alpha\beta\beta_1)$

$= \alpha\beta(\alpha_1, \beta_1)$

(iv) $1.(\alpha_1, \beta_1) = (1.\alpha_1, 1.\beta_1)$

$= (\alpha_1, \beta_1)$

All axioms are satisfied. Hence V is vector space.

Lecture # 2

Example:

Let **F** be a field and $\phi \neq X$. Let $\mathbb{F}^X = \{ f \mid f : X \to \mathbb{F} \}$. Define addition and scalar multiplication in $\mathbb{F}^X$ as

Let $f,g \in \mathbb{F}^X$; $(f + g)(x) = f(x) + g(x)$ (1)

$\forall\ \alpha \in \mathbb{F}$ and $f \in \mathbb{F}^X$

$(\alpha f)(x) = \alpha.f(x)$ (2)

Then show that $\mathbb{F}^X(\mathbb{F})$ is a vector space.

Solution: First we show that $(\mathbb{F}^X,+)$ is an abelian group.

(i) $\mathbb{F}^X$ is closed as

Let $f,g \in \mathbb{F}^X$

$(f + g)(x) = f(x) + g(x)$

(ii) Associativity is trivial.

(iii) Identity

$\forall\ f \in \mathbb{F}^X\ \exists\ I \in \mathbb{F}^X$

such that $I(x) = 0$

Now $(f+I)(x) = f(x) + I(x)$ By (1)

$= f(x) + 0$

$(f+I)(x) = f(x)$

$\Rightarrow f + I = f$

$\Rightarrow$ identity exist in $\mathbb{F}^X$

(iv) Inverse

Let $f \in \mathbb{F}^X\ \exists\ f^{-1} \in \mathbb{F}^X$

Such that $f^{-1}(x) = -f(x)$

Now $(f+f^{-1})(x) = f(x) + f^{-1}(x)$

$= f(x) - f(x) = 0$

$= I(x)$

$\Rightarrow$ $f + f^{-1} = I$

$\Rightarrow$ Inverse exits in $\mathbb{F}^X$

(v) Commutativity

From (1) we have $(f + g)(x) = f(x) + g(x)$

$= g(x) + f(x)$

$= (g+f)(x) \Rightarrow f + g = g + f$

Hence $(\mathbb{F}^X, +)$ is an abelian group.

Now we prove $\mathbb{F}^X(\mathbb{F})$ is a vector space.

(i) Let $\alpha \in \mathbb{F}$ and $f, g \in \mathbb{F}^X$

$[\alpha(f+g)](x) = (\alpha f + \alpha g)(x)$ By (2)

$= (\alpha f)(x) + (\alpha g)(x)$ By (1)

$= \alpha.f(x) + \alpha.g(x)$ By (2)

$\Rightarrow \alpha(f+g) = \alpha f + \alpha g$

(ii) Let $\alpha, \beta \in \mathbb{F}$ and $f \in \mathbb{F}^X$

$[(\alpha+\beta)f](x) = (\alpha f + \beta f)(x)$ By (2)

$= (\alpha f)(x) + (\beta f)(x)$ By (1)

$= \alpha f(x) + \beta f(x)$ By (2)

$\Rightarrow (\alpha + \beta) f = \alpha f + \beta f$

(iii) Let $\alpha, \beta \in \mathbb{F}$ and $f \in \mathbb{F}^X$

$[\alpha(\beta f)](x) = (\alpha\beta f)(x)$ By (2)

$= \alpha\beta.f(x)$ By (2)

$\Rightarrow \alpha(\beta f) = (\alpha\beta)f$

(iv) Let $1 \in \mathbb{F}$ and $f \in \mathbb{F}^X$

$(1.f)(x) = f(x)$

$\Rightarrow 1.f = f$

$\Rightarrow \mathbb{F}^X(\mathbb{F})$ is a vector space.

Subspace:

Let V be the vector space over the field **F.** $V(\mathbb{F})$ be a vector space.

Let $\phi \neq W \subseteq V$ then W is called subspace of V if W itself becomes a vector space under the same define addition and scalar multiplication as in V.

Theorem:

A non-empty subset W of vector space V over the field $\mathbb{F}$ is a subspace of V iff $\alpha u + \beta v \in W$, $\forall$ u,v $\in$ W and $\alpha,\beta \in \mathbb{F}$

Mathematically statement

$\phi \neq W \leq V(\mathbb{F}) \Leftrightarrow \alpha u + \beta v \in W, \forall u,v \in W \ \&\ \alpha,\beta \in \mathbb{F}$

Proof:

Let W be a subspace of $V(\mathbb{F})$

$\Rightarrow$ W is vector space then $\forall$ u,v $\in$ W & $\alpha,\beta \in \mathbb{F}$

$$\alpha u + \beta v \in W$$

Conversely, Let $\alpha u + \beta v \in W$

Take $\alpha = 1$, $\beta = 1$

$\alpha u + \beta v = 1.u + 1.v = u + v \in W$

$\Rightarrow$ (W, +) is closed.

Take $\alpha = 1$, $\beta = 0$ and vice versa

$\Rightarrow \alpha u + \beta v = 1.u + 0.v = u \in W$

$\Rightarrow \alpha u + \beta v = 0.u + 1.v = v \in W$

$\Rightarrow$ (W, .) is closed Hence W is a subspace.

Note: "$\leq$" means subspace, subring, subset.

Question:

Let **F** be a field and $\phi \neq W$. Let $\mathbb{F}^{X} = \{ f \mid f : X \to \mathbb{F} \}$;$Y \subseteq X$ and

$W = \{ f \mid f : Y \to \mathbb{F} \}$ or $W = \{ f \mid f(y) = 0\ \forall\ y \in Y \}$

Then show that W is subspace $\mathbb{F}$**.**

Solution: Let $y_1\ y_2 \in Y$ and α , $\beta \in \mathbb{F}$

Such that $f(y_1) = 0$, $f(y_2) = 0$,

$$\alpha f(y_1) + \beta f(y_2) = \alpha(0) + \beta(0) = 0 \in W$$

Lecture # 3

Example:

Let V be a vector space of all 2×2 matrices over the field R then check either W is subspace or not.

(i) W consists of all 2×2 singular matrices.
(ii) W consists of all 2×2 Idempotent matrices.
(iii) W consists of all 2×2 symmetric matrices.

Solution:

(i) Let W consist of all 2×2 singular matrices i.e. if M ∈ W $\Rightarrow |M| = 0$

Let M and N ∈ W such that

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M + N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But $|M + N| \neq 0 \Rightarrow M + N \notin W$

$\Rightarrow W \nleq V$

(ii) Let W consist of all 2×2 Idempotent matrices i.e. if M ∈ W $\Rightarrow M^2 = M$

Let M ∈ W such that

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow M^2 = M$$

Now $2M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$$(2M)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq 2M \notin W$$

$\Rightarrow W \nleq V$

(iii) Let W consist of all 2×2 symmetric matrices i.e. if A ∈ W $\Rightarrow A^t = A$

And if B ∈ W $\Rightarrow B^t = B$

Let α,β ∈ F = R such that

$(\alpha A + \beta B)^t = (\alpha A)^t + (\beta B)^t \quad = \alpha A^t + \beta B^t$

$\Rightarrow \alpha A + \beta B \in W \qquad \Rightarrow W \leq V$

Example:

Let V = R^3 and $\phi \neq W \subseteq V$

Let W = {(u,v,1) : u,v ∈ R , 1 ∈ R}

Check W is a subspace of V or not.
Solution:

Let x,y ∈ W such that

x = $(u_1, v_1, 1)$ and y = {$(u_2, v_2, 1)$

Now x + y = $(u_1+u_2, v_1+v_2, 1+1)$

= $(u_1+u_2, v_1+v_2, 2) \notin W$

$\Rightarrow W \not\leq V$

Example:

Let V = R^3 and $\phi \neq W \subseteq V$

Let W = {(u,v,ω) : u+v+$w = 0$ } Check W is subspace of V or not.

Solution:

Let x,y ∈ W such that

x = (u_1, v_1, w_1) and y = (u_2, v_2, w_2)

Now let α,β ∈ F

αx+βy = $\alpha(u_1, v_1, w_1) + \beta(u_2, v_2, w_2)$

= $\alpha(u_1+v_1+w_1) + \beta(u_2+v_2+w_2)$

= α(0) + β(0)

= 0 ∈ W Hence W is a vector space of V

Example:

Let V = R^3 and $\phi \neq W \subseteq V$

Let W = {(u,v,w) : u-2v+3$w = 0$ } Check W is subspace of V or not.

Solution:

Let x , y ∈ W such that

x = (u_1, v_1, w_1) and y = (u_2, v_2, w_2)

Now let α,β ∈ F

$\alpha x+\beta y = \alpha(u_1,-2v_1, 3w_1) + \beta(u_2,-2v_2,3w_2)$

$= \alpha(u_1 - 2v_1+ 3w_1) + \beta(u_2 - 2v_2+3w_2)$

$= \alpha(0) + \beta(0)$

$= 0 \in W$ Hence W is a vector space of V

Example:

Let V be a vector space of all real valued function. Let $\phi \neq W \subseteq V$.

Let $W = \{ f : \int_0^1 f = 0 \}$. Check $W \leq V$ or $W \nleq V$.

Solution:

Let u,v ∈ W such that

$u = \int_0^1 f = 0$ and $v = \int_0^1 g = 0$

Now let $\alpha,\beta \in \mathbb{F}$

$\alpha u + \beta v = \alpha\int_0^1 f + \beta\int_0^1 g = \alpha(0) + \beta(0)$

$\alpha u + \beta v = 0 \in W$

$\Rightarrow W \leq V$

Example:

Let $V = R^n$: let $\phi \neq W$

Let $W = \{(x_1, x_2, x_3, \ldots\ldots x_n): x_1+x_2+x_3+\ldots\ldots+x_n=1\}$

Check either W ≤ V or not.

Solution:

Let u,v ∈ W:

u = (1,0,0,………0) and v = (0,1,0……..0)

Now u + v = (1,0,0,………0) + (0,1,0……..0)

= (1,1,0,……..0) ∉W

$\Rightarrow W \nleq V$

Sum of Subspaces:

Let V(F) be a vector space. Let W_1 and W_2 are the subspaces of V(F) then sum of W_1 and W_2 is defined as

$W_1 + W_2 = \{x : x = w_1 + w_2 , w_1 \in W_1 \wedge w_2 \in W_2\}$

This is known as sum of two subspaces.

Note: Sum of two subspaces is again a subspace.

Theorem:

Prove that sum of subspaces is again a subspace.

Proof:

It is clear that $W_1 + W_2 \neq \phi$ as $0 = 0 + 0$

Let $u \in W_1 + W_2 : u = w_1 + w_2 \quad , w_1 \in W_1 , w_2 \in W_2$

$v \in W_1 + W_2 : v = w_1' + w_2' \quad , w_1' \in W_1 , w_2' \in W_2$

Let $\alpha,\beta \in \mathbb{F}$

Now $\alpha u+\beta v = \alpha(w_1 + w_2) + \beta(w_1' + w_2')$

$= \alpha w_1 + \alpha w_2 + \beta w_1' + \beta w_2'$

$= (\alpha w_1+\beta w_1') + (\alpha w_2+\beta w_2') \qquad \in W_1 + W_2$

$\alpha u+\beta v \in W_1 + W_2$

$\Rightarrow W_1 + W_2$ is a subspace of $V(\mathbb{F})$

Direct Sum:

Let $W_1 , W_2 ,\ldots\ldots W_n$ are the subspaces of $V(\mathbb{F})$ then the direct sum of $W_1 , W_2 ,\ldots\ldots W_n$ is denoted by and defined as

$W_1 + W_2 + ,\ldots\ldots +W_n = W_1 \oplus W_2 \oplus ,\ldots\ldots \oplus W_n =$ can be written as

$x = w_1 + w_2 ,\ldots\ldots w_n$ uniquely.

