FLUID MECHANIS I

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Lecture # 1

Introduction:

It is the branch of applied mathematics which deals with fluid at rest or in motion.

There are three categories of fluid mechanics

(i) Fluid Statistics → It deals fluid at rest
(ii) Fluid Kinematics → It deals fluid in motion without considering the force which cause or accompany the motion e.g. speed, velocity, acceleration & displacement etc.
(iii) Fluid Dynamics → It deals with the study of fluid which are in motion.

If one asks why we study the fluid mechanics? We casually look around most things seem to be solids but when one thinks of the oceans, the atmosphere and on out into space it becomes rather obvious that a large portion of the earth surface and of the entire universe is in a fluid state. Therefore, it becomes essential for sciences and engineers to know something about fluid mechanics.

Applications of Fluid Mechanics:

There are many applications of fluid mechanics make it one of the most important and fundamental in almost all engineering and applied scientific studies such as applied mathematics, plasma physics, geo-physics, bio physics and physical chemistry etc. The experimental aspects of fluid mechanics are the studied through various discipline of engineering. The flow of fluids in pipes and channel makes fluid mechanics of importance to civil engineer. They utilize the results of fluid mechanics to understand the transport of river, irrigation channels, the pollution of air and water & to design pipe line systems, flood control systems and dams etc.

The study of fluid machinery such as pumps, fans, blowers, air compressors, heat exchangers, jet and rocket engines, gas turbines, power plants, pollution control equipment etc.

We define a fluid as something which has the property of flowing freely. They are classifying liquid and gases.

Fluid Mechanics:

"The branch of science which is concerned with the study of motion of fluids or those bodies in contact with fluids is called fluid mechanics or hydrostatics".

Or

"The study of forces and flows in fluid is called mechanics".

Fluid:

A substance that has no fixed shape and yields easily to external pressure; a gas or (especially) a liquid.

Historical development of Fluid Mechanics:

Some basic properties of fluids are

(i) Density:
Mass per unit volume is called density
$\rho = \frac{\Delta m}{\Delta V}$ (Δ rate of change)

And ρ = F(x,y,z,t)

Where (x,y,z) are the coordinates of a point and t is the temperature.

(ii) Specific Weight:
It is defined as the weight per unit volume and is denoted by
$$\gamma = \rho g$$

(iii) Specific Volume:
It is defined as volume per unit mass.
$\frac{V}{m} = \frac{1}{\rho}$

(iv) Pressure:
Force per unit area is called pressure.
P = $\lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$
Where F is normal force due to fluid in elementary area.

(v) Viscosity:
It is the property of fluid by which it offers the resistance to sheer (the tangent force per unit area) acting on it i.e. the property of fluid which control the flow of fluid.
Viscosity of liquids decreases with temperature and viscosity of gases increases with temperature.

(vi) Bulk modulus and compressibility:

It is denoted as

$$dp \propto \frac{d\rho}{\rho} \qquad \text{i.e. variation of its density}$$

$$dp = K\frac{d\rho}{\rho} \qquad \text{where K is called Bulk Modulus}$$

Motion of fluid particles:

A fluid consists of innumerable (countless) whose relative position never fix whenever fluid is in motion the particle moves along certain line depending upon the characteristics of fluid and shape of the passage through which the fluid particle moves. It is necessary to observe the motion of fluid particle at various time and point.

Fluid mechanics have two method of fluid motion

(i) Lagrange's method
(ii) Eulerian Method

Lecture # 2

Different types of flow:

(i) Uniform and Non-uniform flow:

The flow in which the velocity of the fluid particle at all section of the particle of pipe and channel are equal i.e. constant.

On the other hand, the fluid particles are said to be non-uniform it the velocity of the particles is not equal i.e. not constant.

(ii) Steady and Unsteady flow:

The flow in which the properties and condition associated with the motion of fluid particle are independent of time where $*\frac{\partial \rho}{\partial t} = 0$

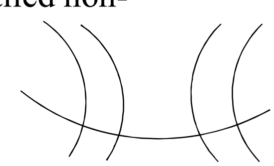
In this flow ρ is density and pressure remain same with the passage of time steady flow may be uniform or non-uniform.

On the other hand, the flow in which the property and condition associated with the dependent of time. In this case the flow pattern changes with time.

(iii) Laminar and Non-Laminar flow or Stream line flow and Turbulent flow:

A flow in which fluid particle have definite path of particle and the two paths of two individuals does not cut each other is called Laminar or stream line flow.

On the other hand, if the flow of each other particle does not trace out a definite path. The path of individuals particle also crosses each other is called non-laminar or turbulent flow.



(iv) Rotational flow & Irrational flow:

Rotational flow is that flow in which fluid particle rotate about their own axis have the same angular velocity.

On the other hand, the fluid particle does not rotate about their own axis and retain their original orientation is called irrational flow.

(v) Compressible and Non-compressible flow:

A flow in which volume and density of fluid changes during the flow is said to be compressible flow.

On the other hand, if volume and density of fluid does not change during the flow is said to be non-compressible flow or incompressible flow.

***Note:** All liquids are generally considered to have incompressible flow.

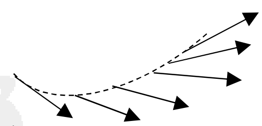
Different types of flow line:

We have discussed that one ever fluid is in motion. Its innumerable particle moves along the central line depending upon the condition of flow as there are many types of flow line that following are important for subject view point.

(i) Path line:

Followed by fluid particles in motion is called a path line. The path line shows the direction of motion of a particle. For a certain period of time or between two given section.

(ii) Stream lines:



The imaginary line drawn in the fluid where the velocity along the tangent.

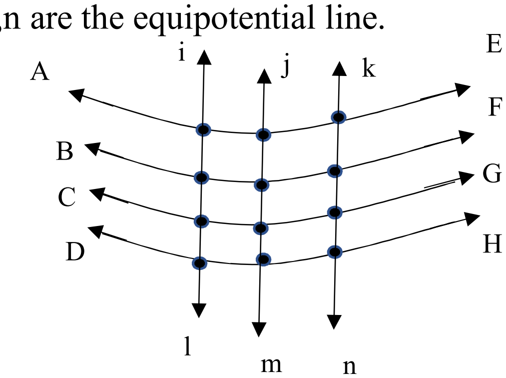
(iii) Filament lines:

The instantaneous pictures of the position of all particle which have passed through a given point at previous time are called filament line.

For example, the line finds by smoke particle exerted from a nozzle of rocket.

(iv) Potential and Equipotential lines:

We know that there is always a loss of head of fluid particles as we proceed along the flow line. If we draw the line joining the points of equipotential on the adjacent flow lines, we get the potential lines. The points where lines A,B,C,D,E,F,G,H are the potential line and i,j,k,l,m,n are the equipotential line.



(v) Flow nets:

The intersection of potential line and stream line of two set of lines are called flow line i.e. intersection with the help of flow nets we can analysis of the behavior of certain phenomenon which cannot be mathematical means. Such a phenomenon is generally analyzed and studied with the joint flow nets.

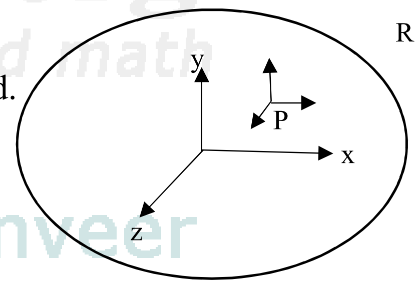
Some results of Vector Analysis:

$\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}$ is called del ∇ operator

$\nabla=\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}=\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]$

Scalar point function:

ψ (x,y,z) is said to be a scalar point function for a region R or of three dimensional space. If corresponding to each point P of R are definite value ϕ(P) of a scalar ϕ can be determined then ϕ is called scalar function or scalar point function of position or a scalar field.



Similarly, if corresponding to each point P(r) a definite value A(P) of a vector A can be determined. Then A is said to be vector function or vector point function or a vector field. As the point P is usually specified P(x,y,z) by the coordinate (x,y,z) then the position vector is equal to

$$\underline{r}=\quad x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

So, $\psi=\psi$ (x,y,z) and A = A(x,y,z) = A(r)

If ψ is a scalar point function then $\psi=\psi$(x,y,z) = k where k is a constant represents a surface called a level surface of a scalar function ψ. But gradient, divergence, curl is denoted as

$$\nabla\phi=\frac{\partial\phi}{\partial x}\hat{\imath}+\frac{\partial\phi}{\partial y}\hat{\jmath}+\frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla.A=\frac{\partial A_1}{\partial x}\hat{\imath}+\frac{\partial A_2}{\partial y}\hat{\jmath}+\frac{\partial A_3}{\partial z}\hat{k}$$

$$\nabla\times A=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$$

Divergence of gradient is called a Laplace operator.

$$\nabla.\nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

The Laplacian operator is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Some Identities:

(i) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(ii) $\nabla(u.v) = (u.\nabla)v+(v.\nabla)u + u\times curlv + v\times curly$

(iii) $\nabla(\phi\overline{u}) = \phi\nabla\overline{u} + \overline{u}\nabla\phi$

(iv) $\nabla.(u\times v) = vcurlu - ucurlv$

(v) $\nabla\times(\phi u) = \phi curlu+ucurl\phi$

(vi) $\nabla\times(u\times v) = u\nabla.v - v\nabla.u+(v.\nabla)u-(u.\nabla)v$

(vii) $\nabla\times(\nabla\phi) = 0$

(viii) $\nabla\times(\nabla\times u) = \nabla(\nabla.u) - \nabla^2 u$

(ix) $\nabla.(\nabla\times u) = 0$

Some Statements:

(i) $\int_s u.nds = \int_v \nabla.udv$

Where s denotes the surface bounded by volume V and n is a unit vector normal to surface s.

(ii) $\int_c u.dr = \int_s curl\, u.nds$; stokes theorem

Where s is a surface bounded by a close curve C and n is unit vector normal to surface s.

***Note:** If $\nabla^2\phi = 0$ then ϕ is called harmonic function.

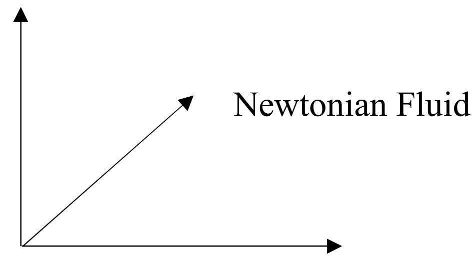
Real Fluid:

A fluid in which the finite viscosity exists and therefore we can exert tangential or sheering stress on a surface with which it is in contact. Real fluid is called viscous fluid. Real fluid can be further divided into Newtonian fluid and Non-Newtonian fluid.

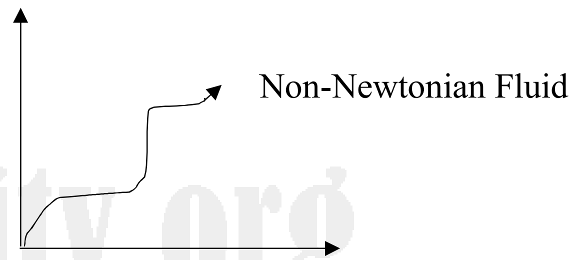
A fluid in which viscosity is independent of the velocity gradient due to single variable. Water and air are Newtonian fluid. They obey law of viscosity

$$\tau = \mu\frac{\partial u}{\partial x}$$

If we draw the graph then we get a straight line. Blood, milk, jellies, butter are example of Newtonian fluid.



A fluid in which viscosity at a given temperature and pressure for which a viscosity is a function of velocity gradient is called Non-Newtonian fluid is not a straight line. Non-Newtonian fluid are very important in fluid mechanics.



Lecture # 03

Non-Viscous and Inviscid Fluid:

A non-viscous and inviscid fluid has zero velocity such that there are no sheers i.e. no deformation. Therefore, an inviscid flow whether are in motion or rest exert only normal stress. Consequently, the sheer stresses in this case is zero. Water and air are treated as inviscid fluid. Non-viscous and inviscid fluid may or may not incompressible.

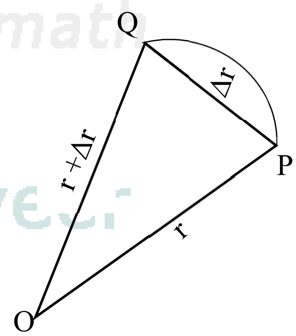
Ideal fluid:

Ideal fluid is a fluid in which both inviscid and incompressible fluid are involved is called Ideal fluid or it is a perfect fluid. These are non-Newtonian fluid. *In ideal fluid the viscosity is zero. There is no internal resistance between them.

Velocity of fluid particle:

$$PQ = \Delta r$$

And therefore, of a fluid particle denoted as 'q'.



$$\vec{q} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{dr}{dt}$$

In cartesian coordinates

$$\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{q} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$u\hat{i} + v\hat{j} + w\hat{k} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

On comparing

$$u = \frac{dx}{dt}, \; v = \frac{dy}{dt}, \; w = \frac{dz}{dt}$$

Material, local and convective derivative: Let a fluid particle moves from P(x,y,z) at time 't' to point Q(x+Δx, y+Δy, z+Δz) at time t+Δt. Further let

F(x,y,z,t) associated with some scalar properties of fluid. Let the total change in F due to the motion of fluid particle from P to Q.

$$\Delta F = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + \frac{\partial f}{\partial t}\Delta t$$

Divide by Δt

$$\frac{\Delta F}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial f}{\partial z}\frac{\Delta z}{\Delta t} + \frac{\partial f}{\partial t}$$

$$\text{Taking } \lim_{\Delta t \to 0} \text{ both side}$$

$$\lim_{\Delta t \to 0}\frac{\Delta F}{\Delta t} = \frac{\partial f}{\partial x}\lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t} + \frac{\partial f}{\partial z}\lim_{\Delta t \to 0}\frac{\Delta z}{\Delta t} + \frac{\partial f}{\partial t}$$

$$\frac{DF}{Dt} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

$$\because u = \frac{dx}{dt},\ v = \frac{dy}{dt},\ w = \frac{dz}{dt}$$

$$= \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v + \frac{\partial f}{\partial z}w + \frac{\partial f}{\partial t}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (q.\nabla)f \qquad \because \vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{D}{Dt} = \frac{d}{dt} = \frac{\partial}{\partial t} + (q.\nabla)$$

Where

* $\frac{D}{Dt}$ is called material derivative.

* $\frac{\partial}{\partial t}$ is called local derivative. It is associated with the variation at the fix position.

* $q.\nabla$ is called convective derivative. It is associated with the change of physical quantity F due to motion of fluid particle.

$* \dfrac{d}{dt}$ is called the differentiation following with the motion of fluid.

$\dfrac{D}{Dt}$ *and* $\dfrac{d}{dt}$ signifies also the differentiation following with the motion of fluid.

