Mathematical Statistics

by

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Partial Contents

These are handwritten notes. We are very thankful to Ms. Iqra Liaqat for sending these notes.

PROBABILITY

Definition:-

i) a quantitative measure of uncertainty

ii) a measure of degree of belief in a particular statement or problem.

iii) Measure of chance

iv) $\dfrac{\text{no. of favourable outcomes}}{\text{no. of total outcomes}}$

if a = no. of favourable outcomes and b = no. of total outcomes then

Probability $= \dfrac{a}{b}$; $0 \leq a \leq b$; $a, b \in Z^+$

and $0 \leq \text{prob} \leq 1$

v) If prob = 0% that means impossible to happening

vi) If Prob = 1 = 100% then means sure.

vii) If Prob $= \frac{1}{2} = 50\%$

Set :-

A set is any well-defined collection or list of distinct objects. or A Collection of well-defined distinct objects.

outcome or any number of outcomes of a random experiment or a trial.

An event is subset of sample space

e.g.) when a die is rolled, then

$S = \{1, 2, 3, 4, 5, 6\}$

For prob of even numbers

Event $A = \{2, 4, 6\}$

For prob of odd numbers

Event $B = \{1, 3, 5\}$

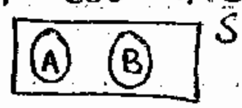
For prob greater than 4

Event $C = \{5, 6\}$

For prob of 6, Event $D = \{6\}$

Mutually Exclusive Events:-

Two events A and B of a single experiment are said to be mutually exclusive or disjoint iff they cannot both occur at the same time.



That is they have no points in common.

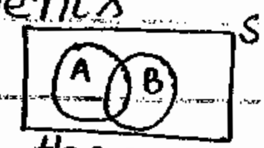
or) Two or more than two events, non-overlapping and $\bigcap_{i=1}^{k} A_i = \phi$ or

$A \cap B = \phi$

Exhaustive Events :-

Events are said to be collectively exhaustive, when the union of mutually exclusive events is entire sample space S.

or) Two or more than two events collectively form S.



e.g) A and B are exhaustive then

$A \cup B = S$; $\bigcup_{i=1}^{k} A_i = S$

Equally Likely Events :-

Two events A and B are said to be equally likely, if they have same chances of occurance in random experiment.

or) when one event is as likely to occur as the other.

e.g) $P(A) = P(B)$

$\Rightarrow$ probability of A = probability of B

Counting Rules :-

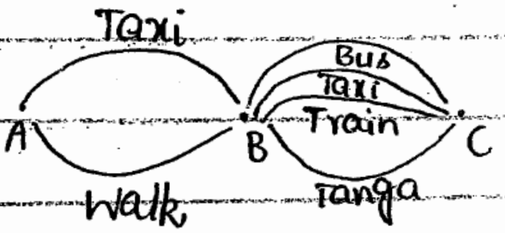
(i) Multiplication Rule :-

* compound experiment
* Ist experiment consist of "n_1" outcomes

★ 2nd experiment consists of "n_2" outcomes

No. of total outcomes in compound experiment $= n_1 \times n_2$

e.g



$2 \times 4 = 8$

TB, TTa, TTr, TTanga

WB, WTa, WTr, WTanga

there are 8 possible ways and are independent to each other.

ii) Permutation Rule :-

A permutation is any ordered subset from a set of n distinct objects.

The number of permutations of r objects, selected in a definite order from n distinct objects is denoted by ${}^nP_r$

and $${}^nP_r = \frac{n!}{(n-r)!}$$

If $n=3$, $r=2$

$${}^3P_2 = \frac{3!}{(3-2)!} = 6$$ no. of arrangements or ways

If A, B and C are 3 objects and we want to select 2 objects then the arrangements are as

AB   or If $n=3$, $r=1$

AC   then $^3P_1 = \frac{3!}{(3-1)!}$

BA

BC $^3P_1 = \frac{3!}{2!} = \frac{3\cdot2\cdot1}{2\cdot1}$

CA $^3P_1 = 3$

CB   then A B C (ways)

(iii) Combination Rule:-

A combination is any subset of r objects, selected without regard to their order, from a set of n distinct objects. The total number of such combinations is denoted by ^nC_r

and $^nC_r = \frac{n!}{(n-r)!\,r!}$

If $n=3$, $r=2$

then $^3C_r = \frac{3!}{(3-2)!\,2!} = \frac{3!}{1!\,2!} = 3$

then we have arrangement to select 2 objects $\Rightarrow$ AB AC BC

Vote:-

To arrange the selected objects we use permutation and combination when the order importance is necessary then we use permutation and when order importance is not necessary then we use combination.

Definitions of Probability:-

Classical Definition:-

(Laplace, a priori def (basic in advance))

If a sample space S consists of "n" mutually exclusive and equally likely outcomes, of which "m" belongs to a certain event A. Then probability of event A written as P(A) and can be defined as

$$P(A) = \frac{m}{n}$$

If $n \to \infty$ then $P(A) = 0$

Example:-

Two fair coins are tossed. Find probability that

(i) no head appears
(ii) all heads
(iii) at least one head
Sol:-

$S = \{HH, HT, TH, TT\}$

$\Rightarrow \quad n(S) = 4$

(i) let Event A denote no head appears
then $A = \{TT\} \Rightarrow n(A) = 1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{4} = 0.25$$

(ii) let Event B denote all heads
then $B = \{HH\} \Rightarrow n(B) = 1$

$$P(B) = \frac{n(B)}{n(S)} = 1/4 = 0.25$$

(iii) let Event C denote at least one head
then $C = \{HH, HT, TH\}$

$\Rightarrow n(C) = 3$

$$P(C) = \frac{n(C)}{n(S)} = \frac{3}{4} = 0.75$$

From the Probability, A and B are equally likely and mutually exclusive.

(iv) 4 head appers

Let Event D denote 4 head appears then $D = \{\} \Rightarrow n(D) = 0$

$$P(D) = \frac{n(D)}{n(S)} = 0/4 = 0$$

Example:-

A coin and a die are tossed together. Find the probability that

(i) even number appears

(ii) even number and head appears

(iii) number 6 appear with a tail

Sol:-

Coin has two possible outcomes and die has 6 possible outcomes then sample space consists $2 \times 6 = 12$ possible outcomes.

$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\} \Rightarrow n(S) = 12$

(i) Let event A denote even number appears

$A = \{H2, H4, H6, T2, T4, T6\}$

$\Rightarrow n(A) = 6$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{12} = \frac{1}{2} = 0.5$$

(ii) Let event B denote even number

and head appears

$B = \{H2, H4, H6\}$

$\Rightarrow n(B) = 3$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{12} = 1/4 = 0.25$$

(iii) Let event C denote number 6 appear with a tail

$C = \{6T\} \Rightarrow n(C) = 1$

$$P(C) = \frac{n(C)}{n(S)} = 1/12 = 0.083$$

Relative Frequency :-

(A posteriori def (after experiment))

If a random experiment is repeated a large number of times say "n" under identical conditions and if an event A is observed to occur "m" times then

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

Example :-

A coin is tossed. What is the probability that a head appears.

Sol :-

By priori def $S = \{H, T\} \Rightarrow n(S) = 2$

and $P(H) = 1/2 = 0.5$
under assumption that H and T are equally likely.
By posteriori def, a coin is tossed 100 times and head appears 57 times then $P(H) = \frac{57}{100} = 0.57$

Subjective approach:- (personal opinion)

The subjective or personalistic probability is a measure of the strength of a person's belief regarding the occurrence of an event A.

Example:-

Six white ball and four black balls which are indistinguishable apart from their colour, are placed in a bag. If six balls are taken at random, what is probability that

(i) 6 white balls are selected
(ii) 3 " " " "
(iii) 0 " " " "
(iv) 3 white and 3 black balls
(v) at least 2 black balls

Sol:-

Total 10 balls are placed in bag and we select 6 balls, then the possible outcomes are

$n(S) = {}^{10}C_6 = 210$ no. of ways.

(i) Let A denote 6 white balls appear then

$n(A) = {}^6C_6 = 1$

$P(A) = \frac{n(A)}{n(S)} = 1/210 = 0.0048$

(ii) Let Event B denote 3 white balls are selected

$n(B) = {}^6C_3 \, {}^4C_3 = 20 \times 4 = 80$

$P(B) = \frac{n(B)}{n(S)} = \frac{80}{210} = 0.38$

(iii) Let Event C denote 0 white balls are selected

$n(C) = {}^4C_6 =$ impossible

$P(C) = 0$

(iv) Let Event D denote 3 white and 3 black balls are selected.

$n(D) = {}^6C_3 \, {}^4C_3 = 20 \times 4 = 80$

$$P(D) = \frac{n(D)}{n(S)} = \frac{80}{210} = 0.38$$

v) Let Event E denote at least 2 black balls.

$$n(E) = {}^4C_2\,{}^6C_4 + {}^4C_3\,{}^6C_3 + {}^4C_4\,{}^6C_2$$
$$= 15\times 6 + 20\times 4 + 1\times 15$$
$$= 90+80+15$$
$$n(E) = 185$$
$$P(E) = \frac{n(E)}{n(S)} = \frac{185}{210} = 0.88$$

Axioms of Probability:-

i) For any event say E_i
$$0 \le P(E_i) \le 1$$

ii) $P(S) = 1$ where S is sure event or sample space

iii) If A and B are 2 mutually exclusive events then
$$P(A \cup B) = P(A) + P(B)$$
$$\text{or } P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Theorem:-

If ϕ is an impossible event then $P(\phi) = 0$

Proof:-

we can write

$S = S \cup \phi$ where S is sure event

then $P(S) = P(S \cup \phi) \rightarrow (1)$

Since, S and ϕ are mutually exclusive

then $P(S \cup \phi) = P(S) + P(\phi)$

So $P(S) = P(S) + P(\phi)$ by (1)

we know $P(S) = 1$

then $1 = 1 + P(\phi)$

$\Rightarrow P(\phi) = 0$

Theorem:-

If $\bar{A}$ is compliment of event A defined on sample space S then

$P(\bar{A}) = 1 - P(A)$ or $P(A) = 1 - P(\bar{A})$

Proof:-

If $\bar{A}$ is compliment of A defined on S then we can write

$A \cup \bar{A} = S$

$\Rightarrow P(A \cup \bar{A}) = P(S)$

$\Rightarrow P(A \cup \bar{A}) = 1$

because $P(S) = 1$ (Axiom of Prob)

Since, A and $\bar{A}$ are mutually exclusive events

$\therefore P(A \cup \bar{A}) = P(A) + P(\bar{A})$

then $P(A) + P(\bar{A}) = 1$

$\Rightarrow P(\bar{A}) = 1 - P(A)$

or $P(A) = 1 - P(\bar{A})$

Example:-

A coin is tossed 4-times in succession what is the prob that at least one head appears.

Sol:-

$S = \{$HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THTH, TTHH, THHT, HTTT, THTT, TTHT, TTTH, TTTT$\}$

$n(S) = 2^4 = 16$

(i) let the event A denote at least one head appears $(H \geq 1)$

$n(A) = 15$,

$$P(A) = \frac{n(A)}{n(S)} = \frac{15}{16}$$

(or)

(ii) A = at least one head appear

$\bar{A}$ = no head appears

$n(\bar{A}) = 1$

$$P(\bar{A}) = \frac{n(\bar{A})}{n(S)}$$

$$P(\bar{A}) = \frac{1}{16}$$

we know $P(A) = 1 - P(\bar{A})$

$$P(A) = 1 - 1/16$$

$$P(A) = \frac{16-1}{16} = 15/16$$

(or)

(iii) As coin is fair then for a single tossed $P(H) = \frac{1}{2} = P(T)$

A = at least one head appears

$\bar{A}$ = no head appears (all tails)

then $P(A) = 1 - P(\bar{A})$

$$= 1 - P(TTTT)$$

"If A and B are independent then $P(AB) = P(A) \cdot P(B)$

$$= 1 - P(T) \cdot P(T) \cdot P(T) \cdot P(T)$$

$$= 1 - 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2$$

$$= 1 - (1/2)^4$$

$$P(A) = 1 - \frac{1}{16} = 15/16$$

Example:-

A coin is so biased that the probability that it falls showing tail is 3/4.

(a) Find the prob of obtaining at

east one head when it is tossed
(i) 5 times (ii) n times
b) How many times the coin be tossed so that the prob of obtaining at least one head is greater than 0.98?

Sol:–

$P(T) = 3/4$

A = at least one head appears

$\bar{A}$ = no head appear (all tails)

$P(\bar{A}) = P(T) = 3/4$

$P(A) = 1 - P(\bar{A})$

(i) when coin is tossed 5-times.
then

$P(A) = 1 - (3/4)^5$

Because $P(A) = 1 - P(TTTTT)$

$= 1 - P(T) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(T)$

$= 1 - (P(T))^5$

$= 1 - (3/4)^5$

$= 1 - 0.237$

$P(A) = 0.762$

(ii) when coin is tossed n-times
then

$P(A) = 1 - (P(\bar{A}))^n$

$\Rightarrow \quad P(A) = 1 - (3/4)^n$

Because $P(\bar{A}) = P(T) = 3/4$.

(b)

$P(A) > 0.98$

and $P(A) = 1 - \left(\frac{3}{4}\right)^n$

then

$$1 - \left(\frac{3}{4}\right)^n > 0.98$$

$$\left(\frac{3}{4}\right)^n < 1 - 0.98$$

$$\left(\frac{3}{4}\right)^n < 0.02$$

$$\log\left(\frac{3}{4}\right)^n < \log(0.02)$$

$$n \log\left(\frac{3}{4}\right) < \log(0.02)$$

$$n(-0.124) < (-1.698)$$

$$n < \frac{-1.698}{-0.124}$$

$$n < 13.69$$

$$n < 14$$

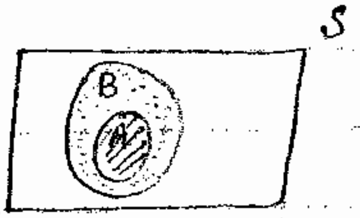
Theorem:-

If A and B are two events defined on sample space S such that $A \subseteq B$ then

$$P(A) \leq P(B)$$

Proof:-

For $A \subseteq B$, then we can write



$$B = (A \cap B) \cup (B \cap \bar{A}) \rightarrow (1)$$

where $A \cap B$ and $B \cap \bar{A}$ are mutually exclusive then

$$P((A \cap B) \cup B \cap \bar{A}) = P(A \cap B) + P(B \cap \bar{A})$$

By (1)

$$P(B) = P((A \cap B) \cup (B \cap \bar{A}))$$

$$P(B) = P(A \cap B) + P(B \cap \bar{A}) \rightarrow (2)$$

As $A \cap B = A$

So $P(A \cap B) = P(A)$

then (2) becomes

$$P(B) = P(A) + P(B \cap \bar{A})$$

According to axioms of probability

$$P(B \cap \bar{A}) \geq 0$$

$$\therefore P(B) \geq P(A)$$

$$\Rightarrow P(A) \leq P(B).$$

Theorem :-

If A and B are two events defined on S then

i) $P(A\cap\bar{B}) = P(A) - P(A\cap B)$.

Proof :-

Consider A and B are non-mutually exclusive events.

we can write.

$A = (A\cap\bar{B})\cup(A\cap B)$

$P(A) = P((A\cap\bar{B})\cup(A\cap B))$

$P(A) = P(A\cap\bar{B}) + P(A\cap B)$

$\Rightarrow P(A\cap\bar{B}) = P(A) - P(A\cap B)$

ii)

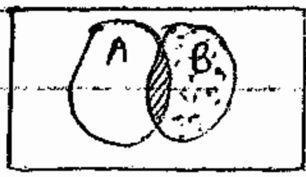
$P(\bar{A}\cap B) = P(B) - P(A\cap B)$

Proof :-

we can write event B as

$B = (B\cap\bar{A})\cup(A\cap B)$

$P(B) = P[(\bar{A}\cap B)\cup(A\cap B)]$



Since, $\bar{A}\cap B$ and $A\cap B$ are mutually exclusive events. So,

$P(B) = P(\bar{A}\cap B) + P(A\cap B)$

$P(\bar{A}\cap B) = P(B) - P(A\cap B)$

Theorem:-

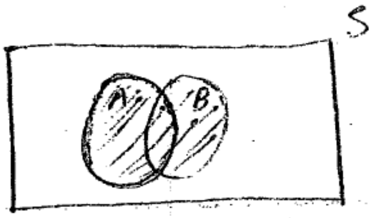
General law of addition

If A and B are two events defined on S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof:-

we can write



$A \cup B = A \cup (\bar{A} \cap B)$

Since, A and $\bar{A} \cap B$ are mutually exclusive

so $P(A \cup B) = P(A \cup (\bar{A} \cap B))$

$P(A \cup B) = P(A) + P(\bar{A} \cap B)$

we know $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B) \rightarrow (1)$

Corrolary:-

If A and B are two mutually exclusive events, then

$A \cap B = \phi$

and $P(A \cap B) = P(\phi) = 0$

$\Rightarrow P(A \cap B) = 0$ put in (1)

$\Rightarrow P(A \cup B) = P(A) + P(B)$.

Corollary :-

If A and B are two events defined on S then

$P(A \cup B) \leq P(A) + P(B)$

Proof :-

Consider A and B are two non-mutually exclusive events.

By General law of addition

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

we know $P(A \cap B) \geq 0 \because 0 \leq \text{Prob} \leq 1$

$P(A \cup B) = P(A) + P(B) + (-P(A \cap B))$

$P(A \cup B) \leq P(A) + P(B) + 0$

$\Rightarrow P(A \cup B) \leq P(A) + P(B)$

Example :-

An integer is choosen at random from the first 200 positive integers. What is the probability that the choosen integer is divisible by 6 or 8?

Sol :-

$n(S) = {}^{200}C_1 = 200$

Let the event A denote the chosen integer is divisible by 6.

Let the event B denote the

Chosen integer is divisible by 8.
then we have to calculate $P(A \cup B)$.
So, we use addition law

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \rightarrow (1)$$

So $n(A) = \frac{n(S)}{6} = \frac{200}{6}$

$$n(A) = 33$$

$$n(B) = \frac{n(S)}{8} = \frac{200}{8} = 25$$

$$n(A \cap B) = \frac{n(S)}{24} = \frac{200}{24}$$

$$n(A \cap B) = 8$$

then $P(A) = \frac{n(A)}{n(S)} = 33/200$

$$P(B) = \frac{n(B)}{n(S)} = 25/200$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = 8/200$$

put all values in (1)

$$P(A \cup B) = \frac{33}{200} + \frac{25}{200} - \frac{8}{200}$$

$$P(A \cup B) = \frac{50}{200} = 1/4 = 0.25$$

Example:-

Three horses A, B and C are in a race, A is twice as likely to win race as B, B is twice as likely to win as C. What is the probability that A or B wins?

Sol:-

$$P(A) = 2P(B)$$
$$P(B) = 2P(C)$$

(i) A, B and C are mutually exclusive

(ii) A, B and C are exhaustive

then $P(A) + P(B) + P(C) = 1 \rightarrow (1)$

As we know

$$P(A) = 2P(B) = 2(2P(C))$$
$$P(A) = 4P(C)$$
$$P(B) = 2P(C)$$

put in (1)

$$4P(C) + 2P(C) + P(C) = 1$$
$$7P(C) = 1$$
$$P(C) = 1/7$$

put this result in (1)

$$P(A) + P(B) + \frac{1}{7} = 1$$

$$\Rightarrow P(A)+P(B) = 1-\frac{1}{7}$$

$$P(A\cup B) = 6/7.$$

Examples:-

Two fair dice are rolled. What is the probability of obtaining i) a total of 7

ii) a doublet (both dice shown same number).

iii) a total of 6 or 10.

iv) a total of 8 or number 6 on any die

v) a total of 13.

