Fluid Mechanics II

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Dedicated

To

My Honorable Teacher

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&

My Parents

Lecture # 01

Fluid:

A fluid is a substance that deform continuously under the application of sheer stress (tangential stress). No matter how small or large the sheer stress.

Examples: Water, milk, oil, jam, lipstick etc.

Stress:

Forcer per unit area (F/A) is called stress. It is denoted by τ. It has two types

(i) Sheer stress / Tangential stress
(ii) Normal Stress

Sheer stress:

Tangent component of force per unit area is called sheer stress.

Normal stress:

Normal component of force per unit area is called Normal stress.

Types of forces:

There are two types of forces

(i) Surface force
(ii) Body force

Surface force:

All the force acting on the boundaries of medium through direct contact. **OR** Force per unit area is called surface force.

The surface force is due to the surrounding fluid on the element under consideration.

Examples: pressure, stress etc.

Body force: All the force develops without physical contact. **OR** Force per unit volume (element of the body) is called body force. The body forces are distributed throughout the volume of the body. Example: gravitational force, magnetic field etc.

Element:

Element is a part of substance that has all the specification of that substance.

Types of fluid:

Newtonian and Non-Newtonian fluid:

If fluid satisfy the Newton's law of viscosity is called Newtonian fluid otherwise called Non-Newtonian fluid.

$$\tau \propto \frac{du}{dy}$$

$$\tau = \mu \frac{du}{dy}$$

Flow:

The quantity of fluid passing through a point per unit time is called flow.

Density:

Mass per unit volume is called density.

Viscosity:

It is the measure of resistance against the motion of fluid. It is denoted by μ. It is also called absolute viscosity and dynamic viscosity.

Kinematic viscosity:

It is the ratio of absolute viscosity to density. It is denoted as $\eta(Eta)$

$$\eta = \frac{\mu}{\rho}$$

Compressibility:

Compressibility is the measure of change in fluid w.r.t volume and density under the action of external forces.

Compressible fluid:

A type of fluid in which change occur due to volume and density changes by the action of pressure (temperature) is called compressible fluid.

Examples: gases.

Incompressible fluid:

A type of fluid in which no change occur due to volume and density changes by the action of pressure (temperature) is called incompressible fluid.

Ideal fluid:

A fluid that have zero viscosity and incompressible is called ideal fluid.

*An incompressible and inviscid fluid are called ideal fluid,

Viscous fluid:

Fluid that have non-zero viscosity or finite viscosity and can exert sheer stress on the surface is called viscous fluid or real fluid.

Inviscid fluid:

Fluid having zero viscosity is called inviscid fluid.

Steady flow:

A type of flow in which velocity of any other fluid property does not change with time.

$$\frac{\partial \rho}{\partial t}=0 \ , \ \frac{\partial P}{\partial t}=0 \ , \ \frac{\partial V}{\partial t}=0$$

Unsteady flow:

A type of flow in which velocity of any other fluid property change with time.

$$\frac{\partial \rho}{\partial t}\neq 0 \ , \ \frac{\partial P}{\partial t}\neq 0 \ , \ \frac{\partial V}{\partial t}\neq 0$$

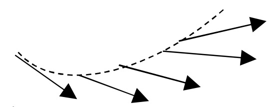
Rotational flow:

A type of flow in which fluid particle rotate about their own axis is called rotational or rotating flow.

Irrotational flow:

A type of flow in which fluid particle does not rotate about their own axis is called irrotational flow.

Stream lines:



The imaginary line drawn in the fluid where the velocity along the tangent.

Potential line:

If we draw the line joining the points of equipotential on the adjacent flow lines, we get potential lines.

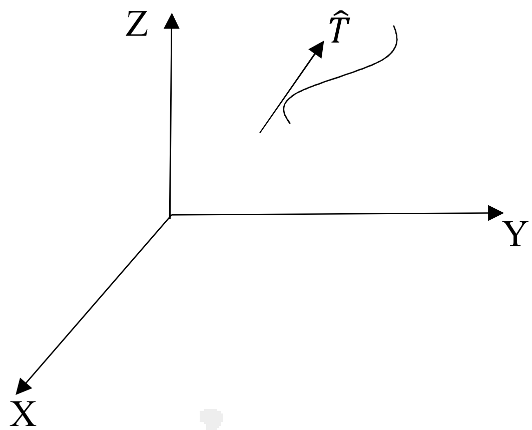
Laminar and Turbulent flow:

A type of flow in which stream line does not cross each other is called Laminar flow otherwise called turbulent flow.

Lecture # 02

Stream lines:

A curve drawn in the fluid such that tangent to every point of it is in the direction of fluid velocity



Steady flow:

The flow does not change with time.

Stream lines have same pattern at all points.

Unsteady flow:

Flow pattern changes with time. Stream line changes from point to point.

Differential Equations of stream lines:

Since the tangent drawn at every point in the fluid motion is in the direction of its velocity. So,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\vec{V} \times \frac{d\vec{r}}{dx} = 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$(vdz - wdy)\hat{i} - (udz - wdx)\hat{j} + (udy - vdx)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

By comparing on both sides

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$ is the equation of stream line.

Vortex motion:

The most general displacement of a fluid involves rotation such that the rotational vector (vortex vector or vorticity) $\xi=\nabla\times q\neq 0 \;\; or \;\; \xi=Curl q\neq 0$ $where \;\; \xi(Xi)$.

Vorticity vector:

Let $\vec{q}=u\hat{i}+v\hat{j}+w\hat{k}$ be the fluid velocity such that $Curl\vec{q}\neq 0$ then

$$\xi=\nabla\times\vec{q} \qquad vorticity\ vector$$

Let $\xi=\xi_x\hat{i}+\xi_y\hat{j}+\xi_z\hat{k}$ i.e. ξ_x,ξ_y,ξ_z are the cartesian components of $\vec{\xi}$

Then $\xi_x\hat{i}+\xi_y\hat{j}+\xi_z\hat{k}=Curl\vec{q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}$

$$\xi_x i+\xi_y j+\xi_z k=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)i+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)j+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)k$$

On comparing

$$\xi_x=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\;\; \xi_y=\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\;\; \xi_z=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

In two dimensions cartesian coordinates vorticity is given as

$$\xi_x\hat{i}+\xi_y\hat{j}+\xi_z\hat{k}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & 0\end{vmatrix} \qquad \Rightarrow \qquad \xi_z=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\vec{k}$$

In polar coordinates $\qquad \xi_z = \frac{1}{r}V_\theta + \frac{\partial}{\partial r}V_\theta - \frac{1}{r}\frac{\partial V_r}{\partial \theta}$

Vortex line:

Vortex line is a curve in the fluid such that tangent to it at every point is in the direction of vorticity vector.

$\xi = \xi_x \hat{i} + \xi_y \hat{j} + \xi_z \hat{k}$ & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of the point P on the vortex line.

Then $\vec{\xi} \,//\, d\vec{r}$ i.e $\vec{\xi} \times d\vec{r} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \xi_x & \xi_y & \xi_z \\ dx & dy & dx \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\left(\xi_y dz - \xi_z dy\right) = 0 \quad , \qquad \left(\xi_x dz - \xi_z dx\right) = 0 \quad , \qquad \left(\xi_x dy - \xi_y dx\right) = 0$$

$\frac{dx}{\xi_x} = \frac{dy}{\xi_y} = \frac{dz}{\xi_z}$ gives the equation of vortex line.

Vortex tube or Vortex filament:

Vortex tube is a bundle of vortex lines. If we draw vortex lines from each point of a closed curve in the fluid, we obtain a tube called a vortex tube.

A vortex tube of infinitesimal cross section is called a vortex filament.

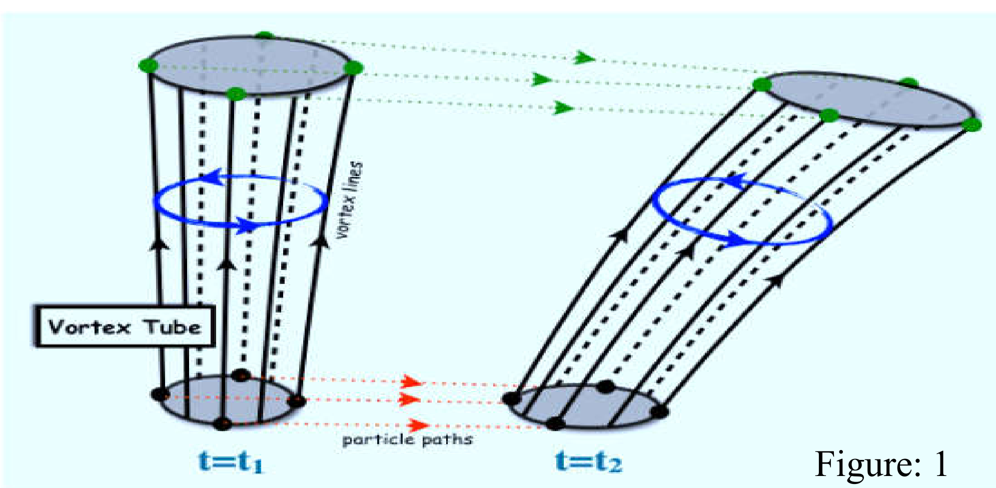


Figure: 1

Figure 1 shows the evolution of a vortex tube.

***Note**: A vortex line or tube cannot terminate or originate at internal points in a fluid. Only for closed curves. They can terminate on boundaries.

★**Question:** If the velocity components are given as u = kx , v = 0 , w = 0

Then show that the motion is not rotational.

Solution: q = [u , v , w] $\Rightarrow$ $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

Here u = kx , v = 0 , w = 0

$$Curl\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx & 0 & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}(kx)\right)\hat{k} = 0$$

The motion is irrotational.

Question: If $\vec{q} = \left[ax^2yt\ , by^2zt, \text{czt}^2\right]$. Find the vorticity vector where a,b,c are constants.

Solution: We know that ξ_x, ξ_y, ξ_z are the cartesian components of vorticity vector.

$$\xi_x = \left(\frac{\partial czt^2}{\partial y} - \frac{\partial by^2zt}{\partial z}\right),\ \xi_y = \left(\frac{\partial ax^2yt}{\partial z} - \frac{\partial czt^2}{\partial x}\right),\ \xi_z = \left(\frac{\partial by^2zt}{\partial x} - \frac{\partial ax^2yt}{\partial y}\right)$$

$$\xi_x = -by^2t\ ,\ \xi_y = 0\ ,\ \xi_z = -ax^2t$$

The vorticity vector is $\left[-by^2t\ ,0, -ax^2t\right]$

Circulation:

If C is a closed curve, then **circulation** about C is given by

$$\Gamma = \oint_C \vec{q}.\vec{dr} = \int_S \hat{n}.curlq dS = \int_S \hat{n}.\vec{\xi}\,dS = \int_S \vec{\xi} d\vec{S}$$

*The quantity $\left|\hat{n}.\vec{\xi}\right| dS$ is called the strength of the vortex tube.

A vortex tube with a unit strength is called a unit vortex tube.

Different types of Vortices:

(i) Forced vortex:

In this type the fluid rotates as a rigid body with constant angular velocity.

(ii) Free cylindrical vortex:

In this type the fluid moves along streamlines which are concentric circles in horizontal planes and there is no variation of total energy with radius.

(iii) Free spiral vortex:

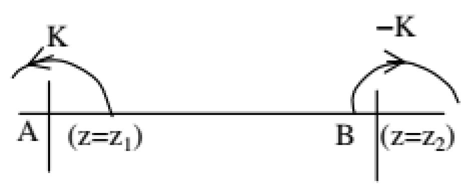
In this type there is a combination the free cylindrical vortex and a source (radial flow).

(iv) Compound vortex:

In this type the fluid rotates as a forced vortex at the centre and as a free vortex.

Vortex pair:

A pair of vortices of equal and opposite strengths is called a vortex pair.



Let K and −K be the strengths of the two vortices at A ($z = z_1$) and B ($z = z_2$) respectively. Then the complex potential is

$$W = iK \log (z-z_1) - iK \log (z-z_2)$$

The velocity at A is due to the presence of the vortex at B and vice-versa.

Vortex Rows:

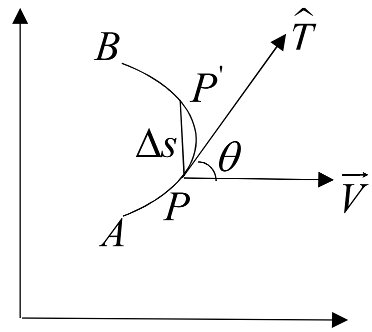
When a body moves slowly through a liquid, rows of vortices are sometimes formed. There vortices can, when stable, be photographed.

Here we consider infinite system of parallel line vortices and two-dimensional flow will be presumed throughout.

Lecture # 03

Flow along a curve:

Let A and B be any two points in the fluid and ABP curve or path joining them lying entirely within the fluid divide the curve ABP into number of small elements.



Let P *and* P' be an element of the curve of length Δs.

Let $\vec{V}$ be the velocity vector and $\hat{T}$ is the flow along the element PP' is defined as the product of tangential component of velocity vector $\vec{V}$ with the length Δs of the element PP'.

$$\text{Flow along } PP' = \left(\vec{V}.\hat{T}\right)$$

$$\text{Flow along } ABP = \lim_{\Delta s\to 0}\sum\left(\vec{V}.\hat{T}\right)\Delta s$$

$$\text{Flow along } ABP = \int_A^B\left(\vec{V}.\hat{T}\right)ds \quad \_\_\_(i)$$

If θ is angle between $\vec{V}$ and $\hat{T}$ then equation (i) becomes

$$\text{Flow along } ABP = \int_A^B\left|\vec{V}\right|\left|\hat{T}\right|\cos\theta\, d\theta$$

$$\text{Flow along } ABP = \int_A^B V\cos\theta\, d\theta \quad \_\_\_(ii)$$

$$\text{Since } \hat{T} = \frac{dr}{ds} \qquad \because \text{ by differential geometry}$$

$$dr = \hat{T}ds \qquad \_\_\_\_\_(iii)$$

Put (iii) in (i)

In general, we can write as

$$Flow\ along\ ABP = \int_A^B \vec{V}.dr \quad \_\_\_(iv)$$

$$Since\ \vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\vec{V}.dr = udx + vdy + wdz$$

$$Flow\ along\ ABP = \int_A^B udx + vdy + wdz$$

Question: The velocity components are $u = x^2 y$, $v = x^2 - y^2$. Find the flow along y = $3x^2$ and y = 3x where $0 \le x \le 1$, $0 \le y \le 3$.

