Mathematical Statistics I

Muzammil Tanveer

mtanveer8689@gmail.com

0316-7017457

Dedicated

To

My Honorable Teacher

&

My Parents

Lecture # 01

Statistics:

Statistics is defined as a science which deals with the collection of facts and data and the drawing conclusions or inference from this data by applying scientific methods **OR** It is defined as a science of estimates and probabilities.

Branches: There are two branches of statistics

(i) Inferential Statistics (ii) Descriptive Statistics

Frequency:

The number of times a thing occur in a definite period or interval of time is called frequency of that thing.

Frequency Distribution:

The process of grouping the data into classes or groups and then determining the frequency of each class is called the frequency distribution.

Example:

Make a grouped frequency distribution to the weights recorded to the nearest grams of 60 apples picket out at from a consignment.

106	107	76	82	109	107	115	93
187	95	123	125	111	92	86	70
126	68	130	129	139	119	115	128
100	186	84	99	113	204	111	141
136	123	90	115	98	110	78	185
162	178	140	152	173	146	158	194
148	90	107	181	131	75	184	104
110	80	118	82				

Solution: R = Range = Max Value – Lowest Value

$$= 204 - 68 = 136$$

Class Interval = h =20

Total number of classes $= \frac{R}{h} = \frac{136}{20} \approx 7$ (approximate)

Classes	Tally	Frequency
65—84	𝍸 ||||	9
85—104	𝍸 𝍸	10
105—124	𝍸 𝍸 𝍸 ||	17
125—144	𝍸 𝍸	10
145—164	𝍸	5
165—184	||||	4
185—204	𝍸	5
		$\sum f = 60$

Measure of central Tendency or Averages:

An average is a single value which is intended to represents a set of data or a distribution as a whole. It is a less or more central value around which the observations tends to cluster such a central value is called the measure of central tendency as it indicates the central position of the distribution. It is also known as measure of location or measure of position.

Types of Averages: There are five types of averages.

(i) Arithmetic Mean (ii) Geometric Mean

(iii). Harmonic Mean (iv) The Median

(v). The Mode

(i) Arithmetic Mean:

It is defined as the value obtained by dividing the sum of observation by their number. $\bar{x} = \frac{x_1+x_2+\cdots+x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$

For grouped data $\bar{x} = \frac{f_1x_1+f_2x_2+\cdots+f_nx_n}{f_1+f_2+\cdots+f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$

Weight (g)	f_i	x_i (Classmarks or midpoint)	f_ix_i
65—84	9	$\frac{65+84}{2} = 74.5$	670.5
85—104	10	94.5	945.0
105—124	17	114.5	1946.5
125—144	10	134.5	1345.0
145—164	5	154.5	772.5
165—184	4	174.5	698.0
185—204	5	194.5	972.5
	$\sum f_i = 60$		$\sum f_ix_i = 7350$

$$\bar{x} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$$

$$\bar{x} = \frac{7350}{60}$$

$$\bar{x} = 122.5$$

Lecture # 2

Geometric Mean: The Geometric mean G of the set n positive values $x_1, x_2, \dots, x_n$ is defined as the nth root of their product.

$$G = \sqrt[n]{x_1 . x_2 . x_3 \dots x_n}$$

Taking log on both sides

$$\log G = \log\,(x_1 . x_2 . x_3 \dots x_n)^{1/n}$$

$$= \frac{1}{n}\log(x_1 . x_2 . x_3 \dots x_n)$$

$$= \frac{1}{n}\left[logx_1 + logx_2 + logx_3 + \cdots + logx_n\right] \Rightarrow \frac{1}{n}\sum_{i=1}^{n} logx_i$$

$$G = \text{anti-log}\left[\frac{1}{n}\sum_{i=1}^{n} logx_i\right]$$

For group data $$G = \text{anti-log}\left[\frac{1}{\sum_{i=1}^{n} f_i}\sum_{i=1}^{n} f_i logx_i\right]$$

Example: Find G.M of data 42,32,37,46,39,36,41,48,36

Solution: n = 9

$$\log G = \frac{1}{9}\left[log45 + log32 + log37 + log46 + log39 + log36 + log41 + log48 + log36\right]$$

log G = 1.59856

G = anti-log (1.59586)

G = 39.68

Example: Find G.M of data given below

Weight (g)	f_i	x_i (Classmarks or midpoint)	$\log x_i$	$f_i \log x_i$
65--84	9	$\frac{65+84}{2}$ = 74.5	1.8722	16.8498
85—104	10	94.5	1.9754	19.7540
105—124	17	114.5	2.0588	34.9996
125—144	10	134.5	2.1287	21.2870
145—164	5	154.5	2.1889	10.9445
165—184	4	174.5	2.2418	8.9672
185—204	5	194.5	2.2889	11.4445
	$\sum f_i = 60$			$\sum_{i=1}^{n} f_i logx_i$ = 124.2466

$$G = \text{anti-log}\left[\frac{1}{\sum_{i=1}^{n} f_i}\sum_{i=1}^{n} f_i logx_i\right]$$

$$= \text{anti-log}\left[\frac{124.2466}{60}\right]$$

$$= \text{anti-log}\,(2.0707)$$

G = 117.7 (grams)

The Harmonic Mean:

The Harmonic mean of a set of n values is defined as the reciprocal of the arithmetic mean of the reciprocal of the values.

$$H.M = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} \qquad \because\ A.M = \frac{\sum_{i=1}^{n} x_i}{n}$$

For the group data

$$H.M = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} f_i\frac{1}{x_i}}$$

Example: Find H.M of 15,20,25

Solution: Here n = 3

$$H.M = \frac{3}{\frac{1}{15}+\frac{1}{20}+\frac{1}{25}} = \frac{3}{0.15667} = 19.15$$

Example: Find the H.M of the given data

Weight (g)	f_i	x_i (Classmarks or midpoint)	$\frac{1}{x_i}$	$f_i\ \frac{1}{x_i}$
65—84	9	$\frac{65+84}{2} = 74.5$	0.01342	0.12078
85—104	10	94.5	0.01058	0.1058
105—124	17	114.5	0.008734	0.165478
125—144	10	134.5	0.007435	0.07435
145—164	5	154.5	0.006472	0.3236
165—184	4	174.5	0.005731	0.022924
185—204	5	194.5	0.005141	0.025705
	$\sum f_i = 60$			$\sum_{i=1}^{n} f_i\ \frac{1}{x_i} = 0.547397$

$$H.M = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} f_i\frac{1}{x_i}} = \frac{60}{0.547397} = 109.61$$

The Median:

The median is a value which divides an ordered set of data into two equal parts.

Median is the middle value when n is odd and mean of two values when n is even.

Example: Find median of data 45,32,37,46,39,36,41,48,36

Solution: Here n = 9

Arrange into ascending order

$$\overline{32,36,36,37},39,\overline{41,45,46,48} \quad \Rightarrow \quad \text{median} = 39$$

Example: Find median of data 4,8,9,10,6,18,24,16,15,19

Solution: Here n = 10

Arrange into ascending order

$$\overline{4,6,8,9},10,15,\overline{16,18,19,24} \quad \Rightarrow \text{median} = \frac{10+15}{2} = 12.5$$

For group data

Median = $l + \frac{h}{f}\left(\frac{n}{2} - c\right)$ where n = $\sum f$

l = lower class boundary of median class

h = class interval

f = frequency of the median class

c = cumulative frequency of the class preceding the median class

Example: Find median of group data given below

Solution:

Marks	f	Class Boundary	Cumulative Frequency (c)
30—39	8	29.5—39.5	8
40—49	87	39.5—49.5	8+87 = 95
50—59	190	49.5—59.5	94+190 = 285
60—69	304	59.5—69.5	285+304 = 589
70—79	211	69.5—79.5	589+211 = 800
80—89	85	79.5—89.5	800+85 = 885
90—99	20	89.5—99.5	885+20 = 905
	$\sum f = 905$		

Median class

$n = \sum f = 905 \Rightarrow n/2 = 905/2 = 452.5$

Median class: A class carrying highest frequency

l = 59.5 , c = 285 , h = 10

Median = $l + \frac{h}{f}\left(\frac{n}{2} - c\right) \Rightarrow 59.5 + \frac{10}{304}(452.5 - 285)$ = 65 marks

The Mode:

The mode is the value in the data which occurs most frequent.

Example: Find mode of 1,4,5,6,1,8,9,1

Solution: mode = 1

Example: Find mode of 4,4,6,18,19,9,9,10,18,8

Solution: mode = 4,9,18

Example: Find mode of 4,6,8,8,6,4

Solution: mode = no

For Group data:

Mode = $l + \frac{f_m - f_{m-1}}{(f_m - f_{m-1}) + (f_m - f_{m+1})} \times h$

Where l = lower class boundary of the modal class

f_m = frequency of the model class

f_{m-1} = frequency of the class preceding the modal class

f_{m+1} = frequency of the class next (following) to modal class

h = class interval

Example: Find mode of the group data

Marks	f	Class Boundary	Cumulative Frequency (c)
30—39	8	29.5—39.5	8
40—49	87	39.5—49.5	8+87 = 95
50—59	190	49.5—59.5	94+190 = 285
60—69	304	59.5—69.5	285+304 = 589
70—79	211	69.5—79.5	589+211 = 800
80—89	85	79.5—89.5	800+85 = 885
90—99	20	89.5—99.5	885+20 = 905

Modal class

$l = 59.5$, $f_{m+1} = 211$, $f_m = 304$, h = 10 $f_{m-1} = 190$

Mode = $l + \frac{f_m - f_{m-1}}{(f_m - f_{m-1}) + (f_m - f_{m+1})} \times h$

$= 59.5 + \frac{304-190}{(304-190)+(304-211)} \times 10 \quad \Rightarrow$ mode = 65 (marks)

Measure of Dispersion:

By Dispersion we mean the extent to which the values are spread from an average. A quantity which measures this characteristic is called the measure of dispersion.

There are two types of measure of dispersion

(i) Absolute measure of dispersion
(ii) Relative measure of dispersion

Example: The marks obtained by 9 students are given

45,32,37,46,39, 36,41,48,36

Find the coefficient of dispersion.

Solution: Here the highest marks = x_m = 48

Lowest marks = x_0 = 32

Coefficient of dispersion = $\frac{x_m - x_0}{x_m + x_0}$

$= \frac{48-32}{48+32} = 0.2$

Lecture # 03

Measure of Dispersion:

(i) The Range
(ii) The Variance
(iii) The Standard Deviation

(i) The Range:

The range is defined as the difference between the highest observation to the lowest observation.

i.e. R = $x_m - x_0$

Where x_m us the maximum value and x_0 is the smallest value/observation.