Sol:- $n(S) = 36$

$$S = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & \cdots & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & \cdots & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & \cdots & (6,6) \end{bmatrix}$$

i) A = a total of 7

$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$n(A) = 6$

$$P(A) = \frac{n(A)}{n(S)}$$

$$P(A) = \frac{6}{36} = 1/6$$

(ii) B = a doublet

$B = \{(1,1), (2,2) (3,3), (4,4) (5,5), (6,6)\}$

$n(B) = 6$

$$P(B) = \frac{n(B)}{n(S)}$$

$$P(B) = \frac{6}{36} = 1/6$$

(iii)

C = a total 6

$C = \{(1,5), (2,4) (3,3), (4,2) (5,1)\}$

$n(C) = 5$

$$P(C) = \frac{n(C)}{n(S)}$$

$P(C) = 5/36$

D: a total 10

$D = \{(4,6), (5,5), (6,4)\}$

$n(D) = 3$

$$P(D) = \frac{n(D)}{n(S)}$$

$P(D) = 3/36$

$P(C \cup D) = P(C) + P(D)$

$P(C \cup D) = \frac{5}{36} + \frac{3}{36} = \frac{8}{36} = \frac{2}{9}$

$C \cup D = \{(1,5)(2,4), (3,3)(4,2)(5,1)(4,6)$
$(5,5), (6,4)\}$

$n(C \cup D) = 8$

$P(C \cup D) = \frac{n(C \cup D)}{n(S)}$

$P(C \cup D) = 8/36 = 2/9$

iv)

E : total of 8

$E = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$n(E) = 5$

$P(E) = \frac{n(E)}{n(S)} = 5/36$

F : number 6 on die (any die)

$F = \{(1,6), (2,6)(3,6) (4,6)(5,6)(6,6)$
$(6,1)(6,2) (6,3)(6,4)(6,5\}$

$n(F) = 11$

$P(F) = \frac{n(F)}{n(S)} = 11/36$

$E \cup F = \{(1,6), (2,6), (3,5), (3,6)(4,4), (4,6)$
$(5,3), (5,6), (6,2)(6,6), (6,1), (6,3), (6,4), (6,5)\}$

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= 5/36 + 11/36 - 2/36$

$P(E \cup F) = 14/36$

(v) G: a total of 13

$G = \{\ \}$

$P(G) = 0$.

Playing Cards:-

Total no. of cards = 52

Black cards = 26

Red cards = 26

No. of suits = 4

(Club, spade, diamond, Heart)

13, 13, 13, 13

Black Red

A, 2, 3, 4, ---, 10, J, Q, K

A: Aces which are 4

2: → 4 J, Q, K : Face Cards

3 → 4 and Face cards = 12

⋮

10 → 4

J → 4

Q → 4

K → 4.

Example:-

If a card is selected at random from a deck of 52 playings cards, what is the prob. that selected card is.

(i) Red (ii) Black (iii) an ace
(iv) Red or king

Sol:-

$$n(S) = {}^{52}C_1 = 52$$

(i) A: Red card

$$n(A) = {}^{26}C_1\,{}^{26}C_0 = 26$$

$$P(A) = \frac{n(A)}{n(S)}$$

$$P(A) = \frac{26}{52} = \frac{1}{2}$$

(iii) B: an ace

$$n(B) = {}^{4}C_1\,{}^{48}C_0 = 4$$

$$P(B) = \frac{n(B)}{n(S)}$$

$$P(B) = 4/52 = 1/13$$

(ii)

C: Black Card

$$n(C) = {}^{26}C_1\,{}^{26}C_0 = 26$$

$$P(C) = \frac{n(C)}{n(S)} = 26/52$$

(iv) Red or king

C: Red , D: king

$$P(C \cup D) = P(C) + P(D) - P(C \cap D) \rightarrow (1)$$

$$n(C) = {}^{26}C_1 \, {}^{26}C_0 = 26$$

$$P(C) = \frac{n(C)}{n(S)}$$

$$P(C) = \frac{26}{52} = 1/2$$

$$n(D) = {}^{4}C_1 \cdot {}^{48}C_0 = 4$$

$$P(D) = \frac{n(D)}{n(S)}$$

$$P(D) = 4/52 = 1/13$$

$$n(C \cap D) = {}^{2}C_1 \, {}^{50}C_0 = 2$$

$$P(C \cap D) = \frac{n(C \cap D)}{n(S)}$$

$$P(C \cap D) = 2/52 = 1/26$$

put all values in (1)

$$P(C \cup D) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52}$$

$$P(C \cup D) = 28/52$$

Theorem:-

If A, B and C are any three events then the probability that at least one of them occurs is given by:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof:-

Let $B \cup C = D$

then $P(A \cup B \cup C) = P(A \cup D)$

then $P(A \cup D) = P(A) + P(D) - P(A \cap D)$

$$P(A \cup B \cup C) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P[(A \cap B) \cup (A \cap C)]$$

using distributive law.

$$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$\because (A \cap B) \cap (A \cap C) = A \cap B \cap C$

Example:-

A card is drawn at random from an ordinary deck of playing cards. What is the probability that the selected card is red, heart or an ace?

Sol:-

$n(S) = 52$

and A: Red card

B: Heart

C: Ace

we have to find the probability

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$n(A) = 26$

$$P(A) = \frac{n(A)}{n(S)} = \frac{26}{52}$$

$n(B) = 13$

$$P(B) = \frac{n(B)}{n(S)} = 13/52$$

$n(C) = 4$

$$P(C) = \frac{n(C)}{n(S)} = 4/52$$

$n(A\cap B) = 13$

$$P(A\cap B) = \frac{n(A\cap B)}{n(S)}$$

$$= 13/52$$

$n(B\cap C) = 1$

$$P(B\cap C) = \frac{n(B\cap C)}{n(S)}$$

$P(B\cap C) = 1/52$

$n(A\cap C) = 2$

$$P(A\cap C) = \frac{n(A\cap C)}{n(S)}$$

$P(A\cap C) = 2/52$

$n(A\cap B\cap C) = 1$

$$P(A\cap B\cap C) = \frac{n(A\cap B\cap C)}{n(S)}$$

$$= 1/52$$

$$P(A\cup B\cup C) = \frac{26}{52} + \frac{13}{52} + \frac{4}{52} - \frac{13}{52} - \frac{1}{52}$$

$$- 2/52 + 1/52$$

$$= \frac{26 + \cancel{13} + 4 - \cancel{13} - \cancel{1} - 2 + \cancel{1}}{52}$$

$P(A\cup B\cup C) = 28/52$

$P(A\cup B\cup C) = 7/13$

Conditional Probability:-

The probability associated with such a reduced sample space is called conditional probability.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$; provided that $P(B) \neq 0$

and B : conditioning event.

Example:-

Two fair dices are tossed. What is the probability of obtaining a total of 7 given that

(i) total is an odd number

(ii) number 6 appears on Ist die.

(iii) total is at least 6.

(iv) no additional information.

Sol:-

$S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$

$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$

$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$

$(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$

- - - - - - - - - - $(6,6)\}$

$n(S) = 36$

Let A event denote a total of 7

$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

; let event B denote a total is an odd number.

$B = \{(1,2), (1,4), (1,6), (2,1), (2,3), (2,5)$
$(3,2), (3,4), (3,6), (4,1), (4,3), (4,5)$
$(5,2), (5,4), (5,6), (6,1), (6,3), (6,5)\}$

as $A \subset B$ then $A \cap B = A$

$A \cap B = \{(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)\}$

$n(A \cap B) = 6$, $n(S) = 36$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$

$$= 6/36 = 1/6$$

$n(B) = 18$

$$P(B) = \frac{n(B)}{n(S)} = 18/36$$

$$\therefore P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A/B) = \frac{6/36}{18/36} = \frac{6}{18} = 1/3$$

(ii) Let event C denote number 6 appears on first die.

$C = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$n(C) = 6$

$P(C) = 6/36 = 1/6$

$$P(A/C) = \frac{P(A \cap C)}{P(C)} \rightarrow (1)$$

$n(A \cap C) = \{(6,1)\} = 1$

$P(A \cap C) = 1/36$ put in (1)

$$P(A/C) = \frac{1/36}{6/36} = 1/6$$

(iv) let event D denote total is at least 6

$D = \{(1,5), (1,6), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6), (4,2), (4,3) (4,4) (4,5) (4,6) (5,1), (5,2), (5,3) (5,4), (5,5), (5,6), (6,1), (6,2), (6,3) (6,4), (6,5), (6,6)\}$

$n(D) = 26$

$$P(D) = \frac{n(D)}{n(S)}$$

$P(D) = 26/36$

$$P(A/D) = \frac{P(A \cap D)}{P(D)} \rightarrow (2)$$

$A \cap D = \{(1,6), (2,5) (3,4) (4,3) (5,2)$
$(6,1)\}$

$n(A \cap D) = 6$

$P(A \cap D) = 6/36$

Put in (2)

$$P(A/D) = \frac{6/36}{26/36} = \frac{6}{26}$$

$$P(A/D) = 6/26 = \frac{3}{13}$$

(iv)

no additional information then it is not a conditional probability it is simple probability of event A which is

$$P(A) = \frac{n(A)}{n(S)}$$

$$P(A) = 6/36 = 1/6$$

Example:-

what is the prob that a randomly selected poker hand (5 cards) contains exactly 3 aces given that

(i) it contains at least 2 aces.

(ii) all face cards (including aces).

Sol:-

$n(S) = {}^{52}C_5 = 2598960$

Let event A denote 3 aces.

(i) Let event B denote at least 2 ace

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \rightarrow (1)$$

$$n(B) = {}^{4}C_2 \, {}^{48}C_3 + {}^{4}C_3 \, {}^{48}C_2 + {}^{4}C_4 \, {}^{48}C_1$$

$n(B) = 6 \times 17296 + 4 \times 1128 + 1 \times 48$

$n(B) = 103776 + 4512 + 48$

$n(B) = 168336$

$$P(B) = \frac{n(B)}{n(S)}$$

$$P(B) = \frac{108336}{2598960}$$

$n(A \cap B) = {}^{4}C_2 \, {}^{48}C_3 = 4512$

$$P(A \cap B) = \frac{4512}{2598960}$$ put in (1)

$$P(A/B) = \frac{4512}{108336} = 0.04$$

(ii) Let event C denote face cards including aces.

$n(C) = {}^{16}C_5 \, {}^{36}C_0 = 4368$

$$P(C) = \frac{4368}{2598960}$$

S: Face cards

$$n(A \cap C) = {}^{4}C_3 \cdot {}^{12}C_2 \cdot {}^{36}C_0$$

$$= 4 \times 66 \times 1$$

$$n(A \cap C) = 264$$

$$P(A \cap C) = \frac{264}{2598960}$$

$$P(A/C) = \frac{264}{4368}$$

$$P(A/C) = 0.06$$

A	J, Q, K
0	5
1	4
2	3
C ← 3	2 → A
4	1

Theorem:- (Multiplication law)

If A and B are two events defined on a sample space S then

$$P(A \cap B) = P(A) \cdot P(B/A)$$

$$P(A \cap B) = P(B) \cdot P(A/B)$$

where $P(A \cap B)$ is joint prob and $P(A)$, $P(B)$ are marginal probs and $P(B/A)$, $P(A/B)$ are conditional probs.

Proof:-

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(B) \cdot P(A/B)$$

Also

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(A) \cdot P(B/A).$$

Total Probability Theorem:-

If a sample space S is divided in "k" classes, say $S_1, S_2, S_3, \ldots, S_k$ and the selection of sample is performed in two steps. i.e.

1) a class S_i is selected at random
2) a sample is selected from S_i selected at step (1). then

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \cdots + P(A \cap S_k)$$

$$P(A) = \sum_{i=1}^{k} P(A \cap S_i)$$

$$= \sum_{i=1}^{k} P(A/S_i) \cdot P(S_i)$$

where $S_1, S_2, S_3, \ldots, S_k$ are mutually exclusive and exhaustive.

Example:-

Box A contains 5 green and 7 red balls. Box B contains 3 green, 3 red and 6 yellow balls. A box is selected at random and a ball is drawn at random from this box. What is the prob that the ball is

$$P(A \cap B) = P(A) \cdot P(B)$$

In general, if $A_1, A_2, \ldots, A_k$ are mutually independent then

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdots P(A_k)$$

$$\Rightarrow P\left(\bigcap_{i=1}^{k} A_k\right) = \prod_{i=1}^{k} P(A_i)$$

Example:-

If A and B are two events such that $P(A) = 1/4$, $P(A/B) = 1/2$, $P(B/A) = 2/3$

(i) Are A and B independent?

(ii) - - - - - - mutually exclusive?

(iii) Find $P(A \cap B)$ and $P(B)$.

Sol:-

(i) $P(A) \neq P(A|B)$

$\frac{1}{4} \neq \frac{1}{2}$

$\Rightarrow$ A and B are not independent.

(ii)

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(A) \cdot P(B/A)$$

$$= \frac{1}{4} \cdot \frac{2}{3}$$

$$P(A \cap B) = \frac{1}{6}$$

As $P(A\cap B) \neq 0$
$\therefore$ A and B are not mutually exclusive.
If any conditional probability is non-zero then two events are non-mutually exclusive events.
If any conditional probability is zero then $P(A\cap B) = 0 \Rightarrow$ A and B are m. exclu

viii)

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

$$P(B) = \frac{P(A\cap B)}{P(A/B)}$$

$$P(B) = \frac{1/6}{1/2}$$

$$P(B) = 2/6$$

$$P(B) = 1/3$$

Theorem:-

If the probability of occurence of an event in a single trial is p, then the probability that this event occurs "k" times on "n" independent trials is given by

$${}^nC_k\, p^k q^{n-k} \quad \text{where } q = 1-p.$$

Proof:-

Consider a possible outcome of random experiment with "n" trials.

$$E = \underbrace{A \cap A \cap A \cap A \cap \cdots \cap A}_{k\text{-times}} \cap \underbrace{\bar{A} \cap \bar{A} \cap \bar{A} \cap \cdots \cap \bar{A}}_{n-k \text{ times}}$$

$$P(E) = P(\underbrace{A \cap A \cap A \cap A \cap \cdots \cap A}_{k\text{-times}} \cap \underbrace{\bar{A} \cap \bar{A} \cap \cdots \cap \bar{A}}_{n-k \text{ times}})$$

As trials are independent, then

$$P(E) = \underbrace{P(A) \cdot P(A) \cdot P(A) \cdots P(A)}_{k\text{-times}} \cdot \underbrace{P(\bar{A}) \cdot P(\bar{A}) \cdots P(\bar{A})}_{n-k \text{ times}}$$

As $P(\text{occurance}) = p = P(A)$

$P(\text{non-occarance}) = q = P(\bar{A})$

So $P(E) = \underbrace{p \cdot p \cdot p \cdots p}_{k\text{-times}}\ \underbrace{q \cdot q \cdot q \cdots q}_{n-k \text{ times}}$

$P(E) = p^k q^{n-k}$ where $q = 1-p$

As "k" times occurence of event A in "n" trials is possible in ${}^nC_k$ mutually exclusive ways.

Thus, probability of k successes in n-trials is

$${}^nC_k\, p^k q^{n-k}$$

Binomial law:-

1) Fixed number of trials, say n.
2) Trials are Independent.
3) Two categories. Occurence and non-occurence or Success and Failure.

Example:-

5 coins are tossed, what is the probability of obtaining 3 heads?

Sol:-

$n = 5$, $p = P(\text{Head}) = \frac{1}{2}$

$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$k = 3$ $\qquad \therefore P(k=3) = {}^5C_3\, p^3 q^{5-3}$

$P(3 \text{ heads}) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{5-3}$

$= 10 \cdot \frac{1}{8} \cdot \frac{1}{4}$

$P(3 \text{ heads}) = \frac{5}{16}$

Theorem (Baye's theorem):-

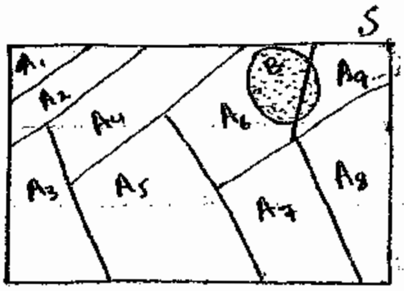
If the events $A_1, A_2, A_3, \ldots, A_k$ are mutually exclusive and exhaustive events defined on S. If B is another event defined on S and also B can occur only if any of $A_i\, (i = 1, 2, \ldots, k)$ has occurred then

$$P(A_i/B) = \frac{P(B/A_i)\,P(A_i)}{\sum_{j=1}^{k} P(B/A_j)\,P(A_j)} \quad ; \; i = 1, \cdots k$$

where $A_i \cap A_j = \phi$

for $i, j = 1, 2, \cdots, k$ and $i \neq j$

$\bigcup_{i=1}^{k} A_i = S$



Proof :-

we can write

$$P(B \cap A_i) = P(A_i) \cdot P(B/A_i) \rightarrow (1)$$

and also

$$P(B \cap A_i) = P(B) \cdot P(A_i/B) \rightarrow (2)$$

From (1) and (2)

$$\Rightarrow \quad P(A_i) \cdot P(B/A_i) = P(B) \cdot P(A_i/B)$$

$$\Rightarrow \quad P(A_i/B) = \frac{P(A_i) \cdot P(B/A_i)}{P(B)} \rightarrow (3)$$

Now, we can write

$$B = S \cap B \qquad \because B \subseteq S$$

$$B = (A_1 \cup A_2 \cup \cdots \cup A_k) \cap B \qquad \because A_i\text{'s are exhaustive}$$

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots \cup (A_k \cap B)$$

As $A_1 \cap B$, $A_2 \cap B$, $\cdots$, $A_k \cap B$ are mutually exclusive.

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_k \cap B)$$

$$P(B) = \sum_{j=1}^{k} P(A_j \cap B)$$

and $P(A_j \cap B) = P(B/A_j) \cdot P(A_j)$

$$P(B) = \sum_{j=1}^{k} P(B/A_j) \cdot P(A_j)$$

put in (3)

$$P(A_i/B) = \frac{P(A_i)P(B/A_i)}{\sum_{j=1}^{k} P(B/A_j)P(A_j)} \; ; \; i = 1, 2 \dots, k$$

Example:-

In a bolt factory, machines A, B and C manufacture 25, 35 and 40% percent of the total production of their outputs, 3, 4 and 5 percent are defective respectively. A bolt is selected at random and found to be is defective (ii) good. What is the prob that bold is manufactured on machine

(a) A (b) B (c) C ?

Sol:-

A : manufactured on machine - A

B : " " " B

C : " " " C

$P(A) = 25\% = 0.25$

$P(B) = 35\% = 0.35$

$P(C) = 40\% = 0.40$

$P(A) + P(B) + P(C) = 1$

D: manufactur unit is defective

$P(D|A) = 3\% = 0.03$

$P(D|B) = 4\% = 0.04$

$P(D|C) = 5\% = 0.05$

i)

$$P(A|D) = \frac{P(A)\cdot P(D|A)}{P(A)\cdot P(D|A) + P(B)\cdot P(D|B) + P(C)\cdot P(D|C)}$$

$$= \frac{(0.25)(0.03)}{(0.25)(0.03) + (0.35)(0.04) + (0.40)(0.05)}$$

$$P(A|D) = \frac{0.0075}{0.0075 + 0.014 + 0.02}$$

$$P(A|D) = \frac{0.0075}{0.0415} = 0.18$$

ii) $$P(B|D) = \frac{P(B)\cdot P(D|B)}{P(B)\cdot P(D|B) + P(C)\cdot P(D|C) + P(A)\cdot P(D|A)}$$

$$= \frac{(0.35)(0.04)}{(0.35)(0.04) + (0.40)(0.05) + (0.25)(0.03)}$$

$$P(B|D) = \frac{0.014}{0.0415} = 0.34$$

iii)

$$P(C|D) = \frac{P(C)\cdot P(D|C)}{P(A)\cdot P(D|A) + P(B)\cdot P(D|B) + P(C)\cdot P(D|C)}$$

$$= \frac{(0.40)(0.05)}{(0.25)(0.03) + (0.35)(0.04) + (0.40)(0.05)}$$

$$P(C|D) = \frac{0.02}{0.0415} = 0.48$$

(ii) G : manufactured unit is good.