The rate of change of physical quantity F associated with some fluid particle is treated as above.

Acceleration of Matrix of fluid particle:

Let a fluid particle moves from point P(x,y,z) at time ‘t’ to Q(x+Δx , y+Δy, z+Δz) at time t+Δt respectively. The particle is at P in time t and after t+Δt the particle is at point Q. Then ‘q’ is the velocity of the fluid particle of P at time t and q+Δq is the velocity of fluid particle at point Q in time t+Δt. Total change in q is

$$\Delta q = \frac{\partial q}{\partial x}\Delta x + \frac{\partial q}{\partial y}\Delta y + \frac{\partial q}{\partial z}\Delta z + \frac{\partial q}{\partial t}\Delta t$$

Divide by Δt

$$\frac{\Delta q}{\Delta t} = \frac{\partial q}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial q}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial q}{\partial z}\frac{\Delta z}{\Delta t} + \frac{\partial q}{\partial t}$$

$$\text{Taking } \lim_{\Delta t\to 0} \text{ both side}$$

$$\lim_{\Delta t\to 0}\frac{\Delta q}{\Delta t} = \frac{\partial q}{\partial x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \frac{\partial q}{\partial y}\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} + \frac{\partial q}{\partial z}\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} + \frac{\partial q}{\partial t}$$

$$\frac{dq}{dt} = \frac{\partial q}{\partial x}\frac{dx}{dt} + \frac{\partial q}{\partial y}\frac{dy}{dt} + \frac{\partial q}{\partial z}\frac{dz}{dt} + \frac{\partial q}{\partial t}$$

$$\because u = \frac{dx}{dt},\ v = \frac{dy}{dt},\ w = \frac{dz}{dt}$$

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (q.\nabla)f \qquad \because \vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + (q.\nabla)q \qquad ...(i)$$

Now for acceleration component

Let $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$$a = \frac{dq}{dt} = ax\hat{i} + ay\hat{j} + az\hat{k} \qquad Put\ in\ eq\,(i)$$

$$ax\hat{i} + ay\hat{j} + az\hat{k} = \frac{\partial}{\partial t}\left(u\hat{i} + v\hat{j} + w\hat{k}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\left(u\hat{i} + v\hat{j} + w\hat{k}\right)$$

$$\Rightarrow \qquad ax = \frac{\partial u}{\partial t} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$ay = \frac{\partial v}{\partial t} + v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$az = \frac{\partial w}{\partial t} + w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

Example: Find the acceleration components at (2,-1,3) of

$$q = \left(x^2 - y^2\right)\hat{i} + x^2yt\,\hat{j} + x^2y^2t^2\,\hat{k}$$

Solution:

Given $q = \left(x^2 - y^2\right)\hat{i} + x^2yt\,\hat{j} + x^2y^2t^2\,\hat{k}$

As $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

On comparing

$$u = x^2 - y^2\ ,\ v = x^2yt\ ,\ w = x^2y^2t^2$$

$$\frac{\partial u}{\partial x} = 2x\ ,\ \frac{\partial v}{\partial y} = x^2t\ ,\ \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} = 0\ ,\ \frac{\partial v}{\partial t} = x^2y\ ,\ \frac{\partial w}{\partial t} = 2x^2y^2t$$

At $(2,-1,3)$ $\qquad u = 3\ ,\ v = -4t\ ,\ w = 4t^2$

$$\frac{\partial u}{\partial x} = 4\ ,\ \frac{\partial v}{\partial y} = 4t\ ,\ \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t}=0\ ,\frac{\partial v}{\partial t}=-4\ ,\frac{\partial w}{\partial t}=8t$$

Now $$ax=\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$= 0 +3(4+4t+0)$

$= 12+12t \quad \Rightarrow 12(1+t)$

$$ay=\frac{\partial v}{\partial t}+v\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$=-4+(-4t)(4+4t+0)$$

$$=-4-16t-16t^2$$

$$az=\frac{\partial w}{\partial t}+w\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$=8t+4t^2(4+4t+0)$$

$$=8t+16t^2+16t^3$$

Lecture # 04

Equation of stream line or stream flow:

As we know that stream line is a curve drawn in the fluid so that tangent at each point is in the direction of motion.

i.e. Fluid velocity at a point.

Let P(r) where r = $x\hat{\imath}+y\hat{\jmath}+z\hat{k}$ so that position vector of the point P on a stream line and let q = $u\hat{\imath}+v\hat{\jmath}+w\hat{k}$ be the fluid velocity and point P then q // dr.

Therefore, equation of stream line is

$$q \times dr = 0$$

$$\begin{vmatrix} i & j & k \\ u & v & k \\ dx & dy & dz \end{vmatrix} = 0$$

$$(vdz - wdy)\hat{i} + (wdx - udz)\hat{j} + (udy - vdx)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

On comparing

$$vdz - wdy = 0 \quad \Rightarrow \quad vdz = wdy \quad \Rightarrow \frac{dz}{w} = \frac{dy}{v} \qquad \_\_\_\_\_(i)$$

$$wdx - udz = 0 \quad \Rightarrow \quad wdx = udz \quad \Rightarrow \frac{dx}{u} = \frac{dz}{w} \qquad \_\_\_\_\_(i)$$

$$udy - vdx = 0 \quad \Rightarrow \quad udy = vdx \quad \Rightarrow \frac{dy}{v} = \frac{dx}{u} \qquad \_\_\_\_\_(i)$$

From (i),(ii) & (iii) we have

$$* \ \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$ are the equation of stream line.

Example:

Find the equation of stream line $q = \left(x^2 - y\right)\hat{i} + \left(x^2 + yz\right)\hat{j} + yz^2\hat{k}$

Solution: As we know that equation of stream line

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$q = \left(x^2 - y\right)\hat{i} + \left(x^2 + yz\right)\hat{j} + yz^2\hat{k}$$

And q = $u\hat{\imath}+v\hat{\jmath}+w\hat{k}$

On comparing

$$u = x^2 - y \ , \ \ v = x^2 + yz \ , \ w = yz^2$$

$$\frac{dx}{x^2 - y} = \frac{dy}{x^2 + yz} = \frac{dz}{yz^2}$$

Equation of path line:

Path line is a curve are trajectory along which a particle travel during its motion is called path line.

Differential equation of path line is $\dfrac{dr}{dt} = q$

Where q = $u\hat{\imath}+v\hat{\jmath}+w\hat{k}$ and r = $x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

$$\frac{d\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{dt} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$* \ \frac{dx}{dt} = u \ , \ \frac{dy}{dt} = v \ , \ \frac{dz}{dt} = w$$

is the required equation of path line.

Difference between stream line and path line:

(i) The stream line is not in general same as the path line.

(ii) Stream line show how each particle is moving at given instant of time whereas path line represents the motion of fluid particle at each instant.

(iii) If the flow is steady the stream lines remain unchanged as the time progressed and hence they are also the path line.

Example: The velocity of fluid is given by

$q = \left(x^2 - y\right)\hat{i} + \left(x^2 + yz\right)\hat{j} + yz^2\hat{k}$ find the equation of path line.

Solution: $q = \left(x^2 - y\right)\hat{i} + \left(x^2 + yz\right)\hat{j} + yz^2\hat{k}$

And $q = u\hat{i} + v\hat{j} + w\hat{k}$ On comparing $u = x^2 - y$, $v = x^2 + yz$, $w = yz^2$

As $\dfrac{dx}{dt} = u$, $\dfrac{dy}{dt} = v$, $\dfrac{dz}{dt} = w$

$\Rightarrow \dfrac{dx}{dt} = x^2 - y$, $\dfrac{dy}{dt} = x^2 + yz$, $\dfrac{dz}{dt} = yz^2$

★ Equation of Continuity or Conservation of mass or Law of conservation of mass:

"Law of conservation of mass state that fluid mass can neither created nor destroyed."

The equation of continuity gives the law of conservation of mass in analytical form or mathematical form

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho q\right) = 0$$

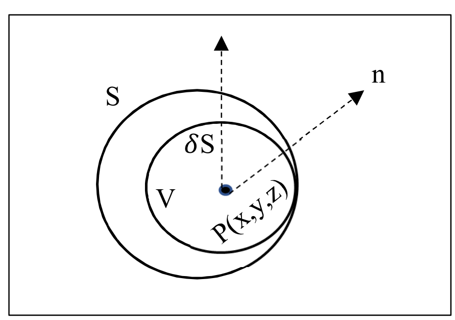
q is the velocity of fluid. Therefore, in a continuous motion the equation of continuity expresses the fact that increase in the mass of fluid with any closed surface drawn in the fluid at any time must be equal to the access of the mass that flows 'in' over the mass of that flows 'out'. Inward flow is equal to outward flow.

★ Equation of Continuity in velocity form or Equation of Continuity by Euler Method or show that the fluid is incompressible or not.

As $$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho q\right) = 0$$

Explanation: Let S be an arbitrary small close surface drawn in the compressible fluid and closing volume V. Let P(x,y,z) be any point of S and P(x,y,z,t) be the

density of the fluid at point P at any time t and the element of the surface δS and q be the velocity of the fluid. Normal component of q is q.$\hat{n}$



Therefore, the set of mass that flows across $\delta S = (q.\hat{n})\, \rho\, \delta S$. Total rate of mass flows across $\delta S = \int_S \rho(q.n)\, dS$

$$\delta S = \int_V div(\rho q)\, dV \quad \because change\ surface\ integral\ into\ volume\ integral$$

Therefore, the total rate of mass flows into volume V $= -\int_V div(\rho q)\, dV$ ___(i)

*Outward flow is positive and inward flow is negative. Again, the mass of fluid in S at time t is given as $= \int_V \rho\, dV$

Total rate of mass increase with in S is $\frac{\partial}{\partial t}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV$ ____(ii)

Let the region V of fluid contains neither source nor sink i.e. not Inlet not outlet. Therefore, from eq (i) & (ii)

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla.(\rho q)\, dV$$

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla.(\rho q)\, dV = 0$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla.(\rho q) \right) dV = 0 \qquad \text{___(iii)}$$

As V is an arbitrary value in which is known as the equation of continuity of Euler form.

$$* \quad \frac{\partial \rho}{\partial t} + \nabla.(\rho q) = 0$$

is the equation of Continuity in velocity form.

*Remark:

For incompressible and homogenous fluid, we know ρ is constant

$$\frac{\partial \rho}{\partial t} = 0$$

Equation (iii) becomes $\Rightarrow$ $\nabla.(\rho q)=0$

$$\rho(\nabla.q)=0$$

$$\nabla.q=0$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

This is called incompressible fluid.

Equation of Continuity in Cartesian coordinate OR differential form of continuity:

Let a fluid particle be at point P. Let ρ and P(x,y,z,t) be the density of at any time t.

Let u,v,w be the velocity component of fluid parallel to x-axis, y-axis and z-axis. We construct a parallelogram with edges δx,δy,δz parallel to three respective axes with point P shown in fig-1.

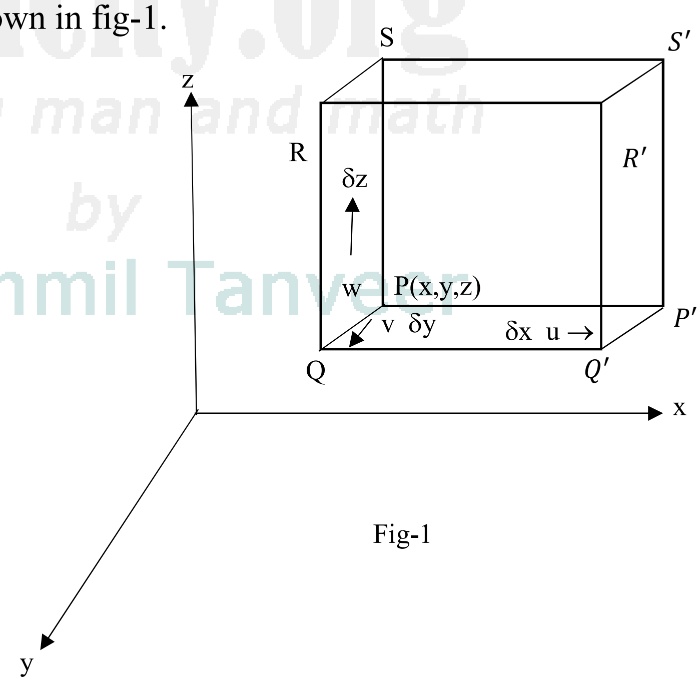


Fig-1

Therefore, we have the mass of fluid passes through the face PQRS is

$$= \rho\,(\delta x\delta y\delta z)\,u$$

$$= F(x,y,z) \qquad \_\_\_\_\_\_(i)$$

Mass of fluid that passes out through the opposite side $P'Q'R'S'$ is

$$= F(x+\delta x,\ \delta y,\ \delta z) \qquad \_\_\_\_\_\_(ii)$$

As x+δx per unit time along x-axis. By using Taylor theorem.

$$= F(x,\mathrm{y},\mathrm{z})+\delta \mathrm{x}\ \frac{\partial}{\partial x}F(x,\mathrm{y},\mathrm{z})+....$$

Again, the net mass per unit time within the element (rectangular parallelepiped) due to the flow through the face PQRS and $P'Q'R'S'$

By using (i) and (ii)

The net rate of mass of the out flow in the x-direction equal to the mass entered through the face PQRS – the mass that leaves through the face $P'Q'R'S'$

$$= F(x,y,z)-\left[F(x,\mathrm{y},\mathrm{z})+\delta \mathrm{x}\ \frac{\partial}{\partial x}F(x,\mathrm{y},\mathrm{z})+....\right]$$

$$=-\delta \mathrm{x}\ \frac{\partial}{\partial x}F(x,\mathrm{y},\mathrm{z})$$

$$=-\delta \mathrm{x}\,\delta y\delta z\ \frac{\partial}{\partial x}(\rho u) \qquad \_\_\_\_\_\_\text{(iii) By (i)}$$

Similarly, Net rate of mass out in y-axis

$$=-\delta \mathrm{x}\,\delta y\delta z\ \frac{\partial}{\partial y}(\rho v) \qquad \_\_\_\_\_\text{(iv)}$$

And in z-axis $=-\delta \mathrm{x}\,\delta y\delta z\ \dfrac{\partial}{\partial z}(\rho w)$ ___(v)

Therefore, total rate of mass flow through element of parallelepiped is

Total net rate of flow $=-\delta \mathrm{x}\,\delta y\delta z\left[\dfrac{\partial}{\partial x}(\rho u)+\dfrac{\partial}{\partial y}(\rho v)+\dfrac{\partial}{\partial z}(\rho w)\right]$ ____(vi)

Again, in the mass of fluid within the chosen element is = ρ(δx,δy,δz)

Total rate of mass increase within the element $=\dfrac{\partial}{\partial t}\rho(\delta \mathrm{x}\,\delta y\delta z)$

$$= \delta \mathrm{x}\,\delta y\delta z\frac{\partial \rho}{\partial t} \qquad \_\_\_\_\_\text{(vii)}$$

From equation (vi) and (vii)

By the law of conservation of mass

$$\delta x \delta y \delta z \frac{\partial \rho}{\partial t} = -\delta x \delta y \delta z \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right]$$

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right]$$

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] = 0 \qquad \text{\_\_\_\_\_(viii)}$$

Now
$$\frac{\partial}{\partial x}(\rho u) = \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x}$$

$$\frac{\partial}{\partial y}(\rho v) = \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y}$$

$$\frac{\partial}{\partial y}(\rho w) = \rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial w}$$

Put in (viii)

$$\frac{\partial \rho}{\partial t} + \left[\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial w} \right] = 0$$

$$\frac{\partial \rho}{\partial t} + \left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial w} \right) + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

$$\frac{\partial \rho}{\partial t} + (q.\nabla)\rho + \rho(\nabla.q) = 0$$

Which is the equation of Continuity in cartesian form.