Example: Find the range from the data

45,32,37,46,39,36,41,48,36

Solution: Highest observation $x_m = 48$

Lowest observation $x_0 = 32$

R = $x_m - x_0$

R = $48 - 32$ = 16 (marks)

(ii) The Mean Deviation:

The mean deviation of n observation of data is defined as the mean of absolute deviation or mod deviations of observations from their mean.

$$M.D = \frac{\sum_{i=1}^{n}|x_i - \bar{x}|}{n} \qquad \textit{for mean}$$

$$M.D = \frac{\sum_{i=1}^{n} f_i|x_i - \bar{x}|}{\sum_{i=1}^{n} f_i} \qquad \textit{for group data}$$

$$M.D = \frac{\sum_{i=1}^{n}|x_i - median|}{n} \qquad \textit{for median}$$

$$M.D = \frac{\sum_{i=1}^{n} f_i \left| x_i - median \right|}{\sum_{i=1}^{n} f_i} \qquad \textit{for group data}$$

Example: Calculate M.D of by mean and median for the data

45,32,37,46,39,36,41,48,36

Solution: $\bar{x} = \dfrac{45+32+37+46+39+36+41+48+36}{9} = 40$

For median 32,36,36,37,39,47,45,46,48

Median = 39

x_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$	$x_i - median$	$\lvert x_i - median \rvert$
32	-8	5	-7	7
36	-4	4	-3	3
36	-4	4	-3	3
37	-3	3	-2	2
39	-1	1	0	0
41	1	1	2	2
45	5	5	6	6
46	6	6	7	7
48	8	8	9	9
		$\sum \left\| x_i - \bar{x} \right\| = 40$		$\sum \left\| x_i - median \right\| = 39$

$$M.D = \frac{\sum \left| x_i - \bar{x} \right|}{n} = \frac{40}{9} = 4.4(marks)$$

$$M.D = \frac{\sum \left| x_i - median \right|}{n} = \frac{39}{9} = 4.3(marks)$$

(iii) **The Variance:** The variance of a set of n observations is defined as mean of squared deviations of the observations from their mean

$$S^2 = \frac{\sum_{i=1}^{n} \left(x_i - \bar{x} \right)^2}{n}$$

For group data $$\sigma^2 = S^2 = \frac{\sum_{i=1}^{n} f_i\left(x_i - \overline{x}\right)^2}{\sum_{i=1}^{n} f_i}$$

(iv) The Standard Deviation:

The positive square root of the variance is called the standard deviation

i.e. $$S = = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n}}$$

For group data $$\sigma = S = \sqrt{\frac{\sum_{i=1}^{n} f_i\left(x_i - \overline{x}\right)^2}{\sum_{i=1}^{n} f_i}}$$

Or $$S^2 = \frac{\sum_{i=1}^{n}\left(x_i^{\,2} - 2x_i\overline{x} - \overline{x^2}\right)}{n}$$

$$S^2 = \frac{\sum_{i=1}^{n} x_i^{\,2} - 2\overline{x}\sum_{i=1}^{n} x_i + n\overline{x^2}}{n}$$

$$S^2 = \frac{\sum_{i=1}^{n} x_i^{\,2} - 2\overline{x}.n\overline{x} + n\overline{x^2}}{n} \qquad \because \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n} \quad \& \; n\overline{x} = \sum_{i=1}^{n} x_i$$

$$= \frac{\sum_{i=1}^{n} x_i^{\,2} - n\overline{x^2}}{n} = \frac{\sum_{i=1}^{n} x_i^{\,2}}{n} - \overline{x^2}$$

$$S^2 = \frac{\sum_{i=1}^{n} x_i^{\,2}}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 \qquad \because \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$S=\sqrt{\frac{\sum_{i=1}^{n}x_i^{2}}{n}-\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^2}$$

For group data

$$S^2=\frac{\sum_{i=1}^{n}f_ix_i^{2}}{\sum_{i=1}^{n}f_i}-\left(\frac{\sum_{i=1}^{n}f_ix_i}{\sum_{i=1}^{n}f_i}\right)^2$$

And

$$S=\sqrt{\frac{\sum_{i=1}^{n}f_ix_i^{2}}{\sum_{i=1}^{n}f_i}-\left(\frac{\sum_{i=1}^{n}f_ix_i}{\sum_{i=1}^{n}f_i}\right)^2}$$

Example: Calculate variance and standard deviation of data

7,8,10,13,14,19,20,25,26,28

Solution: $\bar{x}=\frac{7+8+10+13+14+19+20+25+26+28}{10}=17$

x_i	$x_i-\bar{x}$	$(x_i-\bar{x})^2$
7	-10	100
8	-9	81
10	-7	49
13	-4	16
14	-3	9
19	2	4
20	3	9
25	8	64
26	9	81
28	11	121
		$\sum(x_i-\bar{x})^2=534$

The Variance $S^2=\frac{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{n}=\frac{534}{10}=53.4$

The Standard deviation S = $\sqrt{S^2}=\sqrt{53.4}=7.3$

Example: Calculate the variance and mean deviation for the group data

Weights	f_i	x_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
65—84	9	$\frac{65+84}{2} = 74.5$	670.5	-48	2304	20736
85—104	10	94.5	945.0	-28	74	7840
105—124	17	114.5	1946.5	-8	64	1088
125—144	10	134.5	1345.0	12	144	1440
145—164	5	154.5	772.5	32	1024	5120
165—184	4	174.5	698.0	52	2704	10816
185—204	5	194.5	972.5	72	5184	25920
	$\sum f_i = 60$		$\sum f_i x_i = 7350$			$\sum f_i(x_i - \bar{x})^2 = 72960$

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{7350}{60} = 122.5$$

The Variance $S^2 = \dfrac{\sum f_i\left(x_i - \bar{x}\right)^2}{\sum f_i} = \dfrac{72960}{60} = 1216$

The Standard deviation S = $\sqrt{S^2} = \sqrt{1216} = 34.871$

Moments:

A moment designates the power to which the deviation is raised before averaging them

$$m_1 = \frac{\sum\left(x_i - \bar{x}\right)}{n}$$

$$m_2 = \frac{\sum\left(x_i - \bar{x}\right)^2}{n}$$

$$m_3 = \frac{\sum\left(x_i - \bar{x}\right)^3}{n}$$

And $$m_4 = \frac{\sum\left(x_i - \bar{x}\right)^4}{n}$$

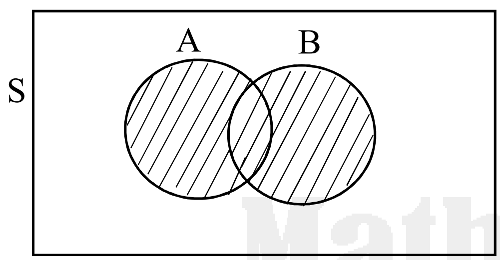
Lecture # 04

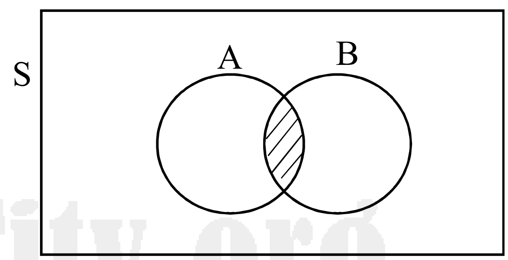
Probability:

Probability is the measure of uncertainty or measure of degree of belief in a particular statement or assumption.

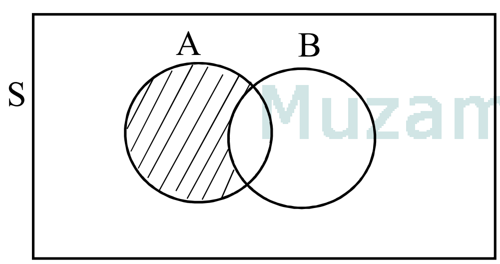
Venn Diagram:

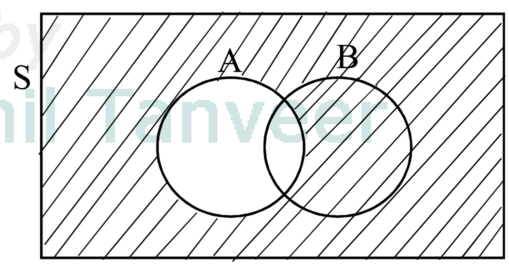
A diagram that is understood to represent the set by circular region or by parts of circular regions or their compliments with respect to a rectangle represent a space S is called the Venn Diagram.





$A \cup B = \{ x \mid x \in A \text{ or } x \in B\}$ $\qquad$ $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$





$A - B = \{x \mid x \in A \text{ and } x \notin B\}$ $\qquad$ $A' = \bar{A} = \{ x \mid x \in S \text{ and } x \notin A\}$

Cartesian Product:

The cartesian product of two sets A and B is denoted by $A \times B$ is a set that contains all ordered pairs (x,y) where $x \in A$ and $y \in B$

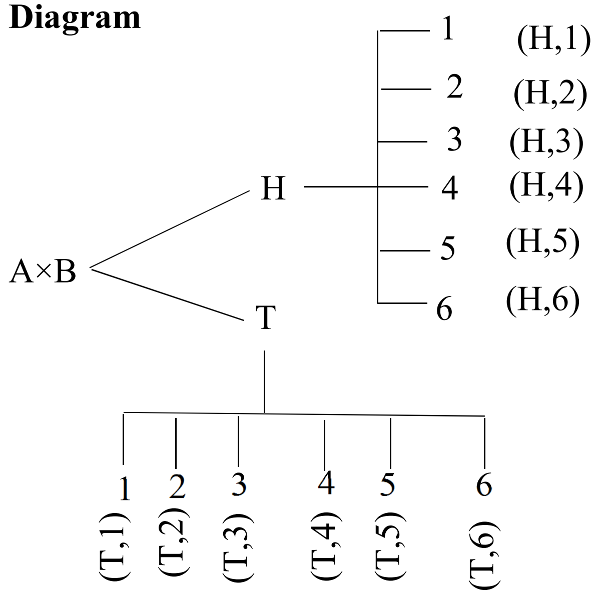
i.e. $\quad A \times B = \{(x,y) \mid x \in A \text{ and } y \in b\}$

Example:

Let $A = \{H,T\}$ & $B = \{1,2,3,4,5,6\}$

$A \times B = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

Or By Tree Diagram



Random Experiment:

The term experiment means a planned activity which results a set of data.

Trail:

A single performance of an experiment is called Trial.

Outcome:

The results obtained from an experiment or a trial is called outcome.

And experiment which produces different results even through it is repeated a large number of time under some essential conditions is called random experiment.

(i) The experiment can be repeated practically or theoretically any number of times.
(ii) The experiment has always two or more outcome.
(iii) The outcome of each repetition is unpredictable.

Sample Space:

A set consisting of all possible outcomes of a random experiment is called the sample space.

$A = \{H,T\}$ for a coin

$S = \{1,2,3,4,5,6\}$ for a dice

Event:

Any subset of a sample space S is called an event

$$A = \{H\} \subseteq \{H,T\}$$

$$B = \{1,4,6\} \subseteq \{1,2,3,4,5,6\}$$

Mutually Exclusive Events:

Two events A and B of single experiment are said to be mutually exclusive or disjoint events iff they cannot occur both together.

Exhaustive Events:

Events are said to be exhaustive if their union make the whole sample space.

$$A = \{H\}\ ,\ B = \{T\}$$

$$A \cup B = \{H,T\}$$

Equally likely events:

Two events A and B are said to be equally likely events if they both have the same chance of occurrence or when one event is as likely to occur as to other.

Permutations:

A permutation is any ordered subset from a set of ‘n’ distinct objects. The number of ‘r’ objects selected in a definite order from ‘n’ distinct is denoted by $^{n}P_{r}$ and given as

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Combinations:

A combination is a set of ‘r’ objects selected without the regard of their order from the set of ‘n’ objects is denoted by $^{n}C_{r}$ and given as

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Example:

A club consist of four members. How many sample points are in a sample space when three officer’s president, secretary and treasurer are chosen.

Solution:

Here n = 4 , r =3

$$^{n}P_{r}=\frac{n!}{(n-r)!}=\frac{4!}{(4-3)!}=\frac{4.3.2.1}{1!}=24$$

Example:

How many sample points are in a sample space S when a person draws a hand of 5 cards from a well shuffled pack of 52 playing cards.