Solution: Given that $u = x^2 y$, $v = x^2 - y^2$

(a) $y = 3x^2 \Rightarrow dy = 6xdx$

$$Flow\ along\ AB = \int_A^B \vec{V}.dr$$

$$Flow\ along\ AB = \int_A^B udx + vdy$$

$$Flow\ along\ AB = \int_A^B \left\{x^2 ydx + \left(x^2 - y^2\right)dy\right\}$$

$$Flow\ along\ AB = \int_A^B x^2\left(3x^2\right)dx + \left(x^2 - 9x^4\right)\left(6xdx\right)$$

$$Flow\ along\ AB = \int_0^1 3x^4 dx + \left(6x^3 - 54x^5\right)dx$$

$$Flow\ along\ AB = \int_0^1 \left(3x^4 + 6x^3 - 54x^5\right)dx$$

$$Flow\ along\ AB = \left(3\frac{x^5}{5} + 6\frac{x^4}{4} - 54\frac{x^6}{6}\right)_0^1$$

$$Flow\ along\ AB = \left(\frac{3}{5} + \frac{6}{4} - \frac{54}{6}\right) - 0 = \frac{3}{5} + \frac{3}{2} - 9$$

$$Flow\ along\ AB = \frac{6+15-90}{10} = \frac{69}{10}$$

(b) y = 3x $\Rightarrow$ dy = 3dx

$$Flow\ along\ AB = \int_A^B \vec{V}.dr$$

$$Flow\ along\ AB = \int_A^B udx + vdy$$

$$Flow\ along\ AB = \int_0^1 \left\{x^2 ydx + \left(x^2 - y^2\right)dy\right\}$$

$$Flow\ along\ AB = \int_0^1 x^2\left(3x\right)dx + \left(x^2 - 9x^2\right)\left(3dx\right)$$

$$Flow\ along\ AB = \int_0^1 \left(3x^3 - 24x^2\right)dx$$

$$Flow\ along\ AB = \left(3\frac{x^4}{4} - 24\frac{x^3}{3}\right)_0^1$$

$$Flow\ along\ AB = \left(\frac{3}{4} - \frac{24}{3}\right) - 0 = \frac{3}{4} - 8 = \frac{3-32}{4}$$

$$Flow\ along\ AB = \frac{-29}{4}$$

Circulation:

The circulation of the fluid along the simple closed curve lying entirely within the fluid is denoted by Γ and is defined as the line integral of tangential component of velocity taken along close curve C.

*Circulation is the measure of rotation of the fluid.

$$\Gamma = \oint_C \vec{V}.dr = \oint \vec{V}.\hat{T}ds = \oint \vec{V}\cos\theta ds$$

Circulation of circuit is equal to the sum of circulation of its sub circuit.

$$\Gamma C = \Gamma C_1 + \Gamma C_2$$

From here we can define the relationship between vorticity and circulation as

$$\Gamma = \oint_C \vec{V}.dr = \iint_S \left(\nabla \times \vec{V}\right) ds \quad \because \left(By\ Stoke's\ Theorem\right)$$

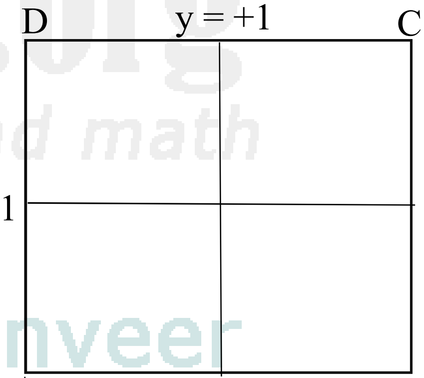
$$where\ vorticity\ = \nabla \times \vec{V}$$

Question: The velocity component for a certain flow field are given by

$$u = x+y\ ,\ \ v = x^2 - y$$

Calculate the circulation around the squares enclosed by the lines $x = \pm 1$, $y = \pm 1$

Solution: The square enclosed by the lines $x = \pm 1$, $y = \pm 1$ as shown in figure.



The circulation around this square is given by

$$\Gamma = \oint_{ABCDA} \vec{V}.dr = \oint_{ABCDA} u\,dx + v dy$$

$$\Gamma = \int (x+y)dx + \left(x^2 - y\right)dy$$

$$Since\ \ (x+y)dx + \left(x^2 - y\right)dy = \alpha$$

$$\Gamma = \int_{AB} \alpha + \int_{BC} \alpha + \int_{CD} \alpha + \int_{DA} \alpha \qquad \_\_\_\_\_(i)$$

Circulation around straight-line AB. So, x varies from −1 to 1.

$$\Gamma = \int_{AB} \alpha = \int_{AB} (x+y)dx + \left(x^2 - y\right)dy$$

$$\because\ y = -1 \Rightarrow dy = 0$$

$$= \int_{-1}^{1} (x-1)dx + 0 = \left|\frac{x^2}{2} - x\right|_{-1}^{1}$$

$$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{2} + 1\right) = \frac{-1}{2} - \frac{-3}{2} = -2$$

Circulation along straight-line BC. So, y varies from −1 to 1.

$$\Gamma = \int_{BC} \alpha = \int_{BC} (x+y)dx + (x^2 - y)dy$$

$$\because x = 1 \Rightarrow dx = 0$$

$$= \int_{-1}^{1} 0 + (1-y)dy = \left| y - \frac{y^2}{2} \right|_{-1}^{1}$$

$$= \left(1 - \frac{1}{2}\right) - \left(-1 - \frac{1}{2}\right) = \frac{1}{2} + \frac{3}{2} = 2$$

Circulation around straight-line CD. So, x varies from 1 to −1.

$$\Gamma = \int_{CD} \alpha = \int_{CD} (x+y)dx + (x^2 - y)dy$$

$$\because y = 1 \Rightarrow dy = 0$$

$$= \int_{1}^{-1} (x-1)dx + 0 = \left| \frac{x^2}{2} - x \right|_{1}^{-1}$$

$$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{2} + 1\right) = \frac{-1}{2} - \frac{-3}{2} = -2$$

Circulation along straight-line DA. So, y varies from 1 to −1.

$$\Gamma = \int_{DA} \alpha = \int_{DA} (x+y)dx + (x^2 - y)dy$$

$$\because x = -1 \Rightarrow dx = 0$$

$$= \int_{1}^{-1} 0 + (1-y)dy = \left| y - \frac{y^2}{2} \right|_{1}^{-1}$$

$$= \left(-1 - \frac{1}{2}\right) - \left(1 - \frac{1}{2}\right) = -\frac{3}{2} - \frac{1}{2} = -2$$

Put in (i) $\quad \Gamma = \int_{ABCDA} \vec{V}.dr = -2 + 2 - 2 - 3 = -4 \quad$ ____(A)

Verification: *Since* $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=2x-1$ By stokes theorem

$$\Gamma=\oint_C \vec{V}.dr=\iint_S\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy$$

$$\Gamma=\int_{-1}^{1}\int_{-1}^{1}(2x-1)dxdy=\int_{-1}^{1}\left(2\frac{x^2}{2}-x\right)_{-1}^{1}dy$$

$$\Gamma=\int_{-1}^{1}\left(x^2-x\right)_{-1}^{1}dy=\int_{-1}^{1}(1-1)-(1+1)dy$$

$$\Gamma=2\int_{-1}^{1}dy=-2|y|_{-1}^{1}=-2(1+1)$$

$$\Gamma=-4 \quad \_\_\_\_(B)$$

From (A) and (B)

$$\Gamma=\oint_C \vec{V}.dr=\iint_S\left(\nabla\times\vec{V}\right)ds$$

★**Question:** The circle u = 3x+y , v = 2x−3y with parametric equation as

x = 1+2cosθ , y = 6+2sinθ

Calculate the circulation around the circle.

Solution: Given that u = 3x+y , v = 2x−3y

x = 1+2cosθ , y = 6+2sinθ

dx = −2sinθdθ , dy = 2cosθdθ

The circulation around the circle is given by

$$\Gamma=\oint\vec{V}.dr=\int udx+vdy$$

$$\Gamma=\int(3x+y)dx+(2x-3y)dy$$

$$\Gamma = \int_0^{2\pi} (3+6\cos\theta+6+2\sin\theta)(-2\sin\theta d\theta)+(2+4\cos\theta-18-6\sin\theta)(2\cos\theta d\theta)$$

$$\Gamma = \int_0^{2\pi} (9+6\cos\theta+2\sin\theta)(-2\sin\theta d\theta)+(-16+4\cos\theta-6\sin\theta)(2\cos\theta d\theta)$$

$$\Gamma = \int_0^{2\pi} \left(-18\sin\theta-12\sin\theta\cos\theta-4\sin^2\theta-32\cos\theta+8\cos^2\theta-12\sin\theta\cos\theta\right)d\theta$$

$$\Gamma = \int_0^{2\pi} \left(-18\sin\theta-24\sin\theta\cos\theta-4\sin^2\theta-32\cos\theta+8\cos^2\theta\right)d\theta$$

$$\Gamma = \int_0^{2\pi} \left(-18\sin\theta-12\sin2\theta-4\left(\frac{1-\cos2\theta}{2}\right)-32\cos\theta+8\left(\frac{1+\cos2\theta}{2}\right)\right)d\theta$$

$$\Gamma = \left|-18\cos\theta-12\frac{\cos2\theta}{2}-2\left(\theta-\frac{\sin2\theta}{2}\right)-32\sin\theta+4\left(\theta+\frac{\sin2\theta}{2}\right)\right|_0^{2\pi}$$

$$\Gamma = \{18+6-2(2\pi-0)-32(0)+4(2\pi+0)\}-\{18+6-0-0+0\}$$

$$\Gamma = 18+6-4\pi+8\pi-18-6$$

$$\Gamma = 4\pi$$

Kelvins Theorem: (For rotation or circulation) or State and prove Kelvins theorem for circulation:

Statement:

For an inviscid (non-viscous) incompressible fluid circulation around any closed curve C moving fluid constants at all times provided that the central forces remain conserved.

Proof:

Let C be the closed curve in fluid such that the curve moves with the fluid so that at all instant circulation consist of same fluid particle. Circulation is defined as

$$\Gamma = \oint \vec{V}.dr$$

To prove that circulation is constant it is sufficient to show $\frac{D\Gamma}{Dt}=0$

Now $$\frac{D\Gamma}{Dt}=\frac{D}{Dt}\oint\vec{V}.dr=\oint\frac{D}{Dt}\left(\vec{V}.dr\right)$$

$$\frac{D\Gamma}{Dt}=\oint\vec{V}.\frac{D}{Dt}\left(dt\right)+dr.\frac{D\vec{V}}{Dt} \qquad \_\_\_\_\_(i)$$

$$Since \quad \frac{D}{Dt}\left(dr\right)=d\left(\frac{Dr}{Dt}\right)=dV \quad \because\left(Bernoulli\ equation\right)$$

$$Similarly\ V.\frac{D}{Dt}\left(dr\right)=V.dr=\frac{1}{2}d\left(\vec{V}.\vec{V}\right)=d\left(\frac{1}{2}V^2\right) \qquad \_\_\_\_(ii)$$

Using equation (ii) in (i)

$$\frac{D\Gamma}{Dt}=\int d\left(\frac{1}{2}V^2\right)+dr.\frac{D\vec{V}}{Dt} \qquad \_\_\_\_(iii)$$

From Euler's equation of motion

$$\frac{DV}{Dt}=F-\frac{1}{\rho}\nabla P \qquad \_\_\_\_(iv)$$

As we know forces are conservative.

$$F=-\nabla\Omega \qquad \_\_\_\_\_\_\_(v)$$ Where Ω is force potential.

Using (v) in (iv)

$$\frac{DV}{Dt}=-\nabla\Omega-\frac{1}{\rho}\nabla P \qquad \_\_\_\_(vi)$$

By taking dot product of equation (vi) with dr

$$\frac{DV}{Dt}.dr=-\nabla\Omega.dr-\frac{1}{\rho}\nabla P.dr \qquad \_\_\_\_(vii)$$

$$\Rightarrow \nabla\Omega.dr=\left(\frac{\partial\Omega}{\partial x}\hat{i}+\frac{\partial\Omega}{\partial y}\hat{j}+\frac{\partial\Omega}{\partial z}\hat{k}\right).\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)$$

$$\nabla\Omega.dr = \frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy + \frac{\partial\Omega}{\partial z}dz$$

$$\nabla\Omega.dr = d\Omega$$

$$\textit{Similarly} \quad \nabla P.\mathrm{dr} = \mathrm{dP}$$

Equation (vii) becomes

$$dr.\frac{DV}{Dt} = -d\Omega - \frac{1}{\rho}dPd = -d\Omega - d\left(\frac{P}{\rho}\right) \quad \_\_\_\_(viii)$$

Since fluid is incompressible i.e. ρ = constant

Using equation (viii) in (iii)

$$\frac{D\Gamma}{Dt} = \int\left(d\left(\frac{1}{2}V^2\right) - d\Omega - d\left(\frac{P}{\rho}\right)\right)$$

$$\frac{D\Gamma}{Dt} = \int d\left(\frac{1}{2}V^2 - \Omega - \frac{P}{\rho}\right)$$

Since V, P and ρ are constant. Therefore, their derivative will also zero.

$$\frac{D\Gamma}{Dt} = \oint d(\text{constant}) = \oint 0 = 0$$

$\Rightarrow$ Γ is constant. Hence circulation remains constant.

Lecture # 04

Remark:

K.E for finite liquid is $K.E = \frac{1}{2}\iint_S \rho\phi \frac{\partial \phi}{\partial n} dS$

The velocity potential is $\vec{V} = -\nabla\phi$

As q = (u,v,w)

$\Rightarrow \quad q = -\nabla\phi = u = v = w$

Acyclic:

Acyclic motion is defined as the irrotational motion in which velocity potential is single valued (as the rectilinear flow of fluid).

Theorem:

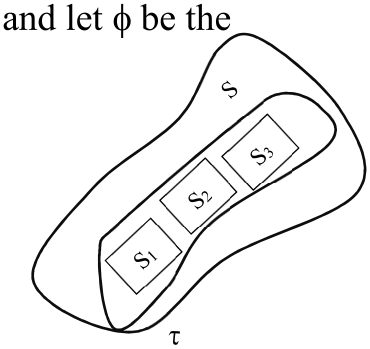
Show that acyclic irrotational motion is impossible in a finite volume of fluid bounded by rigid surfaces at rest

OR

In infinite fluid at rest at infinity and bounded internally by rigid bodies at rest.

Proof:

If possible, suppose that acyclic irrotational motion is possible and let ϕ be the velocity potential. Then, K.E. of the fluid is



$$K.E = T = \frac{\rho}{2}\iiint_\tau \nabla^2\phi d\tau$$

$$\frac{\rho}{2}\iiint_\tau \nabla^2\phi d\tau = \frac{\rho}{2}\iint_S \phi\frac{\partial\phi}{\partial n} dS \qquad \_\_\_\_(i)$$

Where S is the sum of all the rigid boundaries when τ is finite or the sum of internal rigid boundaries when τ is infinite.

