Linear Algebra.

Vector Spaces

Book: Linear Algebra with Applications
by George Nakos, David Joyner

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College Jhang

1) An additive abelian group V is called a vector space over the field F if for all $v, w \in V$ and $a, b \in F$

i) $av \in V$

ii) $a(v+w) = av + aw$

iii) $(a+b)v = av + bv$

iv) $(ab)v = a(bv)$

v) $1v = v$

2) A nonempty subset W of a vector space V is a subspace of V if and only if

i) $u, v \in W \Rightarrow u+v \in W$

ii) for $c \in F$, $v \in W$, $cv \in W$

Example

1) The set D_n of all diagonal matrices of size n is a subspace of M_{nn}.

2) $GL(n, \mathbb{R})$ is not a vector space over $\mathbb{R}$.

3) Let V be a vector space and $v_1, v_2, \ldots, v_k \in V$ then

$$\text{span}\{v_1, v_2, \ldots, v_k\} = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k : a_i \in F\}$$

If $\text{span}\{v_1, v_2, \ldots, v_k\} = V$, then $\{v_1, v_2, \ldots, v_k\}$ is called spanning set of V.

4) Is $-1+x^2$ in $\text{span}\{1+x+x^3, \ -x-x^2-x^3\}$?

Sol:- If $-1+x^2 \in \text{span}\{1+x+x^3, \ -x-x^2-x^3\}$

then there must be $a, b \in \mathbb{R}$ s.t.

$$-1+x^2 = a(1+x+x^3) + b(-x-x^2-x^3)$$

$$-1+x^2 = a + (a-b)x - bx^2 + (a-b)x^3$$

$\Rightarrow$ $\boxed{a=-1}$; $a-b=0$, $-b=1$, $a-b=0$

$$\boxed{b=-1}$$

Values of a and b are consistent, so

$$-1+x^2 \in \text{span}\{1+x+x^3, \ -x-x^2-x^3\}$$

5) Show that $\{(1,2,-1), (-1,1,-2), (1,1,1)\}$ spans $\mathbb{R}^3$

Sol:- Columns of any $n \times n$ non-singular matrix spans $\mathbb{R}^n$. (basis for $\mathbb{R}^n$)

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix}$

then $A \sim \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 0 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & -3 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

So $\{(1,2,-1), (-1,1,-2), (1,1,1)\}$ spans $\mathbb{R}^3$.

6) Let $S \subseteq V$ Then

i) $\text{span}(S)$ is a subspace of V.

ii) $\text{span}(S)$ is the smallest subspace of V that contans S.

7) If one of the vectors $v_1, v_2, \ldots, v_k$ is a linear combination of others, then span remains same if we remove this vector.

8). A subset $X = \{v_1, v_2, \ldots, v_k\}$ of V is called linearly independent (L.I) if the only solution of equation

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

is $a_1 = a_2 = \cdots = a_k = 0$.

Otherwise X is called linearly dependent (L.D).

9). Both $\{1, \cos 2x, \cos^2 x\}$ and $\{1, \cos 2x, \sin^2 x\}$ are linearly dependent.

10). Show that the set $\{x^2, 1+x, -1+x\}$ is L.I.

Sol:- We form a matrix corresponding to polynomials

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

which is invertible. So $\{x^2, 1+x, -1+x\}$ is L.I.

11) Any subset having zero vector is L.D.

12) If X consists of two or more vectors, then X is L.D. if and only if one of vector in X is linear combination of other vectors in X.

13). Any subset of a linearly independent set is itself linearly independent.

14). Any superset of a linearly dependent set is itself linearly dependent.

15). Suppose $\{v_1, v_2, \dots, v_k\}$ is L.I. any vector $v \in \text{span}\{v_1, v_2, \dots, v_k\}$ is uniquely expressible as a linear combination of vectors $v_1, v_2, \dots, v_k$.

16). If $v \notin \text{span}\{v_1, v_2, \dots, v_k\}$, then the set $\{v_1, v_2, \dots, v_k, v\}$ is linearly independent.

17) A subset B of V is called a basis of V if

i) B is linearly independent

ii) span $B = V$

18) $\{1+x, -1+x, x\}$ is not a basis of P_2.

19) If a vector space V has a basis with n elements then n is called dimension of V.

20). The dimension of the subspace

$$\{(2x+y,\ x,\ -x-2y,\ x+y+z) : x, y, z \in \mathbb{R}\}$$

of $\mathbb{R}^4$ is 3.

21). The dimension of the subspace

$$\text{span}\{(1,1,1), (2,1,-1), (1,0,-2)\} \text{ of } \mathbb{R}^3$$

is 2.