Solution:

Here n = 52 , r = 5

$$^{n}C_{r}=\frac{n!}{r!(n-r)!}=\frac{52!}{5!(52-5)!}=\frac{52.51.50.49.48.47!}{5!\,47!}=2{,}598{,}960$$

Definition of Probability:

If a random experiment produces 'n' equally likely and mutually exclusive events are considered favorable for the occurrence of an even A then probability of an event is denoted by P(A) and given as

$$P(A)=\frac{m}{n}=\frac{no.\ favorable\ chances}{Total\ no.of\ chances}$$

(i) $P(A)=\frac{m}{n}\geq 0$

(ii) $P(S)=\frac{n}{n}=1$

(iii) If A and B are two mutually events then $P(A\cup B) = P(A)+P(B)$

Example:

A card is drawn from an ordinary deck of 52 playing cards. Find the probability that

(i) Card is a red card
(ii) Card is a diamond
(iii) Card is 10

Solution:

Here n = 52

(i) Let A represents the event that card is a red card.

$$P(A)=\frac{26}{52}=\frac{1}{2}$$

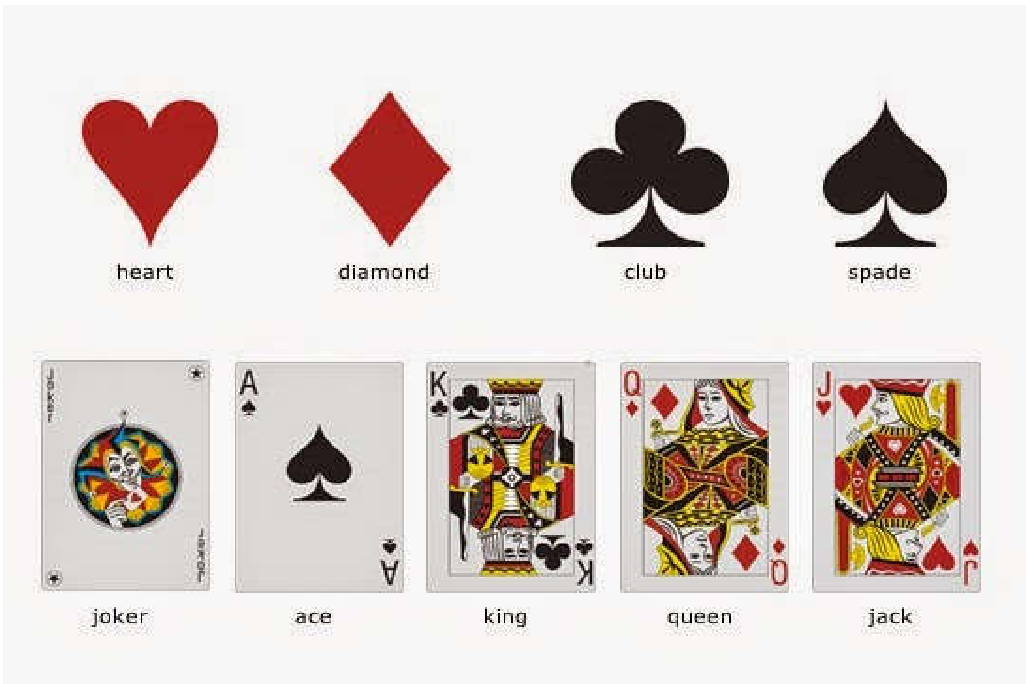
(ii) Let B represent the event that card is a diamond card

$$P(B) = \frac{13}{52} = \frac{1}{4}$$

(iii) Let C represent the event that card is 10

$$P(C) = \frac{4}{52} = \frac{1}{3}$$

For Information:



In deck of card there are total 52 card

<u>4 Ace</u>

<u>4 King</u>

<u>4 Queen</u>

<u>4 Jack</u>

<u>26 Red cards</u>

<u>26 Black cards</u>

<u>13 Heart cards</u>

<u>13 Diamond cards</u>

<u>13 Club cards</u>

<u>13 Spade cards</u>

Lecture # 05

Example:

A fair coin is tossed three times. What is the probability that at least one head appears?

Solution:

The sample space for this experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Thus n(S) = 8

Let A denotes the event that at least one head appears. Then

A = {HHH, HHT, HTH, THH, HTT, THT, TTH}

Therefore, n(A) = 7

Hence $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{7}{8}$

Example: If two fair dice are thrown, what is the probability of getting

(i) A double six

(ii) A sum of 8 or more dots?

Solution: The sample space for this experiment is

S = {(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (2,1) (2,2) (2,3) (2,4) (2,5) (2,6) (3,1) (3,2) (3,3) (3,4) (3,5) (3,6) (4,1) (4,2) (4,3) (4,4) (4,5) (4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

Thus n(S) = 36

(i) Let A represent the event that a double six appears. Then

A = {6,6} $\Rightarrow$ n(A) = 1

Therefore $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{1}{36}$

(ii) Let B denotes the event that a sum of 8 or more dot.

B = {(2,6) (3,5) (3,6) (4,4) (4,5) (4,6) (5,3) (5,4) (5,5) (5,6) (6,2) (6,3) (6,4) (6,5) (6,6) }

n(B) = 15 $P(A) = \dfrac{n(B)}{n(S)} = \dfrac{15}{36} = \dfrac{5}{12}$

Example: Six white balls and four black balls which are indistinguishable apart from color are placed in a bag. If six balls are taken from the bag, find the probability of their being three white and three black.

Solution: Here total balls =10

White balls = 6

Black balls = 4

Let 6 balls are selected at random from ten balls. The possible number if outcomes in which 6 balls are selected from 10 balls.

$$n(\mathrm{S})={}^{10}C_6=\frac{10!}{6!(10-6)!}=210$$

Let A denote the event that three balls are white and three are black

$$n(\mathrm{A})=\binom{6}{3}\times\binom{4}{3}$$

$$n(\mathrm{A})=\frac{6!}{3!(6-3)!}\times\frac{4!}{3!(4-3)!}=\frac{6.5.4.3!}{3.2.1.3!}\times\frac{4.3!}{3!.1!}=80$$

Therefore, $\mathrm{P(A)}=\dfrac{n(\mathrm{A})}{n(\mathrm{S})}=\dfrac{80}{210}=\dfrac{8}{21}$

Example: An employer wishes to hire three people from a group of 15 applicants 8 men and 7 women all of whom are equally qualified to fill the position. If he selects three at random. What is the probability that?

(i) All three will be men
(ii) At least one will be women

Solution: The possible ways that three people are selected from 15 applicants

$$n(\mathrm{S})=\binom{15}{3}=\frac{15!}{3!(15-3)!}=\frac{15.14.13.12!}{3.2.1.12!}=455$$

(i) Let A be the event that all three selected are men. The possible outcomes for this event are

$$n(\mathrm{A})=\binom{8}{3}=\frac{8!}{3!(8-3)!}=\frac{8.7.6.5!}{3.2.1.5!}=56\Rightarrow P(A)=\frac{n(\mathrm{A})}{n(\mathrm{S})}=\frac{56}{455}=\frac{8}{65}$$

(ii) Let B represents the event that at least one women is selected. The total number of ways for this event are

$$n(\mathrm{B})=\binom{7}{1}\times\binom{8}{2}+\binom{7}{2}\times\binom{8}{1}+\binom{7}{3}\times\binom{8}{0}$$

$$n(\mathrm{B})=\frac{7!}{1!(7-1)!}\times\frac{8!}{2!(8-2)!}+\frac{7!}{2!(7-2)!}\times\frac{8!}{1!(8-1)!}+\frac{7!}{3!(7-3)!}\times\frac{8!}{0!(8-0)!}$$

$$n(\mathrm{B})=399 \text{ sample points}$$

$$\Rightarrow P(\mathrm{B})=\frac{n(\mathrm{B})}{n(\mathrm{S})}=\frac{399}{455}=\frac{57}{65}$$

Example:

Four items are taken at random from a box of 12 items and inspected. The box is rejected if more than one item is found to be faulty. If there are three faulty items in the box, find the probability that the box is accepted.

Solution:

The possible number of ways in which 4 items are selected from 12 items

$$n(\mathrm{S})=\binom{12}{4}=\frac{12!}{4!(12-4)!}=495 \text{ sample points}$$

Let A denotes the event that shows no faulty item or one faulty item

$$n(\mathrm{A})=\binom{3}{0}\binom{9}{4}+\binom{3}{1}\binom{9}{3}$$

$$n(\mathrm{A})=\frac{3!}{0!(3-0)!}\times\frac{9!}{4!(9-4)!}+\frac{3!}{1!(3-1)!}\times\frac{9!}{3!(9-3)!}=378$$

$$\text{Therefore, } P(\mathrm{A})=\frac{n(\mathrm{A})}{n(\mathrm{S})}=\frac{378}{495}=0.76$$

Lecture # 06

Laws of Probability:

Theorem: If ϕ is the impossible event then $P(\phi) = 0$

Proof: The sure event S and the impossible event ϕ the mutually exclusive events or $S = S \cup \phi$

$$P(S) = P(S \cup \phi)$$

$$P(S) = P(S) + P(\phi) \qquad \because P(A \cup B) = P(A) + P(B)$$

$$\Rightarrow \quad P(\phi) = 0$$

Laws of Complement:

Theorem: If $\overline{A}$ is the complement of an event A then $P(\overline{A}) = 1 - P(A)$

Proof:

We can write $A \cup \overline{A} = S$

Where A and $\overline{A}$ are both mutually exclusive events.

$$P\left(A \cup \overline{A}\right) = P(S)$$

$$P(A) + P(\overline{A}) = 1$$

$$\because P(A) = \frac{n(A)}{n(S)} \quad \Rightarrow P(S) = \frac{n(S)}{n(S)} = 1$$

$$P(\overline{A}) = 1 - P(A)$$

Or $$P(A) = 1 - P(\overline{A})$$

Example: A coin is tossed 4 times in succession. What is the probability of getting at least on head?

Solution: The sample space for this experiment is

S = {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

$n(S) = 16$

Let A be the event that at least one head appears.

$n(A) = 15 \Rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{15}{16}$

Alternative: Let $\overline{A}$ be the complement of an event A that no head appear

$\overline{A} = \{TTTT\}$

$n(\overline{A}) = 1$

$P(\overline{A}) = \frac{n(\overline{A})}{n(S)} = \frac{1}{16}$

$P(A) = 1 - P(\overline{A})$

$P(A) = 1 - \frac{1}{16} = \frac{15}{16}$

Example:

A coin is biased so that the probability that it falls showing tails is $\frac{3}{4}$

(i) Find the probability of obtaining at least one head when the coin is tossed five times.

(ii) How many times must the coin be tossed so that the probability of obtaining at least one head is greater than 0.98?

Solution: (i) P(tail or no head) = $\frac{3}{4}$

P(head) = $1 - \frac{3}{4} = \frac{1}{4}$ $\qquad \because P(A) = 1 - P(\overline{A})$

The sample space consists of $2^5 = 32$ sample points. Let A be the event that at least one head appears and $\overline{A}$ be the event that no head appears

. $\overline{A} = \{TTTTT\}$

$P(\overline{A}) = \left(\frac{3}{4}\right)^5$

$P(A) = 1 - P(\overline{A}) = 1 - \left(\frac{3}{4}\right)^5 = 0.763$

(ii). Let the coin be tossed n times to obtained the probability of at least one head is greater than 0.98 Then

$$1-\left(\frac{3}{4}\right)^n \geq 0.98$$

$$\left(\frac{3}{4}\right)^n \leq 1-0.98$$

$$\left(\frac{3}{4}\right)^n \leq 0.02$$

Taking log on both sides

$$n\log\left(\frac{3}{4}\right) \leq \log 0.02$$

Dividing both side by $\log\left(\frac{3}{4}\right)$ and reversing the inequality as $\log\left(\frac{3}{4}\right)$ is negative. We have

$$n \geq \frac{\log 0.02}{\log\left(\frac{3}{4}\right)}$$

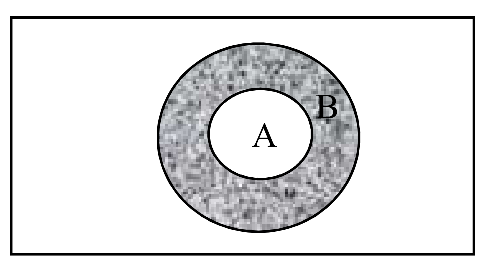
$$n \geq 13.6$$

So n = 14

Probability of subset:

Theorem: If A and B be the two event such that $A \subset B$ then $P(A) \leq P(B)$

Solution: Let the event B can be written as $B = A \cup \left(B \cap \overline{A}\right)$ where A and $B \cap \overline{A}$ are two mutually exclusive events.