Theorem:

$W_1 + W_2 = W_1 \oplus W_2 \Leftrightarrow W_1 \cap W_2 = \{0\}$

or prove that

$V = W_1 + W_2 \Leftrightarrow (i)\ W_1 \oplus W_2 \quad$ (ii) $\quad W_1 \cap W_2 = \{0\}$

Proof:

Let V = $W_1 \oplus W_2$

Let u $\in W_1 \cap W_2$ $\Rightarrow$ u $\in W_1$ and u $\in W_2$

u = u+0 $\in W_1 + W_2$ = V

u = 0+ u $\in W_1 + W_2$ = V

$\therefore$ u has been expressed uniquely as u = u+0 and u = 0+u and the unique which is only possible if u = 0

$\Rightarrow W_1 \cap W_2$={0}

Conversely,

Let $W_1 \cap W_2$={0}

Let v $\in$ V = $W_1 + W_2$

Let v = $u_1 + v_1$ & v $= u_1' + v_1'$

Where $u_1, u_1' \in W_1$ and $v_1, v_1' \in W_2$

$\Rightarrow u_1 - u_1' \in W_1$ and $v_1 - v_1' \in W_2$

$\Rightarrow u_1 - u_1' \in W_2$ and $v_1 - v_1' \in W_1$

$\Rightarrow u_1 - u_1' \in W_1 \cap W_2$ and $v_1 - v_1' \in W_1 \cap W_2$

$\Rightarrow u_1 - u_1' = 0$ and $v_1 - v_1' = 0$

$\Rightarrow u_1 = u_1'$ and $v_1 = v_1'$

Representation of V is unique in V

$\Rightarrow$ V $= W_1 \oplus W_2$

Example:

Let V be vector space of all real valued function
$\mathbb{V}(f : \mathbb{R} \to \mathbb{R})$
$LetX = \{f : fisodd\}, LetY = \{f : fiseven\}$
$ShowthatX \le VandY \le V$
$V = X \oplus Y$

Define addition and scalar multiplication

Let f,g $\in$ V

$(f+g)(x) = f(x) + g(x)$ (1)

Let $\alpha \in \mathbb{F}$ and $f \in V$

$(\alpha f)(x) = \alpha f(x)$ (2)

$X = \{f : f \text{ is odd}\}$ It is clear that $X \neq \phi$ as

$0(-x) = 0 = -0(x)$

$\Rightarrow \quad 0 \in X$

Let $f,g \in X$

$f(-x) = -f(x)$ and $g(-x) = -g(x)$

Let $\alpha,\beta \in \mathbb{F}$ then

$(\alpha f+\beta g)(-x) = (\alpha f)(-x) + (\beta g)(-x)$ $\therefore$ by(1)

$= \alpha.f(-x) + \beta.g(-x)$ $\therefore$ by(2)

$= -\alpha f(x) - \beta g(x)$

$(\alpha f+\beta g)(-x) = -(\alpha f+\beta g)(x)$

$\alpha f+\beta g \in X \quad \Rightarrow \quad X \leq V$

Now $Y = \{f : f \text{ is even}\}$

It is clear that $Y \neq \phi$ as

$0(-x) = 0 = 0(x)$

$\Rightarrow \quad 0 \in Y$

Even Function

$f(-x) = f(x)$

Let $f,g \in Y$

$f(-x) = f(x)$ and $g(-x) = g(x)$

Let $\alpha,\beta \in \mathbb{F}$ then

$(\alpha f+\beta g)(-x) = (\alpha f)(-x) + (\beta g)(-x)$ $\because$ by(1)

$= \alpha.f(-x) + \beta.g(-x)$ $\because$ by(2)

$= \alpha f(x) + \beta g(x)$

$(\alpha f+\beta g)(-x) = (\alpha f+\beta g)(x)$

$\alpha f + \beta g \in Y$

$\Rightarrow \quad Y \leq V$

Now to show X+Y is subspace

$\therefore$ Sum of two subspaces is again subspace.

It is clear that X+Y $\neq \phi$ as

$0 = 0 + 0$

Let u $\in$ X+Y : u $= w_1 + w_2$, $w_1 \in$ X and $w_2 \in$ Y

And v $\in$ X+Y : v $= w_1' + w_2'$, $w_1' \in$ X and $w_2' \in$ Y

Let $\alpha, \beta \in \mathbb{F}$

Now $\quad \alpha u+\beta v = \alpha(w_1 + w_2) + \beta(w_1' + w_2')$

$= \alpha w_1 + \alpha w_2 + \beta w_1' + \beta w_2'$

$= (\alpha w_1 + \beta w_1') + (\alpha w_2 + \beta w_2') \in X+Y$

$\Rightarrow \alpha u+\beta v \in X+Y$

$\Rightarrow$ X+Y is a subspace.

Now we show V = X$\oplus$Y , Let f $\in$ V such that g(x) = f(-x)

$\Rightarrow \quad f = (\frac{1}{2}f + \frac{1}{2}g) + (\frac{1}{2}f - \frac{1}{2}g)$

$\Rightarrow \quad f(-x) = (\frac{1}{2}f + \frac{1}{2}g)(-x) + (\frac{1}{2}f - \frac{1}{2}g)(-x)$

$= (\frac{1}{2}f(-x) + \frac{1}{2}g(-x)) + (\frac{1}{2}f(-x) - \frac{1}{2}g(-x))$

$= (\frac{1}{2}g(x) + \frac{1}{2}f(x)) + (\frac{1}{2}g(x) - \frac{1}{2}f(x))$

$f(-x) = (\frac{1}{2}f + \frac{1}{2}g)(x) - (\frac{1}{2}f - \frac{1}{2}g)(x)$

$\Rightarrow \frac{1}{2}f + \frac{1}{2}g \in Y$ and $\frac{1}{2}f - \frac{1}{2}g \in X$

$\Rightarrow f \in X+Y$

Finally let $f \in X \cap Y \quad \Rightarrow f \in X$ and $f \in Y$

$f(-x) = -f(x) \in X$

$f(-x) = f(x) \in Y$

$\Rightarrow -f(x) = f(x)$

$f(x) + f(x) = 0 \quad \Rightarrow \quad 2f(x) = 0$

$f(x) = 0(x)$

$\Rightarrow \; f = 0$

$\Rightarrow \; X \cap Y = \{0\}$

Hence the result

Lecture # 4

Linear Transformation or Homomorphism:

Let U and V be two vector spaces over the field $\mathbb{F}$ then a mapping

$$T: V \to U$$

is said to be a linear transformation if

(i) $T(v_1+v_2) = T(v_1)+T(v_2)$
(ii) $T(\alpha v) = \alpha T(v)$
$\forall$ v, $v_1, v_2 \in V$ and $\alpha \in \mathbb{F}$
Or A mapping
$T: V \to U$
If $T(\alpha v_1+\beta v_2) = \alpha T(v_1)+ \beta T(v_2)$
And this linear transformation is also known as Homomorphism.

Question:

Let T be a transformation (mapping)

$$T(\alpha,\beta,\gamma) = (\alpha,\beta)$$

Check this transformation is linear or not.

Solution:

Given $T(\alpha,\beta,\gamma) = (\alpha,\beta)$ (1)

$$\text{Let } \left.\begin{aligned} v_1 &= (\alpha_1, \beta_1, \gamma_1) \\ v_2 &= (\alpha_2, \beta_2, \gamma_2) \end{aligned}\right\} \in \mathbb{F}^3$$

Now for any scalar $\alpha,\beta \in \mathbb{F}$

Then $T(\alpha v_1+\beta v_2) = T(\alpha(\alpha_1, \beta_1, \gamma_1)+\beta(\alpha_2, \beta_2, \gamma_2))$

$= T(\alpha\alpha_1+\beta\alpha_2 , \alpha\beta_1+\beta, \beta_2 , \alpha\gamma_1+\beta\gamma_2)$

$= (\alpha\alpha_1+\alpha\beta_1 , \beta\alpha_2 +\beta\beta_2)$ $\therefore$ by (1)

$= (\alpha\alpha_1,\alpha\beta_1)+ (\beta\alpha_2,\beta\beta_2)$

$= \alpha (\alpha_1,\beta_1)+ \beta(\alpha_2, \beta_2)$

$= \alpha T(\alpha_1, \beta_1, \gamma_1)+ \beta T(\alpha_2, \beta_2, \gamma_2)$ $\Rightarrow \alpha T(v_1)+ \beta T(v_2)$

Hence T is linear space

Theorem:

Let T: V $\rightarrow$ U be a linear transformation then

(i) $T(0) = 0$
(ii) $T(-x) = -T(x)$

Proof: (i)

$T(0) = T(0+0)$

$T(0) = T(0) + T(0)$ $\because$ by def.

By cancellation law

$0 = T(0)$

Proof: (ii)

$T(-x)+T(x) = T(-x+x)$ $\because$ by def.

$= T(0$

$T(-x)+T(x) = 0$

$\Rightarrow T(-x) = -T(x)$

Kernel of T or Kernel of Linear Transformation:

Let T: V $\rightarrow$ U be a linear transformation then Kernel of T is

Ker T = { V: T(v) = 0 where v$\in$V and 0 $\in$ U}

Question:

Let u,v $\in$ Ker T such that

$T(u) = 0$ and $T(v) = 0$ $\because$ by def.

Let α , $\beta \in \mathbb{F}$: then

$\alpha u + \beta v = \alpha(u) + \beta(v)$

$= \alpha(T(u)) + \beta(T(v))$

$= \alpha(0) + \beta(0)$

$= 0 \in$ Ker T

Hence Ker T is a subspace.

Theorem:

Let T: V $\rightarrow$ U be a L.T then Ker T ={0} iff T is one-one.

Proof:

Suppose Ker T ={0}

Let T(v_1) = T(v_2)

$\Rightarrow$ T(v_1) $-$T(v_2) = 0

T($v_1 - v_2$) = 0 $\because$ T is L.T

$\Rightarrow$ $v_1 - v_2 \in$ Ker T = 0 $\because$ by def. of Kernel

$\Rightarrow$ $v_1 - v_2 = 0$

$\Rightarrow$ $v_1 = v_2$

$\Rightarrow$ T is one-one

Conversely,

Let T is one-one

If v $\in$ Ker T be any element then by def. of Kernel

T(v) = 0 = T(0)

T(v) = T(0)

Given T is one-one

$\Rightarrow$ v = 0

$\Rightarrow$ Ker T = {0}

Definition:

Let T: V $\rightarrow$ U be a L.T then Range of T is defined as

Range T = T_R= {T(v) : v $\in$ V}

Or Range T = { u: u$\in$U and u = T(v) , v $\in$ V}

Theorem:

Prove that RangeT is a subspace.

Proof:

Let T(0) = 0 , 0 $\in$ V

$\therefore$ T(0) $\in$ RangeT i.e. RangeT $\neq \phi$

Let $\alpha,\beta \in \mathbb{F}$ and T(x) , T(y) $\in$ T(v) be any element. Then

αT(x) + β T(y) = T(αx + βy) $\in$ T(v)

Hence Range T is subspace.

Quotient Space:

Let V be a vector space and W be the subspace V. Define a set

$\frac{V}{W}$ = {v + W: v ∈ V}

If (i) $(v_1$+W) + $(v_2$+W) = (v_1+v_2) +W

(ii). α $(v_1$+W) = αv_1+W

Theorem:

Let T: V → U be a L.T then

$\frac{V}{KerT}$ ≈ T(v) ∵ ≈ (Isomorphic)

Proof:

Let Ker T = K

Define a mapping such that

ϕ: $\frac{V}{K}$ → T(v)

ϕ(v+K) = T(v) ………………(1)

(i) ϕ is well define.

Let v_1+K and v_2+K ∈ $\frac{V}{K}$

Let v_1+K = v_2+ K

$v_1 - v_2$ = K−K ∵ K−K ∈ K

$v_1 - v_2$ ∈ K = Ker T

⇒ T($v_1 - v_2$) = 0

⇒ T(v_1) − T(v_2) = 0 ∵ T is L.T

⇒ T(v_1)= T(v_2)

⇒ ϕ(v_1+K) = ϕ(v_2+K) ∵ by (1)

⇒ ϕ is well define

(ii) ϕ is one-one

Let ϕ(v_1+K) = ϕ(v_2+K)

⇒ T(v_1)= T(v_2) ∵ by (1)

⇒ T(v_1) − T(v_2) = 0

⇒ T($v_1 - v_2$) = 0 ∵ by def. T is L.T

$\Rightarrow \quad v_1 - v_2 \in$ Ker T = K

$\Rightarrow \quad v_1 - v_2 = K - K \qquad \because \quad K - K \in K$

$\Rightarrow \quad v_1 + K = v_2 + K$

$\Rightarrow \quad \phi$ is one-one

(iii) ϕ is Linear

$$\text{Let } \left.\begin{aligned} x &= v_1 + K \\ y &= v_2 + K \end{aligned}\right\} \in \frac{V}{K}$$

Let $\alpha, \beta \in \mathbb{F}$ then

$\phi(\alpha x + \beta y) = \phi[\alpha(v_1 + K) + \beta(v_2 + K)]$

$= \phi[\alpha v_1 + K + \beta v_2 + K] \qquad \because$ by def. of Quotient

$= \phi(\alpha v_1 + \beta v_2 + K) \qquad \because K + K \in K$

$\phi(\alpha x + \beta y) = T(\alpha v_1 + \beta v_2) \qquad \because$ by (1)

$= \alpha T(v_1) + \beta\, T(v_2)$

$= \alpha\, \phi(v_1 + K) + \beta\phi(v_2 + K) \qquad \because$ by (1)

$\Rightarrow \quad \phi$ is Linear

(iv) ϕ is onto

Let $T(v) \in T(V)$ be any element. Then

$\Rightarrow \quad v \in V$ and $\phi(v+K) = T(v)$

$\Rightarrow \quad v + K \in \frac{V}{K}$

$\Rightarrow \quad T$ is onto

Hence $\frac{V}{KerT} \approx T(v)$

Exercise

Check which of the following are linear transformation

Question # 1 T:$R^2 \rightarrow R^2$ s.t T(x_1, x_2) = $(1 + x_1, x_2)$ ____(1)

Solution:

$$\left.\begin{aligned} v_1 &= (x_1', x_2') \\ v_2 &= (x_1'', x_2'') \end{aligned}\right\} \in R^2$$

Let α,β ∈ $\mathbb{F}$ Then

$$T(\alpha v_1 + \beta v_2) = T[\alpha(x_1', x_2') + \beta(x_1'', x_2'')]$$

$$= T[(\alpha x_1' + \beta x_1''), (\alpha x_2' + \beta x_2'')]$$

$$= [(1 + (\alpha x_1' + \beta x_1''), (\alpha x_2' + \beta x_2'')] \quad \because \text{by } (1)$$

$$\neq \alpha T(v_1) + \beta\, T(v_2)$$

Hence T is not linear transformation.