22) Let V be an n-dimensional vector space and let S be a set with m elements.

i) If S is L.I., then $m \leq n$.

ii) If S spans V, then $m \geq n$.

23). Let V be a vector space with dimension n and let S be a set with m elements

i) If S is L.I and $m < n$, then S can be enlarged to a basis.

ii) If $\text{span}(S) = V$, then S contains a basis.

24). If W is a subspace of V, then
$$\dim(W) \leq \dim(V).$$
Equality holds in case $V=W$.

25). The span of columns of a matrix A is called column space of A.

26). Let $S=\{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}^m$. A basis for $\text{span}(S)$ (equivalenty $Col(A)$ where $A=[a_1 \ a_2 \ \ldots \ a_n]$) can be found as follows

i) Form the $m \times n$ matrix $A=[a_1 \ a_2 \ \ldots \ a_n]$.

ii) Row-reduce A to an echelon form B and identify the pivot columns of A.

iii) A basis for $\text{span}(S)$ is the set of pivot columns of A.

27) The span of rows of a matrix A is called row space of A.

28). If $A \sim B$ then $Row(A)=Row(B)$. (not in case column space).

29). The nonzero rows of any echelon form of a matrix A form a basis for $Row(A)$.

30). $\dim(Col(A)) = \dim(Row(A))$ for any matrix A.

31). For any matrix A, we define

$$\text{Rank}(A) = \dim(\text{Col}(A)) = \dim(\text{Row}(A)).$$

i.e. Rank (A) is the number of the pivots of A.

32). For a matrix A of order $m \times n$,

$$\text{Rank}(A) \leq \min(m, n).$$

33). $$\text{Rank}(A) = \text{Rank}(A^T).$$

34). For an $m \times n$ matrix A, we define

$$\text{Null}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

and Nullity of $A = \dim(\text{Null}(A))$.

35). To find a basis for Null(A):

i) Find the general solution vector of $Ax = 0$.
ii) Write the solution vector as linear combination with coefficients the parameters.
iii) The vectors of the linear combination form a basis for Null(A).

36). The nullity of A equals the number of free variables of $Ax = 0$.

37). For a matrix A,

$$\text{Rank}(A) + \text{Nullity}(A) = \text{number of columns of } A$$

38). Suppose that $Ax = 0$ has 20 unknowns and its solution space is spanned by 6 L.I vectors..

i) What is the rank of A?

ii) Can A have size 13×20?

Sol:

(i) Since

$$\text{Nullity}(A) = 6$$

and no. of columns of $A = 20$

so $$\text{rank}(A) = 20 - 6 = 14$$

(ii) $\text{rank}(A) = 14$ means the number of nonzero rows of A are atleast 14 so A can never has 13 rows.

39). The linear system $\underset{m\times n}{A}X = B$ is

i) inconsistent if

$$\text{Rank}(A) \neq \text{Rank}([A : B]).$$

ii) consistent if

$$\text{Rank}(A) = \text{Rank}([A : B]).$$

(a) unique solution if

$$\text{Rank}(A) = \text{Rank}([A : B]) = \text{full rank} = \min(m, n)$$

(b) infinite solution if

$$\text{Rank}(A) = \text{Rank}([A : B]) < \text{full } (< \min(m, n))$$

40). Let V be a vector space with basis $B = \{v_1, v_2, \ldots, v_n\}$, then for $v \in V$, there exist unique scalars $c_1, c_2, \ldots, c_n$ such that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

The matrix

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called coordinate matrix (vector) of v with respect to B. $[v]_B$ changes as the basis B changes. Also $[v]_B$ depends on the order of elements of B.

41). Consider the basis $B = \{(1,0,-1), (-1,1,0), (1,1,1)\}$ of $\mathbb{R}^3$ and $v = (2,-3,4)$. Find $[v]_B$.

Sol:-

$$(2,-3,4) = a(1,0,-1) + b(-1,1,0) + c(1,1,1)$$

Then values of a, b, c are $-3, -4, 1$ respectively.

Thus $$[v]_B = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}$$

42). Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let $u, u_1, \ldots, u_n$ be vectors in V. Then u is a linear combination of $u_1, u_2, \ldots, u_n$ iff $[u]_B$ is a linear combination of $[u_1]_B, [u_2]_B, \ldots, [u_n]_B$. Furthermore for scalars $c_1, c_2, \ldots, c_n$

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

iff

$$[u]_B = c_1 [u_1]_B + c_2 [u_2]_B + \cdots + c_n [u_n]_B.$$

43). Let V be vector space with basis B. Then $\{u_1, u_2, \dots, u_n\}$ is L.I. if and only if $\{[u_1]_B, [u_2]_B, \dots, [u_n]_B\}$ is L.I. in $\mathbb{R}^n$.