*Note: If the fluid is incompressible then ρ is constant

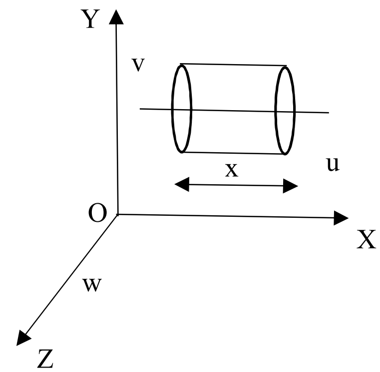
$$\frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \nabla.q = 0$$

Lecture # 05

★Question:

A pulse travelling along a fine straight uniform tube causes the density (ρ) at time t at a distance 'x' from the origin. Then the velocity is u_0 to become $\rho_0\phi(vt-x)$.Prove that u at any time t and distance from origin is given by



$$v+\frac{(u_0-v)\phi(vt)}{\phi(vt-x)}$$

Proof: Let u be the velocity and ρ be the density at a distance x then we are given

$$\rho=\rho_0\phi(vt-x) \qquad \_\_\_\_\_\_\_(1)$$

As we know that

$$\frac{\partial\rho}{\partial t}+\rho\frac{\partial u}{\partial x}+u\frac{\partial\rho}{\partial x}=0 \qquad \_\_\_\_\_\_(2)$$

Diff. equation (1) w.r.t 't'

$$\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial t}\left[\rho_0\phi(vt-x)\right]$$

$$\frac{\partial\rho}{\partial t}=\rho_0\phi'(vt-x)\text{v} \qquad \_\_\_\_\_\_\_(3)$$

Diff. equation (1) w.r.t 'x'

$$\frac{\partial\rho}{\partial x}=\frac{\partial}{\partial x}\left[\rho_0\phi(vt-x)\right]$$

$$\frac{\partial\rho}{\partial x}=\rho_0\phi'(vt-x)(-1) \qquad \_\_\_\_\_(4)$$

Put (3) and (4) in (2)

$$\rho_0\phi'(vt-x)\text{v}+\rho\frac{\partial u}{\partial x}+u\left[\rho_0\phi'(vt-x)(-1)\right]=0$$

$$\rho_0\phi'(vt-x)\,\text{v}+\rho\frac{\partial u}{\partial x}-u\rho_0\phi'(vt-x)=0$$

Put the value of ρ from (1)

$$\rho_0\phi'(vt-x)\,\text{v}+\rho_0\phi(vt-x)\frac{\partial u}{\partial x}-u\rho_0\phi'(vt-x)=0$$

$$\rho_0\phi'(vt-x)\left(v-u\right)+\rho_0\phi(vt-x)\frac{\partial u}{\partial x}=0$$

$$\rho_0\phi'(vt-x)\left(v-u\right)=-\rho_0\phi(vt-x)\frac{\partial u}{\partial x}$$

$$\frac{\phi'(vt-x)}{\phi(vt-x)}=\frac{-1}{v-u}\frac{\partial u}{\partial x}$$

$$\frac{\phi'(vt-x)}{\phi(vt-x)}\partial x=\frac{-1}{v-u}\partial u$$

$$\frac{\phi'(vt-x)}{\phi(vt-x)}\partial x+\frac{1}{v-u}\partial u=0$$

On integrating both sides

$$(-1)\int\frac{(-1)\phi'(vt-x)}{\phi(vt-x)}\partial x+(-1)\int\frac{(-1)}{v-u}\partial u=\int 0$$

$$-\log\phi(vt-x)-\log(v-u)=-\log A$$

$$-\log\left[\phi(vt-x)(v-u)\right]=-\log A$$

$$\left[\phi(vt-x)(v-u)\right]=A \qquad \_\_\_\_\_\_\_\_(5)$$

Now by the condition $u=u_0$ when x = 0

Equation (5) ⇒ $$\left[\phi(vt-0)(v-u_0)\right]=A$$

$$\phi(vt)(v-u_0)=A \qquad \text{put in (5)}$$

$$\phi(vt-x)(v-u)=\phi(vt)(v-u_0)$$

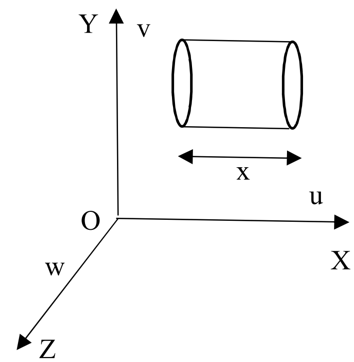
$$v-u=\frac{\phi(vt)(v-u_0)}{\phi(vt-x)}$$

$$u = v - \frac{\phi(vt)(v - u_0)}{\phi(vt - x)}$$

$$u = v + \frac{\phi(vt)(u_0 - \mathrm{v})}{\phi(vt - x)}$$ As required.

Question:

Air obeying Boyles law is in motion in a uniform tube in a small section. Prove that if ρ be the density and v be the velocity of a distance x and from a fixed point at time t then show that



$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[\rho(v^2 + \mathrm{k})\right]$$

Solution:

According to Boyles law

$$P \propto \frac{1}{v}$$

$$P = k.\frac{1}{v}$$

Let $P = k\rho$ ______(i) $\rho = \frac{m}{v}$

We know that

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$ _______(ii) in one dimension

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v)$$

We know general form of Euler equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{-1}{\rho}\frac{\partial P}{\partial x}$$ ______(iii)

Diff. (i) w.r.t ‘x’

$$\frac{\partial \rho}{\partial x} = k\frac{\partial \rho}{\partial x}$$

Put in (iii)

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{-k}{\rho}\frac{\partial P}{\partial x} \qquad \text{\_\_\_\_\_\_(iv)}$$

Diff. (ii) w.r.t 't'

$$\frac{\partial}{\partial t}\left[\frac{\partial P}{\partial t} + \frac{\partial v}{\partial x}(\rho v)\right] = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}(\rho v)\right] = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\left[v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x}\right] = 0$$

Using (iii) and (iv)

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\left[v\left(-\frac{\partial}{\partial x}(\rho v)\right) + \rho\left(\frac{-k}{\rho}\frac{\partial \rho}{\partial x} - v\frac{\partial v}{\partial x}\right)\right] = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial t}\left[v\left(\frac{\partial}{\partial x}(\rho v)\right) + \rho\left(\frac{k}{\rho}\frac{\partial \rho}{\partial x} + v\frac{\partial v}{\partial x}\right)\right] = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x}\left[(\rho v)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial x}(\rho v) + k\frac{\partial \rho}{\partial x}\right]$$

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\rho v.v) + k\frac{\partial \rho}{\partial x}\right]$$

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\rho\left(v^2 + k\right)$$

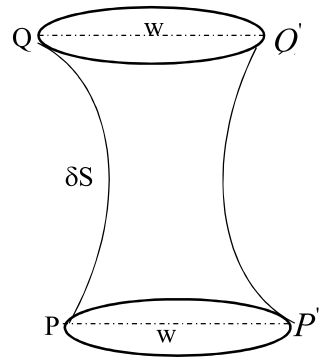
As required

Question:

If w is the area of cross-section of a stream filament. Prove that equation of Continuity is

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial s}(\rho w q)=0$$

Solution:



Given that w is the area of cross-section where δS is the element of the arc of the filament PP' and QQ' in the direction of flow and q is the velocity of fluid.

Let $PP'QQ'$ be a stream filament. Let PQ = δS. The rate of excess of flow out along $PQ=-\delta S\frac{\partial}{\partial s}(\rho w q)$

Again, total mass of fluid within the filament = ρwδS

The rate of increase of the mass of filament = $\frac{\partial}{\partial s}(\rho w\delta S)$

By the equation of Continuity increase in the mass of the fluid must be equal to excess of mass that flows out

$$-\delta S\frac{\partial}{\partial s}(\rho w q)=\frac{\partial}{\partial s}(\rho w\delta S)$$

$$\frac{\partial}{\partial s}(\rho w\delta S)+\delta S\frac{\partial}{\partial s}(\rho w q)=0$$

$$\delta S\frac{\partial}{\partial s}(\rho w)+\delta S\frac{\partial}{\partial s}(\rho w q)=0$$

$$\frac{\partial}{\partial s}(\rho w)+\frac{\partial}{\partial s}(\rho w q)=0$$

As required.

Lecture # 06

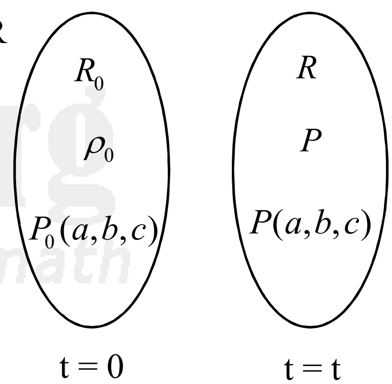
Equation of Continuity by LaGrange method:

Let R_0 be the region occupied by a portion of fluid at time t = 0 and R be the region occupied by the same fluid at time t.

Let (a,b,c) be the initial coordinates of the fluid particle P_0 and ρ_0 is the density at t = 0. Mass of the fluid at t = 0 is $\rho_0 \delta a \delta b \delta c$

Let P be the subsequent position of the fluid (position at P_0 after some time t) at time 't' and ρ is the density of the fluid. Then mass of the fluid at time t is $\rho \delta x \delta y \delta z$. By the law of conservation of mass, the mass contained inside the given volume of the fluid remain same throughout the motion (fluid motion).

Therefore, mass inside R_0 must be equal to mass inside R



$$\iint_{R_0} \rho_0 \delta a \delta b \delta c = \iint_{R} \rho \delta x \delta y \delta z \quad \_\_\_\_\_\_\text{(i)}$$

By the advance Calculus

$$J \delta a \delta b \delta c = \delta x \delta y \delta z \quad \_\_\_\_\text{(ii)}$$

By equation (i)

$$\iint_{R_0} \rho_0 \delta a \delta b \delta c = \iint_{R} \rho J \delta a \delta b \delta c$$

$$\because \; The\,total\,mass\,R_0 = Mass\,inside\,R \quad (R = R_0)$$

$$\Rightarrow \quad \iint_{R_0} \rho_0 \delta a \delta b \delta c = \iint_{R_0} \rho J \delta a \delta b \delta c$$

$$\iint_{R_0} (\rho_0 - \rho J) \delta a \delta b \delta c = 0$$

This is called the equation of continuity in Langrangian form.

Note: From equation (ii)

$$J = \frac{\delta x \delta y \delta z}{\delta a \delta b \delta c}$$

$$J = \frac{(x,y,z)}{(a,b,c)}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}$$

Equation of continuity in cylindrical coordinate (r,θ,z)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\rho\left(qr\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\rho\left(q\theta\right) + \frac{\partial}{\partial z}\rho\left(qz\right) = 0$$

Where qr, qθ, qz are the velocity component along r,θ,z direction respectively.

$\Rightarrow$ In cylindrical symmetry

$$*\frac{\partial}{\partial r} = 0, \quad \frac{\partial}{\partial z} = 0$$

Equation of continuity in spherical coordinate (r,θ,ϕ)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 qr\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\rho \sin\theta q\theta\right) + \frac{1}{r\sin\phi}\left(\rho q\phi\right) = 0$$

Where qr, qθ, qz are the velocity component along r,θ,z direction respectively.

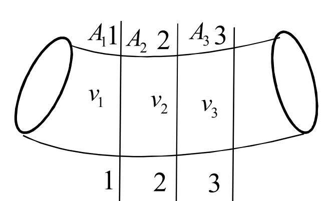
$\Rightarrow$ In cylindrical symmetry

$$*\frac{\partial}{\partial \theta} = 0, \quad \frac{\partial}{\partial \phi} = 0$$

Equation of continuity for liquids for the motion through channels or pipes:

Let an incompressible fluid (liquid) flows through a channel or pipes whose cross-sectional area may or may not be fixed. Then the quantity of fluid passing for a unit (per second) is same at all sections.

Let some fluid (liquid) is flowing through a tempering (مڑا ہوا) pipe as shown in fig.



Let A_1, A_2, A_3 be the area of pipe at section 1–1,2–2,3–3 respectively.

Further let v_1, v_2, v_3 be the velocity of the fluid in the section 1−1,2−2,3−3 respectively. And Q_1, Q_2, Q_3 be the total quantity of liquid through the section 1−1,2−2,3−3 is as

$$Q_1 = A_1 v_1, Q_2 = A_2 v_2, Q_3 = A_3 v_3$$

By the law of conservation of mass, the total quantity of fluid through section 1−1,2−2,3−3 is same

$$Q_1 = Q_2 = Q_3$$

$$A_1 v_1 = A_2 v_2 = A_3 v_3$$

Which is the equation of continuity for liquids for the motion through channels or pipes.

Stream function and potential function:

Stream function:

In steady incompressible plane in two-dimensional flow of practical importance velocity has two component such as u and v when flow is in xy-plane which relates to velocity component.

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x}$$

Then equation of continuity is identically satisfied.

This conclusion can be verified as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

L.H.S $$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

Therefore, velocity components are defined in term of stream function.

We know that the conservation of mass $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ will be satisfied.

*Advantages of stream function:

(i) Stream function is taken only for a flow in plane.

(ii) We have the simplified analysis by having to determined only unknown function ψ(x,y) rather than two function.

(iii). Another advantages of using the stream function to use these function ψ(x,y) to the fact, the lines along which ψ = constant.

ψ = ψ(x,y) are stream line

Here ψ = ψ(x,y)

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

$$-vdx + vdy = 0 \qquad \because \frac{\partial \psi}{\partial y} = u \ , \ -\frac{\partial \psi}{\partial x} = v$$

$$vdx = vdy$$

$$\frac{dx}{u} = \frac{dy}{v}$$

Which are the equation of stream line.