$P(G/A) = 1 - P(D/A) = 0.97$

$P(G/B) = 1 - P(D/B) = 0.96$

$P(G/C) = 1 - P(D/C) = 0.95$

(a)

$$P(A|G) = \frac{P(A)\cdot P(G/A)}{P(A)\cdot P(G/A) + P(B)\cdot P(G/B) + P(C)\cdot P(G/C)}$$

$$P(A/G) = \frac{(0.25)(0.97)}{(0.25)(0.97) + (0.35)(0.96) + (0.40)(0.95)}$$

$$P(A/G) = \frac{0.2425}{0.2425 + 0.336 + 0.38}$$

$$P(A/G) = \frac{0.2425}{0.9585} = 0.252$$

(b)

$$P(B|G) = \frac{P(B)\cdot P(G/B)}{P(A)\cdot P(G/A) + P(B)\cdot P(G/B) + P(C)\cdot P(G/C}$$

$$= \frac{(0.35)(0.96)}{0.9585} = \frac{0.336}{0.9585}$$

$$P(B|G) = 0.35$$

$$P(C/G) = \frac{P(C)\cdot P(G/C)}{P(A)\cdot P(G/A) + P(B)\cdot P(G/B) + P(C)\cdot P(G|}$$

$$P(C/G) = \frac{0.38}{0.9585} = 0.396$$

Theorem:-

If A and B are two independent events defined on S then

$$P(A \cap B) = P(A) \cdot P(B)$$

Proof:-

According to general law of multiplication for any two events A and B, we have

$$P(A \cap B) = P(A) \cdot P(B/A) \rightarrow (1)$$

If A and B are independent events then

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B)$$

Put this result in (1), we get

iff $P(A \cap B) = P(A) \cdot P(B)$

Theorem:-

If A and B are independents events then

(i) A and $\bar{B}$ are independent

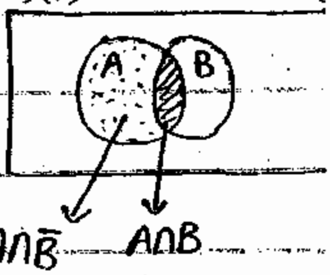
(ii) $\bar{A}$ and B are independent

(iii) $\bar{A}$ and $\bar{B}$ are independent

Proof:-

Given that A and B are independent. So by definition of independent events.

$$P(A\cap B) = P(A) \cdot P(B) \rightarrow (1)$$



(i) To prove A and $\bar{B}$ are independent, we need to prove that

$$P(A\cap\bar{B}) = P(A)\cdot P(\bar{B})$$

we can write

$$A = (A\cap\bar{B})\cup(A\cap B)$$

$$P(A) = P[(A\cap\bar{B})\cup(A\cap B)]$$

As $A\cap\bar{B}$ and $A\cap B$ are mutually exclusive so

$$P(A) = P(A\cap\bar{B}) + P(A\cap B)$$

$$P(A) = P(A\cap\bar{B}) + P(A)\cdot P(B) \quad \text{by (1)}$$

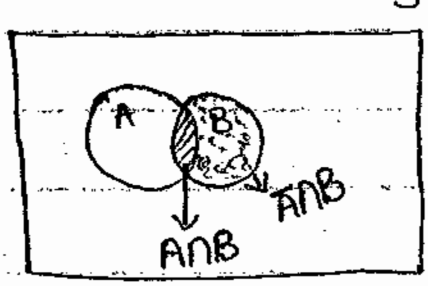
$$\Rightarrow P(A\cap\bar{B}) = P(A) - P(A)\cdot P(B)$$

$$= P(A)[1 - P(B)]$$

$$\Rightarrow P(A\cap\bar{B}) = P(A)\cdot P(\bar{B})$$

(ii)

To prove $\bar{A}$ and $\bar{B}$ are independent, we need to prove that $P(\bar{A}\cap B) = P(\bar{A})\cdot P(B)$



we can write as

$$B = (B\cap\bar{A})\cup(A\cap B)$$

As $A\cap B$ and $(\bar{A}\cap B)$ are mutually exclusive so.

$P((\bar{A}\cap B)\cup(A\cap B)) = P(\bar{A}\cap B) + P(A\cap B)$

then $P(B) = P(\bar{A}\cap B) + P(A\cap B)$

$\Rightarrow P(\bar{A}\cap B) = P(B) - P(A\cap B)$

$= P(B) - P(A).P(B)$

$= P(B)[1 - P(A)]$

$P(\bar{A}\cap B) = P(B)\,P(\bar{A}) = P(\bar{A})P(B)$

iii)

To prove $\bar{A}$ and $\bar{B}$ are independent, we have to prove that $P(\bar{A}\cap\bar{B}) = P(\bar{A}).P(\bar{B})$. For this

Using De-Morgan's law

$$\overline{A\cup B} = \bar{A}\cap\bar{B}$$

$$P(\overline{A\cup B}) = P(\bar{A}\cap\bar{B})$$

$$P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B})$$

$$= 1 - P(A\cup B)$$

$$P(\bar{A}\cap\bar{B}) = 1 - [P(A) + P(B) - P(A\cap B)]$$

$$= 1 - [P(A) + P(B) - P(A).P(B)]$$

$$= 1 - [P(A) + P(B)[1 - P(A)]]$$

$$= [1 - P(A)] - P(B)[1 - P(A)]$$

$$= [1 - P(A)][1 - P(B)]$$

$$P(\bar{A}\cap\bar{B}) = P(\bar{A}).P(\bar{B})$$

$\Rightarrow \bar{A}$ and $\bar{B}$ are independent.

Example:-

The prob that a man will be alive in 25 years is $3/5$, and the prob that his wife will be alive in 25 years is $2/3$. Find the prob that

(i) both will be alive in 25 years

(ii) only man " " " " "

(iii) only wife " " " " "

(iv) at least one " " " " "

(v) neither " " " " "

Sol:-

A: man will be alive in 25 years

B: wife will be alive in 25 years.

(i) Assuming A and B are independent

$$P(A\cap B) = P(A)\cdot P(B)$$

$$= 3/5 \cdot 2/3$$

$$P(A\cap B) = 2/5$$

(ii)

$$P(A\cap \bar{B}) = P(A)\cdot P(\bar{B})$$

$$= 3/5 \cdot 1/3$$

$$P(A\cap \bar{B}) = 1/5$$

(iii)

$$P(\bar{A}\cap B) = P(\bar{A})\cdot P(B)$$

$$= \frac{2}{5} \cdot \frac{2}{3}$$

$$P(\bar{A} \cap B) = 4/15$$

iv)

$$P(A \cap \bar{B}) + P(\bar{A} \cap B) + P(A \cap B) = 1/5 + 4/15 + 2/5$$

$$= 13/15$$

or $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 3/5 + 2/3 - 2/5$$

$$P(A \cup B) = 13/15$$

v)

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$$

$$= 2/5 \cdot 1/3$$

$$P(\bar{A} \cap \bar{B}) = 2/15$$

Random Variable:-

A numerical quantity whose value is determined by the outcome of a random experiment, is called a random variable.

Example:-

Random experiment : 3 coins are tossed then $n(S) = 8$

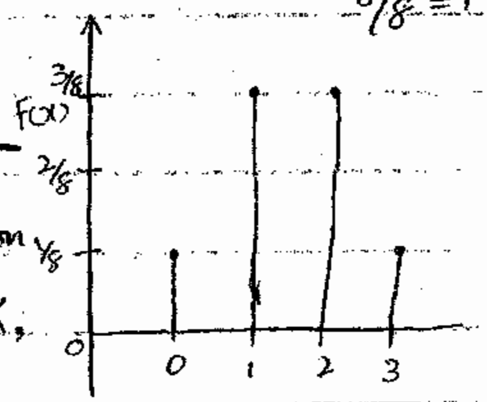
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

X = no. of heads appeared in tossing 3 coins

$X = 0, 1, 2, 3$

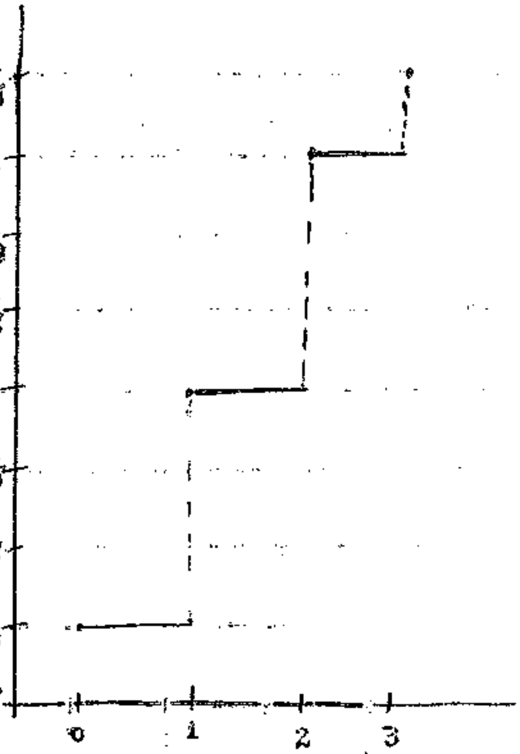
Probability distribution is

x	$P(X=x) = p(x) = f(x)$	$F(x)$
0	$1/8$	$1/8$
1	$3/8$	$4/8$
2	$3/8$	$7/8$
3	$1/8$	$8/8 = 1$
	1	



Distribution Function:-

The distribution function of a random variable X, lenoted by $F(x)$, is lefined by $F(x) = P(X \le x)$. he function $F(x)$ gives he probability of the vent that X takes a value less than or equal to a specified value x. It is also called the umulative distri funct (cdf).



Example:-

Find the prob distribution of the sum of the dots when two fair dice are thrown.

Sol:-

Random experiment : 2 fair dice are thrown

i) Random variable : sum of the dots

So $X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

Because

$$S = \begin{cases} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \} \end{cases}$$

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Probability distribution of X

x	$p(x)$	$F(x)$
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	36/36

(ii)

Random variable : absolute difference of the dots

$$|d_1 - d_2|$$

0	1	2	3	4	5
1	0	1	2	3	4
2	1	0	1	2	3
3	2	1	0	1	2
4	3	2	1	0	1
5	4	3	2	1	0

x	$p(x)$	$F(x)$
0	6/36	6/36
1	10/36	16/36
2	8/36	24/36
3	6/36	30/36
4	4/36	34/36
5	2/36	36/36

Distribution Function:-

Distribution function is denoted by $F(x)$ and defined as

$$F(x) = \sum_{x=-\infty}^{x} P(x) = P(X \leq x)$$

where "$-\infty$" is the smallest value "x" can take.

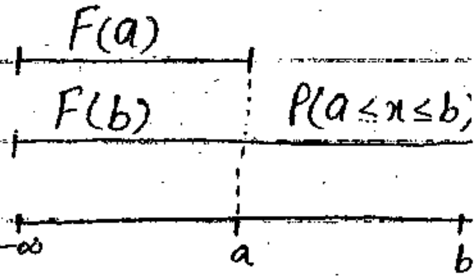
Result:-

$$P(a \leq x \leq b) = \sum_{x=a}^{b} p(x)$$

By using distri-bution function.

$$P(a \leq x \leq b) = F(b) - F(a)$$

e.g) Using Ist example,



$$P(1<x\leq 3)=\sum_{x>1}^{3} P(x)$$

$$P(1<x\leq 3)= p(2)+p(3)$$

$$= \frac{3}{8}+\frac{1}{8} = 4/8 = 1/2$$

or

$$P(1<x\leq 3) = F(3)-F(1)$$

$$= 1-4/8$$

$$P(1<x\leq 3) = 1-1/2 = 1/2$$

$$\Rightarrow P(1<x\leq 3) = \sum_{x>1}^{3} p(x) = F(3)-F(1)$$

If we have to find the prob $P(1\leq x\leq 3)$ then, by using distribution function, we find prob of

$$P(0<x\leq 3) = F(3)-F(0)$$

$$P(0<x\leq 3) = 1-\frac{1}{8} = 7/8$$

$$P(1\leq x\leq 3) = P(1)+P(2)+P(3)$$

$$= 1/8+3/8+3/8$$

$$P(1\leq x\leq 3) = 7/8$$

$$\Rightarrow P(1\leq x\leq 3) = P(0<x\leq 3)$$

Mean and variance of discrete random variable:-

$$S = vt+\varepsilon$$

where vt is signal and ε is

Note:-

If $y = ax + b$ then

1) $p(y) = p(x)$

2) $E(y) = aE(x) + b$

3) $Var(y) = a^2 Var(x)$

Example:-

x	$p(x)$	$xp(x)$	$x^2p(x)$
-1	$3C = 3/12$	$-3/12$	$3/12$
0	$3C = 3/12$	0	0
1	$6C = 6/12$	$6/12$	$6/12$

we know, "sum of all probabilities = 1"

$$\Sigma p(x) = 1$$

$$3C + 3C + 6C = 1$$

$$12C = 1$$

$$C = 1/12$$

$$\text{Mean} = E(x) = \Sigma x p(x)$$

$$= -\frac{3}{12} + 0 + \frac{6}{12}$$

$$= \frac{-3+6}{12} = \frac{3}{12}$$

$$\text{Mean} = E(x) = 1/4 = 0.25$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$= \Sigma x^2 p(x) - \left(\frac{1}{4}\right)^2$$

$$\text{Variance} = \frac{9}{12} - \frac{1}{16} = 0.75 - 0.0625$$
$$= 0.6875$$

If $Y = 2X + 5$

then

x	$p(x)$	$Y = 2x+5$	$P(Y)$
-1	$3/12$	3	$3/12$
0	$3/12$	5	$3/12$
1	$6/12$	7	$6/12$

Probabilities are same for both variables x and Y.

Now we find Mean and Variance of variable Y.

Probability distribution of Y is

Y	$P(Y)$	$YP(Y)$	$Y^2P(Y)$
3	$3/12$	$9/12$	$27/12$
5	$3/12$	$15/12$	$75/12$
7	$6/12$	$42/12$	$294/12$
		$66/12$	$396/12$

$$\text{Mean} = E(Y) = \Sigma Y P(Y)$$
$$= \frac{66}{12} = 5.5$$

$$\text{Variance} = E(Y^2) - [E(Y)]^2$$
$$= \Sigma Y^2 P(Y) - (5.5)^2$$

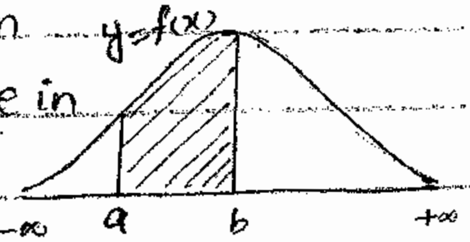
$$\text{variance} = \frac{396}{12} - 30.25$$

$$= 33 - 30.25$$

$$\text{Variance} = 2.75$$

Continuous Random Variable:-

A random variable X is defined to be continuous if it can assume every possible value in an interval [a, b], a<b



f(x): probability density function (p.d.f)

$$P(a \leq x \leq b) = \int_a^b f(x)dx$$

where a and b may be $-\infty$ and $+\infty$ respectively

$P(x = a) = 0$, if x is a continuous random variable.

Properties:-

1) $f(x) \geq 0$.

2) $\int_{-\infty}^{\infty} f(x)dx = 1$

Distribution function:-

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \rightarrow (a)$$

then $P(a \leq x \leq b) = F(b) - F(a)$

From (a)

$$f(x) = \frac{d}{dx} F(x)$$

Example:-

A continuous random variable has the p.d.f given by

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

i)

(a) Check if total area under curve is unity and find

(b) $P(\frac{1}{2} < x < 1)$ (c) $P(x \geq \frac{1}{4})$

(d) $P(x \leq 3/4)$ (e) $P(X = \frac{1}{2})$

Sol:-

(a) Total area $= P(0 < x < 1)$

$$= \int_0^1 f(x)\,dx$$

$$= \int_0^1 2x\,dx$$

$$= 2\frac{x^2}{2}\Big|_0^1$$

$$= x^2\Big|_0^1$$

Total area $= 1 - 0 = 1$

(b)

$$P(\tfrac{1}{2} < x < 1) = \int_{1/2}^1 f(x)\,dx$$

$$= \int_{1/2}^{1} 2x\,dx$$

$$= 2\frac{x^2}{2}\Big|_{1/2}^{1}$$

$$= x^2\Big|_{1/2}^{1}$$

$$= (1)^2 - (1/2)^2 = 1 - \frac{1}{4}$$

$$P(\tfrac{1}{2} < x < 1) = 3/4$$

c)

$$P(X \geq \tfrac{1}{4}) = P(\tfrac{1}{4} \leq x \leq 1)$$

$$= \int_{1/4}^{1} f(x)\,dx$$

$$= \int_{1/4}^{1} 2x\,dx$$

$$= \frac{2x^2}{2}\Big|_{1/4}^{1}$$

$$= (1)^2 - (\tfrac{1}{4})^2$$

$$= 1 - \frac{1}{16}$$

$$P(X \geq 1/4) = 15/16$$

(d)

$$P(X \leq 3/4) = P(0 \leq x \leq 3/4)$$

$$= \int_{0}^{3/4} f(x)\,dx$$

$$P(X < 3/4) = \int_0^{3/4} 2x\,dx$$

$$= 2\frac{x^2}{2}\Big|_0^{3/4}$$

$$= x^2\Big|_0^{3/4} = \left(\frac{3}{4}\right)^2 - (0)^2 = \frac{9}{16}$$

(e)

$P(X = 1/2) = 0$

Because x is continuous r.v.

(ii)

Find distribution function $F(x)$ and use $F(x)$ to find the probabilities in (a), (b), (c), (d) and (e).

Sol:-

$$F(x) = P(X \le x)$$

$$= P(0 \le X \le x)$$

$$= \int_0^x f(x)\,dx$$

$$= \int_0^x 2x\,dx$$

$$= 2\frac{x^2}{2}\Big|_0^x$$

$$F(x) = x^2 - 0 = x^2$$

(a) Total area $= P(0 < x < 1)$

$= F(1) - F(0)$

Total area $= (1)^2 - (0)^2 = 1$

2) $P(1/2 < x < 1) = F(1) - F(1/2)$

$$= (1)^2 - \left(\frac{1}{2}\right)^2$$

$$= 1 - \frac{1}{4}$$

$$= 3/4$$

c)

$$P\left(X \geq \frac{1}{4}\right) = P\left(\frac{1}{4} \leq X \leq 1\right)$$

$$= F(1) - F\left(\frac{1}{4}\right)$$

$$= (1)^2 - \left(\frac{1}{4}\right)^2$$

$$P\left(X \geq \frac{1}{4}\right) = 1 - \frac{1}{16} = 15/16$$

d)

$$P(X \leq 3/4) = P(0 \leq X \leq 3/4)$$

$$= F(3/4) - F(0)$$

$$= (3/4)^2 - (0)^2$$

$$P(X \leq 3/4) = 9/16$$

e)

$P\left(X = \frac{1}{2}\right) = 0$ Because X is a continuous random variable

Example :-

A countinoous r.v X has the following p.d.f

$$f(x) = \begin{cases} kx ; & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find value of k.

Sol:-

we know total area under the curve is "1"

So Total area = $P(0 \le x \le 2)$

$$1 = \int_0^2 f(x)dx$$

$$1 = \int_0^2 kx\,dx$$

$$1 = \frac{kx^2}{2}\Big|_0^2$$

$$1 = \frac{k}{2}\left[x^2\Big|_0^2\right]$$

$$2 = k[(2)^2 - (0)^2]$$

$$2 = k(4-0)$$

$$2 = 4k$$

$$\Rightarrow k = \frac{2}{4}$$

$$\Rightarrow k = 1/2$$

Example:-

A continuous r.v X has the following p.d.f

$$f(x) = \begin{cases} x/2 & 0 < x \le 1 \\ \frac{1}{4}(3-x) & 1 < x \le 2 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{4} & 2 < x \le 3 \\ \frac{1}{4}(4-x) & 3 < x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that total area is one.

b) $P(\frac{1}{2} < x < \frac{1}{4})$ (c) $P(\frac{3}{2} < x < \frac{5}{2})$

d) $P(x > 4)$ (e) Find distribution fn $F(x)$

Sol:-

a) Total area $= P(-\infty < x < \infty)$

$$= P(-\infty < x \le 0) + P(0 < x \le 4) + P(4 < x < \infty)$$

$$= 0 + P(0 < x \le 4) + 0$$

Total area $= P(0 < x \le 4)$

$$= P(0 < x \le 1) + P(1 < x \le 2) + P(2 < x \le 3) + P(3 < x \le 4)$$

$$= \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{4}(3-x)\,dx + \int_2^3 \frac{1}{4}\,dx + \int_3^4 \frac{1}{4}(4-x)\,dx$$

Total area $= \left.\frac{x^2}{4}\right|_0^1 + \left.\frac{(3-x)^2}{-8}\right|_1^2 + \left.\frac{x}{4}\right|_2^3 + \left.\frac{(4-x)^2}{-8}\right|_3^4$

$$\text{Total area} = \frac{1}{4}(1-0) - \frac{1}{8}\left((3-2)^2 - (3-1)^2\right)$$

$$+ \frac{1}{4}(3-2) - \frac{1}{8}\left((4-4)^2 - (4-3)^2\right)$$

$$= \frac{1}{4} - \frac{1}{8}(1-4) + \frac{1}{4} - \frac{1}{8}(0-1)$$

$$= \frac{1}{4} - \frac{1}{8}(-3) + \frac{1}{4} + \frac{1}{8}$$

$$= \frac{1}{4} + \frac{3}{8} + \frac{1}{4} + \frac{1}{8}$$

$$= \frac{2+3+2+1}{8}$$

$$\text{Total area} = 8/8 = 1$$

(b)

$$P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{1/4}^{1/2} \frac{x}{2}\,dx$$

$$= \left.\frac{x^2}{4}\right|_{1/4}^{1/2}$$

$$= \frac{1}{4}\left[\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2\right]$$

$$= \frac{1}{4}\left[\frac{1}{4} - \frac{1}{16}\right]$$

$$= \frac{1}{4}\left[\frac{16-4}{64}\right]$$

$$= \frac{1}{4}\left[12/64 \right]$$

$$P(1/4 < x < 1/2) = 3/64$$

c) $P(3/2 < x < 5/2) = P(3/2 < x \leq 2) + P(2 < x \leq 5/2)$

$$= \int_{3/2}^{2} \frac{1}{4}(3-x)\,dx + \int_{2}^{5/2} \frac{1}{4}\,dx$$

$$= \frac{(3-x)^2}{-8}\Big|_{3/2}^{2} + \frac{x}{4}\Big|_{2}^{5/2}$$

$$= -\frac{1}{8}\left((3-2)^2 - (3-\frac{3}{2})^2\right) + \frac{1}{4}\left(\frac{5}{2} - 2\right)$$

$$= -\frac{1}{8}\left(1 - \frac{3}{2}\right) + \frac{1}{4}\left(\frac{5-4}{2}\right)$$

$$= -\frac{1}{8}\left(-\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{16} + \frac{1}{8}$$

$$P(3/2 < x < 5/2) = \frac{1+2}{16} = 3/16$$

(d)

$P(x > 4) = 0$

Because, it is out of the range.