44). Let $B = \{v_1, v_2, \dots, v_n\}$ and $B' = \{v_1', v_2', \dots, v_n'\}$ be two bases for V. Let P be a matrix

$$P = \left[\, [v_1]_{B'} \; [v_2]_{B'}, \dots, [v_n]_{B'} \,\right]$$

Then P is invertible and it is the only matrix such that for all $v \in V$,

$$[v]_{B'} = P\,[v]_B .$$

This matrix P is called the transition matrix (or change-of-basis matrix) from B to B'.

Quotient Spaces.

Book Fundamentals of Linear Algebra by Dennis B. Ames.

1) Let U be a subspace of a vector space V, the set

$$V/U = \{ v + U : v \in V \}.$$

where $v + U = \{ v + u \quad u \in U \}$

forms a vector space, called quotient space or factor space of V by U.

The set V/U is a vector space under the operations defined by

$$(v + U) + (w + U) = (v + w) + U, \quad \text{for } v, w \in V$$

and

$$c\,(v + U) = cv + U \quad , \quad \text{for } c \in F,\ v \in V.$$

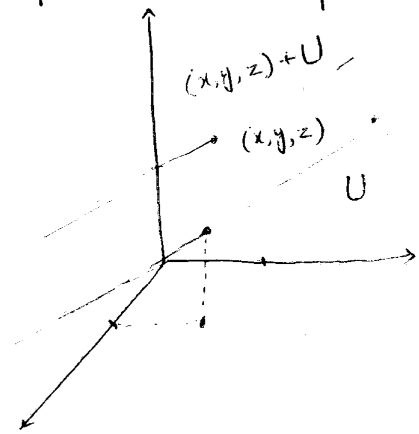
2). Let U be a subspace of $\mathbb{R}^3$ spanned by the vector $(1, 1, 1)$; that is,

$$U = \text{span}\{(1,1,1)\} = \{ k(1,1,1) \quad k \in \mathbb{R} \}.$$

U is the straight line through the origin and the point $(1,1,1)$. For any vector $(x,y,z) \in \mathbb{R}^3$ we can regard the coset $(x,y,z) + U$ as the set of vectors obtained by adding the vector (x,y,z) to each vector of U.

This coset is therefore the set of all

vectors on the line through the point (x, y, z) parallel to the line U. Hence $\mathbb{R}^3/U$ is the collection of lines parallel to U.



3) If $U = \text{span}\{(1,0,0), (0,1,0)\}$

then U is the set of all vectors in the xy-plane, and the cosets are the planes parallel to the xy-plane. Thus the quotient space $\mathbb{R}^3/U$ is the collection of planes parallel to xy-plane.

(i) If V is a finite dimensional vector space and if U is a subspace of V, then

$$\dim(V) = \dim(U) + \dim\left(\frac{V}{U}\right)$$

That is

$$\dim(V/U) = \dim(V) - \dim(U).$$

5). Let $U = \text{span}\{(1,0,0)\}$. Then for any vector $(x,y,z) \in \mathbb{R}^3$, we have

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

and therefore, since $x(1,0,0) \in U$,

$$\begin{aligned}(x,y,z)+U &= U + y\big((0,1,0)+U\big) + z\big((0,0,1)+U\big)\\ &= y(0,1,0) + U + z(0,0,1) + U\\ &= (0,y,z) + U\end{aligned}$$

The vectors $(0,1,0)+U$ and $(0,0,1)+U$ are therefore also independent and hence they form a basis of V/U.

6). The set P_1 is a subspace of P_4. Form the quotient space P_4/P_1

Sol. For $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in P_4$, we have

$$\begin{aligned}p(x) + P_1 &= a_4(x^4+P_1) + a_3(x^3+P_1) + a_2(x^2+P_1) + P_1\\ &= a_4x^4 + a_3x^3 + a_2x^2 + P_1\end{aligned}$$

So x^4+P_1, x^3+P_1, x^2+P_1 spans P_4/P_1.

Moreover these are linearly independent, so a bases for P_4/P_1 is

$$\{x^4+P_1,\ x^3+P_1,\ x^2+P_1\}$$

7). For a vector space V and a subspace U of V, if $v_1+U, v_2+U, \ldots, v_k+U$ is a basis for V/U and if $\beta_1, \beta_2, \ldots, \beta_r$ is a basis for U, then

$$B = \{v_1, v_2, \ldots, v_k, \beta_1, \beta_2, \ldots, \beta_r\}$$ is a basis for V.