S

$B \cap \overline{A}$

$$P(B) = P\left[A \cup B \cap \overline{A}\right]$$

$$P(B) = P(A) + P(B \cap \overline{A})$$

Where $P\left(B \cap \overline{A}\right) \geq 0$

Hence $P(A) \leq P(B)$

Theorem: If A and B be the two events define in a sample space S, then

$$P\left(A \cap \overline{B}\right) = P(A) - P(A \cap B)$$

Solution: The event A can be written as

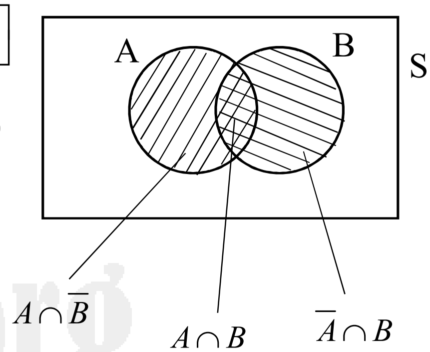
$$A = \left(A \cap \overline{B}\right) \cup \left(A \cap B\right)$$

Where $A \cap \overline{B}$ *and* $A \cap B$ are mutually exclusive events

$$P(\text{A}) = P\left[\left(A \cap \overline{B}\right) \cup \left(A \cap B\right)\right]$$

$$P(\text{A}) = P\left(A \cap \overline{B}\right) + P\left(A \cap B\right)$$

$$P\left(A \cap \overline{B}\right) = P(\text{A}) - P\left(A \cap B\right)$$



Theorem:

If A and B are any two events define in a sample space S, then

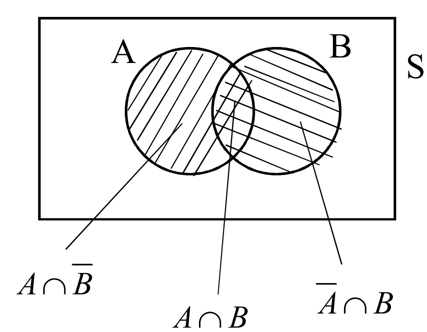
$$P\left(A \cup B\right) = P(A) + P(B) - P(A \cap B)$$

Or "If two events A and B are not mutually exclusive then the probability that at least one of then occurs, is given by the sum of the separate probabilities of event A and B minus the probability of the joint event $A \cap B$"

Proof:

The event $A \cup B$ can be written as

$A \cup B = A \cup \left(\overline{A} \cap B\right)$ where A and $\overline{A} \cap B$ are two mutually exclusive events.



$$P\left(A \cup B\right) = P\left[A \cup \left(\overline{A} \cap B\right)\right]$$

$$P\left(A \cup B\right) = P\left(A\right) + P\left(\overline{A} \cap B\right) \qquad \_\_\_\text{(i)}$$

The event B can be written as

$$B = \left(A \cap B\right) \cup \left(\overline{A} \cap B\right)$$

Where $\left(A\cap B\right) and \left(\overline{A}\cap B\right)$ are two mutually exclusive events.

$$P(B)=P\left[\left(A\cap B\right)\cup\left(\overline{A}\cap B\right)\right]$$

$$P(B)=P\left(A\cap B\right)+P\left(\overline{A}\cap B\right)$$

$$P\left(\overline{A}\cap B\right)=P(B)-P\left(A\cap B\right)$$

Put in (i)

$$P\left(A\cup B\right)=P(A)+P(B)-P\left(A\cap B\right)$$

Lecture # 07

Corollary:

If A and B are mutually exclusive events then

$$P\left(A \cup B\right) = P(A) + P(B)$$

Proof:

Since the events A and B are mutually exclusive so

$$A \cap B = \phi$$

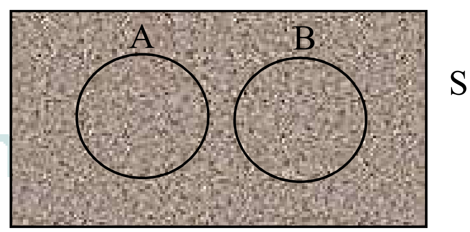
$$P\left(A \cap B\right) = P\left(\phi\right) = 0$$

By addition law $P\left(A \cup B\right) = P(A) + P(B) - P\left(A \cap B\right)$

$$P\left(A \cup B\right) = P(A) + P(B) - 0$$

$$P\left(A \cup B\right) = P(A) + P(B) \quad \text{proved}$$

Alternative:



Let n be the total number of sample points in a sample space S. m_1 be the sample points that contains the event A and m_2 be the sample points that contains event B. The occurrence of $A \cup B$ is the event which consist of all the sample points belonging to either A or b.

Since A and B are mutually exclusive events before they have no sample points in common

$$P\left(A \cup B\right) = \frac{No.\ of\ \ sample\ points\ in\ A \cup B}{Total\ no.of\ sample\ points\ in\ S}$$

$$P\left(A \cup B\right) = \frac{n\left(A \cup B\right)}{n(S)} = \frac{m_1 + m_2}{n}$$

$$P\left(A \cup B\right) = \frac{m_1}{n} + \frac{m_2}{n}$$

$$P\left(A \cup B\right) = P(A) + P(B)$$

Corollary:

If $A_1, A_2, A_3, ...A_k$ are k mutually exclusive events then the probability that one of them occurs is the sum of the probabilities of the separate event.

Proof:

Since $$P(A \cup B) = P(A) + P(B)$$

$$\Rightarrow \quad P(A_1 \cup A_2 \cup A_3 \bigcup ....A_k) = P(A_1) + P(A_2) + P(A_3).... + P(A_k)$$

If events are mutually exclusive and collective exhaustive then

$$\Rightarrow \quad P(A_1) + P(A_2) + P(A_3).... + P(A_k) = 1$$

Corollary:

If A and B are any two events then $P(A \cup B) \leq P(A) + P(B)$

Proof:

These events are not mutually exclusive. Then by addition law

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) + P(A \cap B) = P(A) + P(B)$$

$$\Rightarrow P(A \cup B) \leq P(A) + P(B)$$

In general

$$\Rightarrow \quad P(A_1 \cup A_2 \cup A_3 \bigcup ....A_k) \leq P(A_1) + P(A_2) + P(A_3).... + P(A_k)$$

This is known as **Boole's inequality.**

Example:

If one card is selected at random from a deck of 52 playing cards, what is the probability that the card is club or a face card or both?

Solution:

Let A be the event that the card is club card

n(A) = 13

$$\Rightarrow \quad P(A) = \frac{n(A)}{n(\text{S})} = \frac{13}{52}$$

Let B be the event that the card is a face card

n(B) = 12

$$\Rightarrow \quad P(B) = \frac{n(\text{B})}{n(\text{S})} = \frac{12}{52}$$

Let $A \cap B$ be the event that the card is both club and face card

n($A \cap B$) = 3

P($A \cap B$) = $\frac{3}{52}$

The probability is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}$$

Example:

An integer is chosen at random from the first 200 +ve integers. What is the probability that integer chosen is divided by 6 or by 8?

Solution:

The sample space is S = {1,2,3,…..200}

Let A represent the event that integer chosen is divisible by 6.

$$\text{n(A)} = \frac{200}{6} = 33$$

$$\Rightarrow \quad P(A) = \frac{n(A)}{n(\text{S})} = \frac{3}{200}$$

Let B represent the event that integer chosen is divisible by 8.

$$\text{n(B)} = \frac{200}{8} = 25$$

$$\Rightarrow \quad P(B) = \frac{n(\text{B})}{n(\text{S})} = \frac{25}{200}$$

Let $A \cap B$ be the event that integer chosen is divisible by both 6 and8

$$n(A \cap B) = \frac{200}{24} = 8$$

$$\Rightarrow \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{8}{200}$$

The probability is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{50}{200}$$

$$P(A \cup B) = \frac{1}{4}$$

Example:

A pair of dice are thrown. Find the probability of getting a total of either 5 or 11

Solution: The sample space for this experiment is

S = {(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (2,1) (2,2) (2,3) (2,4) (2,5) (2,6) (3,1) (3,2) (3,3) (3,4) (3,5) (3,6) (4,1) (4,2) (4,3) (4,4) (4,5) (4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

n(S) = 36

Let A be the event that a total of 5 occurs

A = {(1,4) , (2,3) , (3,2) , (4,1)}

n(A) = 4

$$\Rightarrow \quad P(A) = \frac{n(A)}{n(S)} = \frac{4}{36}$$

Let B be the event that a total of 11 occurs

A = {(5,6) , (6,5)}

n(B) = 2

$$\Rightarrow \quad P(B) = \frac{n(B)}{n(S)} = \frac{2}{36}$$

The events A and B are mutually exclusive as a total of 5 and 11 cannot both occur together

$$P\left(A \cup B\right) = P(A) + P(B)$$

$$P\left(A \cup B\right) = \frac{4}{36} + \frac{2}{36} = \frac{1}{6}$$

Example:

Three horses A, B & C are in a race A is twice as likely to win as B and B is twice likely to win as C. What is the probability that A or B wins.

Solution:

According to given condition

P(A) = 2P(B)

P(B) = 2P(C)

Let P be the probability of C

P(C) = P

Then P(B) = 2P(C) = 2P

P(A) = 2P(B) = 2(2P) = 4P

We know that sum of all probabilities is equal to one

4P + 2P + P = 1

7P = 1

$$\Rightarrow \quad P = \frac{1}{7}$$

$$\Rightarrow P(C) = \frac{1}{7}$$

$$\Rightarrow P(\text{B}) = \frac{2}{7}$$

$$\Rightarrow P(\text{B}) = \frac{4}{7}$$

Hence $P\left(A \cup B\right) = P(A) + P(B)$

$$= \frac{4}{7} + \frac{2}{7} = \frac{6}{7}$$

Theorem:

If A, B and C are any three events in a sample space S, then the probability of at least one of them occurring is given as

$$P(A\cup B\cup C)=P(A)+P(B)+\mathrm{P(C)}-\mathrm{P}(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$

Solution:

As $\quad A\cup B\cup C = A\cup(B\cup C)$

$$P(A\cup B\cup C)=P(A\cup(B\cup C))$$

$$=P(A)+P(B\cap C)-P(A\cap(B\cup C))$$

$$=P(A)+P(B)+P(C)-P(B\cap C)-P(A\cap(B\cup C))$$

By distributive law

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(B\cap C)-P((A\cap B)\cup(A\cap C))$$

$$=P(A)+P(B)+P(C)-P(B\cap C)-P(A\cap B)-P(A\cap C)+P((A\cap B)\cup(A\cap C))$$

$$P(A\cup B\cup C)=P(A)+P(B)+\mathrm{P(C)}-\mathrm{P}(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$

Example:

A card is drawn at random from a deck of ordinary playing cards. What is the probability that it is a diamond a face or a king?