(e) Distribution function $F(x)$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

For $0 < x \le 1$, $f(x) = x/2$

$$F(x) = \int_0^x \frac{x}{2}\,dx$$

$$F(x) = \frac{x^2}{4}\Big|_0^x = \frac{x^2}{4}$$

For $1 < x \le 2$, $f(x) = \frac{1}{4}(3-x)$

$$F(x) = \int_1^x \frac{1}{4}(3-x)\,dx$$

$$= \frac{1}{4}\left(3x - \frac{x^2}{2}\right)\Big|_1^x$$

$$= \frac{1}{4}\left(3x - \frac{x^2}{2}\right) - \frac{1}{4}\left(3 - \frac{1}{2}\right)$$

$$F(x) = \frac{1}{4}\left(3x - \frac{x^2}{2}\right) - \frac{5}{8}$$

For $2 < x \le 3$, $f(x) = 1/4$

$$F(x) = \int_2^x \frac{1}{4}\,dx$$

$$F(x) = \frac{1}{4}x \Big|_2^x$$

$$= \frac{1}{4}x - \frac{1}{4}(2)$$

$$F(x) = \frac{1}{4}x - \frac{1}{2}$$

$$F(x) = \frac{x}{4} - \frac{1}{2}$$

where $-\frac{1}{2}$ = constant

For $3 < x \le 4$, $f(x) = \frac{1}{4}(4-x)$

$$F(x) = \int_3^x \frac{1}{4}(4-x)\,dx$$

$$= \frac{1}{4}\left(4x - \frac{x^2}{2}\right)\Big|_3^x$$

$$= \frac{1}{4}\left(4x - \frac{x^2}{2}\right) - \frac{1}{4}\left(12 - \frac{9}{2}\right)$$

$$F(x) = \frac{1}{4}\left(4x - \frac{x^2}{2}\right) - \frac{15}{8}$$

where $\frac{15}{8}$ = constant

For otherwise $f(x) = 0$

$$F(x) = \int 0 \cdot dx$$

$$= C = 1$$

So the distribution function is

$$F(x) = \begin{cases} 0 & x = 0 \\ x^2/4 & 0 < x \leq 1 \\ \frac{1}{4}(3x - x^2/2) - 5/8 & 1 < x \leq 2 \\ \frac{1}{4}x - \frac{1}{2} & 2 < x \leq 3 \\ \frac{1}{4}(4x - x^2/2) - 15/8 & 3 < x \leq 4 \\ 1 & \text{otherwise} \end{cases}$$

Note:-

$$P(a < x < b) = P(a \leq x < b) = P(a < x \leq b)$$
$$= P(a \leq x \leq b)$$

because $P(a \leq x \leq b) = P(x = a) +$
$$P(a < x < b) + P(x = b)$$

$$P(a \leq x \leq b) = 0 + P(a < x < b) + 0$$

$$\Rightarrow P(a \leq x \leq b) = P(a < x < b)$$

Example:-

A countinuous r.v x has the following distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 2x^2/5 & 0 < x \leq 1 \\ \frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right) & 1 < x \leq 2 \\ 1 & x > 2 \end{cases}$$

Find p.d.f and $P(|x| < 1.5)$, $P(|x| > 1.5)$

Sol:-

we know that

$$f(x) = \frac{d}{dx} F(x)$$

For $x < 0$ $F(x) = 0$

then $f(x) = \frac{d}{dx}(0)$

$f(x) = 0$

For $0 < x \le 1$, $F(x) = \frac{2x^2}{5}$

then $f(x) = \frac{d}{dx}\left(\frac{2x^2}{5}\right)$

$= \frac{2}{5}(2x)$

$f(x) = \frac{4}{5}x$

For $1 < x \le 2$, $F(x) = \frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right)$

then

$f(x) = \frac{d}{dx}\left(\frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right)\right)$

$= 0 + \frac{2}{5}\left(3 - \frac{2x}{2}\right)$

$f(x) = \frac{2}{5}(3 - x)$

For $x > 2$ $F(x) = 1$

then $f(x) = \frac{d}{dx}(1)$

$f(x) = 0$

So the probability distribution

function is

$$f(x) = \begin{cases} 0 & x < 0 \\ 4/5\, x & 0 < x \le 1 \\ 2/5(3-x) & 1 < x \le 2 \\ 0 & x > 2, \text{ otherwise} \end{cases}$$

Now find probability of

$$P(|x| < 1.5) = P(x \le \pm 1.5)$$

$$= P(-1.5 < x < 1.5)$$

$$= P(-1.5 < x \le 0) + P(0 \le x < 1.5)$$

$$= 0 + \int_0^{1.5} f(x)\,dx$$

$$= \int_0^{1} 4/5\, x\,dx + \int_1^{1.5} \frac{2}{5}(3-x)dx$$

$$= \frac{4}{5}\left.\frac{x^2}{2}\right|_0^1 + \frac{2}{5}\left.\left(3x - \frac{x^2}{2}\right)\right|_1^{1.5}$$

$$= \frac{2}{5}\left((1.0)^2 - (0)^2\right) + \frac{2}{5}\left(3(1.5) - \frac{(1.5)^2}{2} - \left(3 - \frac{1}{2}\right)\right)$$

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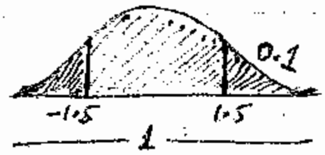
$$= \frac{2}{5}\left((1-0) + (4.5 - 1.125) - 2.5\right)$$

$$= \frac{2}{5}\left(1 + 3.375 - 2.5\right)$$

$$P(|x| < 1.5) = \frac{3.75}{5} = 0.75$$

Now $P(|x| > 1.5)$

$$P(|x| > 1.5) = 1 - P(|x| < 1.5)$$
$$= 1 - 0.75$$
$$P(|x| > 1.5) = 0.25$$



or

$$P(|x| > 1.5) = P(x < -1.5) + P(x > 1.5)$$
$$= 0 + \int_{1.5}^{2} \frac{2}{5}(3-x)\,dx$$
$$= \frac{2}{5}\left(3x - \frac{x^2}{2}\right)\Big|_{1.5}^{2} = \frac{2}{5}\left[(6-2)-(4.5-1.125)\right]$$
$$P(|x| > 1.5) = 0.25$$

Mean and variance of continuous r.v's-

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$\sigma^2 = \text{variance} = E(x-\mu)^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx$$

or $\sigma^2 = E(X^2) - (E(X))^2$

where

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

where $f(x)$ is p.d.f

Standard deviation (S.D) is

$$S.D = \sqrt{\text{variance}} = \sigma$$

Example:-

A continuous r.v X has the p.d.f given by

$$f(x) = \begin{cases} \frac{3}{4}(1+x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) μ (ii) σ (iii) $P(|x-\mu|<\sigma)$.

Sol:-

(i) $\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx$

$$= \int_0^1 x\left(\frac{3}{4}(1+x^2)\right)dx$$

$$= \int_0^1 \frac{3}{4}(x+x^3)\,dx$$

$$= \frac{3}{4}\left(\frac{x^2}{2}+\frac{x^4}{4}\right|_0^1$$

$$= \frac{3}{4}\left(\frac{1}{2}(1-0)+\frac{1}{4}(1-0)\right)$$

$$= \frac{3}{4}\left(\frac{1}{2}+\frac{1}{4}\right)$$

$$= \frac{3}{4}\left(\frac{2+1}{4}\right)$$

$$\mu = \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16} = 0.5625$$

(ii)

$$\sigma^2 = E(x^2) - [E(x)]^2 \rightarrow (1)$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 x^2\left(\frac{3}{4}(1+x^2)\right)dx$$

$$= \int_0^1 \frac{3}{4}(x^2+x^4)\,dx$$

$$E(x^2) = \frac{3}{4}\left(\frac{x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1$$

$$= \frac{3}{4}\left(\frac{1}{3}(1-0) + \frac{1}{5}(1-0)\right)$$

$$= \frac{3}{4}\left(\frac{1}{3} + \frac{1}{5}\right)$$

$$= \frac{3}{4}\left(\frac{5+3}{15}\right)$$

$$= \frac{3}{4} \cdot \frac{8}{15} = \frac{1}{1} \cdot \frac{2}{5}$$

$$E(x^2) = \frac{2}{5}$$

and $E(x) = \frac{9}{16}$

put in (1)

$$\sigma^2 = \frac{2}{5} - \left(\frac{9}{16}\right)^2$$

$$= \frac{2}{5} - \frac{81}{256}$$

$$\sigma^2 = \frac{512-405}{1280}$$

$$\sigma^2 = \frac{107}{1280}$$

$$\sigma = \sqrt{\frac{107}{1280}}$$

$\Rightarrow \quad \sigma = 0.289$

(iii) $P(|x-\mu| < \sigma) = P(-\sigma \le x-\mu \le \sigma)$

$= P(\mu-\sigma \le x \le \mu+\sigma)$

$= P(0.5625 - 0.289 < x < 0.5625 + 0.\ldots)$

$= P(0.2734 < x < 0.8516)$

$$= \int_{0.2734}^{0.8516} f(x)\,dx$$

$$= \int_{0.2734}^{0.8516} \frac{3}{4}(1+x^2)\,dx$$

$$= \frac{3}{4}\left(x + \frac{x^3}{3}\right)\Bigg|_{0.2734}^{0.8516}$$

$$= \frac{3}{4}\left[\left(0.8516 + \frac{(0.8516)^3}{3}\right) - \left(0.2734 + \frac{(0.2734)^3}{3}\right)\right]$$

$$= \frac{3}{4}\left[(0.8516 + 0.2058) - (0.2734 + 0.0068)\right]$$

$$= \frac{3}{4}\left[1.0574 - 0.2802\right]$$

$$= \frac{3}{4}\left[0.7772\right]$$

$$= 0.5829$$

$\Rightarrow P(|x-\mu| < \sigma) = 0.583$

Chebychev's Inequality:-

If X is a random variable with mean $= \mu$ and variance $= \sigma^2 > 0$. Then,

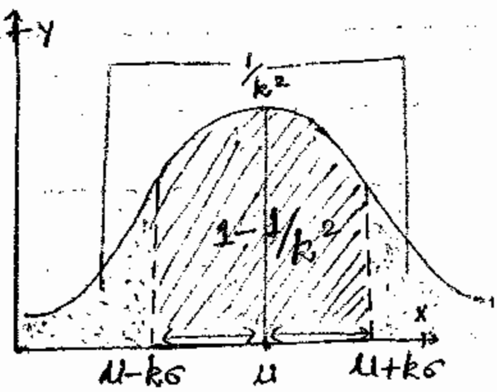
$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

or

$$P(|x - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

where $k > 0$ is a Constant.

$P(\mu - k\sigma < x < \mu + k\sigma)$ gives upper bounds and $P(|x - \mu| \geq k\sigma)$ gives lower bounds.



Example:-

A continuous r.v X has p.d.f

$$f(x) = \begin{cases} \frac{3}{4} x(2-x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$. Compare the result due to Chebychev's inequali

Sol:-

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^2 x\left(\frac{3}{4} x(2-x)\right) dx$$

$$\mu = \int_0^2 \frac{3}{4} x^2 (2-x)\,dx$$

$$\mu = \int_0^2 \frac{3}{4} (2x^2 - x^3)\,dx$$

$$= \frac{3}{4} \left(2\frac{x^3}{3} - \frac{x^4}{4} \right|_0^2$$

$$= \frac{3}{4} \left(\frac{2}{3}(2^3 - 0) - \frac{1}{4}(2^4 - 0) \right)$$

$$= \frac{3}{4} \left(\frac{2}{3}(8) - \frac{1}{4}(16) \right)$$

$$= 4 - \frac{3}{4}(4)$$

$$\mu = 4 - 3 = 1$$

$$\sigma^2 = E(X^2) - [E(X)]^2 \rightarrow (1)$$

$$E(X^2) = \int_0^2 x^2 f(x)\,dx$$

$$= \int_0^2 x^2 \left(\frac{3}{4}(x(2-x)) \right) dx$$

$$= \int_0^2 \frac{3}{4} x^3 (2-x)\,dx$$

$$= \int_0^2 \frac{3}{4} (2x^3 - x^4)\,dx$$

$$= \frac{3}{4}\left(2\frac{x^4}{4} - \frac{x^5}{5}\right|_0^2$$

$$= \frac{3}{4}\left(\frac{2}{4}(2^4 - 0) - \frac{1}{5}(2^5 - 0)\right)$$

$$= \frac{3}{4}\left(\frac{2}{4}(16) - \frac{1}{5}(32)\right)$$

$$= \frac{3}{4}\cdot\frac{2}{4}\cdot 16 - \frac{3}{4}\cdot\frac{1}{5}\cdot 32$$

$$= \frac{6\cdot 16}{16} - \frac{3}{20}\cdot 32$$

$$E(x^2) = 6 - \frac{96}{20} = 6 - 4.8 = 1.2$$

put in (1)

$$\sigma^2 = 1.2 - (1)^2$$

$$\sigma^2 = 1.2 - 1 = 0.2$$

$$\sigma = 0.43$$

$$P(\mu - 2\sigma \le x \le \mu + 2\sigma) = P(0.1 \le x \le 1.9)$$

$$= \int_{0.1}^{1.9} \frac{3}{4} x(2-x)\,dx$$

$$= \frac{3}{4}\left| x^2 - \frac{x^3}{3}\right|_{0.1}^{1.9}$$

$$= \frac{3}{4}\left[\left((1.9)^2 - \frac{(1.9)^3}{3}\right) - \left((0.1)^2 - \frac{(0.1)^3}{3}\right)\right]$$

$= \frac{3}{4}((3.61 - 2.286) - (0.01 - 0.00033))$

$= \frac{3}{4}(1.324 - 0.00967) = \frac{3}{4}(1.31433)$

$= 0.985$

Using Chebychev's inequality

$P(\mu - 2\sigma \le X \le \mu + 2\sigma) \ge 1 - \frac{1}{k^2}$

where $k = 2$

$\ge 1 - \frac{1}{2^2}$

$= 1 - \frac{1}{4}$

$= 0.75$

$\Rightarrow P(\mu - 2\sigma \le X \le \mu + 2\sigma) \ge 0.75$

(iii)

$P(0.325 \le X \le 1.675)$

$\mu - k\sigma = 0.325, \ \mu + k\sigma = 1.675$

$\mu - k\sigma = 0.325$

$\pm \mu \pm k\sigma = \pm 1.675$

$-2k\sigma = -1.35 \Rightarrow k = \frac{-1.35}{-2\sigma}$

$k = \frac{-1.35}{-2(0.43)} = \frac{1.35}{0.86} = 1.56$

$P(0.325 \le X \le 1.675) \ge 1 - \frac{1}{k^2}$

$= 1 - \frac{1}{(1.56)^2} = 1 - 0.410$

$= 0.59$

Joint Distribution:-

If we have p random variables $X_1, X_2, ---, X_p$ defined on sample space S then $f(X_1, X_2, ---, X_p)$ is called joint distribution

$p=2$, $f(X_1, X_2)$ is called bivariate distribution.

$p>2$, $f(X_1, X_2, ---)$ is called multi-variate distribution.

Discrete Bivariate Distribution.

$x_2 \backslash x_1$	x_{11}	x_{21}	x_{31}	x_{41}	----	x_{n1}	h
x_{12}	$f(x_{11}, x_{12})$	$f(x_{21}, x_{21})$	-	-	-	$f(x_{12}, x_{n1})$	$h(x_{12})$
x_{22}	$f(x_{11}, x_{22})$	-	-	-			$h(x_{22})$
x_{32}	$f(x_{11}, x_{32})$	-	-				$h(x_{32})$
$\vdots$	$\vdots$						$\vdots$
x_{n2}	$f(x_{11}, x_{n2})$						$h(n_2)$
Marginal p.d.f	$g(x_{11})$	$g(x_{21})$	$g(x_{31})$		----	$g(x_{n1})$	

of X_1; where h is marginal p.d.f of X_2.

Two events A and B are inde-pendent if $P(A \cap B) = P(A) \cdot P(B)$

Two events X_1 and X_2 are independent if their joint probability = product of margi-nal probability

Example:-

$Y \backslash X$	1	2	3	$h(y)$
1	$1/12$	$1/6$	0	$1/4$
2	0	$1/9$	$1/5$	$14/45$
3	$1/18$	$1/4$	$2/15$	$79/180$
$g(x)$	$5/36$	$19/36$	$1/3$	1

we have to show that x and y are independent or not.

$f(x,y) = f(1,1) = 1/12 = 0.083$

$g(x) \cdot h(y) = \frac{5}{36} \cdot \frac{1}{4} = \frac{5}{144} = 0.034$

$\Rightarrow f(1,1) \neq g(1) \cdot h(1)$

So, X and Y are not independent events.

Results:-

1) $E(X+Y) = \sum_x \sum_y (x+y) f(x,y)$

2) $E(XY) = \sum_x \sum_y (xy) f(x,y)$

3) $E(X+Y) = E(X) + E(Y)$

4) $E(XY) = E(X)E(Y)$

if X and Y are independent.

5) $g(x) = \sum_y f(x,y)$

6) $h(y) = \sum_x f(x,y)$

7) $f(x/y) = \dfrac{f(x,y)}{h(y)}$

8) $f(y|x) = \dfrac{f(x,y)}{g(x)}$

Example:-

An urn contains 3 black, 2 red and 3 green balls If 2 balls are selected at random from it, and X is the no of black balls and Y is the no of red balls selected then find

(i) joint distribution of x and y

(ii) marginal " " of x

(iii) " " " " y

(iv) the conditional p.d $f(x|1)$

(v) $P(X=1|Y=1)$

(vi) $E(X)$, $E(Y)$, $E(X+Y)$, $E(XY)$

(vii) are X and Y independent.