Question # 2: T:$R^2 \rightarrow R^2$ s.t T(x_1, x_2) = (x_2, x_1)

Solution:

$$\left.\begin{aligned} v_1 &= (x_1', x_2') \\ v_2 &= (x_1'', x_2'') \end{aligned}\right\} \in R^2$$

s.t $T(x_1', x_2') = (x_2', x_1')$

$T(x_1'', x_2'') = (x_2'', x_1'')$

Let α,β ∈ $\mathbb{F}$ Then

$$T(\alpha v_1 + \beta v_2) = T[\alpha(x_1', x_2') + \beta(x_1'', x_2'')]$$

$$= T[\, (\alpha x_1' + \beta x_1''),\ (\alpha x_2' + \beta x_2'')]$$

$$= [(\alpha x_2' + \beta x_2''), (\alpha x_1' + \beta x_1'')]$$

$$= \alpha(x_2', x_1') + \beta(x_2'', x_1'')$$

$$= \alpha T(x_1', x_1') + \beta T(x_1'', x_2'')$$

$$= \alpha T(v_1) + \beta\, T(v_2)$$

Hence T is linear.

Question # 3: T: C → C s.t T(z) = $\bar{z}$

Solution:

Let z = x + iy

$$\left.\begin{array}{l} v_1 = z_1 = x_1 + iy_1 \\ v_2 = z_2 = x_2 + iy_2 \end{array}\right\} \in C$$

Such that $T(z_1) = \overline{z_1} = x_1 - iy_1$

$T(z_2) = \overline{z_2} = x_2 - iy_2$

Such that α , β ∈ $\mathbb{F}$ Then

$$T(\alpha v_1 + \beta v_2) = T[\alpha(x_1 + iy_1) + \beta(x_2 + iy_2)]$$
$$= T[\alpha x_1 + i\alpha y_1 + \beta x_2 + i\beta y_2]$$
$$= T[(\alpha x_1 + \beta x_2) + i(\alpha y_1 + \beta y_2)]$$
$$= [(\alpha x_1 + \beta x_2) - i(\alpha y_1 + \beta y_2)]$$
$$= [(\alpha x_1 + \beta x_2 - i\alpha y_1 - i\beta y_2)]$$
$$= [(\alpha(x_1 - iy_1) + \beta(x_2 - iy_2)]$$
$$= \alpha T(z_1) + \beta T(z_2)$$
$$= \alpha T(v_1) + \beta T(v_2)$$

⇒ T is Linear Space.

Question # 4: T: C → C s.t T(z) = $\bar{z}$

Solution: Let $\left.\begin{array}{l} v_1 = z_1 = x_1 + iy_1 \\ v_2 = z_2 = x_2 + iy_2 \end{array}\right\} \in C$

Such that $T(v_1) = T(x_1 + iy_1) = x_1$

$T(v_2) = T(x_2 + iy_2) = x_2$

Such that α , β ∈ $\mathbb{F}$ Then

$$T(\alpha v_1 + \beta v_2) = T[\alpha(x_1 + iy_1) + \beta(x_2 + iy_2)]$$
$$= T[\alpha x_1 + i\alpha y_1 + \beta x_2 + i\beta y_2]$$
$$= T[(\alpha x_1 + \beta x_2) + i(\alpha y_1 + \beta y_2)]$$

$= \alpha x_1 + \beta x_2$

$= \alpha T((x_1 + iy_1) + \beta T(x_2 + iy_2)$

$= \alpha T(v_1) + \beta\, T(v_2)$

$\Rightarrow$ T is Linear Space.

Question # 5: $T: R^3 \to R^3$ s.t $T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3)$

Solution:

$$\left.\begin{matrix} v_1 = (x_1', x_2', x_3') \\ v_2 = (x_1'', x_2'', x_3'') \end{matrix}\right\} \in R^3$$

s.t $T(x_1', x_2', x_3') = (x_1', x_1' + x_2', x_1' + x_2' + x_3', x_3')$

$T(x_1'', x_2'', x_3'') = (x_1'', x_1'' + x_2'', x''_1 + x_2'' + x_3'', x_3'')$

Let $\alpha, \beta \in \mathbb{F}$ Then

$T(\alpha v_1 + \beta v_2) = T[\alpha(x_1', x_2', x_3') + \beta(x_1'', x_2'', x_3'')]$

$= T[\,(\alpha x_1' + \beta x_1''), (\alpha x_2' + \beta x_2''), (\alpha x_3' + \beta x_3'')]$

$= [(\alpha x_1' + \beta x_1''), (\alpha x_1' + \beta x_1'' + \alpha x_2' + \beta x_2''), (\alpha x_1' + \beta x_1'' + \alpha x_2' + \beta x_2'' + \alpha x_3' + \beta x_3''),$
$(\alpha x_3' + \beta x_3'')]$

$= [\alpha x_1', (\alpha x_1' + \alpha x_2'), (\alpha x_1' + \alpha x_2' + \alpha x_3'), \alpha x_3']$

$+[\beta x_1'', (\beta x_1'' + \beta x_2''), (\beta x_1'' + \beta x_2'' + \beta x_3''), \beta x_3'']$

$= \alpha[x_1', x_1' + x_2', x_1' + x_2' + x_3', x_3'] + \beta[x_1'', x_1'' + x_2'', x''_1 + x_2'' + x_3'', x_3'']$

$= \alpha T(x_1', x_2', x_3') + \beta T(x_1'', x_2'', x_3'')$

$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta\, T(v_2)$

$\Rightarrow$ T is Linear Space.

Q6: $T: R^3 \to R^3$ s.t $T(x_1, x_2,) = (x_1, x_1 + x_2, x_2)$

Solution:

$$\left.\begin{matrix} v_1 = (x_1', x_2') \\ v_2 = (x_1'', x_2'') \end{matrix}\right\} \in R^3$$

s.t $T(x_1', x_2') = (x_1', x_1' + x_2', x_2')$

$T(x_1'', x_2'') = (x_1'', x_1'' + x_2'', x_2'')$

Let $\alpha, \beta \in \mathbb{F}$ Then

$T(\alpha v_1 + \beta v_2) = T[\alpha(x_1', x_2') + \beta(x_1'', x_2'')]$

$= T[(\alpha x_1' + \beta x_1''), (\alpha x_2' + \beta x_2'')]$

$= [\alpha x_1' + \beta x_1'', \alpha x_1' + \beta x_1'' + \alpha x_2' + \beta x_2'', \alpha x_2' + \beta x_2'']$

$= [\alpha x_1', \alpha x_1' + \alpha x_2', \alpha x_2'] + [\beta x_1'', \beta x_1'' + \beta x_2'', \beta x_2'']$

$= \alpha[x_1', x_1' + x_2', x_2'] + \beta[x_1'', x_1'' + x_2'', x_2'']$

$= \alpha T(x_1', x_2') + \beta T(x_1'', x_2'')$

$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$

$\Rightarrow$ T is Linear Space.

Question # 7: $T: R \to R^3$ s.t $T(x) = (x, x^2, x^3)$

Solution:

$$\left.\begin{aligned} v_1 &= (x_1) \\ v_2 &= (x_2) \end{aligned}\right\} \in R$$

s.t $T(x_1) = (x_1, x_1^2, x_1^3)$

$T(x_2) = (x_2, x_2^2, x_2^3)$

Let $\alpha, \beta \in \mathbb{F}$ Then

$T(\alpha v_1 + \beta v_2) = T[\alpha(x_1) + \beta(x_2)]$

$= [(\alpha x_1 + \beta x_1), (\alpha x_1 + \beta x_1)^2, (\alpha x_1 + \beta x_1)^3]$

is not a Linear Transformation

Lecture # 5

Theorem:

Let $W \leq V$ then $\exists$ an onto Linear transformation

$V \rightarrow \frac{V}{W}$ with $W = \text{Ker } T$

Proof:

Define a mapping

$T : V \rightarrow \frac{V}{W}$

s.t $T(v) = v + W$ (1)

★ T is well-define.

Let $v_1 = v_2$

$\Rightarrow v_1 + W = v_2 + W$

$\Rightarrow T(v_1) = T(v_2)$ By (1)

We add W

Because W = Ker T

★ T is Linear

Let $v_1, v_2 \in V$, $\alpha, \beta \in \mathbb{F}$

Now $T(\alpha v_1 + \beta v_2) = (\alpha v_1 + \beta v_2) + W$ By (1)

$= (\alpha v_1 + W) + (\beta v_2 + W)$

$= \alpha(v_1 + W) + \beta(v_2 + W)$

$= \alpha T(v_1) + \beta T(v_2)$

T is Linear

$\because$ By def. of Quotient space

T is onto

Let $v+W \in \frac{V}{W}$ $\exists$ $v \in V$

Such that $T(v) = v + W$

$\Rightarrow$ T is onto

Now we show that W = Ker T

Let $v \in \text{Ker}(T) \Leftrightarrow T(v) = W$

$\Leftrightarrow v + W = W$

$\Leftrightarrow$ v $\in$ W

$\Rightarrow$ Ker T = W Proved

★ Why we not use one-one in statement as we use onto. Because W = ker T If W = {0} then we use one-one.

If W = {0}

To show T is one-one

$T(v_1) = T(v_2)$

$\Rightarrow v_1 + W = v_2 + W$

$\Rightarrow v_1 - v_2 = W = 0$

$\Rightarrow v_1 - v_2 = 0$

$\Rightarrow v_1 = v_2$

$\Rightarrow$ T is one-one

Hence $V \cong \frac{V}{W}$

Example:

Let V = $\{c_1e^{2x} + c_2e^{3x} ; c_1, c_2 \in \mathbb{R}\}$ be the vector space of solution of differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6 = 0$ Prove that $V \cong \mathbb{R}^2$

Solution:

$T : V \to \mathbb{R}^2$ defined as

$T(v) = (c_1, c_2)$ where $v = c_1e^{2x} + c_2e^{3x}$

First, we prove that V is vector space

Let $v_1, v_2 \in V$, $\alpha, \beta \in \mathbb{F}$

$v_1 = c_1e^{2x} + c_2e^{3x}$

$v_2 = c_1'e^{2x} + c_2'e^{3x}$ where $c_1, c_1', c_2, c_2' \in \mathbb{R}$

(i) $\alpha(v_1 + v_2) = \alpha(c_1e^{2x} + c_2e^{3x} + c_1'e^{2x} + c_2'e^{3x})$
$= \alpha c_1e^{2x} + \alpha c_2e^{3x} + \alpha c_1'e^{2x} + \alpha c_2'e^{3x})$
$= \alpha(c_1e^{2x} + c_2e^{3x}) + \alpha(c_1'e^{2x} + c_2'e^{3x})$
$= \alpha(v_1) + \alpha(v_2)$

(ii) Let $\alpha, \beta \in \mathbb{F}$, $v_1 = c_1 e^{2x} + c_2 e^{3x} \in V$

$(\alpha+\beta)v_1 = (\alpha+\beta)(c_1 e^{2x} + c_2 e^{3x})$

$= \alpha c_1 e^{2x} + \alpha c_2 e^{3x} + \beta c_1 e^{2x} + \beta c_2 e^{3x}$

$= \alpha(c_1 e^{2x} + c_2 e^{3x}) + \beta(c_1 e^{2x} + c_2 e^{3x})$

$= \alpha(v_1) + \beta(v_1)$

(iii) $\alpha(\beta v_1) = \alpha[\beta(c_1 e^{2x} + c_2 e^{3x})]$

$= \alpha[\beta c_1 e^{2x} + \beta c_2 e^{3x}]$

$= \alpha\beta(c_1 e^{2x} + c_2 e^{3x})$

$= \alpha\beta(v_1)$

(iv) $1 . v_1 = 1 .(c_1 e^{2x} + c_2 e^{3x})$

$= (c_1 e^{2x} + c_2 e^{3x})$

$= v_1$

Hence V is vector space.