Now, since the boundaries are rigid, then at every point of S, the normal velocity is zero i.e. $\frac{\partial\phi}{\partial n} = 0 \qquad \_\_\_\_(ii)$ at each point of S.

From (i) and (ii) we get

$$\frac{\rho}{2}\iiint_{\tau}\nabla^2\phi d\tau = 0$$

$$\Rightarrow \iiint_{\tau}\nabla^2\phi d\tau = 0$$

$$\Rightarrow \iiint_{\tau} q^2 d\tau = 0 \qquad \because q = -\nabla\phi$$

$$\Rightarrow q^2 = 0$$

$$\Rightarrow q = 0$$

Fluid is at rest. Hence there is no motion of fluid. Hence Acyclic irrotational motion is impossible.

Corollary:

If the solid boundaries in motion are instantaneously brought to rest, show that the motion of the fluid will instantaneously cease to be irrotational.

Proof:

If possible, assume that the motion remains irrotational, then the K.E. is given by

$$T = \frac{\rho}{2}\iiint_{\tau} q^2 d\tau = \frac{\rho}{2}\iint_{S}\phi\frac{\partial\phi}{\partial n}dS \qquad \_\_\_\_(i)$$

When the surface S (solid boundary) is brought to rest instantaneously, then q = 0 at each point of S then

$$\because q = 0 \;\; then \quad -\nabla\phi = 0$$

$\Rightarrow$ ϕ = constant at each point of S and

$$\frac{\partial\phi}{\partial n} = 0 = \text{constant at each point of S}$$

Since q = 0 in τ i.e. there is no motion. Thus, the motion is no longer irrotational.

★ Uniqueness Theorem:

If the region occupied by the fluid is finite, then only one irrotational motion of the fluid exists when the boundaries have prescribed velocities.

OR

Show that there cannot be two different forms of acyclic irrotational motion of a given liquid whose boundaries have prescribed velocities.

Proof:

If possible, let ϕ_1 and ϕ_2 be two different velocity potentials representing two motions, then

$$\nabla^2\phi_1 = 0 = \nabla^2\phi_2 \qquad \_\_\_\_\_\_\_(i)$$

Since the kinetic conditions at the boundaries are satisfied by both flows therefore at each point of S

$$\frac{\rho}{2}\iiint_\tau q^2 d\tau = \frac{\rho}{2}\iiint_\tau \nabla^2\phi d\tau = \frac{\rho}{2}\iint_S \phi\frac{\partial\phi}{\partial n}dS$$

$$\phi_1\frac{\partial\phi_1}{\partial n} = 0 \quad \Rightarrow \frac{\partial\phi_1}{\partial n} = 0 \quad \_\_\_\_\_\_(ii)$$

$$\phi_2\frac{\partial\phi_2}{\partial n} = 0 \quad \Rightarrow \frac{\partial\phi_2}{\partial n} = 0 \quad \_\_\_\_\_\_(iii)$$

From (ii) and (iii)

$$\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n} \quad \_\_\_\_\_\_(iv)$$

Let $\phi = \phi_1 - \phi_2$

$$\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2$$

$$\nabla^2\phi = 0 \quad \text{at each point of fluid.}$$

$$\text{And } \frac{\partial\phi}{\partial n} = \frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n} = 0 \text{ at each point of S.}$$

$\Rightarrow \phi$ represents a possible irrotational motion.

Also, the K.E given by

$$\frac{\rho}{2}\iiint_\tau q^2 d\tau = \frac{\rho}{2}\iint_S \phi\frac{\partial\phi}{\partial n}dS = 0$$

Since the boundaries are rigid then at every point of S the normal velocity is zero i.e.

$$\frac{\partial\phi}{\partial n} = 0$$

$\Rightarrow q = 0$ at each point of fluid

$$\Rightarrow -\nabla\phi = 0$$

$$\Rightarrow \nabla \phi = 0 \quad \text{at each point of fluid}$$

$$\Rightarrow \nabla \phi_1 - \nabla \phi_2 = 0$$

$$\Rightarrow \nabla \phi_1 = \nabla \phi_2$$

which shows that the motions are the same. (Moreover ϕ is unique apart from an additive constant).

Theorem-II:

If the region occupied by the fluid is infinite and fluid is at rest at infinity, prove that only one irrotational motion is possible when internal boundaries have prescribed velocities.

Proof:

If possible, let there be two irrotational motions given by two different velocity potentials ϕ_1 & ϕ_2. The conditions on boundaries are

$$\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} \quad \_\_\_\_\_\_(i)$$

And $q_1 = q_2 = 0$ ____(ii) at infinity

Let us write $\phi = \phi_1 - \phi_2$ _____(iii)

$$\nabla^2 \phi = \nabla^2 \phi_1 - \nabla^2 \phi_2$$

$\Rightarrow$ motion given by ϕ is also irrotational.

Further from (iii) we get

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n} = 0 \quad \because \textit{from } (i)$$

$$\Rightarrow \quad q.\hat{n} = 0$$

$$\Rightarrow \quad q = 0 \text{ on the surface}$$

Also $q = -\nabla \phi = -\nabla \phi_1 + \nabla \phi_2$

$q = -\nabla \phi_1 - (-\nabla \phi_2)$

$q = q_1 - q_2$ at infinity.

Hence, we get ϕ = constant

$\phi_1 - \phi_2$ = constant _____(iv)

$\phi_1 - \phi_2 = 0 \quad \Rightarrow \quad \phi_1 = \phi_2$

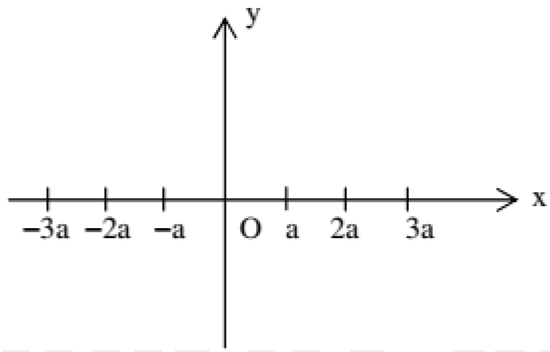
Hence, only one irrotational motion is possible.

***Remark:** The above two uniqueness theorems are useful in finding solutions of $\nabla^2 \phi = 0$ subject to prescribed boundary condition.

Lecture # 05

Single Infinite Row of vortices:

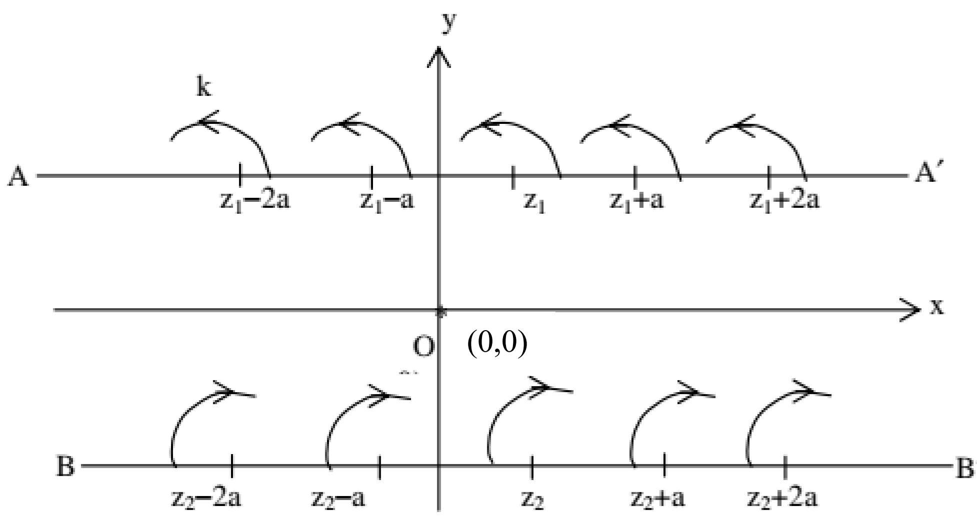
The complex potential of an infinite row of parallel rectilinear vortices (line vortices) of same strength 'K' and a distance 'a' apart. The vortices are placed at points $z = \pm na$; $n = 0, 1, 2, \ldots\ldots$, symmetrical about y-axis. The complex potential due to these vortices is



$W = iK \log z + iK \log (z-a) + iK \log (z-2a) + \ldots\ldots + iK \log (z-na) + iK \log (z+a) + iK \log (z+2a) + \ldots\ldots + iK \log (z+na)$

Double Infinite Row of Vortices:

Let us suppose that we have a system consisting of infinite number of vortices each of strength 'K' evenly placed along a line AA' parallel to x-axis and another system also consisting of infinite number of vortices each of strength '–K' placed similarly along a parallel line BB'. Let the line midway between these two lines of vortices be taken as the x-axis.



Let one vortex on infinite row AA' be at $z = z_1$ and one vortex on infinite row BB' be at $z = z_2$, so that the system consists of vortices K at $z = z_1 \pm na$ and vortices '−K' at $z = z_2 \pm na$, n = 1, 2, ….
The complex potential of the system is

$$W = iK\sum_{n=0}^{\infty}\log\left[\frac{(z - z_1 - na)(z - z_1 + na)}{(z - z_2 - na)(z - z_2 + na)}\right]$$

Velocity potential:

If the flow is irrotational a potential function ϕ can be formulated to represent the velocity field. From vector identity

$$\nabla \times \nabla \phi = 0$$

The velocity of an irrotational flow can be defined by a potential function so that

$$V = -\nabla \phi$$

$$\Rightarrow u = -\frac{\partial \phi}{\partial x}, \; v = -\frac{\partial \phi}{\partial y}, \; w = -\frac{\partial \phi}{\partial z}$$

In polar form $V_r = -\frac{\partial \phi}{\partial r}, V_\theta = -\frac{\partial \phi}{\partial \theta}, V_z = -\frac{\partial \phi}{\partial z}$

★ Kinetic Energy of irrotational motion:

Let S be the surface enclosing the volume τ of the fluid then

$$K.E = \iiint_\tau \frac{1}{2}\rho V^2 d\tau$$

$$\because V^2 = \left|\vec{V}\right|^2 = \vec{V}^2 = \vec{V}.\vec{V}$$

$$K.E = \frac{1}{2}\iiint_\tau \rho\left(\vec{V}.\vec{V}\right)d\tau \qquad \_\_\_\_(i)$$

Since the flow is irrotational therefore

$$\vec{V} = -\nabla\phi$$

$$K.E = \frac{1}{2}\iiint_{\tau} \rho\{(-\nabla\phi).(-\nabla\phi)\}d\tau$$

$$K.E = \frac{1}{2}\iiint_{\tau} \rho(\nabla\phi.\nabla\phi)d\tau \qquad \_\_\_\_\_(ii)$$

Let $\nabla.(\phi\nabla\phi) = \phi\nabla^2\phi + \nabla\phi.\nabla\phi$

$$\nabla.(\phi\nabla\phi) = \nabla\phi.\nabla\phi \qquad \because \nabla^2\phi = 0$$

$$(ii) \Rightarrow \quad K.E = \frac{1}{2}\iiint_{\tau} \rho\nabla.(\phi\nabla\phi)d\tau \qquad \_\_\_\_\_(iii)$$

By using the Gauss Divergence theorem

$$\iiint_V \nabla.\vec{A}\,dV = \iint_S \vec{A}.\hat{n}\,dS$$

Eq (iii) $\quad\Rightarrow\quad K.E = \frac{1}{2}\iint_S \rho(\phi\nabla\phi.\hat{n})dS$

$$K.E = \frac{1}{2}\iint_S \rho\phi\frac{\partial\phi}{\partial n}dS$$

Kelvin's Minimum Energy Theorem:

Statement:

The kinetic energy (K.E) of an irrotational flow for an incompressible fluid occupying the connected region is less than the K.E of any other flow of the fluid having the same normal velocity.

Proof:

Let S be the simply connected region enclosing a volume τ of an incompressible fluid, Let V be the velocity of fluid. Since the flow is irrotational. Therefore,

$$\vec{V} = -\nabla\phi$$

From equation of continuity $\dfrac{D\rho}{Dt}+\rho\left(\nabla.\vec{V}\right)=0 \qquad \_\_\_\_\_(i)$

Since the fluid is incompressible $\Rightarrow \dfrac{D\rho}{Dt}=0$

Eq (i) becomes $\rho\left(\nabla.\vec{V}\right)=0$

$$\nabla.\vec{V}=0 \qquad \_\_\_\_\_(ii)$$

Let T be the kinetic energy for the flow then

$$T=\frac{1}{2}\iiint_{\tau}\rho V^2 d\tau$$

$$\because V^2=\left|\vec{V}\right|^2=\vec{V}^2$$

$$T=\frac{\rho}{2}\iiint_{\tau}\vec{V}^2 d\tau \qquad \_\_\_\_(iii)$$

Let T' and V' be the K.E and velocity of any other flow of the fluid respectively. So, that

$$V'=\vec{V}+\vec{V_0}$$

From equation of continuity

$$\frac{D\rho}{Dt}+\rho\left(\nabla.V'\right)=0$$

Since the fluid is incompressible i.e. $\dfrac{D\rho}{Dt}=0$

$$\Rightarrow \rho\left(\nabla.V'\right)=0$$

$$\Rightarrow \nabla.V'=0$$

$$\Rightarrow \nabla.\left(\vec{V}+\vec{V_0}\right)=0$$

$$\Rightarrow \nabla\vec{V}+\nabla\vec{V_0}=0 \qquad \_\_\_\_(iv)$$

It is also given that the flow has same normal velocity

$$\vec{V}.\hat{n} = V^{'}.\hat{n}$$

$$\vec{V}.\hat{n} = \left(\vec{V} + \vec{V_0}\right).\hat{n}$$

$$\vec{V}.\hat{n} = \vec{V}.\hat{n} + \vec{V_0}.\hat{n}$$

$$\vec{V_0}.\hat{n} = 0 \qquad \_\_\_(v)$$

The K.E $T^{'}$ of any other flow is

$$T^{'} = \frac{1}{2}\iiint_{\tau} \rho V^{'2} d\tau$$

$$T^{'} = \frac{\rho}{2}\iiint_{\tau} \left(\vec{V} + \vec{V_0}\right)^2 d\tau$$

$$T^{'} = \frac{\rho}{2}\iiint_{\tau} \left(\overrightarrow{V^2} + \overrightarrow{V_0^2} + 2\vec{V}.\vec{V_0}\right) d\tau$$

$$T^{'} = \frac{\rho}{2}\iiint_{\tau} \overrightarrow{V^2} d\tau + \frac{\rho}{2}\iiint_{\tau} \overrightarrow{V_0^2} d\tau + \rho\iiint_{\tau} \left(\vec{V}.\vec{V_0}\right) d\tau$$

$$T^{'} = T + T_0 + \rho\iiint_{\tau} \left(\vec{V}.\vec{V_0}\right) d\tau \qquad \_\_\_\_(vi) \qquad \because by\ (iii)$$

Since the flow is irrotational $\quad \vec{V} = -\nabla\phi$

$$T^{'} = T + T_0 + \rho\iiint_{\tau} \left(-\nabla\phi.\vec{V_0}\right) d\tau$$

$$T^{'} = T + T_0 - \rho\iiint_{\tau} \left(\nabla\phi.\vec{V_0}\right) d\tau \qquad \_\_\_\_(vii)$$

$$Since \quad \nabla\left(\phi\vec{V_0}\right) = \phi\nabla.\vec{V_0} + \nabla\phi.\vec{V_0}$$

$$\nabla\left(\phi\vec{V_0}\right) - \phi\nabla.\vec{V_0} = \nabla\phi.\vec{V_0}$$

$$eq\,(vii) \Rightarrow \quad T' = T + T_0 - \rho \iiint_\tau \left(\nabla\left(\phi \overrightarrow{V_0}\right) - \phi \nabla . \overrightarrow{V_0} \right) d\tau$$

From Eq (ii) $\qquad \nabla . V = 0 \Rightarrow \nabla . V_0 = 0$

$$T' = T + T_0 - \rho \iiint_\tau \nabla . \left(\phi \overrightarrow{V_0}\right) d\tau \qquad \_\_\_\_\_ (viii)$$

By using the Gauss Divergence theorem

$$\iiint_V \nabla . \vec{A}\, dV = \iint_S \vec{A} . \hat{n}\, dS$$

$$T' = T + T_0 - \rho \iint_S \phi \overrightarrow{V_0} . \hat{n}\, dS$$

From eq (v) $\overrightarrow{V_0} . \hat{n} = 0$

$$T' = T + T_0$$

$$\Rightarrow T' > T$$

$$Or \quad T < T'$$

Lecture # 06

Laplace equation:

If fluid is an incompressible and ϕ is a velocity potential then $\nabla^2\phi = 0$ is called Laplace equation.