Sol:-

$n(S) = {}^8C_2 = 28$

$y \backslash x$	0	1	2	$h(y)$
0	$3/28$	$9/28$	$3/28$	$15/28$
1	$6/28$	$6/28$	0	$12/28$
2	$1/28$	0	0	$1/28$
$g(x)$	$10/28$	$15/28$	$3/28$	1

$B = 3$

$R = 2$

$G = 3$

2-balls

X: no. of black bal

$x = 0, 1, 2$

Y: no. of red bal

$y = 0, 1, 2$

$$f(x=0, y=0) = \frac{{}^3C_0 \; {}^2C_0 \; {}^3C_2}{{}^8C_2}$$

$$= \frac{1 \cdot 1 \cdot 3}{28}$$

$$f(0,0) = 3/28$$

$$f(x=0, y=1) = \frac{{}^3C_0 \; {}^2C_1 \; {}^3C_1}{{}^8C_2}$$

$$f(0,1) = \frac{1 \cdot 2 \cdot 3}{28} = 6/28$$

$$f(0,2) = \frac{{}^3C_0 \; {}^2C_2 \; {}^3C_0}{{}^8C_2} = \frac{1 \cdot 1 \cdot 1}{28} = 1/28$$

$$f(1,0) = \frac{{}^3C_1 \; {}^2C_0 \; {}^3C_1}{{}^8C_2} = \frac{3 \cdot 1 \cdot 3}{28} = 9/28$$

$$f(1,1) = \frac{{}^3C_1 \; {}^2C_1 \; {}^3C_0}{{}^8C_2} = \frac{3 \cdot 2 \cdot 1}{28} = 6/28$$

$f(x=1, y=2) = 0$ (impossible)

$f(x=2, y=0)$

$f(2,0) = 3/28$

$f(x=2, y=1) = 0$

$f(x=2, y=2) = 0$

(ii)

marginal p.d of x is

x	$g(x)$
0	$10/28$
1	$15/28$
2	$3/28$

(iii)

marginal probability dist of Y is

y	$h(y)$
0	$15/28$
1	$12/28$
2	$1/28$

(iv)

Conditional probability of $x/y=1$

$$f(x/y=1) = \frac{f(x, y=1)}{h(y=1)}$$

x	$f(x/y=1)$
0	$\frac{f(x=0, y=1)}{h(y=1)} = \frac{6/28}{12/28} = \frac{6}{1.}$
1	$\frac{f(x=1, y=1)}{h(y=1)} = \frac{6/28}{12/28} = 6/12$
2	$\frac{f(x=2, y=1)}{h(y=1)} = \frac{0}{12/28} = 0$

(v)

$$P(X=1 \mid y=1) = \frac{f(x=1, y=1)}{h(y=1)}$$

$$= \frac{6/28}{12/28} = 6/12$$

(vi) $$E(X) = \sum_x x g(x) = 0 \cdot \frac{10}{28} + 1 \cdot \frac{15}{28} +$$

$$2 \cdot \frac{3}{28} = \frac{21}{28} = 3/4$$

$$E(Y) = \sum_y y h(y) = 0 \cdot \frac{15}{28} + 1 \cdot \frac{12}{28} + 2 \cdot \frac{1}{28}$$

$$= \frac{14}{28} = 1/2$$

$$E(X+Y) = \sum_x \sum_y (x+y) f(x, y)$$

$$E(X+Y) = (0+0)\cdot\frac{3}{28} + (0+1)\cdot\frac{6}{28} + (0+2)\cdot\frac{1}{28}$$

$$+ (1+0)\cdot\frac{9}{28} + (1+1)\,6/28 + (1+2)\cdot 0 + (2+0)\cdot 3/28$$

$$+ (2+1)\cdot 0 + (2+2)\cdot 0$$

$$E(X+Y) = 0 + 6/28 + 2/28 + 9/28 + \frac{12}{28} + 0$$
$$+ 6/28 + 0 + 0$$

$$E(X+Y) = 35/28 = 5/4$$

or

$$E(X+Y) = E(X) + E(Y)$$
$$= \frac{21}{28} + \frac{14}{28} = 35/28 = 5/4.$$

Now

$$E(XY) = \sum_x \sum_y xy\, f(x,y)$$

$$= (0\cdot 0)\cdot 3/28 + (0\cdot 1)\,6/28 + (0\cdot 2)\frac{1}{28}$$
$$+ (1\cdot 0)\frac{9}{28} + (1\cdot 1)\frac{6}{28} + (1\cdot 2)\cdot 0 + (2\cdot 0)\,3/28$$

$$+ (2\cdot 1)\cdot 0 + (2\cdot 2)\cdot 0$$

$$E(XY) = 0 + 0 + 0 + 0 + \frac{6}{28} + 0 + 0$$

$$E(XY) = 0 + \frac{6}{28} = \frac{6}{28}$$

or

(vii) $E(XY) = E(X).E(Y)$

$= 3/4 \cdot \frac{1}{2}$

$E(XY) = 3/8 \neq E(X).E(Y)$

$\Rightarrow$ X and Y are not independent.

Example 8-

Given the following joint p.d.f

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & 0 \le x \le 2,\ 2 \le y \le 4 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) verify that $f(x,y)$ is a density function.

As $f(x,y) \geq 0$ for $0 \le x \le 2$, $2 \le y \le 4$, there first property of p.d.f holds.

Now take

$$\int_0^2 \int_2^4 \frac{1}{8}(6-x-y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \left| 6y - xy - \frac{y^2}{2} \right|_2^4 dx$$

$$= \frac{1}{8}\int_0^2 [(24-4x-8)-(12-2x-2)]\,dx$$

$$= \frac{1}{8}\int_0^2 (6-2x)\,dx$$

$$= \frac{1}{8}\left(6x - x^2\right|_0^2$$

$$\int_0^2\int^4 \frac{1}{8}(6-x-y)\,dy\,dx = \frac{1}{8}(12-4) = \frac{8}{8} = 1$$

(b) $P(X \le 3/2,\ Y \le 5/2)$

Sol:-

$$P(X \le 3/2,\ Y \le 5/2) = \int_0^{3/2}\int_2^{5/2} \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^{3/2}\left(6y - xy - \frac{y^2}{2}\right|_2^{5/2} dx$$

$$= \frac{1}{8}\int_0^{3/2}\left[6\left(\frac{5}{2}\right) - x\left(\frac{5}{2}\right) - \frac{25/4}{2}\right) - \left(6(2) - 2x - 4/2\right)\right] dx$$

$$= \frac{1}{8}\int_0^{3/2}\left[\left(15 - \frac{5}{2}x - \frac{25}{8}\right) - \left(12 - 2x - 2\right)\right] dx$$

$$= \frac{1}{8}\int_0^{3/2}\left(\frac{15}{8} - \frac{1}{2}x\right) dx$$

$$= \frac{1}{8}\left(\frac{15}{8}x - \frac{1}{4}x^2\right|_0^{3/2}$$

$$= \frac{1}{8}\left(\frac{15}{8}\left(\frac{3}{2} - 0\right) - \frac{1}{4}\left(\frac{9}{4} - 0\right)\right)$$

$$= \frac{1}{8}\left(\frac{45}{16} - \frac{9}{16}\right) = \frac{1}{8}\left(\frac{45-9}{16}\right)$$

$$= \frac{1}{8}\left(\frac{36}{16}\right) = \frac{9}{32}$$

(c)

$$P(X+Y<3) = P(0<x<2,\ 2<y<3-x)$$

$$= \int_0^2 \int_2^{3-x} \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^2 \left(6y - xy - \frac{y^2}{2}\right)\Big|_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^2 \Big[\Big(6(3-x) - x(3-x) - \frac{(3-x)^2}{2}\Big) - \big(6(2) - 2x - 2\big)\Big] dx$$

$$= \frac{1}{8}\int_0^2 \Big[18 - 6x - 3x + x^2 - \frac{(9+x^2-6x)}{2} - 12 + 2x + 2\Big] dx$$

$$= \frac{1}{8}\int_0^2 \Big[18 - 9x + x^2 - \frac{9}{2} - \frac{x^2}{2} + 3x - 10 + 2x\Big]$$

$$= \frac{1}{8}\int_0^2 \Big[\frac{x^2}{2} - 4x + 8 - \frac{9}{2}\Big] dx$$

$$= \frac{1}{16}\int_0^2 (x^2 - 8x + 16 - 9)\,dx$$

$$= \frac{1}{16}\int_0^2 (x^2 - 8x + 7)\,dx$$

$$= \frac{1}{16}\left(\frac{x^3}{3}\Big|_0^2 - 8\frac{x^2}{2}\Big|_0^2 + 7x\Big|_0^2\right)$$

$$= \frac{1}{16}\left[\frac{2^3}{3} - 4(2^2 - 0) + 7(2)\right] = \frac{1}{16}\left[\frac{8}{3} - 16 + 14\right]$$

$$= \frac{1}{16}\left(\frac{8}{3} - 2\right) = \frac{1}{16}\left(\frac{8-6}{3}\right) = \frac{1}{16}\left(\frac{2}{3}\right) = \frac{1}{24}$$

d) Marginal p.d.f of X

$$g(x) = \int_y f(x,y)\,dy$$

$$= \int_2^4 \frac{1}{8}(6-x-y)\,dy$$

$$= \frac{1}{8}\left(6y - xy - \frac{y^2}{2}\right)\Big|_2^4$$

$$= \frac{1}{8}\left[(24-4x-8)-(12-2x-2)\right]$$

$$= \frac{1}{8}\left[24-4x-8-12+2x+2\right]$$

$$= \frac{1}{8}(6-2x)$$

$$g(x) = \frac{1}{4}(3-x) \quad ; \; 0 \le x \le 2$$

$$g(x) = \begin{cases} \frac{1}{4}(3-x) & ; \; 0 \le x \le 2 \\ 0 & \text{elsewhere.} \end{cases}$$

(e) Marginal p.d.f of Y

$$h(y) = \int_x f(x,y)\,dx$$

$$= \int_0^2 \frac{1}{8}(6-x-y)\,dx$$

$$= \frac{1}{8}\left(6x - \frac{x^2}{2} - xy\right)\Big|_0^2$$

$$= \frac{1}{8}\left[(12-2-2y)-0\right]$$

$$= \frac{1}{8}(10-2y)$$

$$h(y) = \frac{1}{4}(5-y) \quad ; \quad 2 \le y \le 4.$$

If $f(x,y) = g(x) \cdot h(y)$

$$= \frac{1}{4}(3-x) \cdot \frac{1}{4}(5-y)$$

$$= \frac{1}{6}[15 - 5x - 3y + xy]$$

$g(x) \cdot h(y) \neq f(x,y)$

Then x and Y are not independent

$\therefore$ x and Y are dependent.

Conditional distribution of X :-

$$f(x/y) = \frac{f(x,y)}{h(y)}$$

$$= \frac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(5-y)}$$

$$f(x/y) = \frac{1}{2}\left(\frac{6-x-y}{5-y}\right)$$

Discrete Probability Distributions

1) Uniform distribution:-

If a random variable X has a uniform distribution defined over interval [a, b] only for integer values of X then

$$p(X = x) = \frac{1}{b-a+1}$$

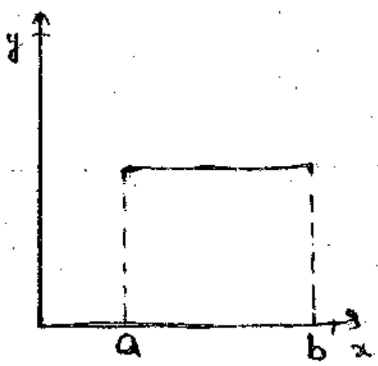
Usually, a uniform distribution is denoted as $X \sim U[a, b]$ where X is discrete random variable.

Example:-

$X \sim U[1, 10]$ and X is d.r.v. find probability distribution of X.

Sol:-

x	$p(x) = \frac{1}{b-a+1}$
0	1/10
1	1/10
2	1/10
3	1/10
4	1/10
5	1/10
6	1/10



7	1/10
8	1/10
9	1/10
10	1/10

$$\text{Mean} = E(X) = \sum_{x=a}^{b} x\,p(x)$$

$$= \sum_{x=a}^{b} x \frac{1}{b-a+1}$$

$$= \frac{1}{b-a+1} \sum_{x=a}^{b} x$$

$$= \frac{1}{10-1+1} \sum_{x=1}^{10} x$$

$$= \frac{1}{10}\left(\frac{10(11)}{2}\right) \qquad \because 1+2+\cdots+n = \frac{n(n+1)}{2}$$

$$E(X) = 11/2 = 5.5$$

Bernoulli Trials:-

A trial is said to be bernoulli trial if
1) can result in two possible outcomes.
2) probability of success remains constant for repeated trials.

Examples:-

1) Tossing a coin.
2) Rolling a die.

(c) Drawing cards from a pack of playing cards when drawn card is replaced after each draw.

Binomial Experiment :-

An experiment consisting of "n" bernoulli trials, is called binomial experiment. or An experiment is called a binomial probability experiment if has (4) proper

Properties of Binomial Experiment :-

: following are properties of BE.

1) Results can be classified in two mutually exclusive categories called success and failure.

2) Probability of success remains constant throughout the experiment.

3) Successive trials are independent.

4) fixed number of trials.

Binomial Distribution :-

A binomial experiment can be modelled using the following function.

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad ; x = 0, 1, 2, \ldots, n$$

where X is binomial random variabl which can take value x, p is the probability of success, $q = 1-p$ is the Probability of failure and n is the number of trials.

n and p are called parameters of binomial distribution

"The shape of any distribution depends on its parameter or parameters".

Example:-

Tossing a fair coin 3 times. Find probability distribution of head

Sol:-

$n = 3$, $p = P(\text{Head}) = \frac{1}{2}$

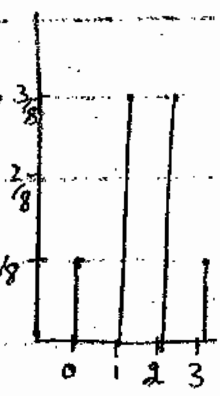
$q = 1 - p = \frac{1}{2}$

$X =$ No. of heads ; $x = 0, 1, 2, 3$

$$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= {}^{3}C_{x}\, p^{x} q^{3-x}$$

x	$P(X = x)$
0	${}^{3}C_{0}(\frac{1}{2})^{0}(\frac{1}{2})^{3} = \frac{1}{8}$
1	${}^{3}C_{1}(\frac{1}{2})^{1}(\frac{1}{2})^{2} = \frac{3}{8}$
2	${}^{3}C_{2}(\frac{1}{2})^{2}(\frac{1}{2})^{1} = \frac{3}{8}$
3	${}^{3}C_{3}(\frac{1}{2})^{3}(\frac{1}{2})^{0} = \frac{1}{8}$



Example:-

A biased coin ($P(\text{Head}) = 3/4$) is tossed 3 times. Find probability distribution of head.

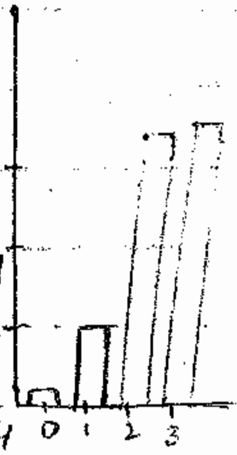
Sol:-

$n = 3$, $p = 3/4$, $q = 1 - 3/4 = 1/4$

$p(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$; $x = 0, 1, 2, 3$

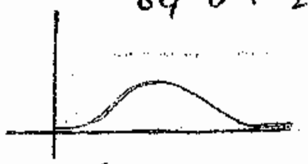
$p(X = x) = {}^{3}C_{x}\, p^{x} q^{3-x}$

x	$P(X = x)$
0	${}^{3}C_{0} (3/4)^{0} (1/4)^{3} = 1/64$
1	${}^{3}C_{1} (3/4)^{1} (1/4)^{2} = 9/64$
2	${}^{3}C_{2} (3/4)^{2} (1/4)^{1} = 27/64$
3	${}^{3}C_{3} (3/4)^{3} (1/4)^{0} = 27/64$



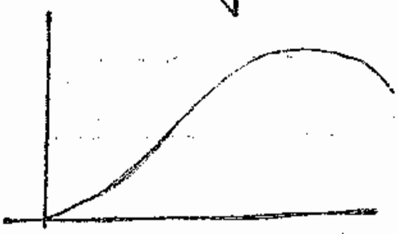
Note:-

If $p = q$



then curve is smooth and symmetric

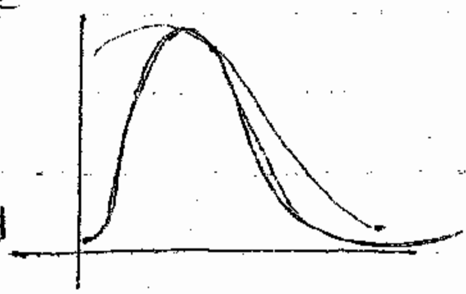
If $p > q$



then curve is below at starting and then go to above

If $p < q$

then If $p < q$



then the curve is above on starting and then go to below.

Example:-

If the probability of getting caught copying someone's else is 0.2. What is the prob of not getting caught in 3 attempts. Assume independence

Sol:-

Random experiment: copying someone's else in exam.

1) two mutually exclusive categories
(i) caught (ii) not caught.
2) fixed no. of trials
3) Independent trials.
4) Prob of success is constant.

$X =$ No. of times getting caught

$P(\text{Not getting caught in 3 attempts}) = P(X=0)$

$= {}^3C_0 (0.2)^0 (0.8)^3$

$= 1 \times 1 \times 0.512$

$= 0.512$

(OR)

Y: No of times not getting caught.

$P(Y=3) = {}^3C_3 (0.8)^3 (0.2)^0$

$= 1 \times 0.512 \times 1$

$= 0.512$

Example:-

If on the average rain falls on 12 days in every thirty, find the prob that
i) rain falls on just 3 days of a given week ii) rain falls on first 3 days of a given week.

Sol:-

(Assuming independent trials)

$P(\text{rain fall on a day}) = \frac{12}{30} = 0.4$

X : no. of days with rain fall.

$p = 0.4 \quad q = 0.6$

i)

$n = 7$ (7 days in a week)

$P(X=x) = {}^{7}C_{x}\, p^{x} q^{7-x} \quad ; x = 0,1,2,\ldots,7$

$P(X=x) = {}^{7}C_{x}\, (0.4)^{x} (0.6)^{7-x}$

$P(X=3) = {}^{7}C_{3}\, (0.4)^{3} (0.6)^{4}$

$= 35 \times 0.064 \times 0.1296$

$P(X=3) = 0.290$

ii)

$P(\text{rain falls on first 3 days}) = (2/5)^{3} (3/5)^{4}$

$= (0.4)^{3} (0.6)^{4} = 0.008$

Note:-

Total Prob = 1

for binomial probability distribution

$$\sum_{x=0}^{n} {}^{n}C_x\, p^x q^{n-x} = 1$$

Properties of binomial p.d:-

1) Show that mean of the binomial distribution $(q+p)^n$ is np.

Proof:-

By definition Mean $= E(x)$

$$= \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x\, {}^{n}C_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{(n-x)!\, x!} p^x q^{n-x} \rightarrow (1)$$

$$= \sum_{x=1}^{\infty} x \cdot \frac{n}{x}\, {}^{n-1}C_{x-1}\, p^x q^{n-x}$$

$$= n \sum_{x=1}^{\infty} {}^{n-1}C_{x-1}\, p^x q^{n-x}$$

put $n-1 = m$, $x-1 = y$

$$E(x) = n \sum_{y=0}^{m} {}^{m}C_y\, p^{y+1} q^{m-y}$$

$$E(x) = np \sum_{y=0}^{m} {}^{m}C_y\, p^y q^{m-y}$$

$$= np\left(\sum_{y=0}^{m} {}^{m}C_y\, p^y q^{m-y} = 1\right)$$

$E(X) = np$

$\Rightarrow$ Mean $= np$

or from (1)

$$E(X) = \sum_{x=1}^{n} x \frac{n(n-1)!}{x(x-1)!(n-x)!} p^{x-1+1} q^{n-x}$$

$$= n\sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} \cdot p q^{n-x}$$

$$= np\sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{n-x} = np$$

2)

Show that variance of binomial p.d $(q+p)^n$ is npq.