*So, using stream function we can find stream lines.

Potential function:

For potential function we use ϕ(x,y) or ϕ(x,y,z)

$$u = \frac{\partial \phi}{\partial x} \ , \ v = \frac{\partial \phi}{\partial y} \ , \ w = \frac{\partial \phi}{\partial z}$$

Theorem: Prove that stream function and potential function are orthogonal.

Proof:

Consider a stream function ψ = ψ(x,y)

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = d\psi$$

For stream function ψ is constant , dψ = 0

$$\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy=0$$

$$-vdx+udy=0 \qquad \because \frac{\partial\psi}{\partial y}=u\ ,\ -\frac{\partial\psi}{\partial x}=v$$

$$vdx=udy$$

$$\frac{dy}{dx}=\frac{v}{u}=m_1(say)$$

For potential function $\phi=\phi(x,y)$

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy=d\phi$$

For potential function ϕ is constant, $d\phi = 0$

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy=0$$

$$udx+vdy=0 \qquad \because \frac{\partial\psi}{\partial x}=u\ ,\ \frac{\partial\psi}{\partial y}=v$$

$$-udx=vdy$$

$$\frac{dy}{dx}=-\frac{u}{v}=m_2\text{ (say)}$$

Now $$m_1.m_2=\frac{v}{u}.\left(\frac{-u}{v}\right)=-1$$

Which show that stream function and potential function is orthogonal.

Comparison between stream function and potential function:

Stream function	Potential function
(i) Stream function is a consequence of conservation of mass.	(i). Potential function is a consequence of irrotationality of the flow.
(ii) Stream function can be defined for two-dimensional.	(ii) Potential function can be defined for two or three-dimensional.
(iii) $\nabla^2\psi=0$	(iii). $\nabla^2\phi=0$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

(iv) ψ(x,y)

$$\frac{\partial\psi}{\partial y}=u\ ,\ -\frac{\partial\psi}{\partial x}=v$$

(iv) ϕ(x,y,z)

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

$$u=\frac{\partial\phi}{\partial x}\ ,\ v=\frac{\partial\phi}{\partial y}\ ,\ w=\frac{\partial\phi}{\partial z}$$

Irrotational flow:

Condition of irrotational flow is

$$\nabla\times Q=0=Curl Q$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)i+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)j+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)k=0$$

Lecture # 07

Polar form of stream function:

$$V_r = q_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$V_\theta = q_\theta = -\frac{\partial \psi}{\partial r}$$

Polar form of potential function:

$$V_r = q_r = \frac{\partial \phi}{\partial r}$$

$$V_\theta = q_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Question:

In two-dimensional flow

$$u = v_x = q_x = x - 4y$$
$$v = u_x = q_y = -y - 4x$$

(i) Show that it satisfies law of conservation of mass.
(ii) Obtain the expression for stream function.
(iii) Show that the flow is potential.
(iv) Find the potential function or obtain the expression for potential function.

Solution: (i)

Given that $\quad u = x - 4y \quad , \quad v = -y - 4x$

$$\frac{\partial u}{\partial x} = 1 \quad , \quad \frac{\partial v}{\partial y} = -1$$

We know the law of conservation of mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(ii). Now the expression for stream function

$$u = \frac{\partial \psi}{\partial y} = x - 4y \qquad \_\_\_\_\_\_(i) \qquad \because u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x} = -y - 4x$$

$$v = \frac{\partial \psi}{\partial x} = y + 4x \qquad \_\_\_\_\_(ii)$$

From (i) $\Rightarrow$ $\frac{\partial \psi}{\partial y} = x - 4y$

$$d\psi = (x - 4y)dy$$

Integrating both sides

$$\int d\psi = \int (x - 4y)dy$$

$$\psi = xy - 2y^2 + f(x) \qquad \_\_\_(iii)$$

Diff. w.r.t x we have

$$\frac{\partial \psi}{\partial x} = y + f'(x) \qquad compare\ \ with\ (ii)$$

$$y + 4x = y + f'(x)$$

$$\Rightarrow \quad f'(x) = 4x$$

Integrating both sides

$$\Rightarrow \int f'(x) = \int 4x\,dx$$

$$\Rightarrow f(x) = 2x^2 \qquad put\ \ in\ \ (iii)$$

$$\psi = xy - 2y^2 + 2x^2 \qquad \text{required result}$$

It will satisfy the Laplace equation

$$\frac{\partial \psi}{\partial x} = y + 4x \quad , \quad \frac{\partial \psi}{\partial y} = x - 4y$$

$$\frac{\partial^2\psi}{\partial x^2}=4 \quad , \quad \frac{\partial^2\psi}{\partial y^2}=-4$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=4+(-4)=0$$

Laplace equation satisfied.

(iii) For potential function

Given that $u=y+4x \quad \Rightarrow \quad \frac{\partial u}{\partial y}=-4$

$$v=-y-4x \quad \Rightarrow \quad \frac{\partial v}{\partial x}=-4$$

For potential flow

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$$

$$-4=-4 \quad \text{satisfied}$$

(iv) Now the expression for potential function

$$u=\frac{\partial\phi}{\partial y}=x-4y \qquad \_\_\_\_\_\_(i) \qquad \because u=\frac{\partial\phi}{\partial y}$$

$$v=\frac{\partial\phi}{\partial x}=-y-4x \qquad \_\_\_\_\_(ii) \qquad \because v=\frac{\partial\phi}{\partial y}$$

From (i) $\Rightarrow \quad \frac{\partial\phi}{\partial y}=x-4y$

$$d\phi=(x-4y)dx$$

Integrating both sides

$$\int d\phi=\int(x-4y)dx$$

$$\phi=\frac{x^2}{2}-4xy+f(y) \qquad \_\_\_\text{(iii)}$$

Diff. w.r.t y we have

$$\frac{\partial \phi}{\partial y} = -4x + f'(y) \qquad compare \ \ with \text{ (ii)}$$

$$-y - 4x = -4x + f'(y)$$

$$\Rightarrow \quad f'(y) = -y$$

Integrating both sides

$$\Rightarrow \int f'(y) = \int -y dy$$

$$\Rightarrow f(y) = \frac{-y^2}{2} \qquad put \ \ in \ \ (iii)$$

$$\phi = \frac{x^2}{2} - 4xy - \frac{y^2}{2} \qquad \text{required result}$$

It will satisfy the Laplace equation

$$\frac{\partial \phi}{\partial x} = x - 4y \quad , \quad \frac{\partial \phi}{\partial y} = -4x - y$$

$$\frac{\partial^2 \psi}{\partial x^2} = 1 \quad , \quad \frac{\partial^2 \psi}{\partial y^2} = -1$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 1 + (-1) = 0$$

Laplace equation satisfied.

Velocity component in cylindrical coordinate in stream function:

$$q_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$q_\theta = -\frac{\partial \psi}{\partial r}$$

Equation of continuity in cylindrical coordinate (2-D)

$$\frac{1}{r} \frac{\partial}{\partial r}(r q_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(q_\theta) = 0$$

$$L.\text{H.S} = \frac{1}{r} \frac{\partial}{\partial r}\left(r \cdot \frac{1}{r} \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r} \frac{\partial}{\partial \theta}\left(-\frac{\partial \psi}{\partial r}\right)$$

$$\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}=0\ (R.H.S)$$

Question:

In a two-dimensional flow the velocity component u = 2y , v = 4x. Find the stream function and also tell about the shape of stream function.

Solution:

For stream function we know

$$u=\frac{\partial\psi}{\partial y}=2y \qquad \_\_\_\_\_\_(i) \qquad \because u=\frac{\partial\psi}{\partial y}$$

$$v=-\frac{\partial\psi}{\partial x}=4x$$

$$v=\frac{\partial\psi}{\partial x}=-4x \qquad \_\_\_\_\_(ii)$$

From (i) $\Rightarrow \dfrac{\partial\psi}{\partial y}=2y$

$$d\psi=2y\,dy$$

Integrating both sides

$$\int d\psi=\int 2y\,dy$$

$$\psi=y^2+f(x) \qquad \_\_\_(iii)$$

Diff. w.r.t x we have

$$\frac{\partial\psi}{\partial x}=f'(x) \qquad compare\ \ with\ (ii)$$

$$-4x=f'(x)$$

$$\Rightarrow\ f'(x)=-4x$$

Integrating both sides

$$\Rightarrow \int f'(x)=\int -4x\,dx$$

$$\Rightarrow f(x)=-2x^2 \qquad put\ \ in\ \ (iii)$$

$$\psi = y^2 - 2x^2$$

$$\frac{\psi}{2} = \frac{y^2}{2} - x^2$$

$$1 = \frac{\frac{y^2}{2}}{\frac{\psi}{2}} - \frac{x^2}{\frac{\psi}{2}}$$

$$1 = \frac{y^2}{\psi} - \frac{x^2}{\frac{\psi}{2}}$$

Which is Hyperbola

Question:

If $\psi = 2r^2 \sin 2\theta$ find V_r, V_θ. Also find V^2. Write the expression for stream function.

Solution: We know that

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$V_r = \frac{1}{r}\frac{\partial\left(2r^2 \sin 2\theta\right)}{\partial \theta}$$

$$V_r = \frac{1}{r} 2r^2 \cos 2\theta .2$$

$$V_r = 4r \cos 2\theta$$

$$V_\theta = -\frac{\partial \psi}{\partial r}$$

$$V_\theta = -\frac{\partial\left(2r^2 \sin 2\theta\right)}{\partial r}$$

$$V_\theta = -2.2r \sin 2\theta$$

$$V_\theta = -4r \sin 2\theta$$

Also $V^2 = \left(V_r\right)^2 + \left(V_\theta\right)^2$

$$Also \qquad V^2 = (4r\cos 2\theta)^2 + (-4r\sin 2\theta)^2$$

$$V^2 = 16r^2\cos^2 2\theta + 16r^2\sin^2 2\theta$$

$$V^2 = 16r^2\left(\cos^2 2\theta + \sin^2 2\theta\right)$$

$$V^2 = 16r^2$$

Also $\psi = 2r^2\sin 2\theta = 2r^2(2\sin\theta\cos\theta)$

$= 4(\mathrm{r}\cos\theta)(r\sin\theta)$

$\psi = 4xy$

is the expression for stream function.

Vorticity Vector:

*For vorticity vector curl q $\neq 0$

Let q = (u,v,w) = ui+vj+wk be the fluid velocity such that curl q $\neq 0$

$$\Omega = \Omega_x i + \Omega_y j + \Omega_z k$$

i.e. $\Omega_x, \Omega_y, \Omega_z$ are the cartesian component of Ω

$$\Omega_x i + \Omega_y j + \Omega_z k = curl\, q$$

$$\Omega_x i + \Omega_y j + \Omega_z k = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\Omega_x i + \Omega_y j + \Omega_z k = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k$$

On comparing

$$\Omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \Omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \Omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

In (2–D) $$\Omega_x i + \Omega_y j + \Omega_z k = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix}$$

$$\Omega_x i + \Omega_y j + \Omega_z k = 0i + 0j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k$$

On comparing

$$\Omega_x = 0 \quad , \quad \Omega_y = 0 \quad , \quad \Omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$\Omega = curl\, q$ is called the Vorticity velocity.

Lecture # 08

Vortex line:

Vortex line is a curve in the fluid such that tangent to it at every point is in the direction of vorticity vector.

Let $$\Omega = \Omega_x i + \Omega_y j + \Omega_z k$$

$$r = xi + yj + zk$$

be the position vector of the point P on a vortex line the $\Omega // dr$

i.e. $$\Omega \times dr = 0$$

$$\begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ dx & dy & dz \end{vmatrix} = 0$$

$$\left(\Omega_y dz - \Omega_z dy \right) i + \left(\Omega_z dx - \Omega_x dz \right) j + \left(\Omega_x dy - \Omega_y dx \right) k = 0$$

$$\Rightarrow \Omega_y dz - \Omega_z dy = 0 \quad \Rightarrow \Omega_y dz = \Omega_z dy \Rightarrow \frac{dz}{\Omega_z} = \frac{dy}{\Omega_y}$$

$$\Rightarrow \Omega_z dx - \Omega_x dz = 0 \Rightarrow \Omega_z dx = \Omega_x dz \Rightarrow \frac{dx}{\Omega_x} = \frac{dz}{\Omega_z}$$

$$\Rightarrow \Omega_x dy - \Omega_y dx = 0 \quad \Rightarrow \Omega_x dy = \Omega_y dx \quad \Rightarrow \frac{dy}{\Omega_y} = \frac{dx}{\Omega_x}$$

From above $* \dfrac{dx}{\Omega_x} = \dfrac{dy}{\Omega_y} = \dfrac{dz}{\Omega_z}$ is the equation of vortex line

Vortex tube or Vortex filament:

If we draw vortex line from each point of a close curve in the fluid we obtained a tube called a vortex tube. A vortex tube of infinitesimal cross-section is called a vortex filament.

Note: A vortex line or tube cannot terminate or originate at internal points in a fluid.

Only for close curve they can terminate boundaries.

*Motion is not rotational or irrotational when curl q = 0

* Motion is rotational when curl q $\neq$ 0

Question: If the velocity component are given as u = kx , v = 0 , w = 0 then show that the motion is not rotational.

Solution: Since q = [u,v,w]

= ui + vj + wk

Here u = kx , v = 0 , w = 0

q = kx i + 0 j + 0 k

Now curl q

$$curl\, q = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx & 0 & 0 \end{vmatrix} = 0i + 0j + 0k = 0 \qquad proved$$

Question: If $q = \left[ax^2yt\ ,\ by^2zt\ ,\ czt^2\right]$ Find the vorticity vector where a,b,c are constants.

Solution: We know that $\Omega_x, \Omega_y, \Omega_z$ are the component of vorticity vector

$$\Omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$= \frac{\partial\left(czt^2\right)}{\partial y} - \frac{\partial\left(by^2zt\right)}{\partial z}$$

$$= 0 - by^2t = -by^2t$$

$$\Omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial\left(ax^2yt\right)}{\partial z} - \frac{\partial\left(czt^2\right)}{\partial x}$$

$$= 0 - 0 = 0$$

$$\Omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\Omega_z = \frac{\partial\left(by^2zt\right)}{\partial x} - \frac{\partial\left(ax^2yt\right)}{\partial y} = -ax^2t$$

Hence the vorticity vectors are $\left[\Omega_x, \Omega_y, \Omega_z\right] = \left[-by^2t, 0, -ax^2t\right]$

Question: If $q = \dfrac{k^2\left(xj - yi\right)}{x^2 + y^2}$ where k is constant. Show that the motion is incompressible. Also find the equation of stream line and test whether the motion is of potential kind and if so determine the velocity potential.