★ Now T is well-define

Let $v_1 = v_2$

$c_1 e^{2x} + c_2 e^{3x} = c_1' e^{2x} + c_2' e^{3x}$

$(c_1 - c_1')\, e^{2x} + (c_2 - c_2')\, e^{3x} \in \text{Ker T}$

$\Rightarrow$ T$[(c_1 - c_1')\, e^{2x} + (c_2 - c_2')\, e^{3x}] = 0$

$\Rightarrow (c_1 - c_1' , c_2 - c_2') = (0,0)$

$\Rightarrow c_1 - c_1' = 0$ and $c_2 - c_2' = 0$

$\Rightarrow c_1 = c_1'$ and $c_2 = c_2'$

$\Rightarrow$ T(v_1) = T(v_2)

★ Now T is one-one

Let T(v_1) = T(v_2)

$\Rightarrow c_1 = c_1'$ and $c_2 = c_2'$

$\Rightarrow c_1 - c_1' = 0$ and $c_2 - c_2' = 0$

$\Rightarrow (c_1 - c_1' , c_2 - c_2') = (0,0)$

$\Rightarrow$ T$[(c_1 - c_1')\, e^{2x} + (c_2 - c_2')\, e^{3x}] = 0$

$(c_1 - c_1')\, e^{2x} + (c_2 - c_2')\, e^{3x} \in \text{Ker } T$

$c_1 e^{2x} + c_2 e^{3x} - c_1' e^{2x} - c_2' e^{3x} = 0$

$c_1 e^{2x} + c_2 e^{3x} = c_1' e^{2x} + c_2' e^{3x}$

$v_1 = v_2$

★ Now T is Linear

Let $\alpha, \beta \in \mathbb{F}$ and $v_1, v_2 \in V$

$T(\alpha v_1 + \beta v_2) = T[\alpha(c_1 e^{2x} + c_2 e^{3x}) + \beta(c_1' e^{2x} + c_2' e^{3x})]$

$= T[\alpha c_1 e^{2x} + \alpha c_2 e^{3x} + \beta c_1' e^{2x} + \beta c_2' e^{3x}]$

$= T[(\alpha c_1 + \beta c_1')\, e^{2x} + (\alpha c_2 + \beta c_2')\, e^{3x}]$

$= (\alpha c_1 + \beta c_1', \alpha c_2 + \beta c_2')$ by (1)

$= (\alpha c_1, \alpha c_2) + (\beta c_1', \beta c_2')$

$= \alpha\, (c_1, c_2) + \beta\, (c_1' + c_2')$

$= \alpha T(v_1) + \beta T(v_2)$

T is Linear

Now T is onto

Let $(c_1, c_2) \in \mathbb{R}^2$ s.t $c_1 e^{2x} + c_2 e^{3x} \in V$

s.t $T(c_1 e^{2x} + c_2 e^{3x}) = (c_1, c_2)$

$\Rightarrow$ T is onto

Hence $V \cong \mathbb{R}^2$

Question:

Let $V = \{c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x};\ c_1, c_2, c_3 \in \mathbb{R}\}$ be the vector space of solution of differential equation $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} - 11\frac{dy}{dx} + 6y = 0$ Prove that $V \cong \mathbb{R}^3$

Solution:

$T : V \to \mathbb{R}^2$ defined as

$T(v) = (c_1, c_2, c_3)$ where $v = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

First, we prove that V is vector space

Let $v_1, v_2 \in$ V , $\alpha, \beta \in \mathbb{F}$

$v_1 = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

$v_2 = c_1' e^x + c_2' e^{2x} + c_3' e^{3x}$ where $c_1, c_1', c_2, c_2', c_3, c_3' \in \mathbb{R}$

(i) $\alpha(v_1 + v_2) = \alpha(c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + c_1' e^x + c_2' e^{2x} + c_3' e^{3x})$

$= \alpha c_1 e^x + \alpha c_2 e^{2x} + \alpha c_3 e^{3x} + \alpha c_1' e^x + \alpha c_2' e^{2x} + \alpha c_3' e^{3x}$

$= \alpha(c_1 e^x + c_2 e^{2x} + c_3 e^{3x}) + \alpha(c_1' e^x + c_2' e^{2x} + c_3' e^{3x})$

$= \alpha(v_1) + \alpha(v_2)$

(ii) Let $\alpha, \beta \in \mathbb{F}$, $v_1 = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \in$ V

$(\alpha+\beta)v_1 = (\alpha+\beta)(c_1 e^x + c_2 e^{2x} + c_3 e^{3x})$

$= \alpha c_1 e^x + \alpha c_2 e^{2x} + \alpha c_3 e^{3x} + \beta c_1 e^x + \beta c_2 e^{2x} + \beta c_3 e^{3x}$

$= \alpha(c_1 e^x + c_2 e^{2x} + c_3 e^{3x}) + \beta(c_1 e^x + c_2 e^{2x} + c_3 e^{3x})$

$= \alpha(v_1) + \beta(v_1)$

(iii) $\alpha(\beta v_1) = \alpha[\beta(c_1 e^x + c_2 e^{2x} + c_3 e^{3x})]$

$= \alpha[\beta c_1 e^x + \beta c_2 e^{2x} + \beta c_3 e^{3x}]$

$= \alpha\beta(c_1 e^x + c_2 e^{2x} + c_3 e^{3x})$

$= \alpha\beta(v_1)$

(iv) $1 . v_1 = 1 .(c_1 e^x + c_2 e^{2x} + c_3 e^{3x})$

$= (c_1 e^x + c_2 e^{2x} + c_3 e^{3x})$

$= v_1$

Hence V is vector space.

Now T is well-define

Let $v_1 = v_2$

$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = c_1' e^x + c_2' e^{2x} + c_3' e^{3x}$

$(c_1 - c_1')\, e^x + (c_2 - c_2')\, e^{2x} + (c_3 - c_3')\, e^{3x} \in$ Ker T

$\Rightarrow$ $T[(c_1 - c_1')\, e^x + (c_2 - c_2')\, e^{2x} + (c_3 - c_3')\, e^{3x}] = 0$

$\Rightarrow$ $(c_1 - c_1', c_2 - c_2'), (c_3 - c_3') = (0,0)$

$\Rightarrow$ $c_1 - c_1' = 0$, $c_2 - c_2' = 0$, $c_3 - c_3' = 0$

$\Rightarrow$ $c_1 = c_1'$, $c_2 = c_2'$, $c_3 = c_3'$

$\Rightarrow$ $T(v_1) = T(v_2)$

★ Now T is one-one

Let $T(v_1) = T(v_2)$

$\Rightarrow \quad c_1 = c_1', \; c_2 = c_2', \; c_3 = c_3'$

$\Rightarrow \quad c_1 - c_1' = 0, \; c_2 - c_2' = 0, \; c_3 - c_3' = 0$

$\Rightarrow \quad (c_1 - c_1', c_2 - c_2'), (c_3 - c_3') = (0,0)$

$\Rightarrow T[(c_1 - c_1')e^x + (c_2 - c_2')e^{2x} + (c_3 - c_3')e^{3x}] = 0$

$(c_1 - c_1')e^x + (c_2 - c_2')e^{2x} + (c_3 - c_3')e^{3x} \in \text{Ker } T$

$c_1 e^x - c_1' e^x + c_2 e^{2x} - c_2' e^{2x} + c_3 e^{3x} - c_3' e^{3x} = 0$

$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = c_1' e^x + c_2' e^{2x} + c_3' e^{3x}$

$v_1 = v_2$

★ Now T is Linear

Let $\alpha, \beta \in \mathbb{F}$ and $v_1, v_2 \in V$

$T(\alpha v_1 + \beta v_2) = T[\alpha(c_1 e^x + c_2 e^{2x} + c_3 e^{3x}) + \beta(c_1' e^x + c_2' e^{2x} + c_3' e^{3x})]$

$= T[\alpha c_1 e^x + \alpha c_2 e^{2x} + \alpha c_3 e^{3x} + \beta c_1' e^x + \beta c_2' e^{2x} + \beta c_3' e^{3x}]$

$= T[(\alpha c_1 + \beta c_1')e^x + (\alpha c_2 + \beta c_2')e^{2x} + (\alpha c_3 + \beta c_3')e^{3x}]$

$= (\alpha c_1 + \beta c_1'), (\alpha c_2 + \beta c_2'), (\alpha c_3 + \beta c_3')$

$= (\alpha c_1, \alpha c_2, \alpha c_3) + (\beta c_1', \beta c_2', \beta c_3')$

$= \alpha(c_1, c_2, c_3) + \beta(c_1', c_2', c_3')$

$= \alpha T(v_1) + \beta T(v_2)$

T is Linear

Now T is onto

Let $(c_1, c_2, c_3) \in \mathbb{R}^2$ s.t $c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \in V$

s.t $\quad T(c_1 e^x + c_2 e^{2x} + c_3 e^{3x}) = (c_1, c_2, c_3)$

$\Rightarrow$ T is onto

Hence $V \cong \mathbb{R}^3$

Assignment:

If X and Y be two subspaces of vector space V over the field $\mathbb{F}$. Then prove that $\frac{X+Y}{X} \cong \frac{Y}{X\cap Y}$

Solution:

Define a mapping

$T : Y \to \frac{X+Y}{X}$

s.t $T(y) = y + X \quad , y \in Y$

(i) T is well-define

Let $y_1 = y_2$

$y_1 + X = y_2 + X$

$T(y_1) = T(y_2)$

(ii). T is Linear

Let $y_1 , y_2 \in Y$ and $\alpha,\beta \in \mathbb{F}$ s.t

$T(\alpha y_1 + \beta y_2) = (\alpha y_1 + \beta y_2) + X$ ∵ By (1)

$= (\alpha y_1 + X) + (\beta y_2 + X)$ ∵ by def. of quotient space

$= \alpha(y_1 + X) + \beta(y_2 + X)$

$= \alpha T(y_1) + \beta T(y_2)$

$\Rightarrow$ T is linear

(iii) T is onto

Let $y + X \in \frac{X+Y}{X}$ s.t $y \in Y$

s.t $T(y) = y + X$

$\Rightarrow$ T is onto

By Fundamental Theorem

$\frac{X+Y}{X} = \frac{Y}{Ker\ T}$

We claim Ker T = $X \cap Y$

Let $a \in$ Ker T

$\Rightarrow \quad T(a) = X$

$a + X = X$

$a \in X$, also $a \in \text{Ker } T \subseteq Y$

$a \in X , a \in Y$

$a \in X \cap Y$

$Ker T \subseteq X \cap Y$ .....(1)

Conversely,

$a \in X \cap Y$

$\Rightarrow$ a $\in$ X , a $\in$ Y

a + X = X

$\Rightarrow$ T(a) = X

$\Rightarrow$ a $\in$ Ker T

$\Rightarrow X \cap Y \subseteq Ker T$ .....(2)

By (1) and (2)

Hence Ker T = X $\cap$ Y

$\frac{X+Y}{X} \cong \frac{Y}{X \cap Y}$ Proved

Lecture # 6

Linear Combination:

Let V be a vector space over the field $\mathbb{F}$.

Let $v_1, v_2,\ldots.. v_n \in$ V

And

$\alpha_1, \alpha_2,\ldots.. \alpha_n \in \mathbb{F}$

Then the element

$\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3\ldots.. +\alpha_n v_n$

is called a linear combination of $v_1, v_2,\ldots.. v_n$ in V

It can be written as

x = $\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3\ldots.. +\alpha_n v_n$

x = $\sum_{i=1}^{n} \alpha_i v_i$

Example:

Write a vector v = (1,-2,5) in the Linear combination (L.C) of e_1= (1,1,1) , e_2= (1,2,3) and e_3= (3,0,-2)

Solution:

v = $\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3$

(1,-2,5) = $\alpha_1(1,1,1) +\alpha_2(1,2,3) +\alpha_3(3,0,-2)$

(1,-2,5) = $(\alpha_1+\alpha_2+3\alpha_3 , \alpha_1+2\alpha_2+0\alpha_3 , \alpha_1+3\alpha_2-2\alpha_3)$

$\alpha_1+\alpha_2+3\alpha_3 = 1$, $\alpha_1+2\alpha_2+0\alpha_3 = -2$, $\alpha_1+3\alpha_2-2\alpha_3 = 5$

In matrix form

$$\begin{bmatrix}1 & 1 & 3\\ 1 & 2 & 0\\ 1 & 3 & -2\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}1\\ -2\\ 5\end{bmatrix}$$

A X B

$$A_B = \left[\begin{array}{ccc|c}1 & 1 & 3 & 1\\ 1 & 2 & 0 & -2\\ 1 & 3 & -2 & 5\end{array}\right]$$

$$A_B = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -5 & 4 \end{array}\right] \quad \sim R_2 - R_1 \quad , \quad \sim R_3 - R_1$$

$$A_B = \left[\begin{array}{ccc|c} 1 & 0 & 6 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & 1 & 10 \end{array}\right] \quad \sim R_1 - R_1 \quad , \quad \sim R_3 - 2R_2$$

$$A_B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -56 \\ 0 & 1 & 0 & 27 \\ 0 & 0 & 1 & 10 \end{array}\right] \quad \sim R_1 - 6R_3 \quad , \quad \sim R_2 + 3R_3$$

$\Rightarrow \quad \alpha_1 = -56 \ , \alpha_2 = 27 \ , \alpha_3 = 10$

Exercise:

Write v = (1, −2,K) in the L.C of $e_1 = (0,1,-2), e_2 = (-2,-1,-5)$ also find the value of 'K'.

Solution:

$$v = \alpha_1 e_1 + \alpha_2 e_2$$

$$= \alpha_1(0,1,-2) + \alpha_2(-2,-1,-5)$$

$$(1,-2,K) = (0\alpha_1 + (-2)\alpha_2 \ , \alpha_1 - \alpha_2 \ , -2\alpha_1 - 5\alpha_2)$$

$$0\alpha_1 + (-2)\alpha_2 = 1 \quad , \alpha_1 - \alpha_2 = -2 \ , -2\alpha_1 - 5\alpha_2 = K$$

$\Rightarrow \ \alpha_2 = -\frac{1}{2}$

And $\qquad \alpha_1 - \alpha_2 = -2$

$\qquad \alpha_1 - (-\frac{1}{2}) = -2$

$\Rightarrow \qquad \alpha_1 = -2 - \frac{1}{2}$

$\Rightarrow \qquad \alpha_1 = -\frac{5}{2}$

Now $\qquad -2\alpha_1 - 5\alpha_2 = K$

$\qquad -2\left(-\frac{5}{2}\right) - 5\left(-\frac{1}{2}\right) = K$

$\Rightarrow \qquad K = 5 + \frac{5}{2} = \frac{10+5}{2}$

$\Rightarrow \qquad K = \frac{15}{2}$

Linearly Dependent:

Let V be a vector space over the field $\mathbb{F}$. Let $v_1, v_2, \ldots. v_n \in$ V and $\alpha_1, \alpha_2, \ldots. \alpha_n \in \mathbb{F}$ then $v_1, v_2, \ldots. v_n$ are said to be linearly dependent if

$$\sum_{i=1}^{n} \alpha_i v_i = 0 \qquad \text{for some } \alpha_i \neq 0$$

Otherwise they are called Linearly independent.