Proof:-

By definition

Variance $= E(X^2) - [E(X)]^2 \rightarrow (1)$

We know $E(X) = np$

Now $E(X^2) = \sum_{x=0}^{n} x^2 p(x)$

$$= \sum_{x=0}^{n} x^2\, {}^{n}C_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (x(x-1)+x)\, {}^{n}C_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\,{}^{n}C_x\, p^x q^{n-x} + \sum_{x=0}^{n} x\,{}^{n}C_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\,{}^{n}C_x\, p^x q^{n-x} + E(X)$$

$$= \sum_{x=2}^{n} x(x-1)\,\frac{n(n-1)}{x(x-1)}\,{}^{n-2}C_{x-2}\, p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} n(n-1)\,{}^{n-2}C_{x-2}\, p^x q^{n-x} + np$$

$$= n(n-1)\sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^x q^{n-x} + np$$

$n-2=m,\ x-2=y$

$$= n(n-1)\sum_{y=0}^{m} {}^{m}C_y\, p^{y+2} q^{m-y} + np$$

$$= n(n-1)p^2\sum_{y=0}^{m} {}^{m}C_y\, p^{y} q^{m-y} + np$$

$$\because \sum_{y=0}^{m} {}^{m}C_y\, p^{y} q^{m-y} = 1$$

$E(x^2) = n(n-1)p^2 + np$, $E(X) = np$

put in (1)

$$\text{variance} = n(n-1)p^2 + np - (np)^2$$

$$= (n^2-n)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= -np^2 + np = np - np^2$$

$= np(1-p)$

Variance $= npq$

Moments:-

In general,

rth moment about origin : $\mu_r' = E(X^r)$

rth moment about mean : $\mu_r = E[(X-\mu)^r]$

where $r = 1, 2, 3, \ldots$

Moments of binomial distribution:-

$\mu_1' = E(X) = np$

$\mu_2' = E(X^2) = n(n-1)p^2 + np$

$\mu_3' = E(X^3) = \sum_{x=0}^{n} x^3 p(x)$

$= \sum_{x=0}^{n} [x(x-1)(x-2) + 3x(x-1) + x]\, p(x)$

$= \sum_{x=0}^{n} x(x-1)(x-2)p(x) + 3\sum_{x=0}^{n} x(x-1)p(x) + \sum_{x=0}^{n} x p(x)$

$= \sum_{x=0}^{n} x(x-1)(x-2)\, {}^nC_x p^x q^{n-x} + 3\sum_{x=0}^{n} x(x-1)\, {}^nC_x p^x q^{n-x} + np$

$= \sum_{x=3}^{n} x(x-1)(x-2) \frac{n(n-1)(n-2)}{x(x-1)(x-2)}\, {}^{n-3}C_{x-3} p^x q^{n-x} + 3\sum_{x=2}^{n} x(x-1) \frac{n(n-1)}{x(x-1)}\, {}^{n-2}C_{x-2} p^x q^{n-x} + np$

$= \sum_{x=3}^{n} n(n-1)(n-2)\, {}^{n-3}C_{x-3} p^{x-3} \cdot p^3 q^{n-x} + 3\sum_{x=2}^{n} n(n-1)\, {}^{n-2}C_{x-2} p^{x-2} p^2 q^{n-x} + np$

$= p^3 n(n-1)(n-2) \sum_{x=3}^{n} {}^{n-3}C_{x-3} p^{x-3} q^{n-x} + 3p^2(n(n-1)) \sum_{x=2}^{n} {}^{n-2}C_{x-2} p^{x-2} q^{n-x} + np$

$= n(n-1)(n-2)p^3(1) + 3n(n-1)p^2(1) + np$

$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

Moments Generating Function (m.g.f):

$$M_x(t) = E(e^{tx})$$

we have $\mu_r' = \frac{d^r}{dt^r} M_x(t)\Big|_{t=0}$

For binomial distribution:-

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x\, (e^t p)^x q^{n-x}$$

$$M_x(t) = (q + pe^t)^n$$

$$\mu_1' = \frac{d}{dt} M_x(t)\Big|_{t=0}$$

$$= n(q+pe^t)^{n-1}\, pe^t\Big|_{t=0}$$

$$= n(q+pe^0)^{n-1}\, pe^0$$

$$= n(q+p)^{n-1}\, p$$

$$= n(1)^{n-1}\, p$$

$$\mu_1' = np$$

$$\mu_2' = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0}$$

$$\mu_2' = \frac{d}{dt}(\mu_1') = \frac{d}{dt}\left(n(q+pe^t)^{n-1} pe^t\right)\Big|_{t=0}$$

$$= np\left[(q+pe^t)^{n-1} e^t + (n-1)e^t(q+pe^t)^{n-2} pe^t\right]\Big|_{t=0}$$

$$= np\left[(q+p)^{n-1}\cdot 1 + (n-1)\cdot 1\,(q+p)^{n-2} p\cdot 1\right]$$

$$= np\left[(1) + (n-1)p\right]$$

$$= np\left[1+(n-1)p\right]$$

$$\mu_2' = np + n(n-1)p^2$$

$$\mu_3' = \frac{d^3}{dt^3} M_x(t)\Big|_{t=0}$$

$$\mu_3' = \frac{d}{dt}\left(\frac{d^2}{dt^2} M_x(t)\Big|_{t=0}\right) = \frac{d}{dt}(\mu_2')\Big|_{t=0}$$

$$\mu_3' = \frac{d}{dt}\left(np\left[(q+pe^t)^{n-1} e^t + (n-1)e^t(q+pe^t)^{n-2} pe^t\right]\right)\Big|_{t=0}$$

$$= np\left[(q+pe^t)^{n-1} e^t + (n-1)(q+pe^t)^{n-2} pe^t\cdot e^t + p(n-1)2e^{2t}(q+pe^t)^{n-2} + (n-1)pe^{2t}(n-2)(q+pe^t)^{n-3} pe^t\right]\Big|_{t=0}$$

$$= np\left[(q+p)^{n-1} e^0 + (n-1)(q+p)^{n-2} pe^0\cdot e^0 + p(n-1)2e^0(q+p)^{n-2} + (n-1)pe^0(n-2)(q+p)^{n-3} pe^0\right]$$

$$= np\left[(1)\cdot(1) + (n-1)(1)p + 2(n-1)p(1) + (n-1)(n-2)p^2(1)\right]$$

$$= np\left[1 + (n-1)p + 2(n-1)p + (n-1)(n-2)p^2\right]$$

$$\mu_3' = np + n(n-1)p^2 + 2n(n-1)p^2 + n(n-1)(n-2)p^3 = np + 3n(n-1)p^2 + n(n-1)(n-2)p^3$$

$$\mu_4' = \frac{d^4}{dt^4} M_X(t)\Big|_{t=0} = \frac{d}{dt}(\mu_3')\Big|_{t=0}$$

$$\mu_4' = \frac{d}{dt}\Big(np[(q+pe^t)^{n-1}e^t + (n-1)(q+pe^t)^{n-2}pe^{2t} + 2p(n-1)e^{2t}(q+pe^t)^{n-2} + (n-1)(n-2)p^2e^{3t}(q+pe^t)$$

$$\mu_4' = np[(q+pe^t)^{n-1}e^t + (n-1)(q+pe^t)^{n-2}pe^{2t} + (n-1)(q+pe^t)^{n-2}p(2e^{2t}) + (n-1)(n-2)(q+pe^t)^{n-3}p^2e^{3t} + 2p(n-1)(2e^{2t})(q+pe^t)^{n-2} + 2p(n-1)(n-2)(q+pe^t)^{n-3}pe^{3t} + (n-1)(n-2)p^2(3e^{3t})(q+pe^t)^{n-3} + (n-1)(n-2)(n-3)(q+pe^t)p^3e^{4t}]_{t=0}$$

$$\mu_4' = np[(q+p)^{n-1}\cdot 1 + (n-1)(q+p)^{n-2}p\cdot 1 + (n-1)(q+p)^{n-2}(2p) + (n-1)(n-2)(q+p)^{n-3}p^2 + 2p(n-1)(2)(q+p)^{n-2} + 2p(n-1)(n-2)(q+p)^{n-3}p\cdot 1 + (n-1)(n-2)p^2(3\cdot 1)(q+p)^{n-3} + (n-1)(n-2)(n-3)(q+p)^{n-4}p^3e^0]$$

$$\mu_4' = np[1\cdot 1 + (n-1)\cdot 1\cdot p + (n-1)\,1\cdot 2p + (n-1)(n-2)\,1\cdot p^2 + 2p(n-1)\,2\cdot 1 + 2p(n-1)(n-2)\,1\cdot p + (n-1)(n-2)p^2\,3\cdot 1 + (n-1)(n-2)(n-3)\,1\cdot p^3\cdot 1]$$

$$\mu_4' = np[1 + (n-1)p + 2(n-1)p + (n-1)(n-2)p^2 + 4(n-1)p + 2(n-1)(n-2)p^2 + 3(n-1)(n-2)p^2 + (n-1)(n-2)(n-3)p^3]$$

$$\mu_4' = np + n(n-1)p^2 + 2n(n-1)p^2 + n(n-1)$$
$$(n-2)p^3 + 4n(n-1)p^2 + 2n(n-1)(n-2)p^3$$
$$+ 3n(n-1)(n-2)p^3 + n(n-1)(n-2)(n-3)p^4$$
$$\mu_4' = np + 7n(n-1)p^2 + 6n(n-1)(n-2)p^3 + n(n-1)(n-2)(n-3)p^4$$

Cumulants:-

The cumulants are a set of parameters of a probability distribution defined by following identity in t; $\sum_{r=0}^{\infty} \mu_r' \frac{t^r}{r!}$ where k_r is rth cumulant.

Cumulants Generating function (c.g.f):-

$$K_x(t) = \log_e M_x(t)$$

In general, rth cumulant k_r is the co-efficient of $\frac{t^r}{r!}$ in the expansion of $k_x(t)$.

C.g.f of binomial distribution:-

We have

$$M_x(t) = (q + pe^t)^n$$

Now c.g.f of binomial distribution can be defined as

$$k_x(t) = \log_e M_x(t)$$

$$= \log_e (q + pe^t)^n$$

$$= n\log_e (q + pe^t)$$

$$= n\log_e \left[q + p\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right]$$

$$= n\log_e\left[q+p+pt+p\frac{t^2}{2!}+p\frac{t^3}{3!}+\cdots\right]$$

$$= n\log_e\left[1+\left(pt+p\frac{t^2}{2!}+p\frac{t^3}{3!}+\cdots\right)\right]$$

$$\because \log_e(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

$$k_x(t) = n\left[\left(pt+p\frac{t^2}{2!}+\cdots\right)-\frac{1}{2}\left(pt+p\frac{t^2}{2!}+p\frac{t^3}{3!}+\cdots\right)^2\right.$$

$$\left.+\frac{1}{3}\left(pt+p\frac{t^2}{2!}+p\frac{t^3}{3!}+\cdots\right)^3-\frac{1}{4}\left(pt+p\frac{t^2}{2!}+p\frac{t^3}{3!}+\cdots\right)^4+\cdots\right]$$

$$k_x(t) = n\left[pt+\left(p\frac{t^2}{2!}-\frac{1}{2}p^2t^2\right)+\left(p\frac{t^3}{3!}-\frac{1}{2}p^2t^3+\frac{1}{3}p^3t^3\right)+\cdots\right]$$

$$= npt+np(1-p)\frac{t^2}{2!}+np\left(\frac{t^3}{3!}-\frac{1}{2}pt^3+\frac{1}{3}p^2t^3+\cdots\right)$$

$$= npt+np(1-p)\frac{t^2}{2!}+np\left(\frac{t^3}{3!}-\frac{3}{3!}pt^3+\frac{2}{3!}p^2t^3\right)+\cdots$$

$$k_x(t) = np\frac{t^1}{1!}+np(1-p)\frac{t^2}{2!}+np(1-3p+2p^2)\frac{t^3}{3!}+\cdots$$

I.M $k_1' = np$, $k_2' = np(1-p)$, $k_3' = np(1-3p+2p^2)$

Examples-

Prove that for binomial distribution $(q+p)^n$, we have recursive rule of moments $\mu_{r+1} = pq\left(nr\mu_{r-1}+\frac{d\mu_r}{dp}\right)$

where μ_r is rth moment about mean.

Sol:-

By definition, rth moment about mean is

$$\mu_r = E(X-\mu)^r$$

$$= \sum_{x=0}^{n} (x-\mu)^r p(x)$$

we have binomial distribution $(q+p)^n$.

As we know for binomial distribution

$\mu = np$ and $p(x) = {}^nC_x \, p^x q^{n-x}$

$$\therefore \mu_r = \sum_{x=0}^{n} (x-np)^r \, {}^nC_r \, p^x q^{n-x}$$

$$\mu_x = \sum_{x=0}^{n} {}^nC_x \, (x-np)^r p^x (1-p)^{n-x}$$

Diff w.r.t p

$$\frac{d}{dp}(\mu_r) = \sum_{x=0}^{n} {}^nC_x \Big[r(x-np)^{r-1}(0-n)p^x$$

$$(1-p)^{n-x} + (x-np)^r x p^{x-1} (1-p)^{n-x} + (x-np)^r p^x$$

$$(n-x)(1-p)^{n-x-1}(-p)\Big]$$

$$\frac{d}{dp}(\mu_r) = \sum_{x=0}^{n} {}^nC_x \Big[-nr(x-np)^{r-1} p^x q^{n-x} + (x-np)^r$$

$$x p^{x-1} q^{n-x} + (x-np)^r p^x$$

$$\frac{d\mu_r}{dp} = -\sum_{x=0}^{n} {}^nC_x (n-x) p^x q^{n-x-1} (x-np)^r + \sum_{x=0}^{n} {}^nC_x (x-np)^r x p^{x-1} q^{n-x} - nr\sum_{x=0}^{n} (x-np)^{r-1}$$

$${}^nC_r p^x q^{n-x}$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x-np)^r [-(n-x)q^{-1} + xp^{-1}] - nr\,\mu_{r-1}$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x-np)^r \left[-\frac{n-x}{q} + \frac{x}{p}\right] - nr\,\mu_{r-1}$$

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x-np)^r \left[\frac{-(n-x)p + xq}{pq}\right] - nr\,\mu_{r-1}$$

$$pq\frac{d\mu_r}{dp} = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x-np)^r (-np + xp + xq) - npqr\,\mu_{r-1}$$

$$pq\frac{d\mu_r}{dp} = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x-np)^r (-np + x(q+p)) - npqr\,\mu_{r-1}$$

$$pq\frac{d\mu_r}{dp} = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x-np)^r (x-np) - npqr\,\mu_{r-1}$$

$$pq\frac{d\mu_r}{dp} = \sum_{x=0}^{n} (x-np)^{r+1}\, {}^nC_x p^x q^{n-x}$$

$-npq r \mu_{r-1}$

$$pq \frac{d\mu_r}{dp} = \mu_{r+1} - pq\, nr \mu_{r-1}$$

$$\mu_{r+1} = pq \frac{d\mu_r}{dp} + pq\, nr \mu_{r-1}$$

$$\mu_{r+1} = pq\left(nr \mu_{r-1} + \frac{d\mu_r}{dp} \right) \rightarrow (1)$$

First, Second and third moment :-

$$\mu_0 = E(x-\mu)^0$$

$$= \sum_{x=0}^{n} (x-\mu)^0 p(x) \quad \because (x-\mu)^0 = 1$$

$$= \sum_{x=0}^{n} 1 \cdot p(x)$$

$$\mu_0 = \sum_{x=0}^{n} p(x) = 1$$

$$\mu_1 = E(x-\mu)^1$$

$$= E(x) - \mu = \mu - \mu$$

$$\mu_1 = 0$$

For μ_2, put $r=1$ in (1)

$$\mu_2 = pq\left(n \cdot 1\, \mu_0 + \frac{d\mu_1}{dp}\right)$$

$$\mu_2 = pq\,(n \cdot 1 \cdot 1 + 0)$$

$$\mu_2 = npq$$

For μ_3, put $r=2$ in (1)

$$\mu_3 = pq\left(n \cdot 2\, \mu_1 + \frac{d\mu_2}{dp}\right)$$

$$\mu_3 = pq\left(2n(0) + \frac{d}{dp}(np(1\right.$$

$$\mu_3 = pq\,(0 + n(1-p) + np(0-1))$$

$$\mu_3 = pq(nq - np) = npq^2 - np^2q$$
$$= npq(q-p)$$

For μ_4, put $r = 3$ in (1)

$$\mu_4 = pq\left(n\cdot 3\cdot \mu_2 + \frac{d}{dp}(\mu_3)\right)$$

$$= pq\left(3n(npq) + \frac{d}{dp}(np(1-p)(1-p-p\right.$$

$$= pq\left(3n^2pq + n\frac{d}{dp}(p(1-p)(1-2p))\right)$$

$$= 3n^2p^2q^2 + n\{(1-p)(1-2p) + p(0-1)$$
$$(1-2p) + p(1-p)(0-2))$$

$$\mu_4 = 3n^2p^2q^2 + n[(1-p)(1-p-p) - p(1-p-p)$$
$$-2p(1-p)]$$

$$\mu_4 = 3n^2p^2q^2 + n[q(q-p) - p(q-p)$$
$$-2pq]$$

$$\mu_4 = 3n^2p^2q^2 + nq(q-p) - np(q-p)$$
$$-2npq$$

$$= 3n^2p^2q^2 + nq^2 - npq - npq + np^2$$
$$-2npq$$

$$= 3n^2p^2q^2 - 4npq + n(p^2+q^2)$$

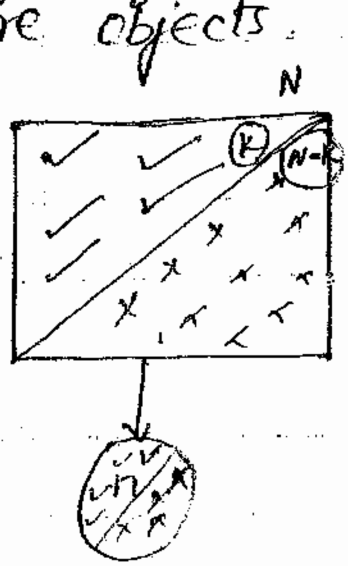
$$\mu_4 = npq(3npq - 4) + n(p^2+q^2)$$

Hyperqeometric Distribution:-

Hypergeometric probability distribution can be used to model a random variable defined on a random experiment possessing the following properties.

1) outcomes can be classified in two mutually exclusive categories.
2) fixed number of trials.
3) Successive trials are dependent.
4) prob of success changes for various trials. (without replacement selection)

Suppose, we have a population of "N" objects. These objects are classified in two mutually exclusive categories (groups) consisting of "k" success and "N-k" failure objects. If we select "n" objects without replacement and "x" is the random variable defined as the no. of objects from the success category selected in the



sample, then

$$P(X=x) = \frac{{}^{k}C_x \; {}^{N-k}C_{n-x}}{{}^{N}C_n}, \quad x=0,1,2,\ldots, \min(n,k)$$

N, n and k are the parameters of hypergeometric distribution.

Example:-

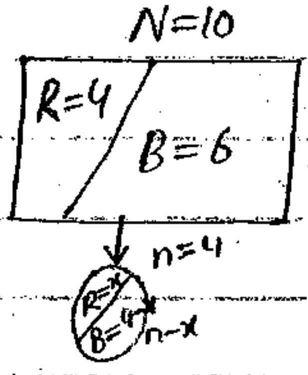
An urn contains 4 red balls and 6 black balls. A sample of 4 balls is selected from this urn.

(a) without replacement

(b) with replacement

Let X be the no. of red balls in the sample then find p.d of X.

Sol:-



$N=10$, $n=4$, $k=4$ (no. of red balls)

(a) without replacement:

$$P(X=x) = \frac{{}^{4}C_x \; {}^{6}C_{4-x}}{{}^{10}C_4} ; \quad x=0,1,2,\ldots \min(n,k)$$

$$x = 0,1,2,3,4$$

The probability distribution is

$$P(X=x) = \frac{{}^3C_x \, {}^4C_{6-x}}{{}^7C_6} \; ; \quad x = 2, 3$$

) 1 daffodil bulb

$P(X=1) = 0$ because $x = 1$ is impossible event

) 3 tulip bulbs

$P(3 \text{ tulip}) = P(3 \text{ daffodil}) \Rightarrow ; k = 3$

$$P(X=3) = \frac{{}^3C_3 \, {}^4C_3}{{}^7C_6} = \frac{1 \cdot 4}{7}$$

$$P(x=3) = 4/7$$

iii) At least 1 tulip:

$P(\text{at least } 1 \text{ tulip}) = P(1 \text{ tulip}) + P(2 \text{ tulip})$
$+ P(3 \text{ tulip}) + P(4 \text{ tulip}) + P(5 \text{ tulip}) + P(6 \text{ tulip})$

$P(\text{at least } 1 \text{ tulip}) = P(5 \text{ daffodil}) + P(4 \text{ dalip})$
$+ P(3 \text{ daffodil}) + P(2 \text{ daffodil}) +$
$P(1 \text{ daffodil}) + P(0\text{-daffodil})$

$$= P(X=5) + P(X=4) + P(X=3)$$
$$+ P(X=2) + P(X=1) + P(X=0)$$
$$= 0 + 0 + \frac{{}^3C_3 \, {}^4C_3}{{}^7C_6} + \frac{{}^3C_2 \, {}^4C_4}{{}^7C_6} + 0 + 0$$

$$P(\text{at least } 1 \text{ tulip}) = \frac{4}{7} + \frac{3}{7} = \frac{7}{7}$$

$$P(\text{at least } 1 \text{ tulip}) = 1$$

Alternative :-

(ii) Y : No of tulip bulbs selected in the sample.