Solution:

First, we prove the motion is incompressible

$$\nabla . q = 0$$

Given that $q = \dfrac{k^2\left(xj - yi\right)}{x^2 + y^2}$

$$q = \frac{k^2xj}{x^2 + y^2} - \frac{k^2yi}{x^2 + y^2}$$

$$q = -\frac{k^2y}{x^2 + y^2}i + \frac{k^2x}{x^2 + y^2}j$$

$$u = -\frac{k^2y}{x^2 + y^2} \quad , \quad \mathrm{v} = \frac{k^2x}{x^2 + y^2}j$$

$$\nabla . q = \frac{\partial}{\partial x}\left(-\frac{k^2y}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{k^2x}{x^2 + y^2}\right)$$

$$\nabla . q = -k^2y\left[(-1)\left(x^2 + y^2\right)^{-2}(2x)\right] + k^2x\left[(-1)\left(x^2 + y^2\right)^{-2}(2y)\right]$$

$$\nabla . q = \frac{2k^2xy}{\left(x^2 + y^2\right)^2} + \frac{-2k^2xy}{\left(x^2 + y^2\right)^2} = 0$$

That the motion is incompressible.

(i) For equation of stream line, we know that

$$\frac{dx}{u}=\frac{dy}{v}$$

$$\frac{dx}{\dfrac{-k^2y}{\left(x^2+y^2\right)}}=\frac{dy}{\dfrac{k^2x}{\left(x^2+y^2\right)}}$$

$$\frac{-\left(x^2+y^2\right)}{k^2}\frac{dx}{y}=\frac{x^2+y^2}{k^2}\frac{dy}{x}$$

$$-\frac{dx}{y}=\frac{dy}{x}$$

(ii) For potential we show $\dfrac{\partial u}{\partial y}=\dfrac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\frac{-k^2y}{x^2+y^2}\right)$$

$$=\frac{\left(x^2+y^2\right).-k^2-\left(-k^2y\right).2y}{\left(x^2+y^2\right)^2}$$

$$=\frac{-k^2x^2-k^2y^2+2k^2y^2}{\left(x^2+y^2\right)^2}=\frac{-k^2x^2+k^2y^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(\frac{k^2x}{x^2+y^2}\right)$$

$$=\frac{\left(x^2+y^2\right).k^2-\left(k^2x\right).2x}{\left(x^2+y^2\right)^2}$$

$$=\frac{k^2x^2+k^2y^2-2k^2x^2}{\left(x^2+y^2\right)^2}=\frac{-k^2x^2+k^2y^2}{\left(x^2+y^2\right)^2}$$

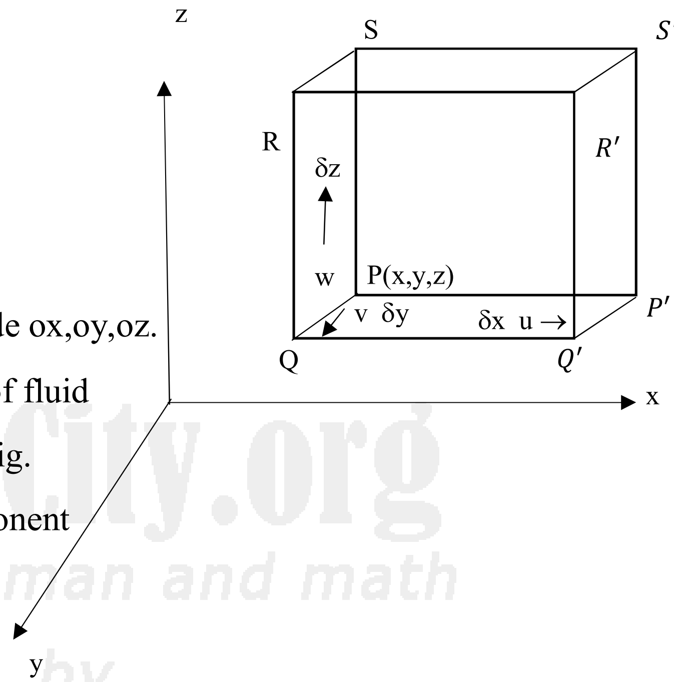
$$\Rightarrow\quad\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$$

Question: Derivation of Euler's equation OR the equation of motion of Euler for an ideal fluid or Euler's equation of motion.

OR Prove that $\rho\dfrac{Du}{Dt} = \rho X - \dfrac{\partial P}{\partial x}$

OR $\dfrac{Du}{Dt} = X - \dfrac{1}{\rho}\dfrac{\partial P}{\partial x}$

OR $\dfrac{\partial q}{\partial t} + \left(q.\nabla\right)q = F - \dfrac{1}{q}\nabla P$



Proof:

Consider a parallelepiped with side ox,oy,oz.

P is the pressure ρ is the density of fluid

P(x,y,z) is the point as shown in fig.

And u ,v ,w be the velocity component

of fluid (q = ui + vj + wk) and

X,Y,Z be the component of

External force on unit mass P = f(x,y,z)

We have force on the face PQRS in X-direction = P δyδz

= f(x,y,z) δyδz

And the force on the face $P'Q'R'S'$ in X-direction = f(x+δx , δy , δz)

as x + δx per unit time by Taylor theorem

$$= f\left(x,y,z\right) + \delta x\frac{\partial}{\partial x}f\left(x,y,z\right)$$

The total force in X-direction

= face PQRS – face $P'Q'R'S'$

$$= f\left(x,y,z\right) - \left[f\left(x,y,z\right) + \delta x\frac{\partial}{\partial x}f\left(x,y,z\right)\right]$$

$$= f\left(x,y,z\right) - f\left(x,y,z\right) - \delta x\frac{\partial}{\partial x}f\left(x,y,z\right)$$

$$= -\delta x \frac{\partial}{\partial x} f(x, y, z)$$

$$= -\delta x \frac{\partial P}{\partial x} \delta y \delta z$$

Total force in X-direction $= -\delta x \frac{\partial P}{\partial x} \delta y \delta z$

The mass of fluid is $\rho \delta x \delta y \delta z$. Hence the external force in X-direction is $= X \rho \delta x \delta y \delta z$

As we know that $\frac{Dq}{Dt}$ is the total acceleration in X-direction.

By Newton second law of motion

Equation of motion = F = ma

Sum of component of external force in X-direction

$$\rho \delta x \delta y \delta z \frac{Dq}{Dt} = X \rho \delta x \delta y \delta z + \left(-\frac{\partial P}{\partial x} \delta x \delta y \delta z \right)$$

$$\rho \frac{Dq}{Dt} = \rho X - \frac{\partial P}{\partial x} \qquad \div by\, \delta x \delta y \delta z$$

$$\frac{Dq}{Dt} = X - \frac{1}{\rho} \frac{\partial P}{\partial x} \qquad \div \; by \; \rho$$

$$\Rightarrow \quad \frac{Du}{Dt} = X - \frac{1}{\rho} \frac{\partial P}{\partial x} \qquad \_\_\_\_(i)$$

$$\Rightarrow \quad \frac{Dv}{Dt} = Y - \frac{1}{\rho} \frac{\partial P}{\partial y} \qquad \_\_\_\_(ii)$$

$$\Rightarrow \quad \frac{Dw}{Dt} = Z - \frac{1}{\rho} \frac{\partial P}{\partial z} \qquad \_\_\_\_(iii)$$

Where q = ui + vj + wk

Since $\nabla P = \frac{\partial P}{\partial x} i + \frac{\partial P}{\partial y} j + \frac{\partial P}{\partial z} k$

From (i), (ii) and (iii) we can write a combine relation

$$\frac{Du}{Dt}i+\frac{Dv}{Dt}j+\frac{Dw}{Dt}k=Xi+Yj+Zk-\frac{1}{\rho}\left(\frac{\partial P}{\partial x}i+\frac{\partial P}{\partial y}j+\frac{\partial P}{\partial z}k\right)$$

$$\frac{Dq}{Dt}=F-\frac{1}{\rho}\nabla P \quad \_\_\_\_\_(iv) \quad \because F=Xi+Yj+Zk \text{ external force}$$

We know $\frac{Dq}{Dt}=\frac{\partial q}{\partial t}+\left(q.\nabla\right)q$

Eq (iv) becomes

$$\frac{\partial q}{\partial t}+\left(q.\nabla\right)q=F-\frac{1}{q}\nabla P$$

is the required Euler equation in fluid.

Lecture # 09

Different form of Euler equation of motion:

We know that Euler equation of motion

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla P$$

Velocity of fluid   external force

*(1) Cartesian tensor form:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = F_i - \frac{1}{\rho}\frac{\partial P}{\partial x_i}$$

Where u_i is the velocity in X-direction.

(2) Cartesian form:

Let u , v ,w are the velocity component in x , y ,z direction respectively.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (q.\nabla)$$

$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

1^{st} $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

2^{nd} $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = F_y - \frac{1}{\rho}\frac{\partial P}{\partial y}$$

3^{rd} $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial P}{\partial z}$$

(3) Cylindrical Form:

Let v_r, v_θ, v_z are velocity component in r,θ , z direction respectively.

In the direction of ‘r’

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_\theta\frac{\partial v_r}{\partial \theta} - v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta}{r^2} = F_r - \frac{1}{\rho}\frac{\partial P}{\partial r}$$

In the direction of 'θ'

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+v_r\frac{v_\theta}{r}=F_\theta-\frac{1}{\rho}\frac{\partial P}{\partial \theta}$$

In the direction of 'z'

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}=F_z-\frac{1}{\rho}\frac{\partial P}{\partial z}$$

(4) Spherical form:

Let v_r, v_θ, v_ϕ are velocity component in r,θ , ϕ direction respectively.

In the direction of 'r'

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2+v_\phi^2}{r}=F_r-\frac{1}{\rho}\frac{\partial P}{\partial r}$$

In the direction of 'θ'

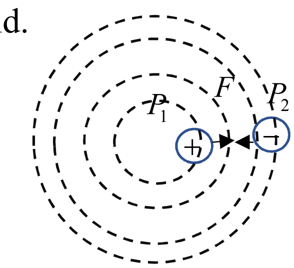
$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}-\frac{\cot\theta}{r}v_\phi^2=F_\theta-\frac{1}{\rho}\frac{\partial P}{\partial \theta}$$

In the direction of 'ϕ'

$$\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\theta}{r}-\cot\theta v_\theta v_\phi=F_\phi-\frac{1}{r\rho\sin\phi}\frac{\partial P}{\partial \phi}$$

Conservative field:

A field in which total word done is zero is called conservative field.



A work done by a force field F is moving a particle from P_1 to P_2 then the force field is said to be conservative.

In such case following statements are equivalent.

*(i) $\int_{P_1}^{P_2} f.dr$ is the independent of path joining any two point P_1 and P_2

*(ii) $\oint f.dr=0$ around a closed path c passing through points P_1 and P_2

* (iii) $F=-\nabla V=-\nabla\phi$

*(iv) $\nabla \times F = 0$

*(v) $F.dr = F_1dx + F_2dy + F_3dz$

is an exact differential equation.

If any of these equations is hold then the field is conservative.

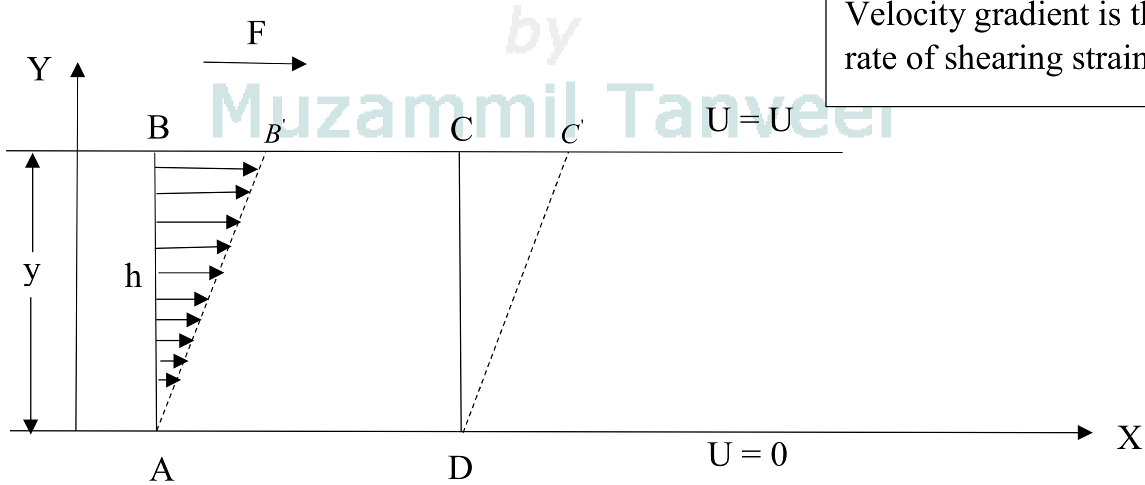
Newton’s law of viscosity or $\tau \propto \frac{du}{dy}$

1st Statement: The viscosity of fluid is a measure of its resistance to a deformation بگاڑ i.e. resistance to a shearing or tangential force when the fluid is in motion.

2nd Statement: Newton law of viscosity state that shearing force τ is directly proportional to $\frac{du}{dy}$, where $\frac{du}{dy}$ is the velocity gradient.

Proof: Consider the motion of fluid between two very large parallel plates. One of which is at rest and the other is moving with constant velocity ‘U’ parallel to itself under the action of a constant force F causes the fluid element occupying the space ABCD to form $AB'C'D$

Velocity gradient is the rate of shearing strain.



Let the distance between two plates be h. *The pressure being constant through out the fluid. This show that velocity at lower plate is zero and at upper plate is ‘U’. We know that the velocity field is one dimensional. Again, the velocity ‘u’ is a function of y only i.e. u = u (y) _____(1)

Further the velocity distribution in the field between these two plates is linear.

$$u = ay + b \qquad \text{\_\_\_\_(ii)}$$

$$u(y) = ay + b \qquad \_\_\_\_(iii) \because by\,(i)$$

By applying the boundary conditions eq (iii) when y = 0 , u(y) = 0

$\Rightarrow \quad 0 = 0 + b \quad \Rightarrow \; b = 0$

Put he value of b in (iii)

$$u(y) = ay + 0$$

$$u(y) = ay \qquad \_\_\_\_(iv)$$

When $\quad y = h \;,\quad u(y) = U(y)$

$$U(y) = ah \qquad \_\_\_\_(v)$$

$$U = ah \qquad \because \; U(y) = U \; by \; (i)$$

$$a = \frac{U}{h} \qquad \text{put in (iv)}$$

$$u(y) = \frac{U}{h}y$$

The rate of change of U in the direction $\perp$ to U is

$$\frac{du}{dy} = \frac{U}{h} \qquad \_\_\_\_(vi)$$

The ratio $\frac{U}{h}$ is the rate of angular deformation of the fluid.