Linear Span:

Let $\phi \neq$ S is a subset of vector space V over the field $\mathbb{F}$ then S is called Linear span if every element of S is a linear combination of finite number of elements of V and it is denoted by

L(S) = <S> = $\{x : x = \sum_{i=1}^{n} \alpha_i v_i \, , v_i \in V\}$

And this set is also known as generating set.

Exercise:

Prove that L(S) is a subspace of V.

Solution:

Let x,y $\in$ L(S) and α , $\beta \in \mathbb{F}$

Then $x = \sum_{i=1}^{n} \alpha_i v_i$, $y = \sum_{i=1}^{n} \beta_i v_i$

Now $\alpha x + \beta y = \alpha \sum_{i=1}^{n} \alpha_i v_i + \beta \sum_{i=1}^{n} \beta_i v_i$

$= \sum_{i=1}^{n} (\alpha\alpha_i) v_i + \sum_{i=1}^{n} (\beta\beta_i) v_i$ $\qquad \because$ T(x) + T(y) = T(x+y)

$= \sum_{i=1}^{n} (\alpha\alpha_i + \beta\beta_i) v_i$

$= \sum_{i=1}^{n} \gamma_i v_i$ $\qquad \because \gamma_i = \alpha\alpha_i + \beta\beta_i$, $1 \leq i \leq n$

$\Rightarrow \quad \alpha x + \beta y \in$ L(S)

Hence L(S) is subspace of V.

Theorem:

L(S) is a smallest subspace of V.

Proof:

First, we prove L(S) $\neq \phi$

Let $s_1 \in S \subseteq V$

$s_1 = 1.\, s_1$, $1 \in \mathbb{F}$

$s_1 \in L(S)$

$\Rightarrow$ $S \subseteq L(S)$

$\Rightarrow$ $L(S) \neq \phi$

Now we prove $L(S) \leq V$

Let x , y $\in$ L(S) , $\alpha , \beta \in \mathbb{F}$

Then $x = \sum_{i=1}^{n} \alpha_i v_i$, $y = \sum_{i=1}^{n} \beta_i v_i$

$\alpha x + \beta y = \alpha \sum_{i=1}^{n} \alpha_i v_i + \beta \sum_{i=1}^{n} \beta_i v_i$

$= \sum_{i=1}^{n} (\alpha\alpha_i) v_i + \sum_{i=1}^{n} (\beta\beta_i) v_i$ $\because T(x) + T(y) = T(x+y)$

$= \sum_{i=1}^{n} (\alpha\alpha_i + \beta\beta_i) v_i$

$= \sum_{i=1}^{n} \gamma_i v_i$ $\because \gamma_i = \alpha\alpha_i + \beta\beta_i$, $1 \leq i \leq n$

$\Rightarrow$ $\alpha x + \beta y \in L(S)$

$\Rightarrow$ $L(S) \leq V(\mathbb{F})$

Now we prove L(S) is smallest subspace of V

Let x $\in$ L(S)

Then $x = \sum_{i=1}^{n} \alpha_i v_i$

Let $v_i \in S$, $\alpha \in \mathbb{F}$

$v_i \in S \subseteq W$ $\forall$ i and W is subspace.

$\Rightarrow$ $\sum_{i=1}^{n} \alpha_i v_i \in W$

$\Rightarrow$ $x \in W$

$\Rightarrow$ $L(S) \subseteq W$

$\Rightarrow$ L(S) is smallest subspace of V.

Remark:

Since L(S) is a subspace and L(T) is subspace then

$L(S) \leq L(T)$

Lemma:

Let $\phi \neq S \subseteq V(\mathbb{F})$ then the following axioms are true.

(i) If $S \subset T$

$\Rightarrow L(S) \subset L(T)$

(ii) $L(S \cup T) = L(S) + L(T)$

(iii) $L(L(S)) = L(S)$

Proof: (i)

Let $S = \{v_1, v_2, \ldots.. v_n\}$ and $T = \{v_1, v_2, \ldots.. v_n, v_{n+1}, \ldots\ldots v_m\}$; $m > n$

Now let $x \in L(S)$

$\Rightarrow \quad x = \sum_{i=1}^{n} \alpha_i v_i \quad \forall \; \alpha_i \in \mathbb{F} \quad , 1 \leq i \leq n$

$\Rightarrow x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_n v_n$

$= \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_n v_n + 0v_{n+1} + 0v_{n+2} + \ldots\ldots. + 0v_m$

$= \sum_{i=1}^{n} \alpha_i v_i = L(T) \qquad \forall \; \alpha_i = 0$ if $i > n$

$\Rightarrow \quad x \in L(T)$

$\Rightarrow \quad L(S) \subset L(T)$

Proof : (ii)

If $\quad S \subset T \quad \Rightarrow L(S) \leq L(T)$ $\qquad$ ∵ by Remark

∵ $\; S \subseteq S \cup T$ where S and T contain distinct element

$\Rightarrow L(S) \subseteq L(S \cup T)$ $\qquad$ ∵ by proof (i)

Also $\quad T \subseteq S \cup T$

$\Rightarrow L(T) \subseteq L(S \cup T)$

$\Rightarrow L(S) + L(T) \subseteq L(S \cup T)$ $\qquad$ …. (1)

∵ $S \subseteq L(S) \subseteq L(S) + L(T)$

And $\qquad T \subseteq L(T) \subseteq L(S) + L(T)$

$\Rightarrow \; S \cup T \subseteq L(S) + L(T)$

Also $\qquad S \cup T \subseteq L(S \cup T)$

$L(S \cup T) \subseteq L(S) + L(T)$ ….(2)

From (1) and (2)

$L(S \cup T) = L(S) + L(T)$

Proof: (iii)

$S \subseteq L(S)$

$\Rightarrow$ $L(S) \subseteq L(L(S))$ ….(1)

Let $x \in L(L(S))$

s.t $x = t_i \sum_{i=1}^{n} \alpha_i v_i$ $\forall\ t_i = 0$ if $i > n$

$= \sum_{i=1}^{n} \alpha_i t_i v_i$

$= \sum_{i=1}^{n} \beta_i v_i$, $\beta_i = \alpha_i t_i$, $1 \leq i \leq n$

$\Rightarrow x \in L(S)$

$\Rightarrow L(L(S)) \subseteq L(S)$ …(2)

From (1) and (2)

$L(L(S)) = L(S)$

Lecture # 7

Theorem:

Let V be a vector space over the field $\mathbb{F}$. Let $v_1, v_2 \in$ V are said to be linearly independent iff $v_1 + v_2$ and $v_1 - v_2$ are linearly independent.

Proof:

Let v_1, v_2 are linearly independent.

Now let $\alpha, \beta \in \mathbb{F}$ Then

$$\alpha(v_1 + v_2) + \beta(v_1 - v_2) = 0$$

$$\Rightarrow \alpha v_1 + \alpha v_2 + \beta v_1 - \beta v_2 = 0$$

$$\Rightarrow (\alpha + \beta)v_1 + (\alpha - \beta)v_2 = 0$$

Since v_1 and v_2 are linearly independent then

$\alpha + \beta = 0$ ….(1)

$\alpha - \beta = 0$ …..(2)

Put $\alpha = \beta$ in (1) $\Rightarrow \beta + \beta = 0$

$\Rightarrow 2\beta = 0 \Rightarrow \beta = 0$

$\Rightarrow \alpha = \beta$

$\Rightarrow v_1 + v_2$ and $v_1 - v_2$ are linearly independent

Conversely,

Let $v_1 + v_2$ and $v_1 - v_2$ are L.I. Now let $\beta v_1 + \gamma v_2 = 0$ where $\beta, \gamma \in \mathbb{F}$

Let $\beta = \beta_1 + \beta_2$, $\gamma = \beta_1 - \beta_2$

$$\Rightarrow (\beta_1 + \beta_2)v_1 + (\beta_1 - \beta_2)v_2 = 0$$

$$\Rightarrow \beta_1 v_1 + \beta_2 v_1 + \beta_1 v_1 - \beta_2 v_2 = 0$$

$$\Rightarrow (v_1 + v_2)\beta_1 + (v_1 - v_2)\beta_2 = 0$$

Since $v_1 + v_2$ and $v_1 - v_2$ are linearly independent then $\beta_1 = \beta_2 = 0$

$\Rightarrow \beta = 0$ and $\gamma = 0$

$\Rightarrow v_1$ and v_2 are L.I

Theorem:

The vectors $v_1, v_2, v_3 \in$ V are said to be linearly independent iff $v_1 + v_2$ and $v_2 + v_3$ and $v_3 + v_1$ are linearly independent.

Proof:

Let v_1, v_2, v_3 are L.I

Let $\alpha, \beta, \gamma \in \mathbb{F}$ Now

$$\alpha(v_1 + v_2) + \beta(v_2 + v_3) + \gamma(v_3 + v_1) = 0$$

$$\Rightarrow \alpha v_1 + \alpha v_2 + \beta v_2 + \beta v_3 + \gamma v_3 + \gamma v_1 = 0$$

$$\Rightarrow \alpha v_1 + \gamma v_1 + \alpha v_2 + \beta v_2 + \beta v_3 + \gamma v_3 = 0$$

$$\Rightarrow (\alpha + \gamma)v_1 + (\alpha + \beta)v_2 + (\beta + \gamma)v_3 = 0$$

Since v_1, v_2, v_3 are L.I then

$\Rightarrow \alpha + \gamma = 0 \quad ..(1) \quad , \alpha + \beta = 0 \quad ....(2) \quad , \beta + \gamma = 0 \quad ...(3)$

$\Rightarrow \alpha = -\gamma$ put in (2)

$\Rightarrow -\gamma + \beta = 0 \Rightarrow \beta = \gamma$ put in (3)

$\Rightarrow \gamma + \gamma = 0 \Rightarrow 2\gamma = 0 \Rightarrow \gamma = 0$

$\Rightarrow \beta = 0 , \quad \gamma = 0$

$\Rightarrow \alpha = \beta = \gamma = 0$

$\Rightarrow v_1 + v_2$ and $v_2 + v_3$ and $v_3 + v_1$ are L.I

Conversely, let $v_1 + v_2$ and $v_2 + v_3$ and $v_3 + v_1$ are L.I

Now $\alpha = \beta_1 + \gamma_1 \quad , \quad \beta = \alpha_1 + \gamma_1 \quad , \quad \gamma = \alpha_1 + \beta_1$

$\Rightarrow (\beta_1 + \gamma_1)v_1 + (\alpha_1 + \gamma_1)v_2 + (\alpha_1 + \beta_1)v_3 = 0$

$\Rightarrow \beta_1 v_1 + \gamma_1 v_1 + \alpha_1 v_2 + \gamma_1 v_2 + \alpha_1 v_3 + \beta_1 v_3 = 0$

$\Rightarrow \beta_1(v_1 + v_3) + (v_1 + v_2)\gamma_1 + (v_2 + v_3)\alpha_1 = 0$

Since $v_1 + v_2$ and $v_2 + v_3$ and $v_3 + v_1$ are linearly independent

$\Rightarrow \alpha_1 = 0 , \beta_1 = 0 , \gamma_1 = 0 \quad \Rightarrow \alpha = 0 , \beta = 0 , \gamma = 0$

$\Rightarrow v_1, v_2$ *and* v_3 are L.I.

Example:

Let A = $\begin{pmatrix} 1 & 2 & -3 \\ 6 & -5 & 4 \end{pmatrix}$, B = $\begin{pmatrix} 6 & -5 & 4 \\ 1 & 2 & -3 \end{pmatrix}$

Prove that A and B are L.I

Solution:

Let $\alpha,\beta \in \mathbb{F}$ then

$\alpha A + \beta B = 0$

$\alpha\begin{pmatrix} 1 & 2 & -3 \\ 6 & -5 & 4 \end{pmatrix} + \beta\begin{pmatrix} 6 & -5 & 4 \\ 1 & 2 & -3 \end{pmatrix} = 0$

$\begin{pmatrix} \alpha & 2\alpha & -3\alpha \\ 6\alpha & -5\alpha & 4\alpha \end{pmatrix} + \begin{pmatrix} 6\beta & -5\beta & 4\beta \\ \beta & 2\beta & -3\beta \end{pmatrix} = 0$

$\begin{pmatrix} \alpha+6\beta & 2\alpha-5\beta & -3\alpha+4\beta \\ 6\alpha+\beta & -5\alpha+2\beta & 4\alpha-3\beta \end{pmatrix} = 0$

$\Rightarrow \quad \alpha + 6\beta = 0 \qquad \ldots(1)$

$2\alpha - 5\beta = 0 \qquad \ldots(2)$

And all others elements are zero

(1) $\Rightarrow \alpha = -6\beta$ put in (2)

$2(-6\beta)-5\beta = 0 \quad \Rightarrow \quad -12\beta-5\beta = 0$

$\Rightarrow \quad -17\beta = 0 \quad \Rightarrow \quad \beta = 0$

$\Rightarrow \quad \alpha = 0$

Hence A and B are L.I

Example:

Let V be a vector space of polynomial over the field $\mathbb{F}$ ($R^3\{x\}$) and let u,v $\in$ V let

u = 2-5t+6t^2-t^3

v = 3+2t-4t^2+5t^3 check either u,v are L.I or not

Solution:

Let $\alpha,\beta \in \mathbb{F}$ then u and v are L.I if $\alpha u+\beta v = 0$

$\alpha(2-5t+6t^2-t^3)+\beta(3+2t-4t^2+5t^3) = 0$

$2\alpha-5\alpha t+6\alpha t^2-\alpha t^3+3\beta+2\beta t-4\beta t^2-5\beta t^3 = 0$

$(2\alpha+3\beta)+(-5\alpha+2\beta)t+(6\alpha-4\beta)t^2+(-\alpha+5\beta)t^3 = 0$

t is L.I then

$2\alpha+3\beta = 0$ ..(1) , $-5\alpha+2\beta = 0$ …(2) , $6\alpha-4B = 0$ …(3) , $-\alpha+5\beta = 0$ ….(4)

(4) $\Rightarrow \alpha = 5\beta$ put in (1)

$2(5\beta) + 3\beta = 0 \quad \Rightarrow \quad 10\beta+3\beta = 0$

$\Rightarrow \quad 13\beta = 0 \quad \Rightarrow \quad \beta = 0$

$\Rightarrow \quad \alpha = 0$

$\Rightarrow$ u and v are L.I

Lemma:

The non-zero vectors are L.D iff one of them say v_i is the L.C of its preceding one's. (L.C پچھلے والے تک سے v_i)

Proof:

Let v_i be the L.C of its preceding vectors i.e.