$k = 4$

$$P(Y=y) = \frac{{}^4C_y \, {}^3C_{6-y}}{{}^7C_6} \quad ; \quad y = 3, 4$$

$$P(Y=3) = \frac{{}^4C_3 \, {}^3C_3}{{}^7C_6} = \frac{4}{7}$$

(iii) At least 1 tulip:

$$P(Y \geq 1) = P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4)$$

$$= \frac{{}^4C_1 \, {}^3C_5}{{}^7C_6} + \frac{{}^4C_2 \, {}^3C_4}{{}^7C_6} + \frac{{}^4C_3 \, {}^3C_3}{{}^7C_6} + \frac{{}^4C_4 \, {}^3C_2}{{}^7C_6}$$

$$= 0 + 0 + \frac{4 \cdot 1}{7} + \frac{1 \cdot 3}{7}$$

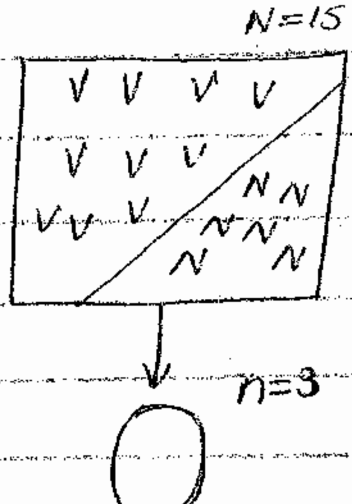
$$P(Y \geq 1) = \frac{4+3}{7} = 1$$

Example :-

To avoid detection at customs, a traveller placed 6 norcotics tablets in a bottle containing 9 vitamin pills that are similar in appearance. If the customs official selects 3 of the tablets at random for analysis, what is the prob that the traveller will be arrested for illegal possession of norcotics?

Sol:-

$N = 15, \quad n = 3$



X: No. of norcotics tablets selected in the sample.

$k = 6$

$$P(X = x) = \frac{{}^{6}C_x \, {}^{9}C_{3-x}}{{}^{15}C_3} \quad ; x = 0, 1, 2, 3$$

P(being arrested) = P(at least 1 norcotics appears) = $P(X \geq 1)$

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{{}^{6}C_1 \, {}^{9}C_2}{{}^{15}C_3} + \frac{{}^{6}C_2 \, {}^{9}C_1}{{}^{15}C_3} + \frac{{}^{6}C_3 \, {}^{9}C_0}{{}^{15}C_3}$$

$$= \frac{6 \cdot 36}{455} + \frac{15 \cdot 9}{455} + \frac{20 \cdot 1}{455}$$

$$P(X \geq 1) = \frac{371}{455}$$

or

P(being arrested) = 1 − P(not being)

$$= 1 - P(X = 0)$$

$$= 1 - \frac{{}^{6}C_0 \, {}^{9}C_3}{{}^{15}C_3}$$

$$= 1 - \frac{1 \cdot 84}{455} = \frac{371}{455}$$

Properties of hypergeometric distribution.

1) Mean of hypergeometric distribution is

Mean of $h(x; N, n, k) = \frac{nk}{N}$

Proof:-

By definition

$$\text{Mean} = E(x)$$

$$= \sum_x x\,P(x)$$

In case of hypergeometric dist $h(x; N, n, k)$

$$\text{Mean} = \sum_{x=0}^{n} x \frac{{}^{k}C_{x}\ {}^{N-k}C_{n-x}}{{}^{N}C_{n}}$$

$$\text{Mean} = \sum_{x=1}^{n} x \frac{\frac{k}{x}\ {}^{k-1}C_{x-1}\ {}^{N-k}C_{n-x}}{{}^{N}C_{n}}$$

$$= k \sum_{x=1}^{n} \frac{1\ {}^{k-1}C_{x-1}\ {}^{N-k}C_{n-x}}{{}^{N}C_{n}}$$

Let $x-1 = y$

$$= k \sum_{y=0}^{n-1} \frac{{}^{k-1}C_{y-1}\ {}^{N-k}C_{n-1-y}}{{}^{N}C_{n}}$$

$$= k \sum_{y=0}^{n-1} \frac{{}^{k-1}C_{y-1}\ {}^{N-k}C_{n-1-y}}{\frac{N}{n}\ {}^{N-1}C_{n-1}}$$

$$= \frac{nk}{N} \sum_{y=0}^{n-1} \frac{{}^{k-1}C_{y}\ {}^{N-k}C_{n-1-y}}{{}^{N-1}C_{n-1}}$$

$$\text{Mean} = \frac{nk}{N}$$

2)

variance of hypergeometric distt

$$h(x; N, n, k) = \frac{nk(N-k)}{N^2}\left(\frac{N-n}{N-1}\right)$$

Proof:-

$$\text{variance} = E(x^2) - [E(x)]^2 \rightarrow (1)$$

we have $E(x) = \frac{nk}{N}$

$$E(x^2) = \sum_{x=0}^{n} x^2 p(x)$$

$$= \sum_{x=0}^{n} [x(x-1) + x] p(x)$$

$$= \sum_{x=0}^{n} x(x-1)p(x) + \sum_{x=0}^{n} x p(x)$$

$$= \sum_{x=0}^{n} x(x-1)p(x) + E(x)$$

$$E(x^2) = \sum_{x=0}^{n} x(x-1) \frac{{}^{k}C_{x} \; {}^{N-k}C_{n-x}}{{}^{N}C_{n}} + \frac{nk}{N}$$

$$= \sum_{x=2}^{n} x(x-1) \frac{\frac{k(k-1)}{x(x-1)} \; {}^{k-2}C_{x-2} \; {}^{N-k}C_{n-x}}{{}^{N}C_{n}} + \frac{nk}{N}$$

$$= k(k-1) \sum_{x=2}^{N} \frac{{}^{k-2}C_{x-2} \; {}^{N-k}C_{n-x}}{\frac{N(N-1)}{n(n-1)} \; {}^{N-2}C_{n-2}} + \frac{nk}{N}$$

$$= \frac{k(k-1)\, n(n-1)}{N(N-1)} \sum_{x=2}^{N} \frac{{}^{k-2}C_{x-2} \; {}^{N-k}C_{n-x}}{{}^{N-2}C} + \frac{nk}{N}$$

$$= \frac{nk(k-1)(n-1)}{N(N-1)} \cdot 1 + \frac{nk}{N}$$

$$E(x^2) = \frac{nk}{N} \frac{(k-1)(n-1)}{N-1} + \frac{nk}{N}$$

put in (1)

$$\text{variance} = \frac{nk}{N} \frac{(k-1)(n-1)}{N-1} + \frac{nk}{N} - \frac{n^2k^2}{N^2}$$

$$= \frac{Nnk(k-1)(n-1) + nkN(N-1) - n^2k^2(N-1)}{N^2(N-1)}$$

$$= \frac{nk}{N^2(N-1)} [N(k-1)(n-1) + N(N-1) - nk(N-1)]$$

$$= \frac{nk}{N^2(N-1)} [(Nk-N)(n-1) + N^2 - N - nkN + nk]$$

$$= \frac{nk}{N^2(N-1)} [\cancel{nNk} - nN - Nk + \cancel{N} + N^2 - \cancel{N} - \cancel{nkN} + nk]$$

$$= \frac{nk}{N^2(N-1)} [+N(N-n) - k(N-n)]$$

$(N^2 - nN) + (Nk - nk)$

$$= \frac{nk}{N^2(N-1)} [(N-n)(N-k)]$$

$$\text{variance} = \frac{nk(N-k)}{N^2} \left(\frac{N-n}{N-1}\right)$$

Note:-

when $N \to \infty$ then $p \to \frac{k}{N}$;

$WOR \simeq WR$

WOR : $\frac{1}{N}, \frac{1}{N-1}, \frac{1}{N-2}, \ldots$

WR : $\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots$

If $N \to \infty$ then $\frac{1}{N-k} \simeq \frac{1}{N}$ where k is finite.

then

$$\text{Mean} = \frac{nk}{N} \to np$$

$$\text{variance} = \frac{nk}{N^2}(N-k)\left(\frac{N-n}{N-1}\right)$$

$$= n\frac{k}{N}\left(1-\frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$$

$$= np(1-p)\left(\frac{N-n}{N-1}\right)$$

$$\text{variance} = npq\left(\frac{N-n}{N-1}\right)$$

"If N is very large and n is small then $\frac{N-n}{N-1} \simeq 1$" $N = 100,000$, $n = 5$

$$\text{variance} = npq\,(1)$$

$$\text{variance} = npq$$

Example:-

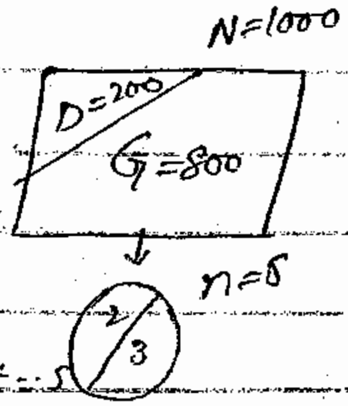
A box contains 1000 bolts of which 200 are defective and 800 are non-defective. If 5 bolts are selected at random, what is the prob. that it contains exactly 2 defective?

Sol:-

$N = 1000, \quad n = 5$

X : no. of defective

$k = 200$



$$P(X=x) = \frac{{}^{200}C_x \; {}^{800}C_{5-x}}{{}^{1000}C_5} \; ; x = 0, \ldots, 5$$

$$P(X=2) = \frac{{}^{200}C_2 \; {}^{800}C_{5-2}}{{}^{1000}C_5}$$

$$P(X=2) = \frac{19900 \cdot 85013600}{8.25 \times 10^{12}}$$

$$P(X=2) = \frac{1.69 \times 10^{12}}{8.25 \times 10^{12}} = 0.20$$

Using binomial distribution.

or. $p = \frac{k}{N} = \frac{200}{1000} = 0.2$

$q = 0.8$

$$P(X=x) = {}^5C_x (0.2)^x (0.8)^{5-x} \; ; x = 0, 1, \ldots, 5$$

$$P(X=2) = {}^5C_2 (0.2)^2 (0.8)^3$$

$= 10 \times 0.04 \times 0.512$

$P(X=2) = 0.20$

Property 3:-

If N becomes infinitely large then hypergeometric p.d tends to the binomial p.d.

i.e. $\lim_{N\to\infty} h(x;N,n,k) \to b(x;n,p)$

where $p = \frac{k}{N}$

Proof:-

Take hypergeometric p.d

$$h(x;N,n,k) = \frac{{}^{k}C_x \; {}^{N-k}C_{n-x}}{{}^{N}C_n}$$

$$= \frac{\frac{k!}{x!(k-x)!} \; \frac{(N-k)!}{(n-x)!(N-k-n+x)!}}{\frac{N!}{n!(N-n)!}}$$

$$= \frac{k!(N-k)!\,n!\,(N-n)!}{x!(k-x)!(n-x)!(N-k-n+x)!\,N!}$$

$$= \frac{n!}{x!(n-x)!} \; \frac{k!(N-k)!(N-n)!}{(k-x)!(N-k-n+x)!N!}$$

put $k = Np$, as $q = 1-p$

$q = 1 - \frac{k}{N}$

$q = \frac{N-k}{N} \Rightarrow qN = N-k$

$$= {}^nC_x \frac{(Np)!\,(Nq)!\,(N-n)!}{(Np-x)!\,(Nq-n+x)!\,N!} \rightarrow (1)$$

$$= {}^nC_x\, Np(Np-1)(Np-2)\cdots(Np-x+1)\cancel{(Np-x)!}$$
$$Nq(Nq-1)(Nq-2)\cdots(Nq-n+x+1)\cancel{(Nq-n+x)!}$$
$$\cancel{(N-n)!} \Big/ \cancel{(Np-x)!}\,\cancel{(Nq-n+x)!}\,N(N-1)(N-$$
$$\cdots(N-n+1)\cancel{(N-n)!}$$

$$= \frac{{}^nC_x\, \cancel{N}p \cdot \cancel{N}(p-\frac{1}{N}) \cdot \cancel{N}(p-\frac{2}{N}) \cdots \cancel{N}(p-\frac{x+1}{N})\; \cancel{N}q\, \cancel{N}(q-\frac{1}{N})\cancel{N}(q-\frac{2}{N})\cdots \cancel{N}(q-\frac{n+x+1}{N})}{\cancel{N} \cdot \cancel{N}(1-\frac{1}{N}) \cdot \cancel{N}(1-\frac{2}{N}) \cdots \cancel{N}(N-\frac{n+1}{N})}$$

$$\lim_{N\to\infty} h(x; N, n, k) = \lim_{N\to\infty} {}^nC_x\, p(p-\tfrac{1}{N})(p-\tfrac{2}{N})\cdots$$
$$(p-\tfrac{x+1}{N})\,q(q-\tfrac{1}{N})(q-\tfrac{2}{N})\cdots$$
$$(q-\tfrac{n+x+1}{N}) \Big/ (1-\tfrac{1}{N})(1-\tfrac{2}{N})\cdots(1-\tfrac{n+1}{N})$$

$$= \frac{{}^nC_x\, p(p-\frac{1}{\infty})(p-\frac{2}{\infty})\cdots(p-\frac{(x-1)}{\infty})\; q(q-\frac{1}{\infty})(q-\frac{2}{\infty})\cdots(q-\frac{(n-x-1)}{\infty})}{(1-\frac{1}{\infty})(1-\frac{2}{\infty})\cdots(1-\frac{(n-1)}{\infty})}$$

$$= {}^nC_x\, p(\underbrace{p \cdot p \cdots p}_{(x-1)}) \cdot q(\underbrace{q \cdot q \cdots q}_{n-x-1})$$

$$= {}^nC_x\, p^x q^{n-x} \quad \text{proved}$$

R) from (1)

$$h(x; N, n, k) = \frac{(Np)!\,(Nq)!\,(N-n)!}{(Np-x)!\,(Nq-n+x)!\,N!} \cdot {}^nC_x$$

put $n! = e^{-n} n^n \sqrt{2\pi n}$ (By Stirling's Approximation)

$$= {}^nC_x\, e^{-Np} (Np)^{Np} \sqrt{2\pi Np}\, e^{-Nq} (Nq)^{Nq} \sqrt{2\pi Nq}\; e^{-(N-n)} (N-n)^{N-n} \sqrt{2\pi (N-n)} \Big/ e^{-(Np-x)} (Np-x)^{Np-x} \sqrt{2\pi(Np-x)}\; e^{-(Nq-n+x)} (Nq-n+x)^{Nq-n+x} \sqrt{2\pi(Nq-n+x)} \cdot e^{-N} N^N \sqrt{2\pi N}$$

$$= \frac{{}^nC_x\, e^{-Np-Nq-N+n-Np+x-Nq-n+x+N}\, N^{Np+Nq+N-N-n-Np+x-Nq+n-x}\, p^{Np} q^{Np} \left(1-\frac{n}{N}\right)^{N-n} p^{1/2} q^{1/2} \left(1-\frac{n}{N}\right)^{1/2}}{\left(p-\frac{x}{N}\right)^{1/2} \left(p-\frac{x}{N}\right)^{Np-x} \left(q-\left(\frac{n-x}{N}\right)\right)^{1/2} \left(q-\frac{n-x}{N}\right)^{Nq-n+x}}$$

$$= \frac{{}^nC_x\, e^0 N^0\, p^{Np+1/2}\, q^{Np+1/2} \left(1-\frac{n}{N}\right)^{N-n+1/2}}{p^{Np-x+\frac{1}{2}}\, q^{Nq-n+x+1/2} \left(1-\frac{x}{Np}\right)^{Np-x+1/2} \left(1-\frac{n-x}{Nq}\right)^{Nq-n-x+1/2}}$$

$$= \frac{{}^nC_x \left(1-\frac{n}{N}\right)^{N-n+\frac{1}{2}}}{p^{-x} q^{-n+x} \left(1-\frac{x}{Np}\right)^{Np-x+1/2} \left(1-\frac{n-x}{Nq}\right)^{Nq-n-x+1/2}}$$

As $N \to \infty$

$$= \frac{{}^nC_x\, (1-0)^{\infty-n+1/2}\, p^x q^{n-x}}{(1-0)^{\infty-x+1/2}\, (1-0)^{\infty-n-x+1/2}}$$

$$= {}^nC_x\, p^x q^{n-x} = b(x; n, p)$$

$$\Rightarrow \lim_{N\to\infty} h(x; N, n, k) = b(x; n, p)$$

Binomial

Example:-

In a binomial distribution, mean = 36 and $\sigma = 4.8$. Find the probability of $x = 2$; $P(x=2)$

Sol:-

$$E(x) = 36$$

$$np = 36 \rightarrow (1)$$

$$\text{variance} = npq$$

$$\sigma = \sqrt{npq}$$

$$4.8 = \sqrt{npq}$$

Taking square

$$23.04 = npq \rightarrow (2)$$

Divide (2) by (1)

$$\frac{npq}{np} = \frac{23.04}{36}$$

$$q = 0.64$$

$$p = 1 - q = 1 - 0.64$$

$$p = 0.36$$

put in (1)

$$n(0.36) = 36$$

$$n = 100$$

Binomial probability distribution

is

$$P(X=x) = {}^nC_x\, p^x q^{n-x} \quad ; x=0,1,2,\ldots 100$$

$$P(X=2) = {}^{100}C_2\,(0.36)^2(0.64)^{98}$$

$$= 4950(0.1296)(1.0\times10^{-19})$$

$$P(X=2) = 6.4990\times10^{-17}$$

$$P(X=2) = 0$$

ii) P(at least 2)

$$P(X\geq2) = P(X=2)+P(X=3)+\cdots+P(X=100)$$

$$P(X\geq2) = 1-\left[P(X=0)+P(X=1)\right] \rightarrow (1)$$

$$P(X=0) = {}^{100}C_0\,(0.36)^0(0.64)^{100}$$

$$= 1\times1\times4.1495\times10^{-20}$$

$$P(X=0) = 4.15\times10^{-20} = 0$$

$$P(X=1) = {}^{100}C_1\,(0.36)^1(0.64)^{99}$$

$$= 100\times0.36\times6.484\times10^{-20}$$

$$= 2.33\times10^{-18}$$

$$P(X=1) = 0$$

put in (1)

$$P(X\geq2) = 1-(0+0)$$

$$= 1-0$$

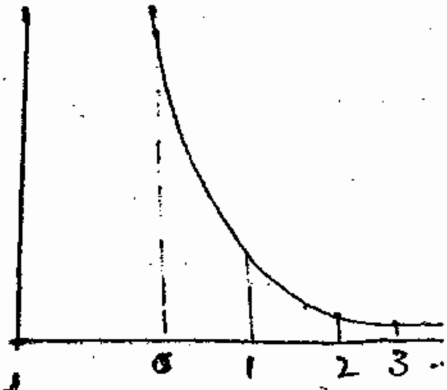
$$P(X\geq2) = 1$$

Poisson Distribution:-

(Model of rare events)

If X is a poisson random variable, then

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; x = 0, 1, \cdots, \infty$$



where λ is mean of poisson distribution and its only parametre.

Example:-

If X follows a poisson distribution with parametre $\lambda = 2$, find the p.d of X.

Sol:-

Poisson distribution is

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; x = 0, 1, 2, \cdots$$

$$P(X=x) = \frac{e^{-2}2^{x}}{x!}$$

x	$P(X=x)$
0	$\frac{e^{-2}2^{0}}{0!} = 0.1353$
1	$\frac{e^{-2}2^{1}}{1!} = 0.2706$
2	0.2706

x	$p(x) =$
3	$e^{-2}2^3/3! = 0.1806$
4	$e^{-2}2^4/4! = 0.0902$
5	$e^{-2}2^5/5! = 0.0361$
6	$e^{-2}2^6/6! = 0.0120$
7	$e^{-2}2^7/7! = 0.0034$
8 or more	$1-[\text{all about}] = 0.0014$
	1

Example:-

Assume that the prob of being killed in an accident in a coal mine during a year is $\frac{1}{1400}$. Use the poisson dist to calculate the prob that in the mine employing 350 miners, there will be at least one fatal accident in a year.

$$\left(\lim_{\substack{n\to\infty \\ p\to 0}} b(x; n; p) = \text{Poisson}(x; np) \right)$$

Sol:-

$p = P(\text{getting killed in a mine accident}) = \frac{1}{1400} = 0.0007$

$n = 350$

$\lambda = np = 350(0.0007) = 0.245$

So, Poisson distribution is (Approxi-

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad ; x = 0, 1, 2, \dots$$

$$P(X=x) = \frac{e^{-0.245}(0.245)^x}{x!}, \quad x = 0,1,2, \ldots 350$$

$$P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) + \cdots + P(X=350)$$

or

$$P(X \geq 1) = 1 - P(X=0) \rightarrow (1)$$

$$P(X=0) = \frac{e^{-0.245}(0.245)^0}{0!} = 0.7827$$

$$P(X \geq 1) = 1 - 0.7827$$

$$P(X \geq 1) = 0.2173$$

So Binomial distribution is (true)

$$P(X=x) = {}^{n}C_x \, p^x q^{n-x}$$

$$P(X=x) = {}^{350}C_x \, (0.0007)^x (0.9993)^{350-x} \quad ; x = 0,1,2, \ldots, 350$$

$$P(X \geq 1) = 1 - P(X=0) \rightarrow (2)$$

$$P(X=0) = {}^{350}C_0 \, (0.0007)^0 (0.9993)^{350}$$

$$P(X=0) = 1 \cdot 1 \cdot 0.7826 = 0.7826$$

$$P(X \geq 1) = 1 - 0.7826$$

$$P(X \geq 1) = 0.2174$$

Note:-

The approximation will start to disturb when n becomes small and p becomes large, then approximation will be poor.