Solution: Here n(S) = 52

Let A represents the event that the card is diamond card

n(A) = 13

$$\Rightarrow \quad P(A)=\frac{n(A)}{n(\mathrm{S})}=\frac{13}{52}$$

Let B represents the event that the card is face card

n(B) = 12

$$\Rightarrow \quad P(B)=\frac{n(\mathrm{B})}{n(\mathrm{S})}=\frac{12}{52}$$

Let C represents the event that the card is king card

$$n(C) = 4$$

$$\Rightarrow \quad P(C) = \frac{n(C)}{n(S)} = \frac{4}{52}$$

Let $A \cap B$ represents the event that the card is king card

$$n(A \cap B) = 3$$

$$\Rightarrow \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{3}{52}$$

Let $B \cap C$ represents the event that the card is king card

$$n(B \cap C) = 4$$

$$\Rightarrow \quad P(B \cap C) = \frac{n(B \cap C)}{n(S)} = \frac{4}{52}$$

Let $A \cap C$ represents the event that the card is king card

$$n(A \cap C) = 1$$

$$\Rightarrow \quad P(A \cap C) = \frac{n(A \cap C)}{n(S)} = \frac{1}{52}$$

Let $A \cap B \cap C$ represents the event that the card is king card

$$n(A \cap B \cap C) = 1$$

$$\Rightarrow \quad P(A \cap B \cap C) = \frac{n(A \cap B \cap C)}{n(S)} = \frac{1}{52}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$= \frac{13}{52} + \frac{12}{52} + \frac{4}{52} - \frac{3}{52} - \frac{4}{52} - \frac{1}{52} + \frac{1}{52}$$

$$= \frac{13 + 12 + 4 - 3 - 4 - 1 + 1}{52}$$

$$= \frac{22}{52}$$

Lecture # 08

Conditional Probability:

The sample space for an experiment must often be changed when some additional information pertaining to the outcome of the experiment is received. The effect of such information is to reduce the sample space by excluding some outcomes as being impossible which before receiving the information where believed possible. The probabilities associated with such a reduce sample space are called conditional probabilities.

Let us consider a die-throwing experiment

The sample space is S = {1,2,3,4,5,6} $\Rightarrow$ n(S) = 6

Let A be the event that only 6 occur.   A = {6}

If there is a condition that 6 occur only when the die shows an even number of dots. Let B be the event that only even number occur.   B = { 2,4,6}

$$\text{P(die shows 6 / die shows even numbers)} = \frac{1}{3}$$

Symbolized as $P\left(A/B\right)$ = probability of A when B occurred and defined as

$$P\left(A/B\right)=\frac{\text{number of sample points in } A\cap B}{\text{number of sample points in } B}$$

$$P\left(A/B\right)=\frac{n\left(A\cap B\right)}{n\left(B\right)}$$

Dividing and multiplying by original sample space i.e n(S)

$$P\left(A/B\right)=\frac{n\left(A\cap B\right)}{n\left(S\right)}\cdot\frac{n(S)}{n(B)}$$

$$P\left(A/B\right)=\frac{P\left(A\cap B\right)}{P\left(B\right)} \quad \text{where } P(B)\neq 0$$

If P(B) = 0 the conditional probability $P\left(A/B\right)$ satisfies all the basic axioms of probability

(i) $0 \le P\left(A/B\right) \le 1$

(ii) $P\left(S/B\right) = 1 \qquad \left[\because S \cap B = B,\ P\left(S/B\right) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1\right]$

(iii) $P\left(A_1 \cup A_2/B\right) = P\left(A_1/B\right) + P\left(A_2/B\right)$

Example:

Two coins are tossed. What is the conditional probability that two heads result, given that there is at least one head?

Solution:

The sample space for this experiment is

S = {HH, HT, TH, TT} $\Rightarrow$ n(S) = 4

Let A represent the event that two heads appears

A = {HH} $\Rightarrow$ n(A) = 1

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$$

Let B be the event that at least one head appears

B = {HT , TH, HH} $\Rightarrow$ n(B) = 3

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{4}$$

We have to calculate $P\left(A/B\right) = \dfrac{P(A \cap B)}{P(B)}$ ______ (i)

$$A \cap B = \{HH\} \quad \Rightarrow \quad n(A \cap B) = 1$$

$$P\left(A/B\right) = \frac{n(A \cap B)}{n(S)} = \frac{1}{4} \qquad put\ \ in\ \ (ii)$$

$$P\left(A/B\right) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Question:

A man tosses two fair dice. What is the conditional probability that the sum of the two dice will be 7 given that

(i) The sum is odd
(ii) The sum is greater than 6
(iii) The two dice had the same outcome

Solution:

The sample space for this experiment is

S = {(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (2,1) (2,2) (2,3) (2,4) (2,5) (2,6) (3,1) (3,2) (3,3) (3,4) (3,5) (3,6) (4,1) (4,2) (4,3) (4,4) (4,5) (4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

n(S) = 36

Let A be the event that the sum is 7

A = {(1,6) , (2,5) , (3,4) , (4,3), (5,2), (6,1)}

n(A) = 6

$$\Rightarrow \quad P(A)=\frac{n(A)}{n(\mathrm{S})}=\frac{6}{36}=\frac{1}{6}$$

Let B be the event that the sum is odd

B = { (1,2) (1,4) (1,6) (2,1) (2,3) (2,5) (3,2) (3,4) (3,6) (4,1) (4,3) (4,5) (5,2) (5,4) (5,6) (6,1) (6,3) (6,5)}

n(B) = 18

$$\Rightarrow \quad P(B)=\frac{n(\mathrm{B})}{n(\mathrm{S})}=\frac{18}{36}=\frac{1}{2}$$

Let C be the event that the sum is greater than 6

C = { (1,6) (2,5) (2,6) (3,4) (3,5) (3,6) (4,3) (4,4) (4,5) (4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

n(C) = 21

$$\Rightarrow \quad P(C)=\frac{n(\mathrm{C})}{n(\mathrm{S})}=\frac{21}{36}=\frac{7}{12}$$

Let D be the event that two dice has same outcome

A = {(1,1) , (2,2) , (3,3) , (4,4), (5,5), (6,6)}

n(A) = 6

$$\Rightarrow \quad P(D)=\frac{n(\mathrm{D})}{n(\mathrm{S})}=\frac{6}{36}=\frac{1}{6}$$

$$A\cap B=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$n(A\cap B)=6$$

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{6}{36}=\frac{1}{6}$$

$$A\cap C=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$n(A\cap C)=6$$

$$P(A\cap C)=\frac{n(A\cap C)}{n(S)}=\frac{6}{36}=\frac{1}{6}$$

$$A\cap D=\phi$$

$$P(A\cap D)=0$$

$$P\left(A/B\right)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{6}}{\frac{1}{2}}=\frac{1}{3}$$

$$P\left(A/C\right)=\frac{P(A\cap C)}{P(C)}=\frac{\frac{1}{6}}{\frac{7}{12}}=\frac{2}{7}$$

$$P\left(A/D\right)=\frac{P(A\cap D)}{P(D)}=\frac{0}{\frac{1}{6}}=0$$

Example: What is the probability that a randomly selected poker hand, contain exactly 3 aces, given that it contains at least 2 aces.

Solution: Here total cards = 52

Total no. of ways in which 5 cards are randomly selected from 52 playing cards

$$= {}^{52}C_5 = \frac{52!}{5!(52-5)!} = 2,598,960 = \binom{52}{5}$$

Let A be the event that exactly 3 aces appear

$$n(\text{A}) = \binom{4}{3}\binom{48}{2}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$$

Let B be the event that at least 2 aces appear

$$n(\text{B}) = \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}$$

$$n(A \cap B) = \binom{4}{3}\binom{48}{2}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}\cdot\frac{\binom{52}{5}}{\binom{4}{2}\binom{48}{3}+\binom{4}{3}\binom{48}{2}+\binom{4}{4}\binom{48}{1}}$$

$$P\left(\frac{A}{B}\right)=\frac{4512}{108336}=0.0416$$

Theorem: Multiplication Law

If A and B are any two events define in a sample space S then

$$P(A\cap B)=P(A)\,.P\left(\frac{B}{A}\right) \quad \textit{provided } P(A)\neq 0$$

$$=P(B)\,.P\left(\frac{A}{B}\right) \quad \textit{provided } P(B)\neq 0$$

Proof: By definition of conditional probability

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)} \qquad ,P(\text{B})\neq 0$$

$$P(A\cap B)=P(B)P\left(\frac{A}{B}\right) \qquad ,P(\text{B})\neq 0$$

$$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)} \qquad ,P(\text{A})\neq 0$$

$$P(A\cap B)=P(A)P\left(\frac{B}{A}\right) \qquad ,P(\text{A})\neq 0$$

Alternative proof:

Let n be the total number of sample points of an experiment. Let m_1 be the number of sample points for the occurrence of event A and m_2 be the sample points for the occurrence of event B. Let m_3 be the sample point for the occurrence of event $A\cap B$

i.e. $$P(A\cap B)=\frac{m_3}{n}$$

The fraction $\frac{m_3}{n}$ may be written as

$$\frac{m_3}{n}=\frac{m_3}{m_1}\cdot\frac{m_1}{n} \qquad \text{But} \quad P(A)=\frac{m_1}{n}$$

And $\frac{m_3}{m_1}$ = conditional probability of B given that A has occurred i.e. $P\left(B/A\right)$

Hence $\quad P(A\cap B)=P(A)P\left(B/A\right)$

Similarly, $\quad P(A\cap B)=P(B)P\left(A/B\right)$

Corollary: In the case of event A,B and C

$$P(A\cap B\cap C)=P(A)P\left(B/A\right)P\left(C/A\cap B\right)$$

Proof: Let $D=A\cap B$

$$A\cap B\cap C=D\cap C$$

$$P(A\cap B\cap C)=P(D\cap C)$$

$$=P(D)P\left(C/D\right)$$

$$=P(A\cap B)P\left(C/A\cap B\right)$$

$$P(A\cap B\cap C)=P(A)P\left(B/A\right)P\left(C/A\cap B\right) \qquad \text{proved}$$

Example: A box contains 15 items 4 of which are defective and 11 are good. Two items are selected what is the probability that the first is god and second is defective.

Solution: Let A be the event that the first item is good $\Rightarrow$ $P(A)=\frac{n(A)}{n(\text{S})}=\frac{11}{15}$

Let B be the event that second item is defective $\Rightarrow$ $P(B)=\frac{n(\text{B})}{n(\text{S})}=\frac{4}{14}$

We have to find $P(A\cap B)$ $\Rightarrow P(A\cap B)=P(A)P\left(B/A\right)=\frac{11}{15}\times\frac{4}{14}=\frac{44}{210}=0.16$

Lecture # 09

Example: Two cards are dealt from a pack of ordinary playing cards. Find the probability that the second card dealt is a heart.

Solution: Let H_1 be the event that first card dealt is a heart and H_2 be the event that the second card dealt is a heart. Then

P (second card is heart = P (first card is heart not heart) + P (first card is not heart and second card is heart)

$$P(H_2) = P(H_1 \cap H_2) + P(\overline{H_1} \cap H_2)$$

$$= P(H_1)P(H_2 / \mathrm{H}_1) + P(\overline{H_1})P(H_2 / \overline{H_1})$$

$$= \frac{13}{52}.\frac{12}{51} + \frac{39}{52}.\frac{13}{51} \qquad \because P(H_1) = \frac{13}{52},\ P(\overline{H_1}) = 1 - \frac{13}{52} = \frac{39}{52}$$

$$= \frac{1}{17} + \frac{13}{68} = \frac{4+13}{68} = \frac{17}{68} = \frac{1}{4}$$

Example: Box A contain 5 green and 7 red balls. Box B contain 3 green, 3 red and 6 yellow balls. A box is selected at random and a ball is drawn at random from it. What is the probability that the ball drawing is green.

Solution: Let G denote the event that green ball is drawn

P(Green ball) = P(Box A is selected and green ball is drawn)

+ P(Box B is selected and green ball is drawn)

$$= P(A \cap B) + P(B \cap G)$$

$$= P(A)P\left(G/A\right) + P(B)P\left(G/B\right)$$

$$= \frac{1}{2}.\frac{5}{12} + \frac{1}{2}.\frac{3}{12}$$

$$= \frac{5}{24} + \frac{3}{24} = \frac{5+3}{24} = \frac{8}{24} = \frac{1}{3}$$

Example: An urn contains 10 white and 3 black balls another urn contains 3 white and 5 black balls. Two balls are transferred from first urn and placed in the second and then one ball is taken from the latter. What is the probability that it is a white ball?