$v_i = \alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3\ldots.. +\alpha_{i-1} v_{i-1}$

$\Rightarrow \quad \alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3\ldots.. +\alpha_{i-1} v_{i-1}+(-1)\, v_i = 0$

As $\alpha_i = -1 \neq 0$

$\Rightarrow$ vectors are L.D

Conversely,

Let the vectors are L.D then $\exists$

$\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots\ldots \alpha_m \in \mathbb{F}$ of which at least one $\alpha_i \neq 0$ s.t

$\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3\ldots.. +\alpha_i v_i + \alpha_{i+1} v_{i+1}+\ldots\ldots+\alpha_m v_m = 0 \qquad \because i < m$

Take $\alpha_{i+1} = \alpha_{i+2} = \ldots\ldots\ldots.= \alpha_m = 0$

$\Rightarrow \quad \alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3\ldots.. +\alpha_i v_i = 0$

$\Rightarrow \quad -\alpha_i v_i = \alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3\ldots.. +\alpha_{i-1} v_{i-1}$

$\Rightarrow \quad v_i = \left(-\frac{\alpha_1}{\alpha_i}\right) v_1+\left(-\frac{\alpha_2}{\alpha_i}\right) v_2+\ldots\ldots\ldots\ldots+\left(-\frac{\alpha_{i-1}}{\alpha_i}\right) v_{i-1}$

$\Rightarrow \quad v_i$ is the L.C of its preceding one's.

Theorem:

The vectors are L.I if each element in their Linear span has unique representation.

Proof:

Let S = $\{v_1, v_2, \ldots\ldots\ldots\ldots v_n\} \subseteq V(\mathbb{F})$

Let $L(S) = \left\{\sum_{i=1}^{n} \alpha_i v_i \ : \ \alpha_i \in \mathbb{F}\right\}$

Let $v \in S$

$\Rightarrow \quad v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_n v_n \ , \quad \forall \ \alpha_i \in \mathbb{F} \ , \quad 1 \leq i \leq n$

Let $v = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 \ldots\ldots + \beta_n v_n \ , \quad \forall \ \beta_i \in \mathbb{F} \ , \quad 1 \leq i \leq n$

be another representation of v

$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 \ldots\ldots + \beta_n v_n$

$\Rightarrow \quad (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \ldots\ldots\ldots\ldots + (\alpha_n - \beta_n) v_n = 0$

Since $v_1, v_2, \ldots\ldots\ldots\ldots v_n$ are Linearly independent then

$\alpha_1 - \beta_1 = 0 \ , \alpha_2 - \beta_2 = 0 \ , \ldots\ldots\ldots\ldots\ldots , \alpha_n - \beta_n = 0$

$\Rightarrow \ \alpha_1 = \beta_1 \ , \alpha_2 = \beta_2 , \ldots\ldots\ldots\ldots\ldots , \ \alpha_n = \beta_n$

$\Rightarrow$ v has unique representation

Theorem:

Let V be a vector space over the field $\mathbb{F}$. Let $S \subseteq V$

$S = \{v_1, v_2, \ldots\ldots\ldots\ldots v_n\}$ then

(i) S is L.I if any of its subset is L.I
(ii) S is L.D if any of its superset is L.D

Proof (i). :

Let S is L.I

Let $T = \{v_1, v_2, \ldots\ldots\ldots\ldots v_n\} \subseteq V$ where i < n

Let $\alpha_i \in \mathbb{F}$

Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_i v_i = 0$

$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_i v_i + \alpha_{i+1} v_{i+1} + \ldots\ldots\ldots + \alpha_n v_n = 0$

Since $v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_n$ are L.I

Take $\alpha_{i+1} + \alpha_{i+2} + \alpha_{i+3} + \ldots\ldots\ldots\ldots \alpha_n = 0$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_i v_i + 0v_{i+1} + 0v_{i+2} + 0v_{i+3} + \ldots\ldots\ldots\ldots 0v_n = 0$

Since $v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_n$ are L.I

$\Rightarrow \quad \alpha_1 = \alpha_2 = \ldots\ldots\ldots = \alpha_i = 0$

$\Rightarrow$ T is L.I

Proof (ii) :

Let S is L.D

$S = \{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_n\}$

$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_n v_n = 0$ for some $\alpha_i \neq 0$

$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_n v_n + 0.v = 0$ for some $\alpha_i \neq 0$

$\Rightarrow \quad T = \{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_n, v\} \supseteq S$ is L.D

Lecture # 8

Basis:

Let V be a vector space over the field $\mathbb{F}$. Let S be non-empty subset of V then S is called basis for V if

(i) S is linearly independent
(ii) V = L(S)

Example:

Let S = {(1,0) ,(0,1)} $\subseteq \mathbb{R}^2(\mathbb{R})$ then prove that S is basis of $\mathbb{R}^2$

Solution:

$$Let \;\; u_1 = (1,0) \quad , \quad u_2 = (0,1)$$
$$and \quad \alpha = 1 \quad \beta = -4 \;\in \mathbb{R} \quad then$$
$$\alpha u_1 + \beta u_2 = 1(1,0) - 4(0,1)$$
$$= (1,0) + (0,-4)$$
$$= (1,-4) \in \mathbb{R}^2$$
$$Hence\ S\ is\ Basis\ of\ \mathbb{R}^2$$

Example:

Let S = {(1,0,0) (0,1,0),(0,0,1)} $\subseteq \mathbb{R}^3(\mathbb{R})$ then prove that S is basis of $\mathbb{R}^3$

$$Let \;\; u_1 = (1,0,0) \quad , \quad u_2 = (0,1,0) \quad , \; u_3 = (0,0,1)$$
$$and \quad \alpha = 1 \quad \beta = 2 \, , \quad \gamma = 3 \quad then$$
$$\alpha u_1 + \beta u_2 + \gamma u_3 = 1(1,0,0) + 2(0,1,0) + 3(0,0,1)$$
$$= (1,0,0) + (0,2,0) + (0,0,3)$$
$$= (1,2,3) \in \mathbb{R}^3$$
$$Hence\ S\ is\ Basis\ of\ \mathbb{R}^3$$

Dimension:

Number of elements in the basis of vector space $V(\mathbb{F})$ is called Dimension.

Theorem:

Every Finite dimensional vector space (F.D.V.S) contain Basis

Proof: Let V be a F.D.V.S over the field $\mathbb{F}$ .Let

T = $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_n\}$ be a finite subset of V which is spanning set (generating set) for V.

Case-I

If T is L.I then there is nothing to prove i.e. Every element of T spans the vector space V (L(T) = V) $\Rightarrow$ T is basis for V

Case-II

If T is L.D then any vector (say) V_r is Linear combination of its preceding ones. Then eliminating that vector from T the remaining vectors are $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_{r-1}\}$ still spans V

Now If $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_{r-1}\}$ is L.I then there is nothing to prove. (Then $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_{r-1}\}$ will be basis of V)

If $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_{r-1}\}$ is L.D then any other vector (say) V_{r-1} is L.C of its preceding one's. By eliminating this vector, the remaining vectors $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_{r-2}\}$ still spans V

Continuing this process until we get as set of vectors $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_n\}$

Where n ≤ r which is L.I. This being a spanning set it will be basis for V

$\Rightarrow$ Every F.D.V.S contain Basis.

Theorem:

Let V be a F.D.V.S of dimension 'n' then any set of n+1 or more vectors is Linearly dependent.

Proof:

Since V be F.D.V.S so it contains basis. Let

B = $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_n\}$ be the basis for V.

Let S = $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_r\}$ where r > n

We need to prove that S is L.D

i.e. $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_r v_r = 0$ ….(1)

$\Rightarrow \quad \alpha_i \neq 0$ for some α_i where $1 \leq I \leq r$

Where $\alpha_i \in \mathbb{F}$ Since B is Basis for V

$\Rightarrow \quad L(B) = V \qquad \because$ by def.

i.e. for all $v_i \in V = L(B)$; $1 \leq i \leq r$ can be expressed uniquely as a L.C of basis vectors

$$\left.\begin{aligned}
\Rightarrow \quad v_1 &= a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n \\
\Rightarrow \quad v_2 &= a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n \\
&\;\; . \quad . \quad . \quad . \quad . \\
&\;\; . \quad . \quad . \quad . \quad . \\
\Rightarrow \quad v_r &= a_{r1}u_1 + a_{r2}u_2 + \ldots + a_{rn}u_n
\end{aligned}\right\} \quad (2)$$

Using (2) in (1) we have

$\alpha_1(a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n) + \alpha_2(a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n) + \ldots\ldots$

$\ldots + \alpha_r(a_{r1}u_1 + a_{r2}u_2 + \ldots + a_{rn}u_n) = 0$

$(\alpha_1 a_{11} + \alpha_2 a_{21} + \ldots + \alpha_r a_{r1})\, u_1 + (\alpha_1 a_{12} + \alpha_2 a_{22} + \ldots + \alpha_r a_{r2})\, u_2 + \ldots\ldots$

$\ldots + (\alpha_1 a_{1n} + \alpha_2 a_{2n} + \ldots + \alpha_r a_{rn})\, u_n = 0$

Since $u_1, u_2, \ldots\ldots\ldots u_n$ are L.I

$$\begin{aligned}
\Rightarrow \quad & \alpha_1 a_{11} + \alpha_2 a_{21} + \ldots + \alpha_r a_{r1} = 0 \\
& \alpha_1 a_{12} + \alpha_2 a_{22} + \ldots + \alpha_r a_{r2} = 0 \\
& . \quad . \quad . \quad . \\
& \alpha_1 a_{1n} + \alpha_2 a_{2n} + \ldots + \alpha_r a_{rn} = 0
\end{aligned}$$

Which is homogeneous system of 'n' equation in r unknowns. Which gives us a non-trivial solution which indicates that one of the scalar is non-zero

$\Rightarrow$ S is L.D

Maximal L.I Set: Let $\phi \neq S \subseteq V$. Let $T \supset S$ if T is L.D then S is called Maximal L.I set.

Minimal Set of generators: Let G be set of generators of a vector space $V(\mathbb{F})$ Then $H \subset G$ is not a generating set for V then G is called Minimal generating set.

Lecture # 9

Theorem:

If V is F.D.V.S and $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_r\}$is L.I subset of V. Then it can be extended to form a basis of V.

Proof:

If $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_r\}$ spans V then it itself forms a basis of V and there is nothing to prove.

Let S = $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_r, v_{r+1}, \ldots\ldots\ldots\ldots\ldots v_n\}$ be the maximal L.I subset of V containing $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_r\}$ we show S is a basis of V for which it is enough to prove that S spans V.

Let v $\in$ V be any element then

T = $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots\ldots, v_n, v\}$ is L.D

Then $\exists$ $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots\ldots \alpha_n$, $\alpha \in \mathbb{F}$ s,t

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_n v_n + \alpha v = 0 \quad \text{where } \alpha \neq 0$$

$$-\alpha v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_n v_n$$

$$v = \left(\frac{-\alpha_1}{\alpha}\right) v_1 + \left(\frac{-\alpha_2}{\alpha}\right) v_2 + \ldots\ldots\ldots + \left(\frac{-\alpha_n}{\alpha}\right) v_n$$

v is a linear combination of $v_1, v_2, \ldots\ldots\ldots\ldots\ldots\ldots, v_n$ which is required result.

Theorem:

Let V be a vector space over the field $\mathbb{F}$. Let B $\subseteq$ V the following statement are equivalent.

(i) B is basis for V
(ii) B is a minimal set of generators for V
(iii) B is maximal L.I set of vectors.

Proof: (i) $\Rightarrow$ (ii)

Suppose B is Basis for V $\Rightarrow$ B is L.I

Let H $\subset$ B let $v_i \in$ B but $v_i \notin$ H

We claim that H is not a set of generators on the contrary, suppose H is generating set of V for $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots\ldots \alpha_i \in \mathbb{F}$ and $v_1, v_2, \ldots\ldots\ldots\ldots\ldots v_i \in$ H s.t $v_i = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots\ldots + \alpha_j v_j$ where $v_i \in$ B and B $\subseteq$ V

But $v_i = 1.\, v_i$ $1 \in \mathbb{F}$

⇒ A contradiction i.e. v_i does not have the unique representation

⇒ H is not a set of generators

⇒ B is a minimal set of generators for V

(ii) ⇒ (iii)

Suppose that B is a minimal set of generators for V

We need to prove that B is maximal L.I set of vectors

⇒ If B is not L.I

Then at least one of the vector is a L.C of its preceding vectors.