Force F is directly proportional to area (A)

$$F \propto AU \qquad \because \; U \text{ is velocity}$$

And $\quad F \propto \frac{1}{h}$

By combining

$$F \propto \frac{AU}{h}$$

$$\frac{F}{A} \propto \frac{U}{h}$$

The force F exerts sheering stress on the fluid between the plates. Here we take

$$\frac{F}{A} = \tau$$

$$\Rightarrow \qquad \tau \propto \frac{U}{h}$$

From (vi) $$\tau \propto \frac{du}{dy}$$

Or $$\tau = \mu \frac{du}{dy}$$

Where μ is called the dynamic viscosity is the proportionality constant between shearing stress and velocity gradient.

This law can be written as $\mu = \dfrac{\tau}{\frac{du}{dy}} = \dfrac{Shearing\ stress}{Velo\text{city gradient}}$

This relation can be regarded as the definition of viscosity. Viscosity μ is a scalar quantity. Viscosity depend upon the nature of the fluid. Its value is small for thin fluid (such as water, milk etc) and is very large for viscous fluid (such as grease, honey etc)

Note:

The viscosity of fluid is very important property in analysis of liquid behavior and fluid motion near the solid boundary. The fluid which obeys the Newton law of viscosity is known as Newtonian fluid. The role of temperature in fluid (viscosity) was neglected by Newton law of viscosity.

Lecture # 10

Bernoulli's Equation for unsteady irrotational flow under conservative forces:

Proof: We know that Euler equation of motion is

$$\frac{\partial V}{\partial t}+\left(V\ .\nabla\right)V=F-\frac{1}{\rho}\nabla P \qquad \_\_\_\_\_(i)$$

For vector analysis

$$\nabla\left(A.B\right)=\left(A.\nabla\right)B+\left(B.\nabla\right)A+A\times\left(\nabla\times B\right)+B\times\left(\nabla\times A\right)$$

For A = B = V

$$\nabla\left(\mathrm{V}.V\right)=\left(V.\nabla\right)V+\left(\mathrm{V}.\nabla\right)V+V\times\left(\nabla\times V\right)+V\times\left(\nabla\times V\right)$$

$$\nabla V^2=2\left(V.\nabla\right)V+2V\times\left(\nabla\times V\right)$$

$$\nabla V^2-2V\times\left(\nabla\times V\right)=2\left(V.\nabla\right)V$$

$$\left(V.\nabla\right)V=\frac{1}{2}\nabla V^2-V\times\left(\nabla\times V\right)$$

Eq (i) becomes

$$\frac{\partial V}{\partial t}+\frac{1}{2}\nabla V^2-V\times\left(\nabla\times V\right)=F-\frac{1}{\rho}\nabla P \qquad \_\_\_\_\_(ii)$$

For irrotational flow

$$\nabla\times V=0\ ,\ V=-\nabla\rho$$

Also, from external forces

$$F=-\nabla V$$

Put in (ii) $\Rightarrow \quad \frac{\partial}{\partial t}\left(-\nabla\rho\right)+\frac{1}{2}\nabla V^2-V\times\left(0\right)=-\nabla V-\frac{1}{\rho}\nabla P$

$$\Rightarrow \quad -\nabla\frac{\partial\rho}{\partial t}+\frac{1}{2}\nabla V^2=-\nabla V-\frac{1}{\rho}\nabla P$$

$$\Rightarrow \quad -\nabla\frac{\partial\rho}{\partial t}+\frac{1}{2}\nabla V^2+\nabla V+\frac{1}{\rho}\nabla P\ =0$$

Taking dot product with dr

$$-\nabla\frac{\partial\rho}{\partial t}.dr+\frac{1}{2}\nabla V^2.dr+\nabla V.dr+\frac{1}{\rho}\nabla P.dr\ =0 \qquad \_\_\_\_\_\_(iii)$$

As $\nabla V.dr=\left(\frac{\partial V}{\partial x}i+\frac{\partial V}{\partial y}j+\frac{\partial V}{\partial z}k\right).\left(dxi+dyj+dzk\right)=\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz$

$$\nabla V.dr=dV$$

Similarly, $\nabla\frac{\partial\rho}{\partial t}.dr=d\frac{\partial\rho}{\partial t}$

$$\nabla\frac{1}{2}V^2.dr=d\frac{1}{2}V^2$$

$$\nabla P.\mathrm{dr}=\mathrm{dP}$$

Put in (iii) $\Rightarrow\ -d\frac{\partial\rho}{\partial t}+d\frac{1}{2}V^2+dV+\frac{1}{\rho}dP\ =0$

On integration $-\int d\frac{\partial\rho}{\partial t}+\int d\frac{1}{2}V^2+\int dV+\frac{1}{\rho}\int dP\ =\int 0$

$$-\frac{\partial\rho}{\partial t}+\frac{1}{2}V^2+V+\frac{1}{\rho}P=F\left(t\right)$$

Where F(t) is an arbitrary function of t. F(t) has same value through the entire flow. F(t) is Bernoulli or pressure equation irrotational and in viscous flow. This equation holds for both incompressible and compressible flow.

Case-I: If the fluid is incompressible then the above equation is

$$-\frac{\partial\rho}{\partial t}+\frac{1}{2}V^2+V+\frac{1}{\rho}P=F\left(t\right)$$

The equation holds for unsteady irrotational in viscous and incompressible/compressible flow.

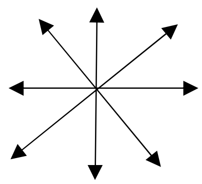
Case-II: If the motion is steady the $\frac{\partial\rho}{\partial t}=0$

$\Rightarrow\qquad \frac{1}{2}V^2+V+\frac{1}{\rho}P=F\left(t\right)$ This equation holds for unsteady, irrotational, in viscous and incompressible/compressible flow.

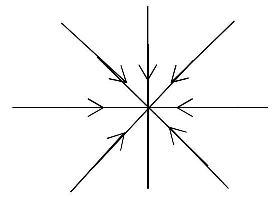
Sources and Sinks:

If the motion of the fluid consists of symmetrical redial motion in all directions proceeding from a point is called a simple source.

However, a flow is such that the flow is directed inward to a point from all directions in a symmetrical manner. The point is called simple sink.



Sources



Sinks

Sources and sinks in 2-D:

In two dimensions a source of strength ‘m’ is such that the flow across any small surrounding is $2\pi m$.

Sink is regarded as source of strength ‘−m’. Consider a circle of radius ‘r’ with source at its center the radial velocity 'V_r' is given as

$$V_r = \frac{-1}{r}\frac{\partial \psi}{\partial \theta} = \frac{-\partial \phi}{\partial r} \quad \_\_\_\_(i)$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$$

The flow across a circle is $2\pi r V_r$

$$\because \quad 2\pi r V_r = 2\pi m$$

$$r V_r = m$$

$$\Rightarrow \quad V_r = \frac{m}{r}$$

$$\Rightarrow \quad \frac{-1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{r}$$

$$\Rightarrow \quad \frac{\partial \psi}{\partial \theta} = -m$$

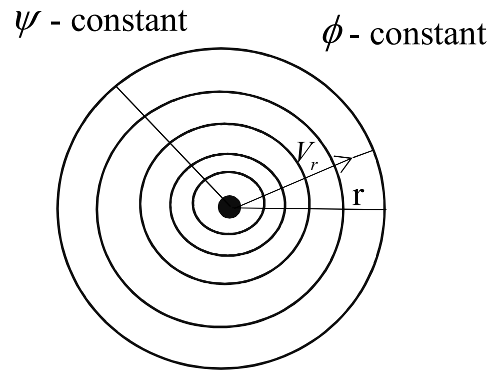
On integration $\Rightarrow \quad \psi = -m\theta \qquad (ii)$

Again from (i)

$$-\frac{\partial \phi}{\partial r} = V_r$$

$$-\frac{\partial \phi}{\partial r} = \frac{m}{r}$$

$$\frac{\partial \phi}{\partial r} = -\frac{m}{r}$$



On integration $\Rightarrow$ $\phi = -m \log r$

Complex Potential due to a source:

Let there be a source of strength m at the origin then

$$w = \phi + i\psi$$

$$w = -m \log r - im\theta$$

$$w = -m\left(\log r + i\theta\right)$$

$$w = -m\left(\log r + \log e^{i\theta}\right)$$

$$w = -m \log re^{i\theta}$$

$$w = -m \log \mathrm{z} \qquad \text{where} \quad \mathrm{z} = \mathrm{re}^{i\theta}$$

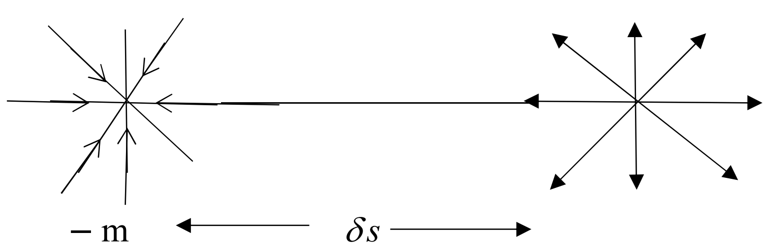
If the source is at point z' then $w = -m \log\left(z - z'\right)$

In general, the source of strength $m_1, m_2, \mathrm{m}_3, ...m_n$ stated at the point $\mathrm{z}_1, \mathrm{z}_2, \mathrm{z}_3, ...\mathrm{z}_n$

$$w = -m_1 \log\left(z - z_1\right) - m_2 \log\left(z - z_2\right)......m_n \log\left(z - z_n\right)$$

Doublet:

A combination of source of strength 'm' and sink of strength '–m' at a distance δs apart

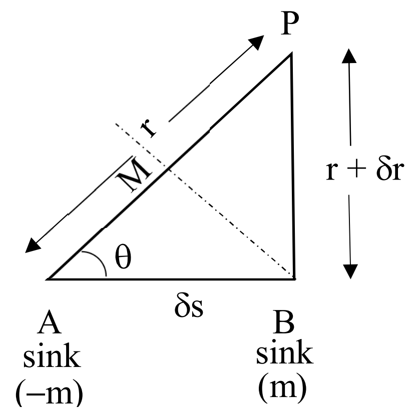


Such that m approaches to infinity and $\delta s \to 0$

The product $m\delta s$ remain finite and is equal to $\mu \Rightarrow (m\delta s = \mu)$ is called Doublet of strength μ. The line δs is called the axis of doublet.

Complex Potential due to doublet in 2-D:

Let A and B denote the position of a sink and source of strength '–m' , 'm' and P be any point.



Let AP = r

BP = r + δr

∠PAB = θ

Let ϕ be the velocity potential due to doublet.

Then $\phi = m\log r - m\log(r+\delta r)$

$= -m[\log(r+\delta r) - \log r]$

$$= -m\left[\log\left(\frac{r+\delta r}{r}\right)\right]$$

$$= -m\left[\log\left(1+\frac{\delta r}{r}\right)\right]$$

$$\phi = -m\,\frac{\delta r}{r} \qquad \_\_\_\_(i) \;\because by\;\; first\ approximation$$

Let BM be the perpendicular drawn from B to AP, then we get

AM = AP – PM

$\because AP = r \;,\; PM \cong PB \cong r+\delta r$

$AM = r-(r+\delta r)$

$AM = -\delta r$

From figure

$$\cos\theta = \frac{AM}{AB} = \frac{-\delta r}{\delta s} \qquad \Rightarrow \delta r = -\delta s\cos\theta$$

Eq (1) becomes

$$\phi = -m\left(\frac{-\delta s\cos\theta}{r}\right) \qquad \Rightarrow \qquad \phi = \frac{m\delta s\cos\theta}{r} \qquad \_\_\_\_\_\_(ii)$$

We know that

$$m\delta s = \mu$$

$$\phi = \frac{\mu\cos\theta}{r} \quad \_\_\_\_\_(iii)$$

$$\Rightarrow \frac{\partial\phi}{\partial r} = \frac{-\mu\cos\theta}{r^2}$$

$$Since \ \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\Rightarrow \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{-\mu\cos\theta}{r^2}$$

$$\Rightarrow \frac{\partial\psi}{\partial\theta} = -\frac{\mu}{r}\cos\theta$$

$$\psi = -\frac{\mu}{r}\sin\theta + c \qquad \_\_\_\_\_(iv) \ \because on \ \text{integration}$$

Let if we put c = 0

$$\psi = -\frac{\mu}{r}\sin\theta$$

Therefore, complex potential due to doublet is given as

$$w = \phi + i\psi$$

$$w = \frac{\mu\cos\theta}{r} + i\left(-\frac{\mu}{r}\sin\theta\right)$$

$$w = \frac{\mu}{r}[\cos\theta - i\sin\theta]$$

$$w = \frac{\mu}{r}e^{-i\theta}$$

$$w = \frac{\mu}{re^{i\theta}} = \frac{\mu}{z} \qquad \because z = re^{i\theta}$$

Note:

(i) Equipotential curve:

$$\phi = \text{constant}$$

$$\frac{\mu\cos\theta}{r} = cons\tan t = c$$

$$\frac{\mu r\cos\theta}{r^2} = c$$

$$r\cos\theta = c\frac{r^2}{\mu}$$

$$r\cos\theta = c' r^2 \qquad \because\ c' = \frac{c}{\mu}$$

$$x = c'\left(x^2 + y^2\right) \quad \because x = r\cos\theta\ , r^2 = x^2 + y^2$$

Which is circle touching Y-axis at (0,0).

(ii) Stream lines:

$$\psi = \text{constant}$$

$$-\frac{\mu\sin\theta}{r} = cons\tan t = c$$

$$-\frac{\mu r\sin\theta}{r^2} = c$$

$$r\sin\theta = -c\frac{r^2}{\mu}$$

$$r\sin\theta = c' r^2 \qquad \because\ c' = -\frac{c}{\mu}$$

$$y = c'\left(x^2 + y^2\right) \quad \because y = r\sin\theta\ , r^2 = x^2 + y^2$$

Which is circle touching X-axis at (0,0).

(iii) If the doublet makes an angle α with X-axis then we have $\theta-\alpha$ for θ. So, that

$$w = \frac{\mu}{z} = \frac{\mu}{re^{i(\theta-r)}}$$

$$w = \frac{\mu}{re^{i\theta}.e^{-ir}} = \frac{\mu e^{ir}}{re^{i\theta}} = \frac{\mu e^{ir}}{z}$$

If doublet is at $A'(x', y')$ *where* $z' = x' + iy'$ *we have*

$$w = \frac{\mu e^{i\alpha}}{z - z'}$$

(iv) If doublets are of strength $\mu_1, \mu_2, \mu_3, ...\mu_n$ and are situated at $z_1, z_2, z_3, ...z_n$ then complex potential is

$$w = \frac{\mu_1 e^{i\alpha_1}}{z - z_1} + \frac{\mu_2 e^{i\alpha_2}}{z - z_2} + ... + \frac{\mu_n e^{i\alpha_n}}{z - z_n}$$

Where $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are angles with X-axis.