Proof:

We know that the standard form of equation of continuity is

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0 \qquad \_\_\_\_(i)$$

$$Since \ \frac{D}{Dt} = \frac{\partial}{\partial t} + V.\nabla$$

$$\Rightarrow \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + V.(\nabla \rho) \qquad \_\_\_\_(ii)$$

From (i) $$\frac{\partial \rho}{\partial t} = -\nabla.(\rho V)$$

Put in (ii) $$\frac{D\rho}{Dt} = -\nabla(\rho V) + V.(\nabla \rho)$$

$$\frac{D\rho}{Dt} = -\{\rho(\nabla.V) + V.(\nabla \rho)\} + V.(\nabla \rho)$$

$$\frac{D\rho}{Dt} = -\rho(\nabla.V) - V.(\nabla \rho) + V.(\nabla \rho)$$

$$\frac{D\rho}{Dt} = -\rho(\nabla.V)$$

$$\frac{D\rho}{Dt} + \rho(\nabla.V) = 0 \qquad \_\_\_\_(iii)$$

Since fluid is incompressible ρ = constant.

$$\Rightarrow \frac{D\rho}{Dt} = 0$$

Equation (iii) $\Rightarrow$ $$\rho \nabla . V = 0$$

$$\Rightarrow \nabla . V = 0 \qquad \_\_\_\_\_ (iv)$$

Also, flow is irrotational $$V = -\nabla \phi$$

Put in (iv) $$\nabla . (-\nabla \phi) = 0$$

$$-\nabla^2 \phi = 0$$

$$\Rightarrow \nabla^2 \phi = 0$$

$$\Rightarrow \nabla^2 \phi = 0$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$ which is required Laplace equation.

Stress:

It is defined as stress in a medium result from forces acting on some portion of medium

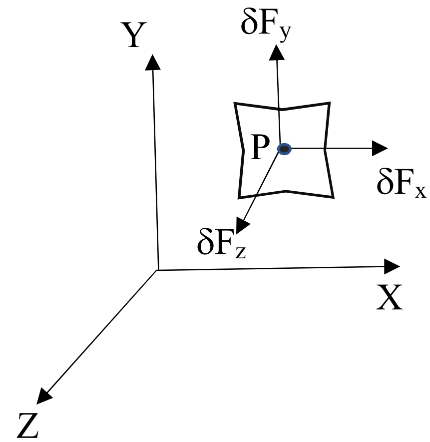
$$stress = \frac{F}{A}$$

Normal stress:

$$\sigma_n = \lim_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n}$$

Tangential stress or sheer stress:

$$\tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n}$$



Let us consider the stress acting on planes whose outward normal are in X,Y,Z directions. Then

$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_x}{\delta A_x}$$

As we have following sheer stress

$$\begin{matrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{matrix}$$

Note: (i) We have double subscript notation to label stresses like τ_{yx} etc.

x denotes the direction in which stress acts and y denotes the plane on which stress acts.

(ii). X-plane = YZ-plane

(iii). Density = $\dfrac{mass}{volume} = \dfrac{m}{V}$

(iv). By Newton second law

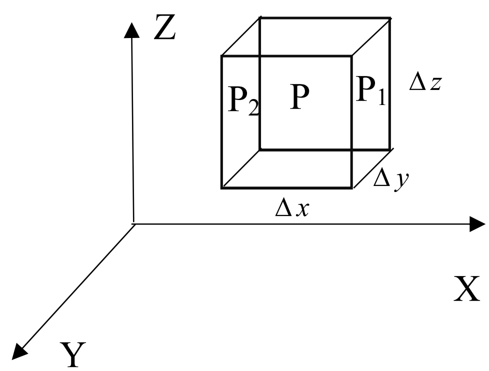
$$F = ma$$

$$F = m\frac{dV}{dt} \quad \because a = \frac{dV}{dt}$$

$$if \;\; \rho = m \;\; then \;\; F = \rho\frac{dV}{dt}$$

Generalization equation of motion:

Consider a fluid element whose center point is P and stress τ_{x_x} . P_1 and P_2 is its right side and left side corner point respectively.



Length element along X-axis is Δx

Length element along Y-axis is Δy

Length element along Z-axis is Δz

At point P_1 $\tau_{xx} + \dfrac{\partial \tau_{xx}}{\partial x}.\dfrac{\Delta x}{2}$

At point P₂ $\tau_{xx}-\frac{\partial\tau_{xx}}{\partial x}.\frac{\Delta x}{2}$

Consider the X-component of surfaces forces

$$dF_{sx}=\left(\tau_{xx}+\frac{\partial\tau_{xx}}{\partial x}.\frac{\Delta x}{2}\right)\Delta y\Delta z-\left(\tau_{xx}-\frac{\partial\tau_{xx}}{\partial x}.\frac{\Delta x}{2}\right)\Delta y\Delta z$$

$$+\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}.\frac{\Delta y}{2}\right)\Delta x\Delta z-\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}.\frac{\Delta y}{2}\right)\Delta x\Delta z$$

$$+\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}.\frac{\Delta z}{2}\right)\Delta x\Delta y-\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}.\frac{\Delta z}{2}\right)\Delta x\Delta y$$

$$dF_{sx}=\left(\tau_{xx}+\frac{\partial\tau_{xx}}{\partial x}.\frac{\Delta x}{2}-\tau_{xx}+\frac{\partial\tau_{xx}}{\partial x}.\frac{\Delta x}{2}\right)\Delta y\Delta z$$

$$+\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}.\frac{\Delta y}{2}-\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}.\frac{\Delta y}{2}\right)\Delta x\Delta z$$

$$+\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}.\frac{\Delta z}{2}-\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}.\frac{\Delta z}{2}\right)\Delta x\Delta y$$

$$dF_{sx}=\left(2\frac{\partial\tau_{xx}}{\partial x}.\frac{\Delta x}{2}\right)\Delta y\Delta z+\left(2\frac{\partial\tau_{yx}}{\partial y}.\frac{\Delta y}{2}\right)\Delta x\Delta z+\left(2\frac{\partial\tau_{zx}}{\partial z}.\frac{\Delta z}{2}\right)\Delta x\Delta y$$

$$dF_{sx}=\frac{\partial\tau_{xx}}{\partial x}\Delta x\,\Delta y\Delta z+\frac{\partial\tau_{yx}}{\partial y}\Delta x\,\Delta y\Delta z+\frac{\partial\tau_{zx}}{\partial z}\Delta x\,\Delta y\Delta z$$

$$dF_{sx}=\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)\Delta x\,\Delta y\Delta z \qquad \_\_\_\_(i)$$

Now for body forces $\qquad dF_{Bx}=mg_x$

Net force along X-component $\quad dF_x = dF_{sx} + dF_{Bx}$

$$dF_x = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \Delta x\, \Delta y \Delta z \ + m g_x$$

$$\because \Delta V = \Delta x\, \Delta y \Delta z$$

$$dF_x = \left(\frac{\partial \tau_{x_x}}{\partial x} + \frac{\partial \tau_{y_x}}{\partial y} + \frac{\partial \tau_{z_x}}{\partial z} \right) \Delta V \ + m g_x$$

By Newton second law of motion

$$dF_x = m a_x$$

$$m a_x = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \Delta V \ + m g_x \qquad \_\_\_\_\_ (ii)$$

$$\because \rho = \frac{m}{\Delta V} \quad \Rightarrow m = \rho \Delta V$$

$$\rho \Delta V a_x = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \Delta V \ + \rho \Delta V g_x$$

$$Since\ a = \left(a_x, a_y, a_z \right) = \frac{dV}{dt} = \left(\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt} \right)$$

$$\rho \Delta V \frac{du}{dt} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \Delta V \ + \rho \Delta V g_x$$

$\Delta V \neq 0$ because if $\Delta V = 0$ then one of our components $\Delta x, \Delta y, \Delta z$ becomes zero and our body can never move. So, $\Delta V \neq 0$ we divide $\Delta V\ and\ \rho$

$$\frac{du}{dt} = \frac{1}{\rho} \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + g_x \qquad \_\_\_\_\_ (iii)$$

Similarly, for y-direction

$$\frac{dv}{dt}=\frac{1}{\rho}\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)+g_y \qquad \_\_\_\_\_(iv)$$

Similarly, for z-direction

$$\frac{dw}{dt}=\frac{1}{\rho}\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+g_z \qquad \_\_\_\_\_(v)$$

$If\ \ u=u(x,y,z,\mathrm{t})\, then$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial t}+\frac{\partial u}{\partial t}.\frac{\partial t}{\partial t}$$

$$\Rightarrow\frac{du}{dt}=\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}v+\frac{\partial u}{\partial z}w+\frac{\partial u}{\partial t}$$

Equation (iii) becomes

$$\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}v+\frac{\partial u}{\partial z}w+\frac{\partial u}{\partial t}=\frac{1}{\rho}\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+g_x \qquad \_\_\_\_\_(vi)$$

Similarly, the equation of motion in $\hat{j}$ and $\hat{k}$ directions are

$$\frac{\partial v}{\partial x}u+\frac{\partial v}{\partial y}v+\frac{\partial v}{\partial z}w+\frac{\partial v}{\partial t}=\frac{1}{\rho}\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+g_y \qquad \_\_\_\_\_(vii)$$

$$\frac{\partial w}{\partial x}u+\frac{\partial w}{\partial y}v+\frac{\partial w}{\partial z}w+\frac{\partial w}{\partial t}=\frac{1}{\rho}\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+g_z \qquad \_\_\_\_\_(viii)$$

Equation (vi),(vii),(viii) provide the equation of motion of fluid element at P(x,y,z)

Euler equation of motion for in-viscus (real) fluid:

We consider X component of general equation of motion

$$\rho\frac{\partial u}{\partial t}=\frac{1}{\rho}\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+g_x \qquad \_\_\_\_(i)$$

We may have some assumption

Set-I:

$$\tau_{xy} = \tau_{yx} = 0$$

$$\tau_{yz} = \tau_{zy} = 0$$

$$\tau_{xz} = \tau_{zx} = 0$$

Set-II:

$$\tau_{xx} = -P + \sigma_{xx}$$

$$\tau_{yy} = -P + \sigma_{yy}$$

$$\tau_{zz} = -P + \sigma_{zz}$$

Set-III:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$$

Diff. set II w.r.t x,y and z

$$\frac{\partial \tau_{xx}}{\partial x} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}(\sigma_{xx}) = -\frac{\partial P}{\partial x} \qquad \because \sigma_{xx} = 0$$

$$\frac{\partial \tau_{yy}}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial y}(\sigma_{yy}) = -\frac{\partial P}{\partial y} \qquad \because \sigma_{yy} = 0$$

$$\frac{\partial \tau_{zz}}{\partial z} = -\frac{\partial P}{\partial z} + \frac{\partial}{\partial z}(\sigma_{zz}) = -\frac{\partial P}{\partial z} \qquad \because \sigma_{zz} = 0$$

Put all these values in (i)

$$\rho \frac{du}{dt} = \left(-\frac{\partial P}{\partial x} + 0 + 0 \right) + \rho g_x$$

$$\rho \frac{du}{dt} = -\frac{\partial P}{\partial x} + \rho g_x \qquad \_\_\_\_\_(ii)$$

Similarly, for y and z component

$$\rho\frac{dv}{dt}=-\frac{\partial P}{\partial y}+\rho g_y \quad \_\_\_\_\_(iii)$$

$$\rho\frac{dw}{dt}=-\frac{\partial P}{\partial z}+\rho g_z \quad \_\_\_\_\_(iv)$$

As we know that

$$\vec{V}=u\hat{i}+v\hat{j}+w\hat{k}$$

$$\frac{d\vec{V}}{dt}=\frac{du}{dt}\hat{i}+\frac{dv}{dt}\hat{j}+\frac{dw}{dt}\hat{k}$$

Multiplying by ρ

$$\rho\frac{d\vec{V}}{dt}=\rho\frac{du}{dt}\hat{i}+\rho\frac{dv}{dt}\hat{j}+\rho\frac{dw}{dt}\hat{k} \quad \_\_\_\_(v)$$

Put equations (ii),(iii) (iv) in (v)

$$\rho\frac{d\vec{V}}{dt}=\left(-\frac{\partial P}{\partial x}+\rho g_x\right)\hat{i}+\left(-\frac{\partial P}{\partial y}+\rho g_y\right)\hat{j}+\left(-\frac{\partial P}{\partial z}+\rho g_z\right)\hat{k}$$

$$\rho\frac{d\vec{V}}{dt}=-\left(\frac{\partial P}{\partial x}\hat{i}+\frac{\partial P}{\partial y}\hat{j}+\frac{\partial P}{\partial z}\hat{k}\right)+\rho\left(g_x\hat{i}+g_y\hat{j}+g_z\hat{k}\right)$$

$$\rho\frac{d\vec{V}}{dt}=-\nabla P+\rho\vec{g} \quad \_\_\_(vi)$$

Since $\frac{d}{dt}$ is a material time derivative, $\frac{d}{dt}=\frac{\partial}{\partial t}+\nabla.V$

Equation (vi) $\Rightarrow$ $\rho\left[\frac{\partial\vec{V}}{\partial t}+\left(\nabla.\vec{V}\right)\vec{V}\right]=-\nabla P+\rho\vec{g}$ is the Euler equation of motion.