Solution: Urn I white = 10 , black = 3

Urn II white = 3 , black = 5

Let A be the event that two ball are transferred from urn I to urn II. Then A can be occur in these possible ways

A_1 = 2 white balls

A_2 = 1 white and 1 black ball

A_1 = 2 black balls

$$P(A_1) = \frac{\binom{10}{2}\binom{3}{0}}{\binom{13}{2}} = \frac{45}{78}$$

$$P(A_2) = \frac{\binom{10}{1}\binom{3}{1}}{\binom{13}{2}} = \frac{30}{78}$$

$$P(A_3) = \frac{\binom{10}{0}\binom{3}{2}}{\binom{13}{2}} = \frac{3}{78}$$

Situation of urn II when two balls are transferred.

(i) 5 white 5 black (when 2 white ball drawn)
(ii) 4 white 6 black (when 1 white 1 black ball drawn)
(iii) 3 white 7 black (when 2 black ball drawn)

Let w be the event that white ball drawn from urn II when 2 balls transferred from urn I

$$P(w) = P(w \cap A_1) + P(w \cap A_2) + P(w \cap A_3)$$

$$= P(A_1)P\left(\frac{w}{A_1}\right) + P(A_2)P\left(\frac{w}{A_2}\right) + P(A_3)P\left(\frac{w}{A_3}\right)$$

$$= \frac{45}{78}\cdot\frac{5}{10} + \frac{30}{78}\cdot\frac{4}{10} + \frac{3}{78}\cdot\frac{3}{10}$$

$$= \frac{59}{130} = 0.4538$$

Example: A card is drawn at random from a deck of ordinary playing cards. What is the probability that it is a diamond, a face card or a king?

Solution: Let A be the event that card is diamond $P(A) = \frac{13}{52}$

Let B be the event that card is a face card $P(B) = \frac{12}{52}$

Let C be the event that card is king card $P(C) = \frac{4}{52}$

$$P(A \cap B) = P(A)P\left(\frac{B}{A}\right) = \frac{13}{52}\cdot\frac{3}{13} = \frac{3}{52}$$

$$P(B \cap C) = P(B)P\left(\frac{C}{B}\right) = \frac{12}{52}\cdot\frac{4}{12} = \frac{4}{52}$$

$$P(A \cap C) = P(A)P\left(\frac{C}{A}\right) = \frac{13}{52}\cdot\frac{1}{13} = \frac{1}{52}$$

$$P(A \cap B \cap C) = P(A)P\left(\frac{B}{A}\right)P\left(\frac{C}{A \cap B}\right) = \frac{13}{52}\cdot\frac{3}{13}\cdot\frac{1}{3} = \frac{1}{52}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$P(A \cup B \cup C) = \frac{13}{52} + \frac{12}{52} + \frac{4}{52} - \frac{3}{52} - \frac{4}{52} - \frac{1}{52} + \frac{1}{52}$$

$$P(A \cup B \cup C) = \frac{22}{52} = 0.423$$

Example: Three urns of same appearance are given as follows

Urn A contain 5 red and 7 white balls

Urn B contain 4 red and 3 white balls

Urn C contain 3 red and 4 white balls

An urn is selected at random and a ball is drawn from the urn

(i) What is the probability that the ball drawn is red?

(ii) If the ball drawn is red, what is the probability that it came from urn A.

Solution (i) : Let R be the event that red ball is drawn

$$P(R)=P(A\cap R)+P(B\cap R)+P(C\cap R)$$

$$=P(A)P\left(R/A\right)+P(B)P\left(R/B\right)+P(C)P\left(R/C\right)$$

$$=\frac{1}{3}.\frac{5}{12}+\frac{1}{3}.\frac{4}{7}+\frac{1}{3}.\frac{3}{7}$$

$$=0.4722$$

(ii). Probability of red ball from urn A is

$$P\left(A/R\right)=\frac{P(A\cap R)}{P(R)} \quad \text{\_\_(i)}$$

$$P(A\cap R)=P(A)P\left(R/A\right)$$

$$=\frac{1}{3}.\frac{5}{12}=\frac{5}{36}=0.1389 \qquad \text{put in (i)}$$

$$P\left(A/R\right)=\frac{0.1389}{0.4722}=0.2941$$

Independent and dependent events:

Two events A and B in the same sample space S are defined to be independent (or statistically independent) if the probability that one event occurs is not affected by whether the other even has or has not occurred.

i.e. $$P\left(A/B\right)=P(A) \quad and \quad P\left(B/A\right)=P(B)$$

It follows that two event A and B are independent iff

$$P(A\cap B)=P(A)P(B)$$

If the two events A and B are mutually exclusive then P(A) P(B) = 0

Which is true when either P(A) = 0 or P(B) = 0.

The events A and B are defined to be dependent if $P(A\cap B)\neq P(A)P(B)$

If three events A,B and C are mutually independent then

$$P(A\cap B\cap C)=P(A).P(B).P(C)$$

Example: Two events A and B are such that

$P(A)=\dfrac{1}{4}$, $P\left(A/B\right)=\dfrac{1}{2}$, $P\left(B/A\right)=\dfrac{2}{3}$

(i) Are A and B independent events?
(ii) Are A and B mutually exclusive events?
(iii) Find $P(A\cap B)$ and $P(B)$

Solution (i): If A and B independent events $P\left(A/B\right)=P(A)$ but given $P(A)=\dfrac{1}{4}$, $P\left(A/B\right)=\dfrac{1}{2}$ $i.e$ $P(A)\neq P\left(A/B\right)$. Hence, A and B are not independent events.

(ii). If A and B are mutually exclusive events then $P\left(A/B\right)=0$ but given $P\left(A/B\right)=\dfrac{1}{2}\neq 0$. Hence, A and B are not mutually exclusive.

(iii). Now $P(A\cap B)=P(A)P\left(B/A\right)=\dfrac{1}{4}.\dfrac{2}{3}=\dfrac{1}{6}$

Also $P(A\cap B)=P(B)P\left(A/B\right)$

$$\Rightarrow P(B)=\frac{P(A\cap B)}{P\left(A/B\right)}=\frac{\frac{1}{6}}{\frac{1}{2}} \qquad \Rightarrow P(B)=\frac{1}{3}$$

Lecture # 10

Example: Two fair dice one red and one green are thrown. Let A denote the event that the red die shows an even number and B the event that the green die shows 5 or 6. Show that the event A and B are independent.

Solution: The sample space for this experiment is

S = {(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (2,1) (2,2) (2,3) (2,4) (2,5) (2,6) (3,1) (3,2) (3,3) (3,4) (3,5) (3,6) (4,1) (4,2) (4,3) (4,4) (4,5) (4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

$$n(S) = 36$$

Let A be the event that the red die shows an even number

A = {(2,1) , (2,2) , (2,3) , (2,4), (2,5), (2,6),(4,1) ,(4,2) ,(4,3),(4,4),(4,5),(4,6), (6,1),(6,2),(6,3)(6,4),(6,5),(6,6)}

$$n(A) = 18$$

$$\Rightarrow \quad P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

Let B be the event that green die shows 5 or 6

B = {(1,5),(1,6),(2,5),(2,6),(3,5),(3,6),(4,5),(4,6),(5,5),(5,6),(6,5),(6,6)}

$$n(B) = 12$$

$$\Rightarrow \quad P(B) = \frac{n(B)}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

The event $A \cap B$ is

$A \cap B$ = {(2,5),(2,6),(4,5),(4,6),(6,5),(6,6)}

$$n(A \cap B) = 6$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

The event A and B are independent of

$$P(A).P(B) = P(A \cap B) \quad \Rightarrow \frac{1}{2}.\frac{1}{3} = \frac{1}{6} \ \textit{hence } A \textit{ and } B \textit{ are independent}$$

Example: Let A be the event that a family has children of both sexes and B be the event that a family has at most one boy. If a family is known to have

(i) Three children then show that A and B are independent events
(ii) Four children then show that A and B are dependent event.

Solution (i): Let b denote the boy and g denote the girl

The sample space is S = {bbb , bbg , bgb , gbb , bgg , gbg , ggb ,ggg}

n(S) = 8

Let A be the event that children of both sexes

A = { bbg, bgb ,gbb , bgg , gbg, ggb}

n(A) = 6

$$\Rightarrow \quad P(A) = \frac{n(A)}{n(S)} = \frac{6}{18} = \frac{3}{4}$$

Let B be the event that family has at most one boy

B = {bgg, gbg, ggb ,ggg}

n(B) = 4

$$\Rightarrow \quad P(B) = \frac{n(B)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

The event $A \cap B$ is

$A \cap B$ = {bgg , gbg , ggb}

$$n(A \cap B) = 3$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{3}{8}$$

Now $P(A).P(B) = P(A \cap B) \quad \Rightarrow \frac{3}{4}.\frac{1}{2} = \frac{3}{8}$ *hence A and B are independent*

(ii). The sample space is S = {bbbb , bbbg , bbgb , bgbb ,gbbb, bbgg , bgbg , bggb ,gbbg, gbgb, ggbb, bggg, gbgg, ggbg ,gggb ,gggg}

n(S) = 16

Let A be the event that children of both sexes

A = {bbbg , bbgb , bgbb ,gbbb, bbgg , bgbg , bggb ,gbbg, gbgb, ggbb, bggg, gbgg, ggbg ,gggb }

n(A) = 14

$$\Rightarrow \quad P(A)=\frac{n(A)}{n(S)}=\frac{14}{16}=\frac{7}{8}$$

Let B be the event that family has at most one boy

B = {bggg, gbgg, ggbg ,gggb ,gggg}

n(B) = 5

$$\Rightarrow \quad P(B)=\frac{n(B)}{n(S)}=\frac{5}{16}$$

The event $A\cap B$ is

$A\cap B=$ {bggg , gbgg , ggbg, gggb}

$$n(A\cap B)=4$$

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{4}{18}=\frac{2}{9}$$

Now $P(A).P(B)=P(A\cap B) \quad \Rightarrow \frac{7}{8}.\frac{5}{16}\neq\frac{2}{9}$ *hence A and B are dependent*

Theorem: If A and B are two independent events then $P(A\cap B)=P(A)P(B)$

Proof: Since A and B are independent event

$$P(A|B)=P(A) \quad and \;\; P(B|A)=P(B)$$

As $\qquad P(A\cap B)=P(A)P(B|A)$

$$P(A\cap B)=P(A)P(B)$$

Theorem: If A and B are two independent events in a sample space S then

(i) A and $\overline{B}$ are independent
(ii) $\overline{A}$ and B are independent
(iii) $\overline{A}$ and $\overline{B}$ are independent

Solution: (i) The event $A\cap B$ *and* $A\cap\overline{B}$ are mutually exclusive and their union is A

$$A=(A\cap B)\cup\left(A\cap\overline{B}\right)$$

$$P(A)=P\left[(A\cap B)\cup\left(A\cap\overline{B}\right)\right]$$

$$P(A)=P(A\cap B)+P\left(A\cap\overline{B}\right)$$

$$P\left(A\cap\overline{B}\right)=P(A)-P(A\cap B)$$

$$P\left(A\cap\overline{B}\right)=P(A)-P(A)P(B) \qquad \because P(A\cap B)=P(A)P(B)$$

$$P\left(A\cap\overline{B}\right)=P(\mathrm{A})\left[1-P(B)\right]$$

$$P\left(A\cap\overline{B}\right)=P(\mathrm{A})\,\mathrm{P}\left(\overline{B}\right)$$

Hence A and $\overline{B}$ are independent

(ii). Similarly, $P(B)=P(B\cap A)+P\left(B\cap\overline{A}\right)$

$$P\left(\overline{A}\cap B\right)=P(B)-P(A\cap B)$$

$$P\left(\overline{A}\cap B\right)=P(B)-P(A)P(B) \qquad \because P(A\cap B)=P(A)P(B)$$

$$P\left(\overline{A}\cap B\right)=P(\mathrm{B})\left[1-P(A)\right]$$

$$P\left(\overline{A}\cap B\right)=P(\mathrm{B})\,\mathrm{P}\left(\overline{A}\right)$$

Hence $\overline{A}$ and B are independent

(iii). Using De-Morgan law

$$\overline{A}\cap\overline{B}=\overline{A\cup B}$$

$$P\left(\overline{A}\cap\overline{B}\right)=P\left(\overline{A\cup B}\right)$$

$$=P\left(A\cup B\right)$$

$$=1-P(A)-P(B)+P(A\cap B) \qquad \because P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=1-P(A)-P(B)+P(A)P(B) \quad \because P(A\cap B)=P(A)P(B)$$

$$=\left[1-P(A)\right]-P(B)\left[1-P(A)\right]$$

$$=\left[1-P(A)\right]\left[1-P(B)\right]$$

$$=P(\overline{A})P(\overline{B})$$

Hence $\overline{A}$ and $\overline{B}$ are independent

Example: Two cards are drawn from a well shuffled ordinary deck of 52 cards. Find the probability that they are both aces if the first card is

(i) Replaced
(ii) Not replaced

Solution: Let A denote the event ace on first draw and B be the event that second card drawn is also ace

(i) P (both are ace) = $P(A\cap B)$

$$P(A\cap B)=P(A)P(B)=\frac{4}{52}.\frac{4}{52}=\frac{16}{2704}=\frac{1}{169}$$

(ii) P (both card are ace) = P (first card is ace) × P(2^{nd} is ace given that 1^{st} is ace)

$$=\frac{4}{52}.\frac{3}{51}=\frac{1}{221}$$

Example: A pair of fair dice is thrown twice. What is the probability of getting total of 5 and 11 ?