Example: What arranges of sources and sinks will give arise to the function

$$w = \log\left(z - \frac{a^2}{z}\right)$$

Draw rough sketch of stream line where ψ = constant. Prove that the stream lines are subdivided into circle of radius a (r = a) and the axis of y.

Solution: Given that the function $w = \log\left(z - \frac{a^2}{z}\right)$

$$w = \log\left(\frac{z^2 - a^2}{z}\right) = \log\left(\frac{(z-a)(z+a)}{z}\right)$$

$$w = \log(z-a) + \log(z+a) - \log z \qquad \_\_\_\_(i)$$

This implies that there are two sources of strength '–1' and two sinks of strength '1' at z = a , z = –a and a source of unit strength at z = 0.

Now $\quad w = \phi + i\psi \quad \_\_\_\_(ii) \qquad \because z = x + iy \quad , \log z = \log|z| + i\, Arg\, z$

Compare (i) and (ii)

$$\phi + i\psi = \log(z-a) + \log(z+a) - \log z$$

$$\phi + i\psi = \log(x + iy - a) + \log(x + iy + a) - \log(x + iy)$$

$$\phi + i\psi = \log[(x-a) + iy] + \log[(x+a) + iy] - \log(x + iy)$$

$$\phi+i\psi=\log\sqrt{(x-a)^2+y^2}+i\tan^{-1}\left(\frac{y}{x-a}\right)+\log\sqrt{(x+a)^2+y^2}+i\tan^{-1}\left(\frac{y}{x+a}\right)-\log\sqrt{x^2+y^2}-i\tan^{-1}\left(\frac{y}{x}\right)$$

For stream lines

$$\psi=\tan^{-1}\left(\frac{y}{x-a}\right)+\tan^{-1}\left(\frac{y}{x+a}\right)-\tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi=\tan^{-1}\left(\frac{\frac{y}{x-a}+\frac{y}{x+a}}{1-\frac{y}{x-a}.\frac{y}{x+a}}\right)-\tan^{-1}\left(\frac{y}{x}\right)\quad\because\tan^{-1}A+\tan^{-1}B=\tan^{-1}\frac{A+B}{1-AB}$$

$$\psi=\tan^{-1}\left(\frac{\frac{y(x+a)+y(x-a)}{x^2-a^2}}{\frac{x^2-y^2-a^2}{x^2-a^2}}\right)-\tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi=\tan^{-1}\left(\frac{xy+ay+xy-ay}{x^2-y^2-a^2}\right)-\tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi=\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)-\tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi=\tan^{-1}\left(\frac{\frac{2xy}{x^2-y^2-a^2}-\frac{y}{x}}{1+\frac{2xy}{x^2-y^2-a^2}.\frac{y}{x}}\right)\quad\because\tan^{-1}A-\tan^{-1}B=\tan^{-1}\frac{A-B}{1+AB}$$

$$\psi=\tan^{-1}\left(\frac{\frac{2xy-y\left(x^2-y^2-a^2\right)}{\left(x^2-y^2-a^2\right)x}}{\frac{\left(x^2-y^2-a^2\right)x+2xy^2}{\left(x^2-y^2-a^2\right)x}}\right)$$

$$\psi=\tan^{-1}\left(\frac{2xy-y\left(x^2-y^2-a^2\right)}{\left(x^2-y^2-a^2\right)x+2xy^2}\right)$$

$$\psi = \tan^{-1}\left(\frac{2x^2y - x^2y + a^2y + y^3}{x^3 - a^2x - xy^2 + 2xy^2}\right)$$

$$\psi = \tan^{-1}\left(\frac{x^2y + a^2y + y^3}{x^3 - a^2x + xy^2}\right)$$

For stream lines ψ = constant

$$\tan^{-1}\left(\frac{x^2y + a^2y + y^3}{x^3 - a^2x + xy^2}\right) = \cos n \tan t = c$$

$$\frac{x^2y + a^2y + y^3}{x^3 - a^2x + xy^2} = \tan c = c$$

$$\frac{y\left(x^2 + y^2 + a^2\right)}{x\left(x^2 + y^2 - a^2\right)} = c$$

Here arises two cases c = 0 , c = ∞

(i) For c = 0

$$\frac{y\left(x^2 + a^2 + y^2\right)}{x\left(x^2 + y^2 - a^2\right)} = 0$$

$$y\left(x^2 + a^2 + y^2\right) = 0$$

$$y = 0\ ,\quad x^2 + a^2 + y^2 = 0$$

$$x^2 + y^2 = -a^2$$

$$r^2 = -a^2$$

The X-axis is a stream line.

(ii) For c = ∞

$$\frac{y\left(x^2 + a^2 + y^2\right)}{x\left(x^2 + y^2 - a^2\right)} = \infty = \frac{1}{0}$$

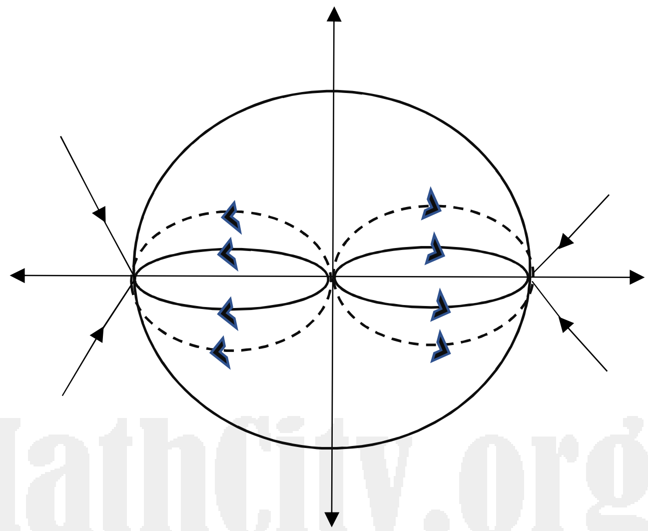
$$x\left(x^2 + y^2 - a^2\right) = 0$$

$$x = 0\ ,\ x^2 + y^2 - a^2 = 0$$

$$x^2 + y^2 = a^2$$

$$r^2 = a^2$$
$$\Rightarrow \quad r = a$$

Y-axis is a stream line.



Lecture # 12

Some definition

(i) Stokes stream function:

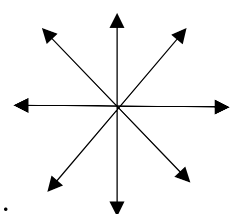
In fluid dynamics, the stokes stream function is used to describe the streamlines and flow velocity in a three-dimensional incompressible flow with axis symmetry. A surface with a constant value of the stokes stream function encloses a stream tube, everywhere tangential to the flow velocity vectors.

$$w_r = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\mu_\phi \sin\theta\right) - \frac{\partial \mu_\theta}{\partial\phi}\right)\hat{r}$$

$$w_\theta = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \mu_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r\mu_\phi\right)\right)\hat{\theta}$$

$$w_\phi = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\mu_\theta\right)\right)\hat{\phi}$$

(ii) Line sources and Line sinks:



Two dimensional sources and sinks is generated by a line source. Coincident with the axis stream lines of the flow generated by dipole line. Source coincident with the axis and aligned along the axis.

(iii) Axisymmetric flow:

Using cylindrical coordinates (r,θ,z) where r = 0 is the axis of axisymmetric flow and $\left(\mu_r, \mu_\theta, \mu_z\right)$ are the velocities in those (r,θ,z) directions the continuity equation is $$\frac{1}{r}\frac{\partial\left(r\mu_r\right)}{\partial r} + \frac{\partial \mu_z}{\partial z} = 0$$

And this allows the definition of another stream function, ψ known as Stokes stream function (different from the stream function used in plan flow) defined as

$$*\begin{cases} u_z = \dfrac{1}{r}\dfrac{\partial\psi}{\partial r} \\ u_r = -\dfrac{1}{r}\dfrac{\partial\psi}{\partial z} \end{cases}$$

And whose definition automatically assures that the continuity equation is satisfied.

(iv) Line doublets:

We know that $$w = \frac{\mu e^{i\alpha}}{z - z_0}$$

If $\alpha = 0$ then the line sources are on X-axis and thus

$$w = \frac{\mu}{z - z_0}$$

If there are number of line doublets of strengths $\mu_1, \mu_2, \mu_3, ...\mu_n$ per unit length with line sinks at points $z_1, z_2, z_3, ...z_n$ and their axis being inclined at angles $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ with the positive direction of X-axis, then the complex potential is given by

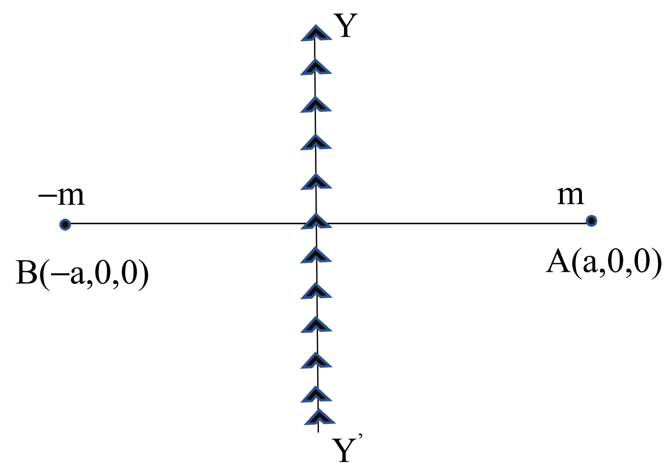
$$w = \mu_1 \frac{e^{i\alpha_1}}{z - z_1} + \mu_2 \frac{e^{i\alpha_2}}{z - z_2} + ... + \mu_n \frac{e^{i\alpha_n}}{z - z_n}$$

Images in rigid infinite plane and solid spheres:

(i) Images in rigid infinite plane:

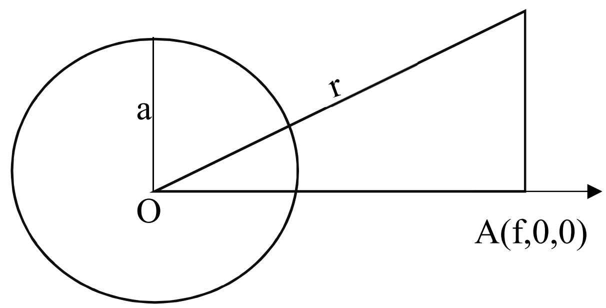
Consider a simple source of strength 'm' and sink of '−m' situated at A(0,0) and B(−a,0,0) at a distance 'a' from an infinite plane yy'.

We shall know that appropriate image system for this is as shown in the figure.



(ii) Image of a source in a sphere:

Suppose a source of strength 'm' is situated at point. A at a distance f(>a) from the center of the sphere of radius 'a'.



Example What arranges of sources and sinks will give arise to the function

$$w = m\log\left(z - \frac{1}{z}\right)$$

Draw rough sketch of stream line where ψ = constant. Prove that the stream lines are subdivided into circle of radius a (r = a) and the axis of y.

Solution: Given that the function $w = m\log\left(z - \frac{1}{z}\right)$

$$w = m\log\left(\frac{z^2 - 1}{z}\right) = m\log\left(\frac{(z-1)(z+1)}{z}\right)$$

As $\quad w = m\log(z-1) + m\log(z+1) - m\log(z-0)$

$Put \quad z = x + iy \quad and \quad w = \phi + i\psi$

$$\phi + i\psi = m\log(x + iy - 1) + m\log(x + iy + 1) - m\log(x + iy)$$

$$\phi + i\psi = m\log(x + iy - 1) + m\log(x + iy + 1) - m\log(x + iy)$$

$$\phi + i\psi = m\log[(x-1) + iy] + m\log[(x+1) + iy] - m\log(x + iy)$$

$$\phi + i\psi = m\left[\log\sqrt{(x-1)^2 + y^2} + i\tan^{-1}\left(\frac{y}{x-1}\right)\right] + m\left[\log\sqrt{(x+1)^2 + y^2} + i\tan^{-1}\left(\frac{y}{x+1}\right)\right] - m\left[\log\sqrt{x^2 + y^2} + i\tan^{-1}\left(\frac{y}{x}\right)\right]$$

For stream lines ψ comparing imaginary parts

$$\psi = m\tan^{-1}\left(\frac{y}{x-1}\right) + m\tan^{-1}\left(\frac{y}{x+1}\right) - m\tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi = m\left[\tan^{-1}\left(\frac{y}{x-1}\right) + \tan^{-1}\left(\frac{y}{x+1}\right)\right] - m\tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi = m\tan^{-1}\left(\frac{\frac{y}{x-1} + \frac{y}{x+1}}{1 - \frac{y}{x-1}.\frac{y}{x+1}}\right) - m\tan^{-1}\left(\frac{y}{x}\right) \quad \because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$$

$$\psi = m\tan^{-1}\left(\frac{\frac{y(x+1)+y(x-1)}{(x-1)(x+1)}}{\frac{(x-1)(x+1)-y^2}{(x-1)(x+1)}}\right) - \tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi = m\tan^{-1}\left(\frac{xy+y+xy-y}{x^2-1-y^2}\right) - \tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi = m\tan^{-1}\left(\frac{2xy}{x^2-y^2-1}\right) - \tan^{-1}\left(\frac{y}{x}\right)$$

$$\psi = m\tan^{-1}\left(\frac{\frac{2xy}{x^2-y^2-1} - \frac{y}{x}}{1 + \frac{2xy}{x^2-y^2-1}.\frac{y}{x}}\right) \quad \because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}$$

$$\psi = m\tan^{-1}\left(\frac{\frac{2xy - y(x^2-y^2-1)}{(x^2-y^2-1)x}}{\frac{(x^2-y^2-1)x+2xy^2}{(x^2-y^2-1)x}}\right)$$

$$\psi = m\tan^{-1}\left(\frac{2xy - y(x^2-y^2-1)}{(x^2-y^2-1)x+2xy^2}\right)$$

$$\psi = m\tan^{-1}\left(\frac{2x^2y - x^2y + y + y^3}{x^3 - x - xy^2 + 2xy^2}\right)$$

$$\psi = m\tan^{-1}\left(\frac{y\left(x^2+y^2+1\right)}{x\left(x^2+y^2-1\right)}\right)$$