Lecture # 07

Bernoulli Equation:

We know that Euler equation of motion is

$$\rho\left[\frac{\partial\vec{V}}{\partial t}+\left(\nabla.\vec{V}\right)\vec{V}\right]=\rho\vec{g}-\nabla P \qquad \_\_\_\_(i)$$

From vector analysis, we know that

$$\nabla\left(V^2\right)=\nabla\left(\vec{V}.\vec{V}\right)=2\left(\nabla.\vec{V}\ \right)\vec{V}+2\vec{V}\times\left(\nabla\times\vec{V}\right)$$

$$\nabla\left(\vec{V}.\vec{V}\right)-2\vec{V}\times\left(\nabla\times\vec{V}\right)=2\left(\nabla.\vec{V}\ \right)\vec{V}$$

$$\Rightarrow\ \left(\nabla.\vec{V}\ \right)\vec{V}=\frac{1}{2}\nabla\left(\vec{V}.\vec{V}\right)-\vec{V}\times\left(\nabla\times\vec{V}\right)$$

$$Let\ \vec{g}=-g\hat{k}=-g\nabla z$$

$$\rho\left[\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla\left(\vec{V}.\vec{V}\right)-\vec{V}\times\left(\nabla\times\vec{V}\right)\right]=\rho\left(-g\nabla z\right)-\nabla P$$

$$\rho\left[\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla\left(\vec{V}.\vec{V}\right)\right]-\rho\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]=-\rho g\nabla z-\nabla P$$

$$\rho\left[\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla\left(\vec{V}.\vec{V}\right)\right]+\rho g\nabla z+\nabla P=\rho\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]$$

Divide by ρ $$\left[\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla\left(\vec{V}.\vec{V}\right)\right]+g\nabla z+\frac{1}{\rho}\nabla P=\vec{V}\times\left(\nabla\times\vec{V}\right)$$

Rearranging $$\frac{1}{\rho}\nabla P+g\nabla z+\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla V^2=\vec{V}\times\left(\nabla\times\vec{V}\right) \qquad \_\_\_(ii)$$

is called the Bernoulli equation for unsteady flow.

Bernoulli Equation for steady flow:

For steady flow $\dfrac{\partial \vec{V}}{\partial t}=0$

Put in (ii) $$\Rightarrow \frac{1}{\rho}\nabla P+g\nabla z+0+\frac{1}{2}\nabla V^2=\vec{V}\times\left(\nabla\times\vec{V}\right)$$

$$\Rightarrow \frac{1}{\rho}\nabla P+g\nabla z+\frac{1}{2}\nabla V^2=\vec{V}\times\left(\nabla\times\vec{V}\right)$$

Taking dot product on both side with ds

$$\frac{1}{\rho}\nabla P.ds+g\nabla z.ds+\frac{1}{2}\nabla V^2.ds=\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right].ds \quad \_\_\_\_(iii)$$

As $\vec{V}\times\left(\nabla\times\vec{V}\right)\perp ds \Rightarrow \left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right].ds=0$

$$Also\ \ \nabla P.ds=dP$$

$$\nabla z.ds=dz$$

$$\nabla V^2.ds=dV^2$$

Put in (iii) $$\frac{1}{\rho}dP+gdz+\frac{1}{2}dV^2=0$$

Now integrate above equation $\displaystyle\int\frac{1}{\rho}dP+\int gdz+\frac{1}{2}\int dV^2=\int 0$

$$\frac{1}{\rho}P+gz+\frac{1}{2}V^2=\text{constant}$$

*This is called Bernoulli equation for in viscous, incompressible, steady and rotational flow along the stream line.

*This equation is also true for both rotational $\left(\nabla\times\vec{V}\neq 0\right)$ and irrotational $\left(\nabla\times\vec{V}=0\right)$ flow.

Navier-Stokes equation:

As we know that the X-component of general equation of motion is

$$\rho\frac{\partial u}{\partial t}=\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+\rho g_x \quad \_\_\_\_(i)$$

Now we will make following assumptions

Set-I:

$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \quad \_\_\_\_\_(ii)$$

$$\tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \quad \_\_\_\_\_(iii)$$

$$\tau_{zx}=\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \quad \_\_\_\_\_(iv)$$

Set-I:

$$\tau_{xx}=-P-\frac{2}{3}\nabla.\vec{V}+2\mu\frac{\partial u}{\partial x} \quad \_\_\_\_(v)$$

$$\tau_{yy}=-P-\frac{2}{3}\nabla.\vec{V}+2\mu\frac{\partial v}{\partial y} \quad \_\_\_\_(vi)$$

$$\tau_{zz}=-P-\frac{2}{3}\nabla.\vec{V}+2\mu\frac{\partial w}{\partial z} \quad \_\_\_\_(vii)$$

$$As \quad \vec{V}=u\hat{i}+v\hat{j}+w\hat{k}$$

$$\nabla.\vec{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

Equation (v) becomes $\tau_{xx}=-P-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}$

Diff. w.r.t ‘x $\quad \dfrac{\partial \tau_{xx}}{\partial x} = -\dfrac{\partial P}{\partial x} - \dfrac{2}{3}\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) + 2\mu \dfrac{\partial^2 u}{\partial x^2}$

$$\frac{\partial \tau_{xx}}{\partial x} = -\frac{\partial P}{\partial x} - \frac{2}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right) + 2\mu \frac{\partial^2 u}{\partial x^2} \quad \_\_\_(viii)$$

Diff. equation (ii) w.r.t ‘y’ $\quad \dfrac{\partial \tau_{yx}}{\partial y} = \mu\left(\dfrac{\partial^2 v}{\partial y \partial x} + \dfrac{\partial^2 v}{\partial y^2}\right)\_\_\_(ix)$

Diff. equation (iv) w.r.t ‘z’ $\quad \dfrac{\partial \tau_{zx}}{\partial y} = \mu\left(\dfrac{\partial^2 u}{\partial z^2} + \dfrac{\partial^2 w}{\partial x \partial z}\right)\_\_\_(x)$

Using equation (viii), (ix), (x) in (i)

$$\rho\frac{\partial u}{\partial t} = \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right) + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\left(\frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right)$$

Rearranging

$$\rho\frac{\partial u}{\partial t} = \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right) + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\left(\frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right)$$

$$\rho\frac{\partial u}{\partial t} = \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right) + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2}\right) + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right)$$

$$\rho\frac{\partial u}{\partial t} = \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\nabla^2 u + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\rho\frac{\partial u}{\partial t} = \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\frac{\partial}{\partial x}\left(\nabla.\vec{V}\right) + \mu\nabla^2 u + \mu\frac{\partial}{\partial x}\left(\nabla.\vec{V}\right)$$

For incompressible fluid $\nabla.\vec{V} = 0$

$$\rho\frac{\partial u}{\partial t} = \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\frac{\partial}{\partial x}(0) + \mu\nabla^2 u + \mu\frac{\partial}{\partial x}(0)$$

$$\Rightarrow \rho \frac{\partial u}{\partial t} = \rho g_x - \frac{\partial P}{\partial x} + \mu \nabla^2 u$$ is the X-component of Navier-Stokes equation.

Similarly, for Y and Z components.

$$\rho \frac{\partial v}{\partial t} = \rho g_y - \frac{\partial P}{\partial y} + \mu \nabla^2 v$$

$$\rho \frac{\partial w}{\partial t} = \rho g_z - \frac{\partial P}{\partial z} + \mu \nabla^2 w$$

Parallel flows:

A flow is called parallel if there is only one velocity component. If
$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ then $\vec{V} = u\hat{i} \quad when \ \ v = w = 0$

The practical application of this simple case if the flow between parallel flat plates (planes). Circular pipes and concentric rotating cylinder in such one component flow the Navier-Stokes equation simplify, consider by and infect permit and exact solution e.g. $\nabla.\vec{V} = 0$

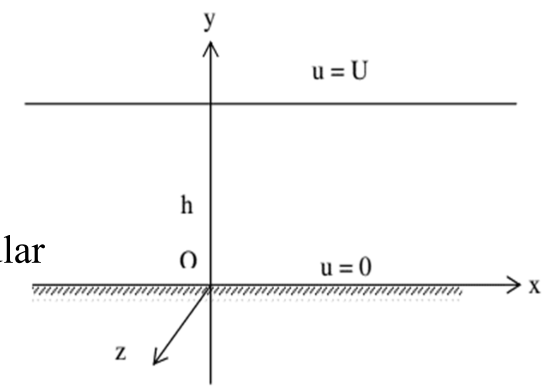
$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \ \ becomes \ \frac{\partial u}{\partial x} = 0$$

Lecture # 08

Couette flow:

The simple Couette flow or simple sheer flow is the flow between two parallel plates one which y = 0 is at rest and other is y = h moving with the uniform constant velocity 'u' parallel to itself.

Consider the steady laminar flow of inviscous, incompressible fluid between the two infinite horizontal parallel flat plates. Let X-axis be the direction of the flow and Y-axis perpendicular to the direction of flow. Consider the distance between the plates be 'h' and the width of the plates in Z-direction be finite.



Case-I: The X-component of Navier-Stokes equation is

$$\rho \frac{du}{dt} = \rho g_x - \frac{\partial P}{\partial x} + \mu \nabla^2 u \quad \_\_\_(i)$$

*The assumptions are

(i) One dimensional flow i.e u = u(y) , v = w = 0

(ii) Viscous medium i.e $\mu \neq 0$

(iii) Incompressible flow i.e. $\rho \neq 0$
(iv) Steady flow i.e. independent of time
(v) No pressure i.e pressure gradient is zero.
(vi) No body force i.e. $g_x = 0$

From equation (i)

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\nabla^2 u + \frac{\rho}{\rho} g_x$$

$$\frac{du}{dt}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]+g_x$$

According to these above assumptions

$$0=\left(\frac{\partial^2 u}{\partial y^2}\right)\frac{\mu}{\rho}+0$$

$$\Rightarrow \frac{d^2u}{dy^2}=0$$

Integrating w.r.t 'y'

$$\Rightarrow \frac{du}{dy}=c_1$$

Again integrating

$$y=c_1y+c_2 \quad \_\_\_(ii)$$

According to boundary condition

u = 0 at y = 0 ____ (iii)

u = U at y = h ____(iv)

Using (iii) in (ii) we have

$0 = c_1(0) + c_2 \Rightarrow c_2 = 0$

(ii). $\Rightarrow$ $u = c_1y$ ___(v)

Using (iv) in (v) we have

$U = c_1 h \Rightarrow c_1=\frac{U}{h}$

Put in (v)

$$u=\frac{U}{h}.y$$

$\frac{u}{U}=\frac{y}{h}$ is the required velocity field for Couette flow.

Case-I: When both plated moves with uniform velocity i.e

According to boundary condition

$u = u_1$ at $y = 0$ ____ (vi)

$u = u_2$ at $y = h$ ____(vii)

From equation (ii)

$$y = c_1 y + c_2 \quad \_\_\_(viii)$$

Using (vi) in (viii) we have

$u_1 = c_1(0) + c_2 \Rightarrow c_2 = u_1$

Put in (viii) $\Rightarrow u = c_1 y + u_1$ ___(ix)

Using (vii) in (ix) we have

$u_2 = c_1 h + u_1 \quad \Rightarrow c_1 = \dfrac{u_2 - u_1}{h}$

Put in (ix)

$$u = \frac{u_2 - u_1}{h} y + u_1$$

$$u = \frac{(u_2 - u_1) y + u_1 h}{h}$$ which is the required solution.

★ Generalization of Couette flow:

It is simple Couette flow with non-zero pressure gradient. Therefore, the boundary conditions are same. The X-component of Navier-Stokes equation is

$$\frac{du}{dt} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + g_x \quad \_\_\_(i)$$

According to assumptions

(i) One dimensional flow i.e $u = u(y)$, $v = w = 0$

(ii) Viscous medium i.e $\mu \neq 0$

(iii) Incompressible flow i.e. $\rho \neq 0$

(iv) Steady flow i.e. independent of time

(v) No body force i.e. $g_x = 0$

Equation (i) $\Rightarrow 0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)$

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dP}{dx}$$

On integrating w.r.t 'y'

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dP}{dx}y + c_1$$

Again, integrating w.r.t 'y'

$$u = \frac{1}{\mu}\frac{dP}{dx}\frac{y^2}{2} + c_1 y + c_2$$

$$u = \frac{1}{2\mu}\frac{dP}{dx}y^2 + c_1 y + c_2 \quad \_\_\_\_ (ii)$$

Using boundary condition

u = 0 at y = 0 _____(iii)

u = U at y = h ____ (iv)

Using (iii) in (ii) we have

$$0 = \frac{1}{2\mu}\frac{dP}{dx}(0)^2 + c_1(0) + c_2 \Rightarrow c_2 = 0$$

Put in (ii) $u = \frac{1}{2\mu}\frac{dP}{dx}y^2 + c_1 y \quad \_\_\_ (v)$

Using (iv) in (v) we have $U = \frac{1}{2\mu}\frac{dP}{dx}h^2 + c_1 h$

$$c_1 = \frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx}$$

Put in (v) $$u = \frac{1}{2\mu}\frac{dP}{dx}y^2 + \left(\frac{U}{h} - \frac{h}{2\mu}\frac{dP}{dx}\right)y$$

$$u = \frac{1}{2\mu}\frac{dP}{dx}y^2 + \frac{U}{h}y - \frac{h}{2\mu}\frac{dP}{dx}y$$

$$u = \frac{U}{h}y + \frac{hy}{2\mu}\left(-\frac{dP}{dx}\right) + \frac{1}{2\mu}\frac{dP}{dx}y^2$$

$$u = \frac{U}{h}y + \frac{hy}{2\mu}\left(-\frac{dP}{dx}\right)\left[1 - \frac{y}{h}\right] \quad \text{\_\_\_\_}(vi)$$

Which is the equation for the velocity field of generalized Couette flow.