8). Let $B = \{u_1, u_2, \ldots, \omega_n\}$ be a basis for a subspace U of V, and extend it to a basis

$$\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_k\}$$

of V. Then

$$\{v_1+U, v_2+U, \ldots, v_k+U\}$$

is a basis for V/U.

Linear Transformations.

Book Linear Algebra with Applications
by George Nakos, David Joyner.

1). Let V and W be vector spaces. A linear transformation from V to W is a map $T: V \longrightarrow W$ such that for all $u, v \in V$ and scalar c,

i) $T(u+v) = T(u) + T(v)$.

ii) $T(cu) = cT(u)$.

2). Show that $T: \mathbb{R}^3 \to \mathbb{R}^2$, defined by

$$T(x,y,z) = (x-z, y+z)$$

is linear.

Sol:- Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$. Then

$$T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$$

$$= ((x_1+x_2)-(z_1+z_2), (y_1+y_2)+(z_1+z_2))$$

$$= (x_1 - z_1 + x_2 - z_2, y_1 + z_1 + y_2 + z_2)$$

$$= (x_1 - z_1, y_1 + z_1) + (x_2 - z_2, y_2 + z_2)$$

$$= T(u) + T(v)$$

Akhtar Abbas
Lecturer (Mathematics)
at Ghazali Degree College Jhang

and

$$T(cu) = T(cx_1, cy_1, cz_1)$$

$$= (cx_1 - cz_1, cy_1 + cz_1)$$

$$= c(x_1 - z_1, y_1 + z_1) = cT(u)$$

So T is linear.

3) Every matrix transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T(x) = Ax$$

where A is an $m \times n$ matrix, is linear.

4) A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is called

i) reflection about x-axis, if T is defined by

$$T(x,y) = (x,-y)$$

ii) reflection about y-axis, if T is defined by

$$T(x,y) = (-x,y)$$

iii) Compression along x-axis if

$$T(x,y) = (cx, y) \quad . \quad 0 < c < 1$$

iv) expansion along x-axis if

$$T(x,y) = (cx, y) \quad , \quad c > 1$$

v) compression along y-axis if

$$T(x,y) = (x, cy) \quad , \quad 0 < c < 1$$

vi) expansion along y-axis if

$$T(x,y) = (x, cy) \quad , \quad c > 1$$

vii) shear along x-axis if

$$T(x,y) = (x+cy, y) \quad , \quad c - \text{constant}$$

viii) shear along y-axis if

$$T(x,y) = (x, cx+y)$$

ix) counterclockwise θ rad rotation if $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

5). Show that the transformation $T : P_2 \to P_1$ defined by

$$T(a+bx+cx^2) = b+2cx$$

is linear.

Sol:- Let $a_1+b_1x+c_1x^2$, $a_2+b_2x+c_2x^2 \in P_1$. Then

$$T\left((a_1+b_1x+c_1x^2)+(a_2+b_2x+c_2x^2)\right) = T\left((a_1+a_2)+(b_1+b_2)x+(c_1+c_2)x^2\right)$$

$$= (b_1+b_2)+2(c_1+c_2)x^2$$

$$= (b_1+2c_1x^2)+(b_2+2c_2x^2)$$

$$= T(a_1+b_1x+c_1x^2)+T(a_2+b_2x+c_2x^2).$$

6). Let $T: V \to W$ be a linear transformation and let $B = \{v_1, v_2, \ldots, v_n\}$ spans V Then the set $T(B) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans the range of T.

7) Let $T: V \to W$ be a linear transformation. The kernel $K(T)$ and range $R(T)$ of T are defined by

$$K(T) = \{v \in V : T(v) = 0\}$$

$$R(T) = \{T(v) : v \in V\}$$

Moreover $K(T)$ is a subspace of V and $R(T)$ is a subspace of W.

8). Find the kernel of $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-z \\ y+z \end{bmatrix}$$

Sol:

$$\text{Ker}(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - z = 0, \quad y + z = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = z, \quad y = -z \right\}$$

$$= \left\{ \begin{bmatrix} z \\ -z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Akhtar Abbas
Lecturer (Mathematics)
Govt. Ghazali Degree College Jhang

9). We define

nullity of $T = \dim(\text{Ker}(T))$

and rank of $T = \dim(R(T))$.