For stream lines ψ = constant

$$m\tan^{-1}\left(\frac{y\left(x^2+y^2+1\right)}{x\left(x^2+y^2-1\right)}\right) = \cos n\tan t = c$$

$$\frac{y\left(x^2+y^2+1\right)}{x\left(x^2+y^2-1\right)} = c$$

Here arises two cases c = 0 , c = ∞

(i) For c = 0

$$\frac{y\left(x^2+y^2+1\right)}{x\left(x^2+y^2-1\right)} = 0$$

$$y\left(x^2+y^2+1\right) = 0$$

$$y = 0\ ,\quad x^2+y^2+1 \neq 0$$

Stream line on X-axis

(ii) For c = ∞

$$\frac{y\left(x^2+y^2+1\right)}{x\left(x^2+y^2-1\right)} = \infty = \frac{1}{0}$$

$$x\left(x^2+y^2-1\right) = 0$$

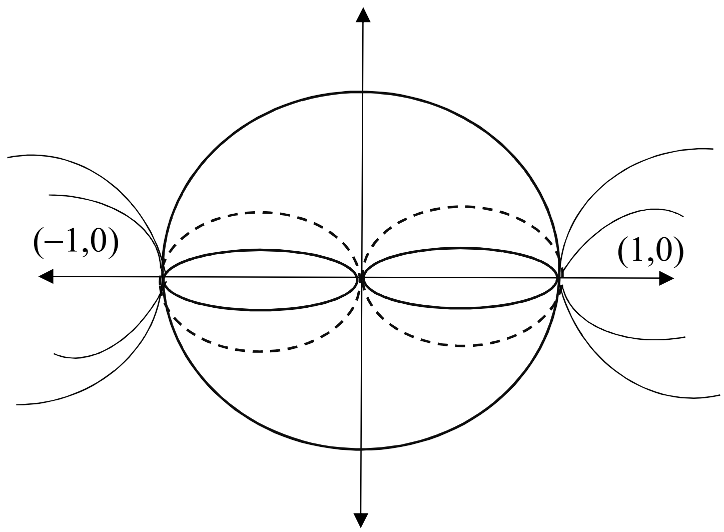
$$x = 0\ ,\ x^2+y^2-1 = 0$$

$$x^2+y^2 = 1$$

$$r^2 = 1$$

$$\Rightarrow \quad r = \pm 1$$

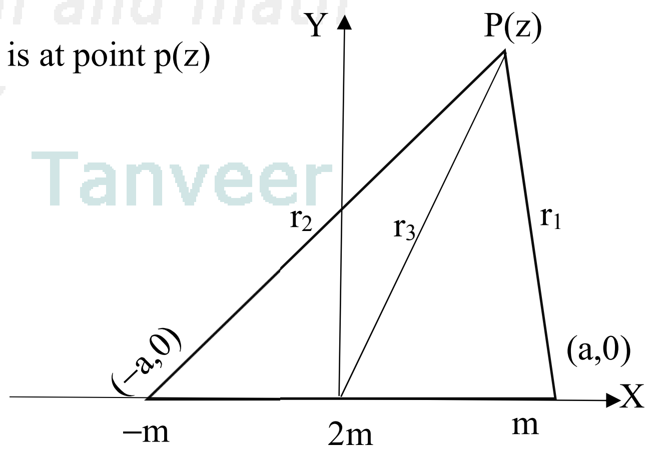
The stream line is on Y-axis



Example: Two sources each of strength ‘m’ are placed at the points (–a,0), (a,0) and a sink of strength ‘2m’ is placed at (0,0). Show that stream line are curves where λ is parameter. Show that the fluid speed at any point is

$$q = \frac{2ma^2}{r_1 r_2 r_3} \ ; where \ r_1 = |z-a| \ , r_2 = |z+a| \ , r_3 = |z|$$

Solution: Complex velocity potential w is at point p(z)



$$w = m\log(z-a) + m\log(z+a) - 2m\log z$$

$$w = m\left[\log(z-a) + \log(z+a)\right] - m\log z^2$$

$$w = m\log(z^2 - a^2) - m\log z^2$$

$$z = x + iy \ , \ and \ \ w = \phi + i\psi$$

$$\phi + i\psi = m\log(x^2 - y^2 + 2ixy - a^2) - m\log(x^2 - y^2 + 2ixy)$$

$$\phi + i\psi = m\log\left[\left(x^2 - y^2 - a^2\right) + i\left(2xy\right)\right] - m\log\left[\left(x^2 - y^2\right) + i\left(2xy\right)\right]$$

$$\phi+i\psi=m\left[\log\sqrt{\left(x^2-y^2-a^2\right)^2+\left(2xy\right)^2}+i\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)\right]-m\left[\log\sqrt{\left(x^2-y^2\right)^2+\left(2xy\right)^2}+i\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)\right]$$

For ψ $\qquad \psi = m\tan^{-1}\left(\frac{2xy}{x^2 - y^2 - a^2}\right) - m\tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$

$$\psi = m\left[\tan^{-1}\left(\frac{2xy}{x^2 - y^2 - a^2}\right) - \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)\right]$$

$$\psi = m\tan^{-1}\left[\frac{\frac{2xy}{x^2 - y^2 - a^2} - \frac{2xy}{x^2 - y^2}}{1 + \frac{2xy}{x^2 - y^2 - a^2}.\frac{2xy}{x^2 - y^2}}\right]$$

$$\psi = m\tan^{-1}\left[\frac{\frac{2xy\left(x^2 - y^2\right) - 2xy\left(x^2 - y^2 - a^2\right)}{\left(x^2 - y^2 - a^2\right)\left(x^2 - y^2\right)}}{\frac{\left(x^2 - y^2 - a^2\right)\left(x^2 - y^2\right) + 4x^2y^2}{\left(x^2 - y^2 - a^2\right)\left(x^2 - y^2\right)}}\right]$$

$$\psi = m\tan^{-1}\left[\frac{2xy\left(x^2 - y^2 - x^2 + y^2 + a^2\right)}{x^4 - x^2y^2 - x^2y^2 + y^4 - a^2x^2 + a^2y^2 + 4x^2y^2}\right]$$

$$\psi = m\tan^{-1}\left[\frac{2xy\left(a^2\right)}{x^4 + y^4 + 2x^2y^2 - a^2\left(x^2 + y^2\right)}\right]$$

$$\psi = m\tan^{-1}\left[\frac{2xy\left(a^2\right)}{\left(x^2 + y^2\right)^2 - a^2\left(x^2 + y^2\right)}\right]$$

$$\psi = m\tan^{-1}\left(\frac{2}{\lambda}\right) \quad where\ \frac{1}{\lambda} = \frac{xya^2}{\left(x^2 + y^2\right) - a^2\left(x^2 - y^2\right)}$$

$$a^2xy\lambda = \left(x^2 + y^2\right) - a^2\left(x^2 - y^2\right)$$

$$a^2xy\lambda + a^2\left(x^2 - y^2\right) = \left(x^2 + y^2\right)$$

$$\left(x^2+y^2\right)=a^2\left[x^2-y^2+\lambda xy\right]$$

Now $$w=m\log(z-a)+m\log(z+a)-2m\log z$$

$$\frac{dw}{dz}=\frac{m}{z-a}+\frac{m}{z+a}-\frac{2m}{z}$$

$$\frac{dw}{dz}=\left[\frac{z(z+a)+z(z-a)-2(z-a)(z+a)}{(z)(z-a)(z+a)}\right]m$$

$$\frac{dw}{dz}=\left[\frac{z^2+az+z^2-az-2z^2+2a^2}{(z)(z-a)(z+a)}\right]m$$

$$\frac{dw}{dz}=\frac{2ma^2}{(z)(z-a)(z+a)}$$

$$q=\frac{dw}{dz}=\frac{2ma^2}{(z)(z-a)(z+a)}\ or\ q=\frac{2ma^2}{|z-a||z+a||z|}$$

$$q=\frac{2ma^2}{r_1r_2r_3}\ ;\ where\ r_1=|z-a|\ ,\ r_2=|z+a|\ ,\ r_3=|z|$$

Theorem: State and prove theorem of Blasius.

Proof: In a steady irrotational two dimensional flow given by the complex potential w = f(z) if the pressure force on the fixed cylindrical surface C and is represented by a force (X,Y) and a couple of Moment M about the origin of co-ordinates then neglecting the external forces

$$X-iY=\frac{i\rho}{2}\int_C\left(\frac{dW}{dz}\right)^2dz$$

$$\text{M = real part of}\left[-\frac{\rho}{2}\int_C z\left(\frac{dW}{dz}\right)^2dz\right]$$

Where ρ is the density of the fluid.

Proof: Let ds be an element of arc at point P(x,y) and the tangent at P makes an angle θ with x-axis. The pressure at P(x,y) is Pds, P is the pressure unit length. Pds acts along the inward normal to the cylindrical surface and its components along the co-ordinates along the co-ordinates axes are

Pds cos(90+θ) , Pds cosθ

i.e − Pds sinθ , Pds cosθ

The pressure at the element ds is

dF = dX + i dY

= − Pds sinθ +iPds cosθ

= I P (cosθ + i sinθ) ds

Pds sinθ along negative x-axis ⇒ −Pds sinθ along positive x-axis

$$dF = iP\left(\frac{dx}{ds} + i\frac{dy}{ds}\right)ds \quad \because \cos\theta = \frac{dx}{ds} \ , \ \sin\theta = \frac{dy}{ds}$$

$$dF = iP\left(dx + idy\right)ds \quad = iPdz \quad \_\_\_\_(i)$$

The pressure equation in the absence of external forces is

$$\frac{P}{\rho} + \frac{1}{2}q^2 = constant = k$$

$$\text{Or} \quad P = -\frac{1}{2}\rho q^2 + k \quad \_\_\_\_\_(ii)$$

$$Further \ \frac{dW}{dz} = -u + iv = -q\cos\theta + iq\sin\theta$$

$$\frac{dW}{dz} = -q\left(\cos\theta - i\sin\theta\right) = -qe^{i\theta} \quad \_\_\_\_(iii)$$

And $$dx + idy = dz = \left(\frac{dx}{ds} + i\frac{dy}{ds}\right)ds = \left(\cos\theta + i\sin\theta\right)ds = e^{i\theta}ds \quad \_\_\_(iv)$$

The pressure on the cylinder is obtained by integrating (i) Therefore

$$F = \int_C iPdz = \int_C i\left(k - \frac{1}{2}\rho q^2\right)dz$$

$$F = \frac{-i\rho}{2}\int_C q^2 dz \quad \because \int_C ik = 0$$

$$X + iY = \frac{-i\rho}{2}\int_C q^2 e^{i\theta} ds \quad by\ (iv)$$

From here $X - iY = \frac{i\rho}{2}\int_C q^2 e^{-i\theta} ds \quad \Rightarrow X - iY = \frac{i\rho}{2}\int_C \left(q^2 - e^{-2i\theta}\right) e^{i\theta} ds$

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dW}{dz}\right)^2 dz \quad using\,(iii)\ \&\,(iv)$$

The moment M is given by $M = \int_C \left|\vec{r} \times \vec{F}\right|$

$$\vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ x & y & 0 \\ -Pds\sin\theta & Pds\cos\theta & 0 \end{vmatrix} = (0-0)i - (0-0)j + \left(xPds\cos\theta + yPds\sin\theta\right)k$$

$$\vec{r} \times \vec{F} = k\left[xPds\cos\theta + yPds\sin\theta\right] \quad \Rightarrow \quad \vec{r} \times \vec{F} = k\left[xP(dx) + yP(dy)\right]$$

$$\left|\vec{r} \times \vec{F}\right| = xP(dx) + yP(dy)$$

$$M = \int_C \left|\vec{r} \times \vec{F}\right| = \int_C \left[(Pdy)y + (Pdx)x\right]$$

$$M = \int_C \left[\left(k - \frac{1}{2}\rho q^2\right)(xdx + ydy)\right]$$

$$M = \int_C d\left[\frac{1}{2}\left(x^2 + y^2\right)\right] - \frac{\rho}{2}\int_C \left[q^2\left(xdx + ydy\right)\right]$$

$$M = -\frac{\rho}{2}\int_C \left[q^2\left(xdx + ydy\right)\right] \ \because 1st\ integral\ vanishes$$

$$M = -\frac{\rho}{2}\int_C \left[q^2\left(x\cos\theta + y\sin\theta\right)ds\right] \because dx = \cos\theta ds,\ dy = \sin\theta ds$$

$$M = \text{Re}\,al\ part\ of\left[-\frac{\rho}{2}\int_C \left[q^2\left(x + iy\right)\left(\cos\theta + y\sin\theta\right)ds\right]\right]$$

$$M = \text{Real part of}\left[-\frac{\rho}{2}\int_C \left[q^2 z e^{-i\theta} ds\right]\right]$$

$$M = \text{Real part of}\left[-\frac{\rho}{2}\int_C \left[z\left(q^2 e^{-2i\theta}\right)e^{i\theta} ds\right]\right]$$

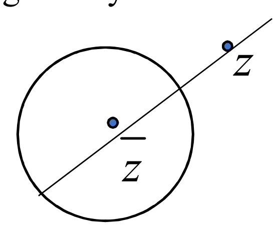
$$M = \text{Real part of}\left[-\frac{\rho}{2}\int_C \left[z\left(\frac{dW}{dz}\right)^2 e^{i\theta} ds\right]\right]$$ Hence theorem is proved.

Theorem: State and prove Milne-Thomson Circle theorem.

Proof: Let f(z) be the complex potential function for a flow having no rigid boundaries and such that there are no singularities with in the circle $|z| = a$.

Then on on introducing a solid circular cylinder with impermeable into the flow the new complex potential for the fluid outside the cylinder is given by

$$w = f(z) + \overline{f}\left(a^2 / z\right),\ |z| \geq a$$



Solution: Let C be the cross section of the cylinder with equation $|z| = 1$. Therefore, on the circle C

$|z| = a$, $z\overline{z} = a^2$, $\overline{z} = a^2 / z$ where $\overline{z}$ is the image of z w.r.t the circle. If z is outside of the circle then $\overline{z} = a^2 / z$ is inside the circle. Further all the singularities of f(z) lie out the circle and the singularities of $f\left(a^2 / z\right)$ and therefore those of $\overline{f}\left(a^2 / z\right)$ lie inside C. Therefore $\overline{f}\left(a^2 / z\right)$ introduce no singularities outside the cylinder. Thus, the function f(z) and $f(z) + \overline{f}\left(a^2 / z\right)$ both have the same singularities outside C. Therefore, the condition satisfied by f(z) in the absence of cylinder are satisfied by $f(z) + \overline{f}\left(a^2 / z\right)$ in the presence of cylinder. Further the complex potential after the intersection of the cylinder $|z| = a$ is $w = f(z) + \overline{f}(\overline{z}) \quad \Rightarrow \quad w = f(z) + \overline{f(z)}$

w = a purely real quantity

But we know that $\quad w = \phi + i\,\psi \quad$ It follows that $\psi = 0$

This proves that the circular cylinder $|z| = a$ is a stream line i.e C is a stream line. Therefore, the new complex potential justifies the fluid motion and hence the circle theorem.