Equation (vi) can be written as

$$\frac{u}{U} = \frac{y}{h} + \frac{hy}{2\mu U}\left(-\frac{dP}{dx}\right)\left[1 - \frac{y}{h}\right]$$

$$\frac{u}{U} = \frac{y}{h} + h^2\left(\frac{y}{h}\right)\frac{1}{2\mu U}\left(-\frac{dP}{dx}\right)\left[1 - \frac{y}{h}\right] \quad \text{\_\_\_\_}(vii)$$

*Let $\alpha = \frac{h^2}{2\mu}U\left(\frac{-dP}{dx}\right)$ be the dimensionless pressure gradient. Equation (vii) becomes

$$\frac{u}{U} = \frac{y}{h} + \alpha\left(\frac{y}{h}\right)\left[1 - \frac{y}{h}\right] \quad \text{\_\_\_\_}(viii)$$

Case-I: If $\alpha > 0 \Rightarrow \frac{dP}{dx} < 0$ *Pressure is decreasing in the direction of flow.

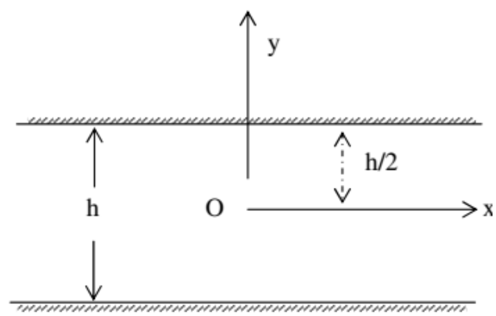
Case-II: If $\alpha < 0 \Rightarrow \frac{dP}{dx} > 0$ *Pressure is increasing in the direction of flow.

Case-III: If $\alpha = 0 \Rightarrow \frac{dP}{dx} = 0$ equation (viii) becomes $\frac{u}{U} = \frac{y}{h}$ which is the solution of simple Couette flow.

Lecture # 09

Plane Poiseuille flow:

If two parallel plates are stationary, the fully developed between the plates is generally referred to as place Poiseuille flow.



Let plane is situated at

$$y = \frac{-h}{2} \, and \; y = \frac{h}{2}.$$

The X-component of Navier-Stokes equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{dP}{dx} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g_x$$

Now without body forces. (Apply assumption)

$$0 = \frac{-1}{\rho}\frac{dP}{dx} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dP}{dx} \quad \_\_\_\_\_ (i)$$

Integrate w.r.t 'y'

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dP}{dx} y + c_1$$

Again, integrate w.r.t 'y'

$$u = \frac{1}{\mu}\frac{dP}{dx}\frac{y^2}{2} + c_1 y + c_2$$

$$u = \frac{1}{2\mu}\frac{dP}{dx} y^2 + c_1 y + c_2 \quad \_\_\_\_\_ (ii)$$

Boundary conditions are

$$u = 0 \;\; at \; y = \frac{h}{2} \quad \_\_\_\_ (iii)$$

$$u = 0 \;\; at \; y = -\frac{h}{2} \qquad \_\_\_\_ (iv)$$

Using equation (iii) and (iv) in (ii)

$$0 = \frac{1}{2\mu}\frac{dP}{dx}\frac{h^2}{4} + c_1\frac{h}{2} + c_2 \qquad \_\_\_\_\_ (v)$$

$$0 = \frac{1}{2\mu}\frac{dP}{dx}\frac{h^2}{4} - c_1\frac{h}{2} + c_2 \qquad \_\_\_\_\_ (vi)$$

Adding equation (v) and (vi)

$$0 = 2\left(\frac{1}{2\mu}\frac{dP}{dx}\frac{h^2}{4}\right) + 2c_2$$

$$2c_2 = -\frac{h^2}{4\mu}\frac{dP}{dx}$$

$$\Rightarrow \; c_2 = -\frac{h^2}{8\mu}\frac{dP}{dx}$$

On subtracting (v) and (vi)

$$0 = 0 + 2\left(\frac{h}{2}c_1\right) + 0 \quad \Rightarrow \; c_1 = 0$$

Equation (ii) becomes

$$u = \frac{1}{2\mu}\frac{dP}{dx}y^2 + 0 - \frac{h^2}{8\mu}\frac{dP}{dx}$$

$$u = \frac{-h^2}{8\mu}\frac{dP}{dx}\left(1 - \frac{4y^2}{h^2}\right)$$

Which is velocity profile of the fully developed laminar flow between two parallel plates is parabolic. Thus, if the pressure gradient viscosity and place spacing are specified then the velocity distribution can be determined.

Poiseuille flow or General Poiseuille flow:

Steady viscous fluid flow drives by an effect of pressure gradient established between the ends of a long straight pipe of uniform circular cross-section or between two parallel plates both are at rest. This flow is symmetric and axis symmetric. If v = (u,v,w) then u ≠ 0 and v = w = 0. Also u = u(y,z).

X-component of Navier-Stokes equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{dP}{dx} + \nu\nabla^2 u + g_x \quad where\ \nu = \frac{\mu}{\rho}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{dP}{dx} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g_x$$

Without body forces (by assumption)

$$0 = \frac{-1}{\rho}\frac{dP}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \because u = u(y,z)$$

Similarly, for y-component

$$0 = \frac{-1}{\rho}\frac{dP}{dy} \quad \Rightarrow \frac{dP}{dz} = 0 \Rightarrow P \neq P(z)$$

⇒ pressure is also independent of z.

So, P = P(x). The X-component becomes

$$0 = \frac{-1}{\rho}\frac{dP}{dx} + \nu\ \nabla^2 u \qquad \because u = u(y,z)$$

$$0 = \frac{-1}{\rho}\frac{dP}{dx} + \frac{\mu}{\rho}\nabla^2 u$$

$$\frac{dP}{dx} = \mu\nabla^2 u$$

$$\frac{1}{\mu}\frac{dP}{dx} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad \_\_\_\_\_\_\_(i)$$

Now we make some substitutions

$$y^* = \frac{y}{h},\; z^* = \frac{z}{h},\; \mathrm{u}^* = \frac{\mu u}{h^2\left(\frac{-dP}{dx}\right)}$$

$$\Rightarrow\; y = hy^*,\; z = z^* h, \mathrm{u} = \frac{\mathrm{u}^* h^2\left(\frac{-dP}{dx}\right)}{\mu}$$

Putting these values in (i)

$$\frac{1}{\mu}\frac{dP}{dx} = \frac{\partial^2}{\partial\left(hy^*\right)^2}\left[\mathrm{u}^* h^2\left(\frac{-dP}{dx}\right)\right] + \frac{\partial^2}{\partial\left(hz^*\right)^2}\left[\mathrm{u}^* h^2\left(\frac{-dP}{dx}\right)\right]$$

$$\frac{1}{\mu}\frac{dP}{dx} = \frac{h^2\left(\frac{-dP}{dx}\right)}{h^2\mu}\left[\frac{\partial^2}{\partial y^{*2}} + \frac{\partial^2}{\partial z^{*2}}\right]$$

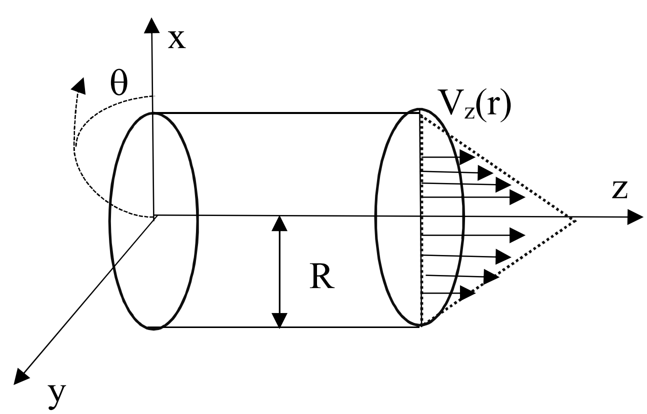
$$1 = -\nabla^2 u^*$$

$\Rightarrow\; \nabla^2 u^* + 1 = 0$ Which is called Poiseuille equation.

Steady laminar flow through a circular pipe (The Hagen-Poiseuille flow):

Consider the steady laminar flow of a viscus incompressible fluid in an infinitely long straight, horizontal circular pipe of radius R as shown in figure.

Let z-axis be along the axis of the pipe and r denote the radial direction measured outward from the z-axis. Let the direction of the flow be along the axis of pipe i.e z-axis. The axially symmetric flow in a circular Flow. Clearly the flow is one-dimensional.



The velocity component in the radial and tangential direction are zero. $V_r = V_\theta = 0$. Under these assumptions the equation of continuity in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r.V_r\right)+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial V_z}{\partial z}=0$$

Reduce to $\frac{\partial V_z}{\partial z}=0$ _____ (i) $\because V_r=V_\theta=0$

Showing that V_z is independent of z due to axial symmetry of the flow. V_z will be independent of θ. Also, V_z is a function of r only i.e. $V_z = V_z(r)$ _____(ii)

The Navier-Stokes equation without body forces in cylindrical coordinates reduce to

$$\left.\begin{aligned} 0&=\frac{-1}{\rho}\frac{\partial P}{\partial r}\\ 0&=\frac{-1}{\rho r}\frac{\partial P}{\partial \theta}\\ 0&=\frac{-1}{\rho}\frac{\partial P}{\partial z}+\nu\left[\frac{\partial^2 V_z}{\partial r^2}+\frac{1}{r}\frac{\partial V_z}{\partial r}\right]\end{aligned}\right\}\_\_\_\_\_\_\_\_(iii)$$

Equation (iii) can be written as

$$\frac{\partial P}{\partial r}=\frac{\partial P}{\partial \theta}=0$$

P = P(z) or P is a function of z alone and

$$\frac{\partial P}{\partial z}=\mu\left[\frac{\partial^2 V_z}{\partial r^2}+\frac{1}{r}\frac{\partial V_z}{\partial r}\right]$$

$$\Rightarrow \frac{\partial P}{\partial z}=\frac{\partial^2 V_z}{\partial r^2}+\frac{\mu}{r}\frac{\partial V_z}{\partial r}$$

$$Multiply\, by\, \frac{r}{\mu}$$

$$\frac{r}{\mu}\frac{dP}{dz}=r\frac{d^2 V_z}{dr^2}+\frac{dV_z}{dr}$$

$$\frac{r}{\mu}\frac{dP}{dz}=\frac{d}{dr}\left(r\frac{dV_z}{dr}\right)$$

Integrate w.r.t 'r'

$$r\frac{dV_z}{dr}=\frac{r^2}{2\mu}\frac{dP}{dz}+A$$

$$\frac{dV_z}{dr}=\frac{r}{2\mu}\frac{dP}{dz}+\frac{1}{r}A \quad \because \text{divide by } r$$

Again integrating

$$V_z=\frac{r^2}{4\mu}\frac{dP}{dz}+A\ln r+B \quad \_\_\_\_\_(iv)$$

Where the arbitrary constant A and B are to be determined from the boundary condition. The first boundary condition is found from the symmetry of the flow which requires that V_z must be finite on the axis of the pipe (r = 0). It follows that we must take A= 0 because otherwise V_z would be infinite at r = 0. Thus equation (iv) reduce to

$$V_z=\frac{r^2}{4\mu}\frac{dP}{dz}+B$$

The second boundary condition $V_z = 0$ at r = R. With this boundary condition the constant B is obtained from (v)

$$0=\frac{R^2}{4\mu}\frac{dP}{dz}+B \Rightarrow B=-\frac{R^2}{4\mu}\frac{dP}{dz}$$

Put the value of B in (v) we get the axial velocity distribution of Hagen Poiseuille flow through pipe as

$$V_z=\frac{r^2}{4\mu}\frac{dP}{dz}-\frac{R^2}{4\mu}\frac{dP}{dz}$$

$V_z=-\frac{R^2}{4\mu}\frac{dP}{dz}\left[1-\frac{r^2}{R^2}\right] \Rightarrow V_z=-\frac{R^2}{4\mu}\frac{dP}{dz}\left[1-\left(\frac{r}{R}\right)^2\right]$ which has the form of paraboloid of revolution.

Lecture # 10

Couette-Poiseuille flow:

As we have $\overline{V}=(u,v,w)$. For one dimension (parallel flow) we can write as $\overline{V}=(0,0,0)$ i.e. v = 0 , w = 0 and u ≠ 0. Also, the equation of continuity in 2-D is

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \text{ where u,v are component of } \overline{V} \text{ and we have v = 0}$$

$$\Rightarrow \frac{\partial u}{\partial x}=0$$

So, u = u (y) , u ≠ u(x) , u is a function of y and independent of x i.e. there is no change in u w.r.t x.

Now from the Navier-Stokes equation in 2-D x-component

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \_\_\_(i)$$

y-component $$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left[\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial y^2}\right] \_\_\_(ii)$$

As we have $\frac{\partial u}{\partial x}=0, v=0 \ and \ u=u(y)$

Using these values in equation (i) and (ii)

(i) $\Rightarrow$ $$0=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\nu\frac{\partial^2 u}{\partial y^2}=\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\Rightarrow \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}=\frac{1}{\rho}\frac{\partial P}{\partial x} \quad \because \nu=\frac{\mu}{\rho}$$

$$\Rightarrow \mu\frac{\partial^2 u}{\partial y^2}=\frac{\partial P}{\partial x} \quad \_\_\_\_\_(iii)$$

(ii) $\Rightarrow \qquad 0 = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} \Rightarrow \dfrac{\partial P}{\partial y} = 0$

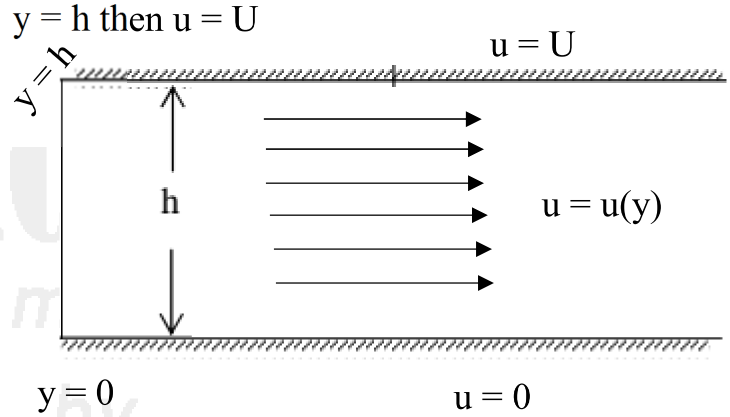
It means $P \neq P(y)$, $P = P(x)$. P is a function of x. Thus, from equation (iii)

$$\mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial x} \qquad \_\_\_\_\_\_(iv)$$

Now we take Poiseuille and Couette at a time. For example, the equation (iv) is of Poiseuille but conditions are of Couette. The boundary conditions are

y = 0 then u = 0

y = h then u = U



Equation (iv) can be written as

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dP}{dx}$$

$$\left(\frac{du}{dy}\right) = \frac{1}{\mu}\frac{dP}{dx}$$

$$d\left(\frac{du}{dy}\right) = \frac{1}{\mu}\frac{dP}{dx}.dy$$

On integration

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dP}{dx}.y + c_1 \qquad \_\_\_\_\_\_(v)$$

Again, on integration

$$u = \frac{1}{2\mu}\frac{dP}{dx}.y^2 + c_1 y + c_2 \qquad \_\_\_\_\_\_(vi)$$

By applying boundary conditions

When y = 0 , u = 0

(vi) $\Rightarrow 0 = 0 + 0 + c_2 \Rightarrow c_2 = 0$

$$\Rightarrow u = \frac{1}{2\mu}\frac{dP}{dx}.y^2 + c_1 y \quad \_\_\_\_\_ (vii)$$

When y = h , u = U

(vii)

$$\Rightarrow U = \frac{1}{2\mu}\frac{dP}{dx}.\mathrm{h}^2 + c_1 h$$

$$c_1 h = U - \frac{\mathrm{h}^2}{2\mu}\frac{dP}{dx}$$

$$c_1 = \frac{U}{h} - \frac{\mathrm{h}}{2\mu}\frac{dP}{dx}$$

Put in (vii)

$$u = \frac{1}{2\mu}\frac{dP}{dx}.y^2 + \left(\frac{U}{h} - \frac{\mathrm{h}}{2\mu}\frac{dP}{dx}\right)y$$

$$u = \frac{1}{2\mu}\frac{dP}{dx}.y^2 + \frac{U}{h}y - \frac{\mathrm{h}}{2\mu}\frac{dP}{dx}y$$

$$u = \frac{U}{h}y - \frac{\mathrm{h}}{2\mu}\frac{dP}{dx}y\frac{1}{2\mu}\frac{dP}{dx}.y^2$$

$$u = \frac{y}{h}U - \frac{\mathrm{h}}{2\mu}\frac{dP}{dx}y\left(1 - \frac{y}{h}\right)$$

$$u = \frac{y}{h}U - \frac{\mathrm{h}^2}{2\mu}\frac{dP}{dx}\frac{y}{h}\left(1 - \frac{y}{h}\right)$$

Which is combine Couette Poiseuille equation.