10) Find bases for the kernel of

$T : \mathbb{R}^4 \to \mathbb{R}^3$, $T(x, y, z, w) = (x+3z, y-2z, w)$

Sol: First we find basis for Ker(T).

$$\text{Ker}(T) = \{(x, y, z, w) : T(x, y, z, w) = (0, 0, 0)\}$$

$$= \{(x, y, z, w) : x + 3z = 0, \quad y - 2z = 0, \quad w = 0\}$$

$$= \{(-3z, 2z, z, 0)\} = \{z(-3, 2, 1, 0) : z \in \mathbb{R}\}$$

$$= \text{span}\{(-3, 2, 1, 0)\}$$

So Basis for Ker(T) is $\{(-3, 2, 1, 0)\}$

11) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a matrix transformation with standard matrix A. Then

i) $Ker(T) = Null(A)$

ii) $R(T) = Col(A)$

iii) $Nullity(T) = Nullity(A)$

iv) $Rank(T) = Rank(A)$

12) Find the range of $T: \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T(x,y,z,w) = (x+3z,\, y-2z,\, w)$$

Sol: We write T as a matrix transformation with matrix

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is in reduced echelon form with pivot columns 1, 2, 4. Hence the vectors $\{(1,0,0), (0,1,0), (0,0,1)\}$ form a basis for $Col(A) = R(T) = \mathbb{R}^3$.

13). If $T: V \to W$ is a linear transformation. Then

$$\dim(V) = Nullity(T) + Rank(T)$$

14) Determine the Nullity and Rank of $T: \mathbb{R}^4 \to P_2$

$$T(a,b,c,d) = (a-b) + (c+d)x + (2a+b)x^2$$

Sol:

$$\begin{aligned} Ker(T) &= \{(a,b,c,d) : T(a,b,c,d) = 0 + 0x + 0x^2\} \\ &= \{(a,b,c,d) : a-b=0,\ c+d=0,\ 2a+b=0\} \\ &= \{(a,b,c,d) : a=b,\ c=-d,\ 3a=0\} \\ &= \{(0,0,c,-c)\} = span\{(0,0,1,-1)\} \end{aligned}$$

Thus $Nullity(T) = \dim(Ker(T)) = 1$. So $Rank(T) = 4-1 = 3$.

15). Let $T : V \to W$ be a linear transformation. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of V and let $B' = \{v_1', v_2', \ldots, v_n'\}$ be a basis of W. The $m \times n$ matrix A with columns

$$[T(v_1)]_{B'}, \; [T(v_2)]_{B'}, \; \ldots, \; [T(v_n)]_{B'}$$

is the only matrix that satisfies

$$[T(v)]_{B'} = A\,[v]_B$$

The matrix A is called the matrix of T with respect to B and B'. (sometimes denoted $[T]_{B,B'}$).

16). Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x-y \\ x+4y \end{bmatrix}$$

and let $B = \{e_2, e_1\}$ and $B' = \{e_3, e_2, e_1\}$ be the basis of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Find the matrix of T with respect to B and B'.

Sol:- We have

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 4e_3 - e_2 + e_1$$

and $$[T(e_2)]_{B'} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

Similarly

$$[T(e_1)]_{B'} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Hence

$$A = [T]_{B,B'} = \begin{bmatrix} 4 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$$

17). Let $T: V \to V$ be a linear transformation from a finite-dimensional vector space V to itself. Let B and B' be two bases of V and let P be the transition matrix from B' to B.
If A is the matrix of T w.r.t. B and A' is the matrix of T w.r.t. B' then

$$A' = P^{-1} A P.$$

18). Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -5x + 6y \\ -3x + 4y \end{bmatrix}$$

and let B and B' be bases

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

a) Compute the matrix A of T w.r.t. B.
b) Compute the transition matrix P from B' to B.
c) Using matrix A of T w.r.t. B and transition matrix P find the matrix A' of T w.r.t. B'.
d) Compute the matrix A' of T w.r.t. B' directly from B'.

Sol:. (a)
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ -3 \end{bmatrix} = -5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence
$$A = [T]_B = \begin{bmatrix} -5 & 6 \\ -3 & 4 \end{bmatrix}$$

(b) Since $\begin{bmatrix}1\\1\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix}$

and $\begin{bmatrix}2\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix}$

So
$$P = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$$

(c)
$$A' = P^{-1}AP = \begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$$

(d)
$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}2\\1\end{bmatrix}$$

and
$$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\-2\end{bmatrix} = 0\begin{bmatrix}1\\1\end{bmatrix} - 2\begin{bmatrix}2\\1\end{bmatrix}$$

So
$$A' = \begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$$

19). Two matrice A and B are said to be similar if there exists an invertible matrix P such that
$$B = P^{-1}AP$$

20). Let A be an $m \times n$ matrix. An affine transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a transformation of the form
$$T(x) = Ax + b$$
for some fixed m-vector b.

If $A = I$, then affine transformation is called a translation by b.

These translations are non-linear if $b \neq 0$.