Solution: Let A be event of getting total of 5 and B be the event that getting total of 11

Then event A can be occur in the following way

A_1 = { a total of 5 occur in first throw}

A_2 = { a total of 5 occur in second throw}

$$n(A_1)=n(A_2)=4$$

$$P(A_1)=P(A_2)=\frac{4}{36}=\frac{1}{9}$$

Then event B can be occur in the following way

B_1 = { a total of 11 occur in first throw}

B_2 = { a total of 5 occur in second throw}

$$n(B_1) = n(B_2) = 2$$

$$P(B_1) = P(B_2) = \frac{2}{36} = \frac{1}{18}$$

Therefore,
$$P(A \cap B) = P(A_1 \cap B_1) + P(A_2 \cap B_2)$$
$$P(A \cap B) = P(A_1)P(B_1) + P(A_2)P(B_2)$$
$$P(A \cap B) = \frac{1}{9}.\frac{1}{18} + \frac{1}{18}.\frac{1}{9} = \frac{1}{162} + \frac{1}{162} = \frac{1}{81}$$

Example: The probability that a man will be alive in 25 years 3/5 and the probability that his wife will be alive in 25 years 2/3. Find the probability that

(i) Both will be alive
(ii) Only the man will be alive
(iii) Only the wife will be alive
(iv) At least one will be alive
(v) Neither will be alive in 25 years

Solution: Let A be the event that the man will be alive and B be the event that wife will be alive in 25 years

$$P(A) = \frac{3}{5} \quad , \quad P(B) = \frac{2}{3}$$

(i) Both will be alive i.e $P(A \cap B)$

Since A and B are independent then

$$P(A \cap B) = P(A).P(B) = \frac{3}{5}.\frac{2}{3} = \frac{2}{5}$$

(ii) Only the man will be alive i.e $P(A \cap \overline{B})$

Since A and $\overline{B}$ are independent and $P(\overline{B}) = 1 - P(B)$

$$P(A \cap \overline{B}) = P(A).P(\overline{B}) = P(A).(1 - P(B)) = \frac{3}{5}\left(1 - \frac{2}{3}\right) = \frac{3}{5}.\frac{1}{3} = \frac{1}{5}$$

(iii) Only the wife will be alive i.e $P(\overline{A} \cap B)$

Since $\overline{A}$ and B are independent and $P(\overline{A}) = 1 - P(A)$

$$P(\overline{A} \cap B) = P(\overline{A}).P(B) = (1 - P(A))P(B) = \left(1 - \frac{2}{3}\right).\left(\frac{2}{3}\right) = \frac{2}{5}.\frac{2}{3} = \frac{4}{15}$$

(iv) At least one will be alive i.e $P(A \cup B)$

Since A and B are independent and not mutually exclusive

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{3}{5} + \frac{2}{3} - \frac{2}{5} = \frac{9 + 10 - 4}{15} = \frac{13}{15}$$

(v) Neither will be alive i.e $P(\overline{A} \cap \overline{B})$

Since $\overline{A}$ and $\overline{B}$ are independent

$$P(\overline{A} \cap \overline{B}) = P(\overline{A}).P(\overline{B})$$

$$P(\overline{A} \cap \overline{B}) = (1 - P(A))(1 - P(B))$$

$$P(\overline{A} \cap \overline{B}) = \left(1 - \frac{3}{5}\right)\left(1 - \frac{2}{3}\right)$$

$$P(\overline{A} \cap \overline{B}) = \frac{2}{5}.\frac{1}{3} = \frac{2}{15}$$

Lecture # 11

Bayes Theorem: If the events $A_1, A_2, \ldots A_k$ from a partition of sample space S that is the event A_i are mutually exclusive and their union is S and if B is any other event of S such that it can occur only if one of the A_i occurs then for any i

$$P(A_i / B) = \frac{P(A_i)P(B/A_i)}{\sum_{i=1}^{k} P(A_i)P(B/A_i)} \quad \text{for } i = 1, 2, \ldots, k$$

Proof: By the definition of conditional probability $\because$ *events are* A_i

$$P(A_i \cap B) = P(A_i)P(B/A_i) \quad \_\_\_(i)$$

And

$$P(A_i \cap B) = P(B)P(A_i/B) \quad \_\_\_(ii)$$

$$\Rightarrow P(A_i)P(B/A_i) = P(B)P(A_i/B) \quad by (i) \,\&\, (ii)$$

$$P(A_i/B) = \frac{P(A_i)P(B/A_i)}{P(B)} \quad (iii)$$

We can write the event B

$$B = S \cap B$$

$$= (A_1 \cup A_2 \cup \ldots \cup A_k) \cap B$$

$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B)$$

$$P(B) = P\left[(A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B)\right]$$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_k \cap B) \quad \because A_i \cap B \text{ are mutually exclusive}$$

$$P(B) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2) + \ldots + P(A_k)P(B/A_k)$$

$$P(B) = \sum_{i=1}^{k} P(A_i)P(B/A_i)$$

Put in (iii) $\Rightarrow$

$$P(A_i/B) = \frac{P(A_i)P(B/A_i)}{\sum_{i=1}^{k} P(A_i)P(B/A_i)}$$

Example: In a bolt factory machines A, B and C manufacture 25 ,35 and 40 percent of the total output respectively. Of their outputs 5,4,2 percent respectively are defective bolts. A bolt is selected at random and found to be defective. What is the probability that the bolt came from machine A,B,C?

Solution:

$$P(A)=25\%=0.25 \quad , P(B)=35\%=0.35 \;, P(C)=40\%=0.40$$

Let E be the event that bolt is defective

$$P(E/A)=5\%=0.05 \quad , P(E/B)=4\%=0.04 \;, P(E/C)=2\%=0.02$$

By Bayes Theorem

$$P(\mathrm{A}|E)=\frac{P(A)P(\mathrm{E/A})}{\sum_{i=1}^{3}P(A_i)P(\mathrm{E}/A_i)} \quad i=A,B,C$$

$$=\frac{P(A)P(\mathrm{E/A})}{P(A)P(E|A)+P(B)P(E|B)+P(C)P(E|C)}$$

$$=\frac{(0.25)(0.05)}{(0.25)(0.05)+(0.35)(0.04)+(0.40)(0.02)}=\frac{0.0125}{0.0345}=0.362$$

Similarly,

$$P(B|E)=\frac{P(B)P(\mathrm{E/B})}{P(A)P(E|A)+P(B)P(E|B)+P(C)P(E|C)}$$

$$=\frac{(0.35)(0.04)}{(0.25)(0.05)+(0.35)(0.04)+(0.40)(0.02)}=\frac{0.014}{0.0345}=0.406$$

$$P(C|E)=\frac{P(C)P(\mathrm{E/C})}{P(A)P(E|A)+P(B)P(E|B)+P(C)P(E|C)}$$

$$=\frac{(0.40)(0.02)}{(0.25)(0.05)+(0.35)(0.04)+(0.40)(0.02)}=\frac{0.008}{0.0345}=0.232$$

Example: An urn contains four balls which are known to be either

(i) All white
(ii) Two white and two black

A ball is drawn at random and is found to be white. What is the probability that all balls are white?

Solution: Let A_1 be the assumption that all the balls are white and A_2 be the assumption that two balls are white and two black. Then

$$P(A_1) = P(A_2) = \frac{1}{2}$$

Let B be the event that ball drawn is white

$$P(B/A_1) = \frac{{}^4C_1}{{}^4C_1} = 1 \quad , \quad P(B/A_1) = \frac{{}^2C_1 \; . \; {}^2C_0}{{}^4C_1} = \frac{2}{4} = \frac{1}{2}$$

By Bayes theorem

$$P(\mathrm{A}_1/\mathrm{B}) = \frac{P(A_1)P(\mathrm{B}/\mathrm{A}_1)}{P(A_1)P(\mathrm{B}/\mathrm{A}_1) + P(A_2)P(\mathrm{B}/\mathrm{A}_2)}$$

$$P(\mathrm{A}_1/\mathrm{B}) = \frac{\left(\frac{1}{2}\right)(1)}{\left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \frac{1}{2}.\frac{4}{3} = \frac{2}{3}$$

$$P(\mathrm{A}_2/\mathrm{B}) = \frac{P(A_2)P(\mathrm{B}/\mathrm{A}_2)}{P(A_1)P(\mathrm{B}/\mathrm{A}_1) + P(A_2)P(\mathrm{B}/\mathrm{A}_2)}$$

$$P(\mathrm{A}_2/\mathrm{B}) = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \frac{1}{4}.\frac{4}{3} = \frac{1}{3}$$

Hence the first assumption is preferred.

Lecture # 12

Random Variable:

A numerical quantity whose value is determined by the outcome of some random experiment is called random variable. It is also known as chance variable/ stochastic variable or variate.

The random variables are denoted by capital Latin letter such as X,Y,Z. While the values taken by them are represented by small letter such as x,y,z.

Distribution function:

The distribution function of a random variable X is denoted by $F(x) = P(X \le x)$. The function F(x) gives the probability of the event that X can take the value less than or equal to the specified value of x. The distribution function is abbreviated to d.f and is also called the cumulative distribution (cdf) as it is the cumulative probability function of the X from the smallest up to special value of x. Since F(x) is probability, it is quite obvious that

$$F(-\infty) = P(\phi) = 0 \text{ and } F(+\infty) = P(s) = 1$$

Let a and b be two real numbers such that a < b then

$$\begin{aligned} F(b) - F(a) &= P(X \le b) - P(X \le a) \\ &= P(a < X \le b) \end{aligned}$$

Which is non-negative and hence F(x) is non-decreasing function of x.

Again $\lim_{h \to 0} F(x+h) = F(x)$, i.e. the function F(x) is continuous on the right at each value of X.

A d.f F(x) thus has following properties

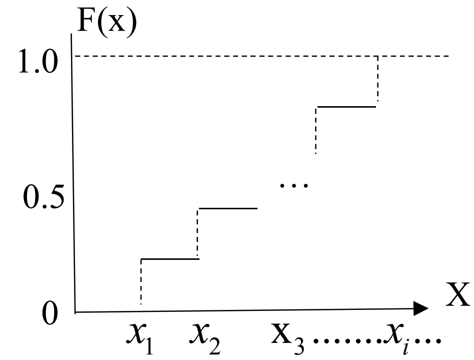
(i) $F(-\infty) = 0$ and $F(+\infty) = 1$
(ii) F(x) is non-decreasing function of x i.e. $F(x_1) \le F(x_2)$ if $x_1 \le x_2$
(iii) F(x) is continuous at least on the right of each x.