If we delete this vector then the remaining set of vectors (subset of B) still span V and producing a contradiction against the minimality of B

Now we prove that B is maximal set ($H \supset B$) H is superset of B

Let $h \in H$ but $h \notin B$

⇒ $h = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_n v_n$

Because B is minimal set of generators

⇒ $h \in H$ ⇒ H is L.D

⇒ B is maximal

(iii). ⇒ (i)

Suppose that B is maximal L.I set of vectors we need to prove that B is basis for V. Let $v \in V$ and $v \neq \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_k v_k$

Where $\alpha_i \in \mathbb{F}$ and $1 \leq i \leq k$ & $v_i \in B$; $1 \leq i \leq k$

⇒ $B \cup \{v\}$ is L.I

As none of the vectors of $B \cup \{v\}$ is a L.C of its preceding one's which implies contradiction with the fact B is maximal L.I set of vectors

∵ $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_k v_k$

⇒ $v \in L(B)$

⇒ $V = L(B)$

Lecture # 10

Theorem:

Let V be a F.D.V.S over the field $\mathbb{F}$. Let W ≤ V then

(i) W is F.D and dim(W) ≤ dim(V)
Moreover, if dim(W) = dim(V) then W = V

(ii) dim(V/W) = dim(V) – dim (W)

Proof: (i)

Let V be of dimension 'n' or let dim(V) = n

Let W ≤ V($\mathbb{F}$)

Let $\{w_1, w_2, \dots\dots\dots\dots w_k\}$ be the largest set of L.I vectors of W. Now we show that $\{w_1, w_2, \dots\dots\dots\dots w_k\}$ is a basis for W.

Let w ∈ W such that w ≠ w_i ∀ i ; 1 ≤ i ≤ k

Then the set $\{w_1, w_2, \dots\dots\dots\dots w_k\}$ is L.D

i.e. $w = \sum_{i=1}^{k} a_i w_i$

$w = a_1 w_1 + a_2 w_2 + a_3 w_3 \dots\dots + a_k w_k$

⇒ w ∈ L ($\{w_1, w_2, \dots\dots\dots\dots w_k\}$)

Now when w = w_i for 1 ≤ i ≤ k

Then w = $0.w_1 + 0.w_2 + \dots\dots + 1.w_i + 0.w_{i+1} + \dots\dots + 0.w_k$

⇒ w ∈ L ($\{w_1, w_2, \dots\dots\dots\dots w_k\}$)

So in each case w ∈ L ($\{w_1, w_2, \dots\dots\dots\dots w_k\}$)

⇒ $\{w_1, w_2, \dots\dots\dots\dots w_k\}$ spans W

⇒ w = L ($\{w_1, w_2, \dots\dots\dots\dots w_k\}$)

⇒ $\{w_1, w_2, \dots\dots\dots\dots w_k\}$ is a basis for W

⇒ W is F.D

Since dim(V) = n (maximal)

And dim(W) = k < dim(V) = n

⇒ dim(W) ≤ dim(V)

Now if dim(W) = dim(V)

$\Rightarrow$ Every basis of W is a basis of V

$\Rightarrow$ W = V

Proof (ii)

Let $\{w_1, w_2, \ldots\ldots\ldots\ldots\ldots w_k\}$ be the basis for W.

Let $\{w_1, w_2, \ldots\ldots\ldots\ldots\ldots w_k, v_1, v_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, v_m\}$ be the basis for V then

$\{v_1 + W, v_2 + W, \ldots\ldots\ldots, v_m + W\}$ be the basis for V/W

First we show that the set

$\{v_1 + W, v_2 + W, \ldots\ldots\ldots, v_m + W\}$ is L.I

Let $\alpha_1 (v_1 + W) + \alpha_2 (v_2 + W) + \ldots\ldots + \alpha_m (v_m + W) = 0 + W$ ....(1)

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_m v_m + W = W$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_m v_m \in W$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_m v_m = w$ for some $w \in W$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_m v_m = a_1 w_1 + a_2 w_2 + a_3 w_3 \ldots.. + a_k w_k$

because $\{w_1, w_2, \ldots\ldots\ldots\ldots\ldots w_k\}$ are the basis for W.

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \ldots.. + \alpha_m v_m + (-a_1 w_1) + (-a_2 w_2) + \ldots.. + (-a_k w_k) = 0$

Since

$\{w_1, w_2, \ldots\ldots\ldots\ldots\ldots w_k, v_1, v_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, v_m\}$ are basis for V

$\Rightarrow \alpha_1 = \alpha_2 = \ldots\ldots\ldots = \alpha_m = (-a_1) = (-a_2) \ldots\ldots = (-a_k) = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \ldots\ldots\ldots = \alpha_m = (a_1) = (a_2) \ldots\ldots = (a_k)$

$\Rightarrow \alpha_1 = \alpha_2 = \ldots\ldots\ldots = \alpha_m = 0$

$\Rightarrow \{v_1 + W, v_2 + W, \ldots\ldots\ldots, v_m + W\}$ is L.I

Let $v + W \in V/W$ by def of quotient

$\because v \in V$ therefore

$v = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 \ldots.. + \alpha_k w_k + a_1 v_1 + a_2 v_2 + a_3 v_3 \ldots.. + a_m v_m$

$\Rightarrow v + W = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 \ldots.. + \alpha_k w_k + a_1 v_1 + a_2 v_2 + a_3 v_3 \ldots.. + a_m v_m + W$

$\Rightarrow$ v+W = $a_1 v_1 + a_2 v_2 + a_3 v_3 \ldots.. + a_m v_m$+W

Because $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 \ldots.. + \alpha_k w_k \in$ W

$\Rightarrow$ W+W = W

$\Rightarrow$ $\{v_1 + W, v_2 + W, \ldots\ldots....., v_m$+W$\}$ spans V/W

$\Rightarrow$ v + W $\in$ L($\{v_1 + W, v_2 + W, \ldots\ldots....., v_m$+W$\}$)

$\Rightarrow$ v+W = L($\{v_1 + W, v_2 + W, \ldots\ldots....., v_m$+W$\}$)

$\Rightarrow$ $\{v_1 + W, v_2 + W, \ldots\ldots....., v_m$+W$\}$ is basis for V/W

$\Rightarrow$ dim(V/W) = m

= m+k-k

Dim(V/W) = dim(V) – dim(W)

Theorem:

Let T be an isomorphism of V_1 and V_2. Then basis of V_1 maps onto the basis of V_2.

Proof:

Let T: $V_1 \rightarrow V_2$ be an isomorphism where V_1 and V_2 are vector space over $\mathbb{F}$

Let $\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots\ldots\}$ be the basis for V_1 then we need to show that $\{T(v_1), T(v_2), \ldots\ldots\ldots\ldots\ldots\ldots\}$ are the basis for V_2

(i) Let $\alpha_1 T(v_1) + \alpha_1 T(v_2) + \ldots\ldots\ldots = 0$ …..(1)

$\Rightarrow$ Since T is linear

$\Rightarrow$ T($\alpha_1 v_1$)+ T($\alpha_2 v_2$)+………. = 0 $\because$ T is linear

$\Rightarrow$ T($\alpha_1 v_1 + \alpha_2 v_2$+……….) = 0 $\because$ T is linear

$\Rightarrow$ $\alpha_1 v_1 + \alpha_2 v_2$+………. $\in$ KerT = {0}

$\Rightarrow$ $\alpha_1 v_1 + \alpha_2 v_2$+………. = 0

Since $v_1, v_2, \ldots\ldots\ldots\ldots\ldots\ldots$ are the basis for V_1

$\Rightarrow$ $\alpha_1 = \alpha_2 = \ldots\ldots\ldots = 0$

From (1) $\{T(v_1), T(v_2), \ldots\ldots\ldots\ldots\ldots\ldots\}$ are L.I

(ii) Let w $\in V_2$ then $\exists$ an element v $\in V_1$ such that T(v) = w

$\Rightarrow$ T($\alpha_1 v_1 + \alpha_2 v_2$+………) = w

$\Rightarrow$ T($\alpha_1 v_1$+$\alpha_2 v_2$+……….) = w

$\Rightarrow$ T($\alpha_1 v_1$)+ T($\alpha_2 v_2$)+………. = w ∵ T is linear

$\Rightarrow$ $\alpha_1 T(v_1)$+ $\alpha_1 T(v_2)$+……… = w ∵ T is linear

$\Rightarrow$ w $\in$ L({T(v_1)+$T(v_2)$+………. })

$\Rightarrow$ V_2 = L({T(v_1)+$T(v_2)$+………. })

$\Rightarrow$ {T(v_1)+$T(v_2)$+………. } are the basis for V_2

Exercise:

If A and B are F.D.V.S then A+B is also F.D Morever

Dim(A+B) = dim(A)+dim(B)-dim(A $\cap$ B)

Proof:

First we prove that

$$\frac{A+B}{A} \cong \frac{B}{A \cap B}$$

Define a mapping

T : B $\rightarrow \frac{A+B}{A}$

s.t T(b) = b+A ; b $\in$B ……..(1)

(i) T is well define

Let $b_1 = b_2$

$\Rightarrow$ b_1 +A = b_2+A

$\Rightarrow$ $T(b_1)$ = T(b_2)

(ii). T is linear

Let b_1 , b_2 $\in$ B and α,β $\in \mathbb{F}$ s.t

T(αb_1+βb_2) = αb_1 +βb_2+A ∵ by (1)

= (αb_1+A) + (βb_2+A)

= α(b_1+A) + β(b_2+A)

= αT(b_1) + βT(b_2)

(iii) T is onto

Let b+A $\in \frac{A+B}{A}$ s.t b $\in$ B

T(b) = b+A $\Rightarrow$ T is onto

By Fundamental Theorem

$\frac{A+B}{A} = \frac{B}{KerT}$

We claim KerT = A $\cap$ B

Let $\alpha \in$ KerT $\Rightarrow$ T(α) = A

α+A = A $\quad\because\quad$ by (1)

$\Rightarrow$ $\alpha \in$ A $\quad$ Also $\alpha \in$ KerT $\subseteq$ B

$\Rightarrow$ $\alpha \in$ A and $\alpha \in$ B $\Rightarrow$ $\alpha \in$ A $\cap$ B

$\Rightarrow$ Ker T $\subseteq$ A $\cap$ B ……….(2)

Conversely

Let $\alpha \in$ A $\cap$ B

$\Rightarrow$ $\alpha \in$ A and $\alpha \in$ B

$\Rightarrow$ α+A = A $\Rightarrow$ T(α) = A

$\Rightarrow$ $\alpha \in$ KerT $\Rightarrow$ A $\cap$ B $\subseteq$ Ker T (3)

From (2) and (3) KerT = A $\cap$ B

Hence $\frac{A+B}{A} \cong \frac{B}{A\cap B}$

Now $\dim(\frac{A+B}{A}) = \dim(\frac{B}{A\cap B})$ $\quad\because$ dim(V/W= dimV-dimW

$\Rightarrow$ dim(A+B) – dim A = dimB – dim(A $\cap$ B)

$\Rightarrow$ dim(A+B) = dim A +dimB – dim(A $\cap$ B) proved

Lecture # 11

★ Theorem:

Two F.D.V.S are isomorphic to each other iff they are of same dimensions.

Proof:

Let V and W be the two-finite dimensional vector space over the field $\mathbb{F}$.

Let dimV = n = dimW (same dimensions) we need to prove that V is isomorphic to W.