For non-dimensional

$$\frac{u}{U} = \frac{y}{h} + \frac{\mathrm{h}^2}{2\mu}\left(-\frac{dP}{dx}\right).\frac{1}{U}\frac{y}{h}\left(1 - \frac{y}{h}\right)$$

$$\frac{u}{U} = \frac{y}{h} + \overline{P}\frac{y}{h}\left(1 - \frac{y}{h}\right) \quad where\ \overline{P} = \frac{\mathrm{h}^2}{2\mu}\left(-\frac{dP}{dx}\right).\frac{1}{U}$$

$\overline{P}$ is non-dimensional pressure.

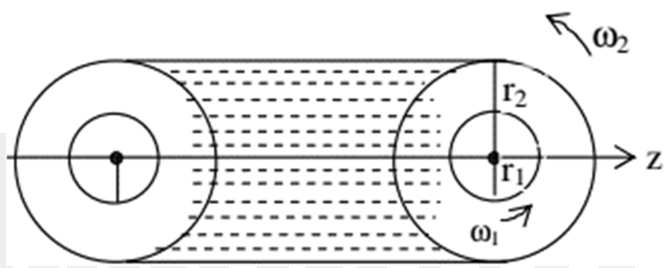
Non-dimensional equation of Couette Poiseuille at a time.

Let $u^* = \dfrac{u}{U}, y^* = \dfrac{y}{h} \Rightarrow u^* = y^* + \overline{P} y^* \left(1 - y^*\right)$

This is the required Couette Poiseuille flow at a time.

Flow between two concentric rotating cylinders:

Consider the steady laminar flow of a viscous incompressible fluid between two infinitely long concentric rotating cylinder with radii R_1 and R_2 ($R_2 > R_1$). Let ω_1 and ω_1 be the steady angular velocities (speed / rotating speed) of the inner and outer cylinder respectively as shown in figure.



Assume the flow between the cylinders to be peripheral (circular or round about) so that we have only the tangential component of velocity V_θ i.e. $V_r = V_z = 0$. The equation of continuity in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rV_r\right) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

Reduces to $\dfrac{\partial V_\theta}{\partial \theta} = 0$ ____(i) $\because V_r = V_z = 0$

So, that V_θ does not depend on θ and $V_\theta = V_\theta(r,z)$. Also, since the cylinders are infinitely long. So, V_θ cannot be a function of z. Thus, we have

$$V_\theta = V_\theta(r) \text{\_\_\_\_(ii)}$$

The Navier Stokes equation in cylindrical coordinates are

R – component

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial t} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_r$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_r}{\partial z^2}\right]$$

θ-Component

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial t}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\rho g_\theta$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]$$

z-component

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial t}+\frac{u_\theta}{r}\frac{\partial u}{\partial\theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial z}+\rho g_z$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]$$

The Navier Stokes equation in cylindrical polar coordinates for present case reduces to

$$-\rho\frac{V_\theta^2}{r}=-\frac{\partial P}{\partial r}\quad\_\_\_\_\_(iii)$$

$$0=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left[\frac{\partial^2 V_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r^2}\right]\quad\_\_\_\_\_(iv)$$

$$0=-\frac{\partial P}{\partial z}\quad\_\_\_\_\_(v)$$

Equation (v) shows that P is independent of z. So, P = P(r,θ).

Since V_θ is a function of 'r' only. It follows that form equation (iii) the pressure must be function of 'r' only i.e. P = P(r). Hence the term $\frac{\partial P}{\partial\theta}$ in (iv) is zero. The equation (iii) and (iv) can be written as

$$\rho\frac{V_\theta^2}{r}=\frac{dP}{dr}\quad\_\_\_\_\_(vi)$$

$$\frac{d^2V_\theta}{dr^2}+\frac{1}{r}\frac{dV_\theta}{dr}-\frac{V_\theta}{r^2}=0 \quad \_\_\_\_\_(vii)$$

Equation (vii) can be written as

$$\frac{d^2V_\theta}{dr^2}+\frac{d}{dr}\left(\frac{V_\theta}{r}\right)=0$$

On integration

$$\frac{dV_\theta}{dr}+\frac{V_\theta}{r}=2A \qquad \because \text{2A is constant}$$

$$\frac{1}{r}\left(r\frac{dV_\theta}{dr}+V_\theta\right)=2A$$

$$\frac{d}{dr}\left(rV_\theta\right)=2Ar$$

Again, on integration

$$rV_\theta=2A.\frac{r^2}{2}+B$$

$$rV_\theta=Ar^2+B$$

$$\Rightarrow \quad V_\theta=Ar+\frac{B}{r} \quad \_\_\_\_\_(viii)$$

Where A and B are constant of integration. The boundary conditions of this rotating cylinder are

$$V_\theta=R_1\omega_1 \quad at\ r=R_1 \quad \because v=r\omega$$

$$V_\theta=R_2\omega_2 \quad at\ r=R_2 \quad \because v=r\omega$$

Using these conditions equations (viii) becomes

$$R_1\omega_1=R_1A+\frac{B}{R_1} \quad \_\_\_\_(ix)$$

$$R_2\omega_2=R_2A+\frac{B}{R_2} \quad \_\_\_\_(x)$$

Solving these equation (ix) and (x)

$$A = \frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2}$$

$$B = \frac{-R_1^2 R_2^2\left(\omega_2 - \omega_1\right)}{R_2^2 - R_1^2}$$

Put these values of A and B in (viii)

$$V_\theta = r\left(\frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2}\right) + \frac{1}{r}\left(\frac{-R_1^2 R_2^2\left(\omega_2 - \omega_1\right)}{R_2^2 - R_1^2}\right)$$

Lecture # 11

We know that

$$V_\theta = r\left(\frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2}\right) + \frac{1}{r}\left(\frac{-R_1^2R_2^2(\omega_2-\omega_1)}{R_2^2 - R_1^2}\right)$$

$$V_\theta = \frac{1}{R_2^2 - R_1^2}\left[\left(R_2^2\omega_2 - R_1^2\omega_1\right)r - \left(\frac{R_1^2R_2^2(\omega_2-\omega_1)}{r}\right)\right] \quad \_\_\_\_(A)$$

Angular Velocity:

Let ω be the angular velocity of the fluid then $V_\theta = r\omega \Rightarrow \omega = \frac{V_\theta}{r}$ from equation (A) we get

$$\omega = \frac{1}{r}\left[\frac{1}{R_2^2 - R_1^2}\left\{\left(R_2^2\omega_2 - R_1^2\omega_1\right)r - \left(\frac{R_1^2R_2^2(\omega_2-\omega_1)}{r}\right)\right\}\right]$$

$$\omega = \frac{1}{R_2^2 - R_1^2}\left\{\left(R_2^2\omega_2 - R_1^2\omega_1\right) - \left(\frac{R_1^2R_2^2(\omega_2-\omega_1)}{r^2}\right)\right\}$$

$$\omega = \frac{1}{R_2^2 - R_1^2}\left[\frac{\left(R_2^2\omega_2 - R_1^2\omega_1\right)r^2 - R_1^2R_2^2(\omega_2-\omega_1)}{r^2}\right]$$

$$\omega = \frac{1}{R_2^2 - R_1^2}\left[\frac{R_2^2\omega_2 r^2 - R_1^2\omega_1 r^2 - R_1^2R_2^2\omega_2 + R_1^2R_2^2\omega_1}{r^2}\right]$$

Re-arranging

$$\omega = \frac{R_1^2\left(R_2^2 - r^2\right)\omega_1 - R_2^2\left(R_1^2 - r^2\right)\omega_2}{\left(R_2^2 - R_1^2\right)r^2} \quad \_\_\_\_(B)$$

Pressure distribution:

The radial pressure distribution resulting from the peripheral motion can be determined form the equation

$$\rho\frac{V_\theta^2}{r}=\frac{dP}{dr}$$

$$\frac{dP}{dr}=\frac{\rho}{r}V_\theta^2$$

$$\frac{dP}{dr}=\frac{\rho}{r}\cdot\frac{1}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)r-\left(\frac{R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r}\right)\right]^2$$

$$\frac{dP}{dr}=\frac{\rho}{r}\cdot\frac{1}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)^2r^2+\left(\frac{R_1^4R_2^4\left(\omega_2-\omega_1\right)^2}{r^2}\right)-2\left(R_2^2\omega_2-R_1^2\omega_1\right)r.\left(\frac{R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r}\right)\right]$$

$$\frac{dP}{dr}=\frac{\rho}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)^2r+\left(\frac{R_1^4R_2^4\left(\omega_2-\omega_1\right)^2}{r^3}\right)-\frac{2\left(R_2^2\omega_2-R_1^2\omega_1\right).R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r}\right]$$

On integration

Since $P = P_1$ at $r = R_1$ we get

$$P_1=\frac{\rho}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)\frac{R_1^2}{2}-\frac{R_1^4R_2^4\left(\omega_2-\omega_1\right)^2}{2R_1^2}-2\left(R_2^2\omega_2-R_1^2\omega_1\right).R_1^2R_2^2\left(\omega_2-\omega_1\right)\ln R_1\right]+c_1$$

$$c_1=P_1-\frac{\rho}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)\frac{R_1^2}{2}-\frac{R_1^4R_2^4\left(\omega_2-\omega_1\right)^2}{2R_1^2}-2\left(R_2^2\omega_2-R_1^2\omega_1\right).R_1^2R_2^2\left(\omega_2-\omega_1\right)\ln R_1\right]$$

Put the value of c_1 in equation (C)

$$P=\frac{\rho}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)\frac{r^2}{2}-\frac{R_1^4R_2^4\left(\omega_2-\omega_1\right)^2}{2r^2}-2\left(R_2^2\omega_2-R_1^2\omega_1\right).R_1^2R_2^2\left(\omega_2-\omega_1\right)\ln r\right]$$

$$+P_1-\frac{\rho}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)\frac{R_1^2}{2}-\frac{R_1^4R_2^4\left(\omega_2-\omega_1\right)^2}{2R_1^2}-2\left(R_2^2\omega_2-R_1^2\omega_1\right).R_1^2R_2^2\left(\omega_2-\omega_1\right)\ln R_1\right]$$

$$P=P_1+\frac{\rho}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)\left(\frac{r^2}{2}-\frac{R_1^2}{2}\right)-\frac{R_1^4R_2^4\left(\omega_2-\omega_1\right)^2}{2}\left(\frac{1}{r^2}-\frac{1}{R_1^2}\right)-2\left(R_2^2\omega_2-R_1^2\omega_1\right).R_1^2R_2^2\left(\omega_2-\omega_1\right)\left(\ln r-\ln R_1\right)\right]$$

$$P=P_1+\frac{\rho}{\left(R_2^2-R_1^2\right)^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)\left(\frac{r^2-R_1^2}{2}\right)-\frac{R_1^4R_2^4\left(\omega_2-\omega_1\right)^2}{2}\left(\frac{1}{r^2}-\frac{1}{R_1^2}\right)-2\left(R_2^2\omega_2-R_1^2\omega_1\right).R_1^2R_2^2\left(\omega_2-\omega_1\right)\left(\frac{\ln r}{\ln R_1}\right)\right]$$

____(D)

This equation is the required pressure distribution and can be used to find the pressure of rotating cylinder.

Maximum Velocity:

The maximum velocity will occur at the position r where $\frac{dV_\theta}{dr}=0$

Now from equation (A) $\Rightarrow \quad V_\theta=\frac{1}{R_2^2-R_1^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)r-\left(\frac{R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r}\right)\right]$

$$\frac{dV_\theta}{dr}=\frac{1}{R_2^2-R_1^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)+\left(\frac{R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r^2}\right)\right]$$

Put $\frac{dV_\theta}{dr}=0$

$$\frac{1}{R_2^2-R_1^2}\left[\left(R_2^2\omega_2-R_1^2\omega_1\right)+\left(\frac{R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r^2}\right)\right]=0$$

$$\frac{1}{R_2^2-R_1^2}\left[\frac{\left(R_2^2\omega_2-R_1^2\omega_1\right)r^2+R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r^2}\right]=0$$

$$\left(R_2^2\omega_2-R_1^2\omega_1\right)r^2+R_1^2R_2^2\left(\omega_2-\omega_1\right)=0$$

$$\left(R_2^2\omega_2-R_1^2\omega_1\right)r^2=-R_1^2R_2^2\left(\omega_2-\omega_1\right)$$

$$\left(R_2^2\omega_2-R_1^2\omega_1\right)r^2=R_1^2R_2^2\left(\omega_1-\omega_2\right)$$

$$r^2=\frac{R_1^2R_2^2\left(\omega_1-\omega_2\right)}{\left(R_2^2\omega_2-R_1^2\omega_1\right)}$$

$$r=R_1R_2\sqrt{\frac{\left(\omega_1-\omega_2\right)}{R_2^2\omega_2-R_1^2\omega_1}}$$

$$r=R_1R_2\sqrt{\frac{\left(\omega_1-\omega_2\right)}{R_2^2\left(\omega_2-\frac{R_1^2\omega_1}{R_2^2}\right)}}$$

$$r=R_1\sqrt{\frac{\left(\omega_1-\omega_2\right)}{\omega_2-\frac{R_1^2\omega_1}{R_2^2}}}\quad\text{\_\_\_\_}(E)$$

For objectives

- Several possible situations can arises depending on the value of angular velocities ω_1 *and* ω_2.
- If $\omega_1>\omega_2$ the numerator is negative. Then since $R_2>R_1$ we have $\omega_2-\left(\frac{R_1}{R_2}\right)^2\omega_1>0$ and there is no real value of r. This implies that fluid velocity increases continuously from $V_\theta=R_1\omega_1$ at the inner surface to $V_\theta=R_2\omega_2$ at the outer surface.
- If $\omega_2<\omega_1$, the numerator is positive. However, there are three possibilities depending on the denominator $\left[\omega_2-\left(\frac{R_1}{R_2}\right)^2\omega_1\right]$ being positive, negative or zero.