Discrete random variable and its probability distribution:

A random variable X is said to be discrete if it can assume values which are finite or countably infinite i.e $x_1, x_2, ...x_n, ...$ are probability points or jump point.

Distribution function

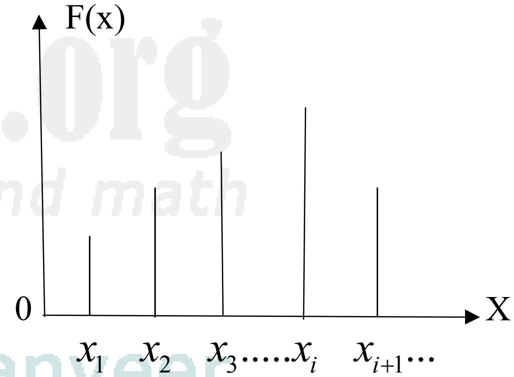
$F(x) = \sum f(x_i)$



Vaues (x_i)	x_1 x_2 x_3…….x_n…
Probability $f(x_i)$	$f(x_1)$ $f(x_2)$ $f(x_3)$…..$f(x_n)$…

"The tabular arrangement of the data/random variable along with corresponding probability is called the probability distribution of that random variable".

The graph of probability distribution is



Example: Find the probability distribution and distribution function for the number of heads when 3 balanced coins are tossed. Construct a probability histogram and graph of the distribution.

Solution: The sample space for three coins is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let X be a random variable that denote the number of heads. Then X can take the value 0,1,2,3.

The corresponding probability are

f(0) = P(X=0) = P[{TTT}] = 1/8

f(1) = P(X=1) = P[{HTT,THT,TTH}] = 3/8

f(2) = P(X=2) = P[{HHT,HTH,THH}] = 3/8

f(3) = P(X=3) = P[{HHH}] = 1/8

X	0	1	2	3
f(x)	1/8	3/8	3/8	1/8

Now we find distribution function

For $x < 0$ $\quad F(x) = P(X \le x)$

$F(0) = 0$

For $0 \le x < 1$ we have $P(X < x) = P(X = 0) = 1/8$

For $1 \le x < 2$, we have $P(X < x) = P(X = 0) + P(X = 1)$

$= 1/8 + 3/8 = 4/8$

For $2 \le x < 3$, we have $P(X < x) = P(X = 0) + P(X = 1) + P(X = 2)$

$= 1/8 + 3/8 + 3/8 = 7/8$

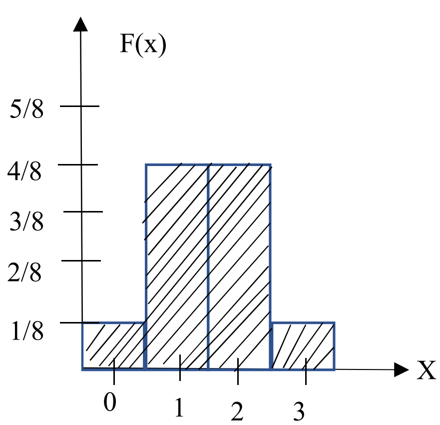
Finally For $x \ge 3$ $\; P(X < x) = P(X = 0) + P(X = 1) + P(X = 2) + P(X=3)$

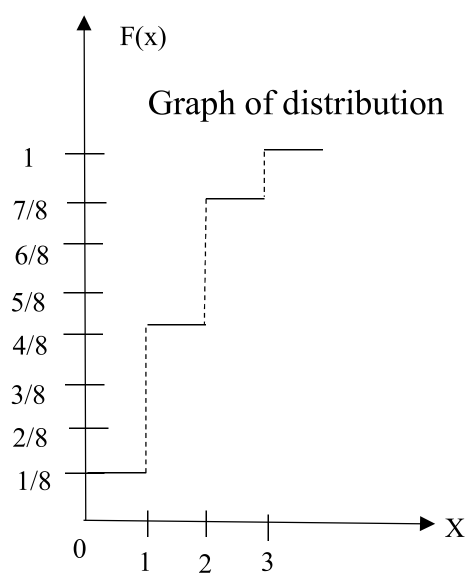
$= 1/8 + 3/8 + 3/8 + 1/8 = 8/8 = 1$

Hence the desired distribution function is

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{8} & \text{for } 0 \le x < 1 \\ \frac{4}{8} & \text{for } 1 \le x < 2 \\ \frac{7}{8} & \text{for } 2 \le x < 3 \\ 1 & \text{for } x \ge 3 \end{cases}$$

Histogram





Lecture # 13

Example: (a) Find the probability distribution of the sum of the dots when two fair dice are thrown.

(b) Use the probability distribution to find the probabilities of obtaining

(i) a sum of 8 or 11

(ii) a sum is greater than 8

(iii). a sum that is greater than 5 but less than or equal to 10

Solution: The sample space for this experiment is

S = {(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (2,1) (2,2) (2,3) (2,4) (2,5) (2,6) (3,1) (3,2) (3,3) (3,4) (3,5) (3,6) (4,1) (4,2) (4,3) (4,4) (4,5) (4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

(a) Let X denote the random variable that denote the sum of dots which appear on the dice. Then X cab take the value 2,3,4,5,….,12

The corresponding probabilities are

f(2) = P(X=2)=P[{(1,1)}] = 1/36

f(3) = P(X=3)=P[{(1,2),(2,1)}] = 2/36

f(4) = P(X=4)=P[{(1,3),(2,2),(3,1)}] = 3/36

Similarly, f(5) = 4/36 , f(6) = 5/36 , f(7) = 6/36, f(8) = 5/36 , f(9) =4/36 , f(10)=3/36 , f(11) = 2/36 , f(12) = 1/36

x_i	2	3	4	5	6	7	8	9	10	11	12
$f(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

(b) Using the probability distribution, we get the required probabilities as follows

(i) P(a sum of 8 or 11) = P [(X=8) or (X=11)]
= P(X=8)+P(X=11)
= 5/36 + 2/36 = 7/36

(ii) P (sum is greater than 8) = P(X>8)
= P(X=9)+P(X=10)+ P(X=11)+P(X=12)
= 4/36+3/36+2/36+1/36 = 10/36

(iii) P (a sum is greater than 5 but less or equal to 10)
$= P(5 < X \leq 10)$

= P(X=6)+P(X=7)+ P(X=8)+P(X=9)+ P(X=10)

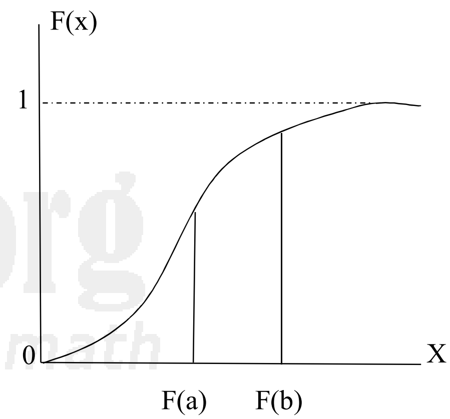
= 5/36 + 6/36 +5/36+4/36+ 3/36 = 23/36

Continuous Random variable and its probability Density function:

A random variable X is defined to be continuous if it can assume every possible value in an interval [a,b] , a < b, where a and b may be $-\infty$ and $+\infty$ respectively.

A random variable X is defined to be continuous if its distribution function F(x) is continuous and differentiable everywhere except at the isolated points in a given range.

The graph of F(x) has no jumps.



Let the derivative of F(x) be denoted by f(x)

i.e. $\frac{d}{dx}F(x)=f(x)$

Since F(x) is a non-decreasing function of x, we have

(i) $f(x) \geq 0$

(ii) $F(x) = \int_{-\infty}^{+\infty} f(x)dx \quad , \forall x$

The function is called probability density function or p.d.f.

A p.d.f has the following properties

(i) $f(x) \geq 0 \qquad \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) The probability that X can take the value in the interval [c,d] , c < d is given by
$P(c < x \leq d) = F(d) - F(c)$

$$= \int_{-\infty}^{d} f(x)dx - \int_{-\infty}^{c} f(x)dx$$

$$= \int_{-\infty}^{d} f(x)dx + \int_{c}^{-\infty} f(x)dx$$

$$= \int_{c}^{-\infty} f(x)dx + \int_{-\infty}^{d} f(x)dx$$

$$= \int_{c}^{d} f(x)dx$$

Example: (a) Find the value of k so that the function f(x) defined as follow may be a density function

$f(x) = kx \quad , 0 \le x \le 2$

$f(x) = 0 \quad , \text{otherwise}$

(b). Find also the probability that both of two sample value will exceed 1

(c) Compute the distribution function F(x).

Solution: (a) The function f(x) will be a density function, if

(i) $f(x) \ge$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

The first condition is satisfied when $k \ge 0$. The second condition will be satisfied if

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{a} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$

$$0 + \int_{0}^{2} kx \quad dx + 0 = 1$$

$$k\frac{x^2}{2}\Big|_0^2 = 1 \qquad \Rightarrow \qquad k\frac{4}{2} = 1 \quad \Rightarrow k = \frac{1}{2}$$

$$Hence \quad f(x) = \begin{cases} \frac{x}{2} & for \quad 0 \le x \le 2 \\ 0 & otherwise \end{cases}$$

(b). $$P(X>1)=\int_1^2 f(x)\,dx$$

$$=\int_1^2 \frac{x}{2}dx=\left.\frac{x^2}{4}\right|_1^2$$

$$=\frac{4}{4}-\frac{1}{4}=\frac{3}{4}$$

(c) To compute distribution function, we find

$$F(x)=P(X<x)=\int_{-\infty}^{x} f(x)\,dx \qquad for\ any\ x$$

$$for\ -\infty<x\le 0$$

$$F(x)=\int_{-\infty}^{x} f(x)\,dx=\int_{-\infty}^{x} 0dx=0$$

$for\ 0<x\le 2$ $$F(x)=\int_{-\infty}^{x} f(x)\,dx=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{x} f(x)\,dx$$

$$=0+\int_0^x \frac{x}{2}dx=\frac{x^2}{4}$$

$$for\ 2<x\le\infty$$

$$F(x)=\int_{-\infty}^{x} f(x)\,dx=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{2} f(x)\,dx+\int_{2}^{\infty} f(x)\,dx$$

$$=0+\int_0^2 \frac{x}{2}dx+0$$

$$=\left.\frac{x^2}{4}\right|_0^2=\frac{4}{4}=1$$

$$F(x)=\begin{cases}0 & for\ -\infty<x\le 0\\ \dfrac{x^2}{2} & for\ \ 0<x\le 2\\ 1 & for\ 2<x\le\infty\ \ or\ x>2\end{cases}$$

Mathematical expectation of a random variable:

Let X be a discrete random variable having possible values $x_1, x_2, ...x_n, ...$ with the corresponding probabilities $f(x_1), f(x_2), ...f(x_n), ...$ such that $\sum f(x) = 1$

Then the mathematical expectation or the expectation or the expected value of X, denoted by E(X) is defined as

$$E(X) = x_1 f(x_1) + x_2 f(x_2) + ... + x_n f(x_n), ...$$

$$= \sum_{i=1}^{\infty} x_i f(x_i) \text{ , provided it converge absolutely.}$$

The sum converges absolutely iff $\sum |x| f(x)$ is finite.

If X is a continuous random variable with p.d.f f(x) then

$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$ provided the integral converges absolutely

i.e. $\int_{-\infty}^{\infty} |x|\, f(x)\, dx$ is finite.