Let $\{v_1, v_2, \ldots\ldots\ldots, v_n\}$ and $\{w_1, w_2, \ldots\ldots\ldots\ldots w_n\}$ be the basis for V and W respectively. Define a mapping

$$\phi : V \to W$$

s.t $\phi(v) = w$ where $v \in V$, $w \in W$

$\Rightarrow$ we can write as

$$a_1 w_1 + a_2 w_2 + a_3 w_3 \ldots\ldots + a_n w_n = \phi(a_1 v_1 + a_2 v_2 + a_3 v_3 \ldots\ldots + a_n v_n) \quad \ldots.(1)$$

$\forall \ a_i \in \mathbb{F}$, $1 \le i \le n$

Now we show that ϕ is Homomorphism (Linear)

Let $\alpha, \beta \in \mathbb{F}$ and $v, v' \in V$

Then

$$\phi(\alpha v+\beta v') = \phi[\alpha(a_1 v_1 + a_2 v_2 + a_3 v_3 .. + a_n v_n) + \beta(b_1 v_1 + b_2 v_2 + .. + b_n v_n)]$$

Where $a_i, b_i \in \mathbb{F}$, $1 \le i \le n$

$$\phi(\alpha v+\beta v') = \phi[\alpha a_1 v_1 + \alpha a_2 v_2 + \ldots. + \alpha a_n v_n + \beta b_1 v_1 + \beta b_2 v_2 + .. + \beta b_n v_n]$$

$$\Rightarrow \phi(\alpha v+\beta v') = \phi\,[(\alpha a_1+\beta b_1)v_1 + (\alpha a_2+\beta b_2)v_2 + \ldots\ldots\ldots + (\alpha a_n+\beta b_n)v_n]$$

$$= (\alpha a_1+\beta b_1)w_1 + (\alpha a_2+\beta b_2)w_2 + \ldots\ldots\ldots + (\alpha a_n+\beta b_n)w_n \qquad \text{by (1)}$$

$$= \alpha(a_1 w_1 + a_2 w_2 + a_3 w_3 \ldots\ldots + a_n w_n) + \beta(b_1 w_1 + b_2 w_2 + b_3 w_3 \ldots\ldots + b_n w_n)$$

$$\phi(\alpha v+\beta v') = \alpha\phi(v) + \beta\phi(v')$$

$\Rightarrow \quad \phi$ is linear

Now by def. we have

$\forall \ w \in W \ \exists \ v \in V$ s.t

$\phi(v) = w$

$\Rightarrow$ ϕ is onto

Let $\phi(v) = \phi(v')$

$\phi(a_1v_1 + a_2v_2 + a_3v_3 \ldots + a_nv_n) = \phi(b_1v_1 + b_2v_2 + \ldots + b_nv_n)$

$\Rightarrow a_1w_1 + a_2w_2 + a_3w_3 \ldots + a_nw_n = b_1w_1 + b_2w_2 + b_3w_3 \ldots + b_nw_n$

$\Rightarrow (a_1 - b_1)w_1 + (a_2 - b_2)w_2 + \ldots + (a_n - b_n)w_n = 0$

Since $\{w_1, w_2, \ldots\ldots w_n\}$ is basis for W

So are linearly independent

$\Rightarrow$ $a_1 - b_1 = 0$, $a_2 - b_2 = 0$, …………, $a_n - b_n = 0$

$\Rightarrow$ $a_1 = b_1$, $a_2 = b_2$, …………., $a_n = b_n$

$\Rightarrow$ $v = v'$

$\Rightarrow$ ϕ is 1-1

$\Rightarrow$ ϕ is an isomorphism b/w V an W

$\Rightarrow$ $V \cong W$ (isomorphic $\cong$)

Conversely,

Let $V \cong W$

Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be the basis for V

We prove that $\{\phi(v_1), \phi(v_2), \ldots \phi(v_n)\}$ are the basis of W

Let $B = \{\phi(v_1), \phi(v_2), \ldots \phi(v_n)\}$

First we prove that B is L.I

Let $\alpha_i \in \mathbb{F}$, $1 \le i \le n$ s.t

$\sum_{i=1}^{n} \alpha_i \phi(v_i) = 0$

$\Rightarrow$ $\sum_{i=1}^{n} \phi(\alpha_i v_i) = 0$ $\because$ ϕ is linear

$\Rightarrow$ $\phi \sum_{i=1}^{n} (\alpha_i v_i) = 0$ $\because$ ϕ is linear

$\because$ v_i where $1 \le i \le n$ are the basis for V are L.I

$\alpha_i = 0$; $1 \le i \le n$

$\Rightarrow$ B is linearly independent

Secondly, we show that L(B) = W

Let w $\in$ W and v $\in$ V

s.t w = ϕ(v)

$\Rightarrow$ w = $\phi(a_1v_1+a_2v_2+a_3v_3 \ldots. +a_nv_n)$

$\Rightarrow$ w = $\phi(a_1v_1) + \phi(a_2v_2)+\ldots\ldots+\phi(a_nv_n)$ $\because$ ϕ is linear

$\Rightarrow$ w = $a_1\phi(v_1) + a_2\phi(v_2)+\ldots\ldots+ a_n\phi(v_n)$ $\because$ ϕ is linear

$\Rightarrow$ W = L(B)

$\Rightarrow$ B is basis for W

$\Rightarrow$ dimW = n = dimV

$\Rightarrow$ dimV = dimW

Lecture # 12

Internal direct sum:

Let V($\mathbb{F}$) be a vector space. Let $u_1, u_2, \dots\dots\dots, u_n$ be the subspace of V. Then V is called the internal direct sum of $u_1, u_2, \dots\dots\dots, u_n$ if $\forall$ v $\in$ V written in one and only one way as

$$v = u_1 + u_2 + \dots\dots\dots + u_n \quad , \quad u_i \in U_i \; ; \quad 1 \le i \le n$$

External direct sum:

Let $v_1, v_2, \dots\dots\dots, v_n$ be the vector space over the same field $\mathbb{F}$. Let V be the set of all ordered n-tuple i.e. $(v_1, v_2, \dots\dots\dots, v_n)$; $v_i \in$ V then we can say that two elements are equal $(v_1, v_2, \dots\dots\dots, v_n)$ and $(v_1', v_2', \dots\dots\dots, v_n')$ where

v_i , $v_i' \in$ V ; $1 \le i \le n$

We can define addition and scalar multiplication in V

$x + y = (v_1, v_2, \dots\dots\dots, v_n) + (v_1', v_2', \dots\dots\dots, v_n')$

$= (v_1 + v_1' , v_2 + v_2', \dots\dots\dots v_n + v_n') \quad \dots\dots\dots(1)$

$\alpha.x = \alpha\, (v_1, v_2, \dots\dots\dots, v_n)$

$= (\alpha v_1, \alpha v_2, \dots\dots\dots, \alpha v_n) \quad \dots\dots\dots(2)$

Then V is called external direct sum of $(v_1, v_2, \dots\dots\dots, v_n)$

$$v = v_1 \oplus v_2 \oplus \dots\dots\dots \oplus v_n$$

Direct Sum:

A vector space V is said to be direct sum of its subspace U and W if

(i) V = U+W
(ii) U $\cap$ W = {0}

Theorem:

If V is the internal direct sum of $u_1, u_2, \dots\dots\dots, u_n$ the V is isomorphic to the external direct sum of $u_1, u_2, \dots\dots\dots, u_n$

Proof:

Let v $\in$ V

$\Rightarrow \quad v = u_1 + u_2 + \dots\dots\dots + u_n \quad \dots..(1) \qquad u_i \in$ U ; $1 \le i \le n$

Define a mapping

$T : V \to u_1\oplus u_2\oplus\ldots\ldots\ldots\oplus u_n$ s.t $T(v) = (u_1, u_2, \ldots\ldots\ldots, u_n)$

i.e. $T(u_1 + u_2 + \ldots\ldots\ldots + u_n) = (u_1, u_2, \ldots\ldots\ldots, u_n)$ …(2)

(1) Now mapping is well-defined because each element of V is written one and only one way (unique representation)

(2) Mapping is linear

Let $\alpha, \beta \in \mathbb{F}$ 1 $v, w \in V$

$T(\alpha v + \beta w) = T(\alpha(u_1 + u_2 + \ldots\ldots\ldots + u_n) + \beta (u'_1, u'_2, \ldots\ldots\ldots, u'_n)$

u_i , $u'_i \in U_i$; $1 \le i \le n$

$= T(\alpha u_1 + \alpha u_2 + \ldots\ldots + \alpha u_n + \beta u'_1 + \beta u'_2 + \ldots\ldots\ldots + \beta u'_n)$

$= T((\alpha u_1 + \beta u'_1 + \alpha u_2 + \beta u'_2 + \ldots\ldots\ldots + \alpha u_n + \beta u'_n)$

$= (\alpha u_1 + \beta u'_1, \alpha u_2 + \beta u'_2, \ldots\ldots\ldots, \alpha u_n + \beta u'_n)$

$= (\alpha u_1, \alpha u_2, \ldots\ldots, \alpha u_n) + (\beta u'_1, \beta u'_2, \ldots\ldots\ldots, \beta u'_n)$

$= \alpha (u_1, u_2, \ldots\ldots\ldots, u_n) + \beta(u'_1, u'_2, \ldots\ldots\ldots, u'_n)$

$= \alpha T(v) + \beta T(w)$

$\Rightarrow$ T is linear

(3). $\forall$ $u_1, u_2, \ldots\ldots\ldots, u_n \in u_1\oplus u_2\oplus\ldots\ldots\ldots\oplus u_n$
$\exists$ $v = v_1, v_2, \ldots\ldots\ldots, v_n \in V$ s.t
$T(v) = u_1, u_2, \ldots\ldots\ldots, u_n$

Which shows that each element of $u_1\oplus u_2\oplus\ldots\ldots\ldots\oplus u_n$ is the image of some element of V $\Rightarrow$ T is surjective (onto)

(4) Let $T(v) = T(w)$

$T(u_1 + u_2 + \ldots\ldots\ldots + u_n) = T(u'_1 + u'_2 + \cdots \ldots\ldots + u'_n)$

$(u_1, u_2, \ldots\ldots\ldots, u_n) = (u'_1, u'_2, \ldots\ldots\ldots, u'_n)$

$u_1 = u'_1$, $u_2 = u'_2$,……… $u_n = u'_n$

$\Rightarrow$ $u_i = u'_i$ $\forall$ i , $1 \le i \le n$

$\Rightarrow$ v = w

$\Rightarrow$ T is injective (one-one)

$\Rightarrow$ T is isomorphism Hence $V \cong u_1\oplus u_2\oplus\ldots\ldots\ldots\oplus u_n$

Lecture # 13

Non-Singular Linear Transformation:

A linear transformation is said to be non-singular if its inverse exists or A linear transformation is non-singular (invertible) if it is one-one or A linear transformation is non-singular if it is an isomorphism.

The set of all non-singular linear transformation is denoted by L(V,V)

Theorem:

Prove that the set L(V,W) is a semi-group under the composition.

Proof:

First, we prove that composition of two linear transformation is also L.T

$T_1 o T_2\ (\alpha v_1+\beta v_2) = T_1(T_2\ (\alpha v_1+\beta v_2))$

$= T_1(T_2\ (\alpha v_1)+ T_2(\beta v_2))$ $\because$ T_2 is linear

$= T_1(\alpha T_2(v_1)+ \beta T_2(v_2))$ $\because$ T_2 is linear

$= T_1(\alpha T_2(v_1))+ T_1\ (\beta T_2(v_2))$ $\because$ T_1 is linear

$= \alpha T_1(T_2(v_1))+ \beta T_1(T_2(v_2))$ $\because$ T_1 is linear

$= \alpha . T_1 o T_2(v_1) + \beta . T_1 o T_2(v_2)$

$\Rightarrow$ $T_1 o T_2$ is Linear

$\Rightarrow$ $T_1 o T_2 \in$ L(V,W)

$\Rightarrow$ L(V,W) is closed under composition

(ii). Associativity is trivial

$\Rightarrow$ L(V,W) is a semi-group under composition

Exercise:

The set L(V,W) of all linear transformation from V to W is abelian group then prove it is a vector space.

Solution:

First we prove L(V,W) is abelian group then vector space

(i) Closure law
$T_1 o T_2\ (\alpha v_1+\beta v_2) = T_1(T_2\ (\alpha v_1+\beta v_2))$

$= T_1(T_2\ (\alpha v_1)+ T_2(\beta v_2))$ $\because$ T_2 is linear

$= T_1(\alpha T_2(v_1)+ \beta T_2(v_2))$ $\because$ T_2 is linear

$= T_1(\alpha T_2(v_1))+ T_1\ (\beta T_2(v_2))$ $\because$ T_1 is linear

$= \alpha T_1(T_2(v_1))+ \beta T_1(T_2(v_2))$ $\because$ T_1 is linear

$= \alpha.T_1 o T_2(v_1) + \beta.T_1 o T_2(v_2)$

$\Rightarrow$ $T_1 o T_2$ is Linear

$\Rightarrow$ $T_1 o T_2 \in$ L(V,W)

$\Rightarrow$ L(V,W) is closed under composition

(ii) Associative law
Associativity is trivial

(iii) Identity law
I:V→W is linear s.t
I(v) = v where v ∈ V , v ∈ W
Becomes I:V→V is identity element of L(V,W)
$\Rightarrow$ identity exist in L(V,W)

$\Rightarrow$ L(V,W) is monoid

(iv) Inverse law
The regular element of this monoid are the non-singular linear transformation i.e. every element has its inverse.
$\Rightarrow$ inverse exist in L(V,W)
$\Rightarrow$ L(V,W) become group
Now we define addition and scalar multiplication
(T_1+T_2)(v) = T_1(v) +T_2(v) …….. (i)
(αT)(v) = α.T(v) ……..(ii)

(v) Commutative law

(T_1+T_2)(v) = T_1(v) +T_2(v)

= T_2(v) +T_1(v)

= (T_2+T_1)(v)

$\Rightarrow$ Commutative law holds in L(V,W)

$\Rightarrow$ L(V,W) become abelian group

Now we show L(V,W) is vector space

(i) Let $\alpha \in \mathbb{F}$, $T_1, T_2 \in$ L(V,W)

$\alpha(T_1+T_2)(v) = (\alpha T_1+\alpha T_2)(v)$ ∵ by (ii)

$\alpha[(T_1+T_2)(v)] = \alpha.T_1(v)+\alpha\, T_2(v)$ ∵ by (i)

(ii) $\alpha,\beta \in \mathbb{F}$ and $T \in$ L(V,W)

$(\alpha+\beta)T(v) = (\alpha T+\beta T)(v)$ ∵ by (ii)

$= \alpha T(v)+\beta T(v)$ ∵ by (i)

(iii) $\alpha,\beta \in \mathbb{F}$ and $T \in$ L(V,W)

$\alpha(\beta T)(v) = (\alpha\beta T)(v)$ ∵ by (ii)

$= \alpha\beta.T(v)$ ∵ by (ii)

(iv) $1 \in \mathbb{F}$ and $T \in$ L(V,W)

1. $T(v) = (1.T)(v)$

$= T(v)$

All axioms are satisfied. Hence L(V,W) is vector space.

★ A set which is ring as well as vector space that set is called **Algebra**.