(i) If $\omega_2>\left(\frac{R_1}{R_2}\right)^2\omega_1$ the denominator is positive and there is a real value occurs at a definite radius r.

(ii) If $\omega_2<\left(\frac{R_1}{R_2}\right)^2\omega_1$ the denominator is negative and there is no real value of radius r.

(iii) If $\omega_2 = \left(\frac{R_1}{R_2}\right)^2 \omega_1$ the value of radius r is indeterminate.

To summarize the tangential velocity attains a Maximum value at some radius attains a maximum value at some radius $R_1 < r < R_2$ only if $\omega_1 > \omega_2 > \left(\frac{R_1}{R_2}\right)^2 \omega_1$

Shearing Stress:

The shearing stress in this case can be determined from

$$\tau_{r\theta} = \mu\left[r\frac{d}{dr}\left(\frac{V_\theta}{r}\right) - \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right]$$

Since $V_r = 0$

$$\Rightarrow \tau_{r\theta} = \mu\left[r\frac{d}{dr}\left(\frac{V_\theta}{r}\right)\right]$$

$$\Rightarrow \tau_{r\theta} = \mu\left[r\frac{d}{dr}\left[\frac{1}{r}.\frac{1}{R_2^2 - R_1^2}\left\{\left(R_2^2\omega_2 - R_1^2\omega_1\right)r - \left(\frac{R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r}\right)\right\}\right]\right]$$

$$\Rightarrow \tau_{r\theta} = \frac{\mu r}{R_2^2 - R_1^2}.\frac{d}{dr}\left\{\left(R_2^2\omega_2 - R_1^2\omega_1\right) - \left(\frac{R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r^2}\right)\right\}$$

$$\Rightarrow \tau_{r\theta} = \frac{\mu r}{R_2^2 - R_1^2}.\left(0 + \frac{2R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r^3}\right)$$

$$\Rightarrow \tau_{r\theta} = \frac{\mu r}{R_2^2 - R_1^2}.\left(\frac{2R_1^2R_2^2\left(\omega_2-\omega_1\right)}{r^3}\right)$$

$$\Rightarrow \tau_{r\theta} = \frac{2\mu R_1^2R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2 - R_1^2\right)r^2} \qquad \_\_\_\_(F)$$

The shearing stress at the walls of inner cylinder is

$$\left(\tau_{r\theta}\right)_{r=R_1}=\frac{2\mu R_1^2R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)R_1^2}=\frac{2\mu R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)} \qquad \_\_\_\_\_(G)$$

The shearing stress at the walls of outer cylinder is

$$\left(\tau_{r\theta}\right)_{r=R_2}=\frac{2\mu R_1^2R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)R_2^2}=\frac{2\mu R_1^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)} \qquad \_\_\_\_\_(H)$$

Torque on the cylinder:

Let us determined the torque or moment of shearing forces acting on the cylinders. The shearing stress at the walls of inner cylinder is given as

$$\left(\tau_{r\theta}\right)_{r=R_1}=\frac{2\mu R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)}$$

And the shearing force per unit length of the inner cylinder is

$$F=\left(\tau_{r\theta}\right)_{r=R_1}\times 2\pi R_1$$

$$F=\frac{2\mu R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)}\times 2\pi R_1$$

$$F=\frac{4\pi\mu R_1R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)}$$

The torque experience by a unit length of the inner cylinder is given by

$$T_1=F\times r$$

$$T_1=\frac{4\pi\mu R_1R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)}\times R_1 \qquad \because r=R_1$$

$$T_1=\frac{4\pi\mu R_1^2R_2^2\left(\omega_2-\omega_1\right)}{\left(R_2^2-R_1^2\right)}$$

The torque of the shearing forces acting on the outer cylinder is

$$T_2=-T_1$$

Note that torque is independent of r. The moment or torque exerted by the cylinders upon each other is of interest in viscometery by knowing the geometry and measuring $T(T_1,T_2)$ at either cylinder. One can calculate the viscosity of the fluid, as first suggested by Couette (1890). This is still a popular method in viscometery.

Lecture # 12

Flow through a cylinder of uniform cross-section:

Consider the steady laminar flow of viscous incompressible fluid through a cylinder of orbitrary but uniform cross-section as shown in figure below. Let z-axis be taken as the axes of the pipe. Since the flow is parallel to z-axis. The velocity components u = v = 0 everywhere. Moreover, the flow being steady so

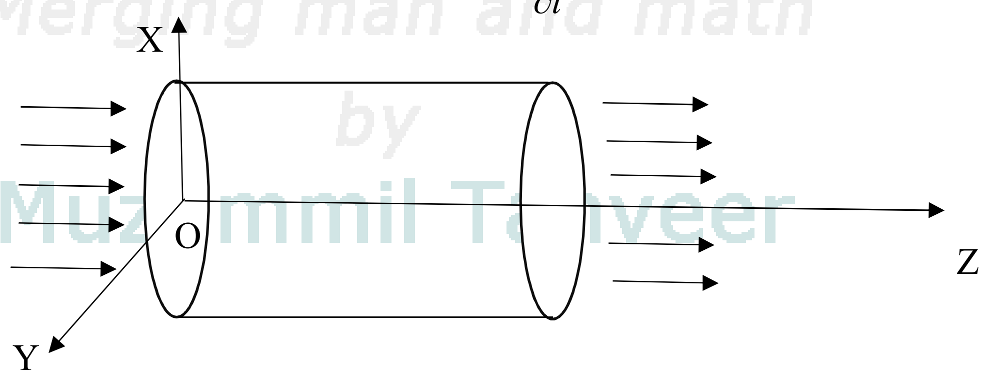
$$\frac{\partial}{\partial t}=0$$

The equation of continuity thus reduces to

$$\frac{\partial w}{\partial z}=0$$

So, that w = w (x,y). Thus, for the present problem

$$u=v=0\ ,\ w=w\,(x,y)\ ,\ \frac{\partial}{\partial t}=0$$



The Navier-Stokes equation without body forces becomes

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial x}\qquad \_\_\_\_\_(i)\ ,\ 0=-\frac{1}{\rho}\frac{\partial P}{\partial y}\qquad \_\_\_\_\_(ii)$$

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\qquad \_\_\_\_\_(iii)$$

From (i) and (ii) $\qquad -\frac{1}{\rho}\frac{\partial P}{\partial x}=-\frac{1}{\rho}\frac{\partial P}{\partial y}=0$

$$\frac{\partial P}{\partial x}=\frac{\partial P}{\partial y}=0 \qquad \because\ P=P(z)$$

From equation (iii)

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \qquad \because \nu=\frac{\mu}{\rho}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial z}=\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$

$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial P}{\partial z}$$

Moreover, $$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial P}{\partial z} \qquad \because \frac{\partial P}{\partial z}\approx\frac{dP}{dz}$$

The L.H.S of this equation is a function of x and y only while R.H.S is a function of z only and since these are equal. Each side must be constant (say) −P. The minus being taken as we except P to decreases as z increases. Thus,

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{-P}{\mu} \quad \_\_\_(iv) \qquad Where\ P=-\frac{dP}{dz}$$

Along with w = 0 on the walls of the cylinder. Hence the problem of finding the velocity distribution reduces to that of finding the solution of equation (iv) subject to boundary condition w = 0 on the cross-section of the pipe (cylinder) cuts the XY-Plane.

The problem can be further simplifying if we write

$$w=w_1-\frac{P}{4\mu}\left(x^2+y^2\right) \qquad \_\_\_\_\_(v)$$

Then $$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w_1}{\partial x^2}-\frac{P}{2\mu} \quad \_\_\_(vi)$$

And $$\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 w_1}{\partial y^2}-\frac{P}{2\mu} \quad \_\_\_(vii)$$

Substituting these partial derivatives in equation (iv) we find that w_1 has to satisfy the two-dimensional Laplace equation.

$$\frac{\partial^2 w_1}{\partial x^2}-\frac{P}{2\mu}+\frac{\partial^2 w_1}{\partial y^2}-\frac{P}{2\mu}=\frac{-P}{\mu}$$

$$\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}-\frac{P}{\mu}=-\frac{P}{\mu}$$

$$\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}=0$$

With boundary condition w = 0 equation (v) becomes

$$0=w_1-\frac{P}{4\mu}\left(x^2+y^2\right)$$

$$\Rightarrow \quad w_1=\frac{P}{4\mu}\left(x^2+y^2\right)$$

In cylindrical polar coordinates (r,θ,z) equation (iv) can be written as

$$\frac{\partial^2 V_z}{\partial z^2}+\frac{1}{r}\frac{\partial V_z}{\partial z}+\frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2}=-\frac{P}{\mu}$$

Where $V_z=V_z\left(r,\theta\right)$ and $P=\dfrac{-dP}{dz}$ is pressure gradient.

Reynold Transport Theorem:

$$\frac{D}{Dt}\iiint\limits_V G\ dV=\iiint\limits_V \frac{\partial G}{\partial t}\,dV+\iint\limits_s G\ \vec{q}.\hat{n}\,ds$$

Where G is any fluid property per unit volume.

Transport of mass:

Assume the fluid property G with density ρ and there is no sink or source of mass inside the system, then

$\iiint\limits_V \rho\, dV$ is the mass of fluid with volume V.

$$\Rightarrow \frac{D}{Dt}\iiint_V \rho \, dV = 0 \qquad \_\_\_\_(i)$$

By using Reynold Transport theorem

$$\frac{D}{Dt}\iiint_V \rho \, dV = \iiint_V \frac{\partial \rho}{\partial t} \, dV + \iint_s \rho \;\; \vec{q}.\hat{n}\, ds \qquad \because G = \rho$$

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV + \iint_s \rho \;\; \vec{q}.\hat{n}\, ds = 0 \qquad \because by \;\; (i)$$

Now by using Gauss Divergence theorem

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV + \iiint_V \nabla.(\rho q)\, dV = 0$$

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla.(\rho q)\right) dV = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho q) = 0$$

$$\frac{\partial \rho}{\partial t} + q - \nabla \rho + \rho \nabla.q = 0 \qquad \because \;\; \nabla \rho = 0 \;\; \text{By Kelvins theorem}$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla.q = 0$$

For incompressible

$$\frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \quad 0 + \rho \nabla.q = 0 \;\; \Rightarrow \;\; \rho \nabla.q = 0$$

$$\Rightarrow \quad \nabla.q = 0$$

Transport of any dynamical:

Let G = ρF be any fluid property per unit mass then prove that

$$\frac{D}{Dt}\iiint_V \rho F\ dV = \iiint_V \rho \frac{DF}{Dt}\, dV$$

Proof: We know that the Reynold theorem

$$\frac{D}{Dt}\iiint_V G\ dV = \iiint_V \frac{\partial G}{\partial t}\, dV + \iint_s G\ \vec{q}.\hat{n}\, ds \qquad \_\_\_\_(i)$$

Put G = ρF

$$\frac{D}{Dt}\iiint_V \rho F\ dV = \iiint_V \frac{\partial}{\partial t}(\rho F)\, dV + \iint_s (\rho F)\ \vec{q}.\hat{n}\, ds$$

By using Gauss divergence theorem

$$\frac{D}{Dt}\iiint_V \rho F\ dV = \iiint_V \frac{\partial}{\partial t}(\rho F)\, dV + \iiint_V \nabla.(\rho F q)\, dV$$

$$\frac{D}{Dt}\iiint_V \rho F\ dV = \iiint_V \left(F\frac{\partial \rho}{\partial t} + \rho\frac{\partial F}{\partial t}\right) dV + \iiint_V \left(\rho F\ \nabla.q + q.\nabla(\rho F)\right) dV$$

$$\frac{D}{Dt}\iiint_V \rho F\ dV = \iiint_V F\frac{\partial \rho}{\partial t} dV + \iiint_V \rho\frac{\partial F}{\partial t} dV + \iiint_V \rho F\ \nabla.q\, dV + \iiint_V q.\nabla(\rho F)\, dV$$

$$\frac{D}{Dt}\iiint_V \rho F\ dV = \iiint_V F\frac{\partial \rho}{\partial t} dV + \iiint_V \rho\frac{\partial F}{\partial t} dV + \iiint_V \rho F\ \nabla.q\, dV + \iiint_V q.(\rho\nabla F + F\nabla\rho)\, dV$$

Rearranging

$$\frac{D}{Dt}\iiint_V \rho F\ dV = \iiint_V F\left(\frac{\partial \rho}{\partial t} + \rho\ \nabla.q + q.\nabla\rho\right) dV + \iiint_V \rho\left(\frac{\partial F}{\partial t} + q.\nabla F\right) dV$$

$$\frac{D}{Dt}\iiint_V \rho F\ dV = \iiint_V F\left(\frac{\partial \rho}{\partial t} + \rho\ \nabla.q + 0\right) dV + \iiint_V \rho\left(\frac{\partial F}{\partial t} + q.\nabla F\right) dV \qquad \because \nabla\rho = 0$$

$$\frac{D}{Dt}\iiint_V \rho F\,dV = \iiint_V F\left(\frac{D\rho}{Dt}+\rho\,\nabla.q\right)dV + \iiint_V \rho\frac{DF}{Dt}dV \quad \because \frac{DF}{Dt}=\frac{\partial F}{\partial t}+q.\nabla F \ \ \& \frac{\partial\rho}{\partial t}\approx\frac{D\rho}{Dt}$$

By Equation of continuity

$$\frac{D\rho}{Dt}+\rho\,\nabla.q=0$$

$$\frac{D}{Dt}\iiint_V \rho F\,dV = 0+\iiint_V \rho\frac{DF}{Dt}dV$$

$$\frac{D}{Dt}\iiint_V \rho F\,dV = \iiint_V \rho\frac{DF}{Dt}dV$$ Hence Proved.