Example:-

Suppose that customers enter a shop at the rate of 30 persons an hour. Find the prob that no customer will enter the shop in a 3-minute interval.

Sol:-

$\lambda = 30$ persons an hour

$= \frac{30}{60} \times 3$ persons per 3-minutes

$\lambda = 1.5$ person / 3-minutes.

So, Poisson distribution is

$$P(X = x) = \frac{e^{-1.5}(1.5)^x}{x!} \quad ; x = 0, 1, 2, \ldots$$

$$P(X = 0) = \frac{e^{-1.5}(1.5)^0}{0!} = 0.223$$

Note:-

$$\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x = e^{-1}$$

Properties of Poisson distribution:-

1) Show that $\lim_{\substack{n \to \infty \\ p \to 0}} b(x; n, p) = P_o(x; \lambda)$ where $\lambda = np$.

Proof:-

Consider the binomial model

$$b(x; n, p) = {}^nC_x \, p^x q^{n-x}$$

As $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$$q = 1 - p = 1 - \frac{\lambda}{n}$$

So

$$b(x;n,p) = {}^{n}C_x \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \left[\frac{n!}{(n-x)!}\, n^{-x} \left(1-\frac{\lambda}{n}\right)^{n-x}\right.$$

$$= \frac{\lambda^x}{x!} \frac{n(n-1)(n-2)\cdots(n-x+1)(n-x)!}{(n-x)!}$$

$$n^{-x} \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\; n \cdot n\left(1-\frac{1}{n}\right) \cdot n\left(1-\frac{2}{n}\right) \cdots n\left(1-\frac{(x-1)}{n}\right)$$

$$n^{-x} \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\, n^{1} \cdot n^{x-1} \cdot n^{-x} \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)$$

$$= \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)$$

Applying $n \to \infty$, $p \to 0$ (such that $\lambda = np$ remains constant)

$$\lim_{\substack{n\to\infty \\ p\to 0}} b(x;n,p) = \frac{\lambda^x}{x!} \lim_{\substack{n\to\infty \\ p\to 0}} \left(1-\frac{\lambda}{n}\right)^n \cdot 1 \cdot 1 \cdots 1$$

$$= \frac{\lambda^x}{x!} e^{-\lambda} \quad ; \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$= \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\lim_{\substack{n\to\infty \\ p\to 0}} b(x;n,p) = P_0(x;\lambda)$$

2) Mean of Poisson distribution $P_0(x;\lambda)$ is λ

Proof:-

By definition

$$\text{Mean} = E(x)$$

$$= \sum_{x=0}^{\infty} x\, p(x)$$

Poisson distribution is

$$f(x) = p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

then

$$E(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x(x-1)!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^{x-1} \cdot \lambda}{(x-1)!}$$ or put $x-1=y$

$$\Rightarrow \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^{y}}{y!} = \lambda$$

$$= \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!}$$

Since, $\sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^{x}}{x!} = 1$

So $E(x) = \lambda . 1$

$\Rightarrow E(X) = \lambda$

3)

Variance of poisson dist $P_0(x;\lambda)$ is λ

Proof:-

variance $= E(x^2) - [E(x)]^2 \rightarrow (1)$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty} (x(x-1)+x)p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)p(x) + \sum_{x=0}^{\infty} xp(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!} + E(x)$$

$$= \sum_{x=2}^{\infty} \cancel{x(x-1)} \frac{e^{-\lambda}\lambda^{x}}{\cancel{x(x-1)}(x-2)!} + \lambda$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2} \lambda^2}{(x-2)!} + \lambda$$

$$= \lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2}}{(x-2)!} + \lambda$$

as $x-2 = y$
at $x=2$
$y=0$
at $x=\infty$
$y=\infty$.

put $x-2 = y$

$$= \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^{y}}{y!} + \lambda$$

$$= \lambda^2 \cdot 1 + \lambda$$

$$E(x^2) = \lambda^2 + \lambda$$

put in (1)

$$\text{variance} = \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\text{variance} = \lambda$$

Recurrence Rule for poisson Probabilities

Suppose, we have a poisson distt $P_0(x;\lambda)$ then

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

we can define

$$P(X=x+1) = \frac{e^{-\lambda} \lambda^{(x+1)}}{(x+1)!}$$

$$= \frac{e^{-\lambda} \lambda^{x} \cdot \lambda}{(x+1)\, x!}$$

$$P(X=x+1) = \left(\frac{\lambda}{x+1}\right) \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$\Rightarrow P(X=x+1) = \frac{\lambda}{(x+1)} P(X=x)$$

This rule relates to successive probabilities.

x	$p(x)$
0	$P(X=0)$
1	$\frac{\lambda}{1} P(X=0)$
2	$\frac{\lambda}{2} P(X=1)$
3	$\frac{\lambda}{3} P(X=2)$ and so on.

Moment Generating function:-

The m.g.f of Poisson dist is

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

Poisson dist is $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$$M_x(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

put $\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{\lambda e^t}$

$$= e^{-\lambda} e^{\lambda e^t}$$

As $\sum_{x}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$

$$M_x(t) = e^{-\lambda+\lambda e^t}$$

$\frac{\lambda^x}{x!} = e^{\lambda}$

$$M_x(t) = e^{\lambda(e^t-1)}$$

Now,

$$\mu_1' = \frac{d}{dt} M_x(t)\Big|_{t=0}$$

$$= \frac{d}{dt} (e^{\lambda(e^t-1)}\Big|_{t=0}$$

$$= e^{\lambda(e^t-1)} \cdot \lambda e^t \Big|_{t=0}$$

$$= e^{\lambda(e^0-1)} \cdot \lambda e^0 = e^{\lambda(1-1)} \cdot \lambda$$

$$= e^0 \cdot \lambda \cdot 1 = 1 \cdot \lambda \cdot 1$$

$$\mu_1' = \lambda$$

Similarly, $\mu_2' = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0}$

$$\mu_2' = \frac{d}{dt} \left(\frac{d}{dt} M_x(t)\right)\Big|_{t=0} = \frac{d}{dt} (\mu_1')\Big|_{t=0}$$

$$\mu_2' = \frac{d}{dt}(e^{\lambda(e^t-1)} \cdot \lambda e^t)\Big|_{t=0}$$

$$\mu_2' = (e^{\lambda(e^t-1)} (\lambda e^t)^2 + e^{\lambda(e^t-1)} \lambda e^t \Big|_{t=0}$$

$$= e^{\lambda(e^0-1)} (\lambda e^0)^2 + e^{\lambda(e^0-1)} \lambda e^0$$

$$= e^{\lambda(1-1)} \lambda^2 + e^{\lambda(1-1)} \lambda$$

$$\mu_2' = e^0 \lambda^2 + e^0 \lambda = \lambda^2 + \lambda$$

$$\mu_2' = \lambda^2 + \lambda$$

Comulant Generating function:-

The c.g.f is defined as

$$k_x(t) = \log_e M_x(t)$$
$$= \log_e e^{\lambda(e^t-1)}$$

$$k_x(t) = \lambda(e^t - 1)$$

$$k_x(t) = \lambda\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots - 1\right)$$

$$k_x(t) = \lambda\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$

As k_r = co-efficient of $\frac{t^r}{r!}$

Therefore

$$k_1 = k_2 = k_3 = \cdots = \lambda$$

Ex:-

Show that for Poisson dist. $P_0(x;\lambda)$

$$\mu_{r+1} = r\lambda\mu_{r-1} + \lambda\frac{d\mu_r}{d\lambda}$$

where μ_r is the rth central moment

Sol:-

By definition

$$\mu_r = E(X - \text{Mean})^r \; ; \; r = 0, 1, 2, 3, \cdots$$

$$= \sum_{x} (x-\lambda)^r p(x)$$

$$\mu_r = \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}\lambda^x}{x!}$$

Differentiate w.r.t 'λ'.

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{1}{x!} \left\{ (x-\lambda)^r e^{-\lambda} x\lambda^{x-1} \right\} (x-\lambda)^r e^{-\lambda}(-1) + \lambda^x + r(x-\lambda)^{r-1}(-1) e^{-\lambda}\lambda^x \Big]$$

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{1}{x!} (x-\lambda)^{r-1} e^{-\lambda} \lambda^{x-1} \left[x(x-\lambda) - (x-\lambda)\lambda - r\lambda\right]$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^{r-1} \frac{e^{-\lambda}\lambda^{x-1}}{x!} \left[x^2 - \lambda x - \lambda x + \lambda^2 - r\lambda\right]$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^{r-1} \frac{e^{-\lambda}\lambda^{x-1}}{x!} \left[(x-\lambda)^2 - r\lambda\right]$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^{r-1+2} \frac{e^{-\lambda}\lambda^{x-1}}{x!} - r\sum_{x=0}^{\infty} (x-\lambda)^{r-1} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \frac{1}{\lambda} \sum_{x=0}^{\infty} (x-\lambda)^{r+1} \frac{e^{-\lambda}\lambda^x}{x!} - r\sum_{x=0}^{\infty} (x-\lambda)^{r-1} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\frac{d\mu_r}{d\lambda} = \frac{1}{\lambda}\mu_{r+1} - r\mu_{r-1}$$

$$\Rightarrow \quad \mu_{r+1} = \lambda\left[\frac{d\mu_r}{d\lambda} + r\mu_{r-1}\right]$$

$$\mu_{r+1} = \lambda \frac{d\mu_r}{d\lambda} + r\lambda\mu_{r-1}$$

Central moments of Poisson distribution

we know $\mu_1 = 0$ (always)

$r=1$ $\quad \mu_2 = \mu_{r+1} = 1 \cdot \lambda \mu_{1-1} + \lambda \frac{d\mu_1}{d\lambda}$

$\Rightarrow \mu_2 = \lambda \mu_0 + \lambda (0)$

$\mu_2 = \lambda$ which is variance

for $r = 2$

$\mu_3 = 2 \cdot \lambda \mu_{2-1} + \lambda \frac{d}{d\lambda} \mu_2$

$\mu_3 = 2 \lambda \mu_1 + \lambda \frac{d}{d\lambda} (\lambda)$

$\mu_3 = 2 \cdot \lambda (0) + \lambda \cdot 1$

$\Rightarrow \mu_3 = \lambda$

for $r = 3$

$\mu_4 = 3\lambda \mu_{3-1} + \lambda \frac{d}{d\lambda} \mu_3$

$\mu_4 = 3\lambda \mu_2 + \lambda \frac{d}{d\lambda} (\lambda)$

$\mu_4 = 3\lambda \cdot \lambda + \lambda (1)$

$\mu_4 = 3\lambda^2 + \lambda$

Negative Binomial Distribution:-

Negative binomial distribution is used to model an experiment possessing the following properties.

1) outcomes can be classified in two mutually exclusive categories; namely Success and failure.

2) prob. of success remains constant.

3) successive trials are independent.

4) No. of trials is variable but no. of success is fixed.

If X is a random variable for no. of trials to achieve "k" Successes then

$$P(X=x) = {}^{x-1}C_{k-1}\, p^k q^{x-k} \; ; x = k, k+1, \ldots$$

Example:-

If fair coin is tossed repeatedly, what is the prob of obtaining total 3 heads on the fifth toss?

Sol:-

$$p = P(H) = \frac{1}{2} = 0.5 \;, \; q = 1-p$$

$$q = 1 - 0.5 = 0.5$$

$$k = 3$$

Negative Binomial Distribution is

$$P(X=x) = {}^{x-1}C_2\,(0.5)^3\,(0.5)^{x-3} \;;\; x=3,4,\ldots$$

$$P(X=5) = {}^4C_2\,(0.5)^3\,(0.5)^2$$

$$= 6\cdot(0.5)^5$$

$$= 6\cdot\left(\tfrac{1}{2}\right)^5 = \frac{6}{32} = \frac{3}{16}$$

$$P(X=5) = 0.1875$$

Ex:-

The prob. that a person will install a black telephone in a residence estimated to be 0.3. Find the prob that the 10th phone installed in the area is the 5th black telephone?

Sol:-

$p = P(\text{installing black-telephone})$

$p = 0.3$

$k = 5$

$$P(X=x) = {}^{x-1}C_{5-1}\,(0.3)^5\,(0.7)^{x-5}$$

$$P(X=10) = {}^9C_4\,(0.3)^5\,(0.7)^5 \;;\; x=5,6,7,8,\ldots$$

$$P(X=10) = 126 \times 0.00243 \times 0.16807$$

$$P(X=10) = 0.05$$

Geometric Distribution :-

For negative binomial distribution if $k=1$, then it is called geometric distribution.

Negative binomial distribution is

$$P(X=x) = {}^{x-1}C_{k-1}\, p^k q^{x-k} \quad ; x=k, k+1, \ldots$$

put $k=1$ $\quad P(X=x) = {}^{x-1}C_{1-1}\, p^1 q^{x-1} = pq^{x-1}$

So, $P(X=x) = pq^{x-1}$ is geometric distribution.

Note :-

Negative binomial p.d.f for of random variable x

$$P(X=x) = {}^{x-1}C_{k-1}\, p^k q^{x-k} \quad ; x=k, k+1, \ldots \rightarrow (1)$$

Equivalently:

$$P(X=x) = {}^{x+k-1}C_{k-1}\, p^k q^x \quad ; x=0,1,2,\ldots$$

for example

if $x=0$, $P(X=0) = {}^{k-1}C_{k-1}\, p^k q^0 = p^k$

if $x=k$, $P(X=k) = {}^{k-1}C_{k-1}\, p^k q^0 = p^k$

By putting in (1)

Properties :-

(Negative binomial dist)

1) Moment generating function :-

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx}\, p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx}\ {}^{x+k-1}C_{k-1}\ p^k q^x$$

$$= p^k \sum_{x=0}^{\infty} {}^{x+k-1}C_{k-1}\ (qe^t)^x$$

$$= p^k \left[{}^{k-1}C_{k-1} + {}^{k}C_{k-1}(qe^t) + {}^{k+1}C_{k} (qe^t)^2 + \cdots \right]$$

$$= p^k \left[1 + k(qe^t) + \frac{k(k+1)}{2!}(qe^t)^2 + \cdots \right]$$

$$M_x(t) = p^k \left[1 - qe^t\right]^{-k}$$

2) **Mean**

$$\text{Mean} = E(x)$$

$$= \frac{d}{dt}\left(M_x(t)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(p^k(1-qe^t)^{-k}\right)\Big|_{t=0}$$

$$= -k\,p^k(1-qe^t)^{-k-1}(-qe^t)\Big|_{t=0}$$

$$= -k\,p^k(1-qe^0)^{-k-1}(-qe^0)$$

$$= -k\,p^k(1-q)^{-k-1}(-q)$$

$$= k\,p^k\,p^{-k-1}q$$

$$= k\,p^{-1}q$$

$$\text{Mean} = \frac{kq}{p}$$

3) Variance

$$\text{variance} = E(x^2) - \{E(x)\}^2 \rightarrow (1)$$

$$E(X^2) = \frac{d^2}{dt^2}\left(M_x(t) \right|_{t=0}$$

$$= \frac{d}{dt}\left(E(X) \right) \Big|_{t=0}$$

$$= \frac{d}{dt}\left(-k p^k (1-qe^t)^{-k-1} (-qe^t) \right|_{t=0}$$

$$= \frac{d}{dt}\left(k p^k (1-qe^t)^{-k-1} qe^t \right|_{t=0}$$

$$= kqp^k \frac{d}{dt}\left((1-qe^t)^{-k-1} e^t \right|_{t=0}$$

$$= kqp^k \Big[(1-qe^t)^{-k-1} e^t + (-(k-1)) (1-qe^t)^{-k-2} (-qe^t) e^t \Big]_{t=0}$$

$$= kqp^k \left[(1-qe^0)^{-k-1} e^0 + (-k-1) (1-qe^0)^{-k-2} (-qe^0) e^0 \right]$$

$$= kqp^k \left[(1-q)^{-k-1} + (-k-1) (1-q)^{-k-2} (-q) \right]$$

$$= kqp^k \left[p^{-k-1} + (k+1) p^{-k-2} q \right]$$

$$= kqp^{k-k-1} + k(k+1) q^2 p^{k-k-2}$$

$$= kqp^{-1} + k(k+1) q^2 p^{-2}$$

$$= \frac{kq}{p} + \frac{k(k+1)q^2}{p^2}$$

Put in (1)

$$\text{variance} = \frac{kq}{p} + \frac{k(k+1)}{p^2}q^2 - \left(\frac{kq}{p}\right)^2$$

$$= \frac{kq}{p} + \frac{k(k+1)}{p^2}q^2 - \frac{k^2q^2}{p^2}$$

$$= \frac{kq}{p} + \frac{k^2q^2}{p^2} + \frac{kq^2}{p^2} - \frac{k^2q^2}{p^2}$$

$$= \frac{kq}{p} + \frac{kq^2}{p^2} = \frac{kpq + kq^2}{p^2}$$

$$\text{variance} = \frac{kq(p+q)}{p^2} = \frac{kq}{p^2}$$

Geometric Distribution:-

For negative binomial experiment ; if $k=1$ then

$$P(X=x) = {}^{x-1}C_{1-1}\, p^1 q^{x-1} ;\ x = 1, 2, \ldots$$

$$= {}^{x-1}C_0\, p^1 q^{x-1}$$

$$= pq^{x-1} ;\ x = 1, 2, \ldots$$

This special case of negative binomial dist. is called a geometric distribution with parametre 'p'.

Example:-

In flipping an unbaised coin, what is prob of obtaining first head on third toss?

Sol:-

$p = P(\text{Head}) = 1/2$

$q = 1 - p = 1/2 \quad ; \; k = 1$

Let X be the no. of tosses

$P(X = x) = pq^{x-1} \quad ; \; x = 1, 2, \ldots$

$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{x-1}$

$P(X = 3) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{2}$

$= 1/8$

Its probability distribution

x	$P(x)$	
1	$(1/2)^1 = 0.5$	
2	$(1/2)^2 = 0.25$	$0.94 = 94\%$
3	$\left(\frac{1}{2}\right)^3 = 0.125$	
4	$(1/2)^4 = 0.0625$	
5	$(1/2)^5 = 0.03125$	
6	$(1/2)^6 = 0.015625$	
7	$(1/2)^7 = 0.0078125$	6%
⋮	⋮	
∞		

Note:-

In negative binomial dist. last trial can never be failure.

CONTINUOUS DISTRIBUTIONS

Function $f(x)$ of a continuous random variable x is called probability density function. Following are properties of c.

1) $f(x_i) \geq 0$ for all x_i.

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Uniform Distribution:-

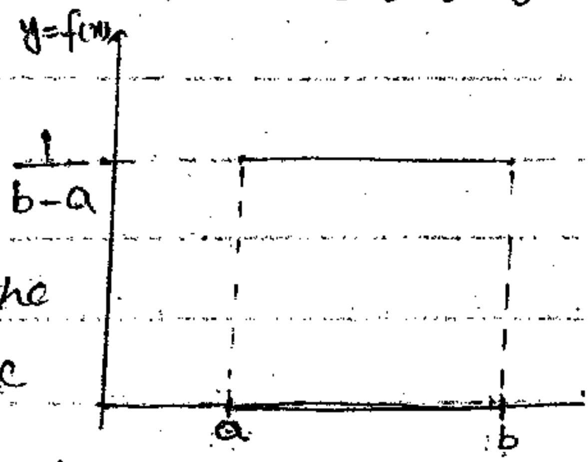
If we have a random variable X which follows a uniform distribution over all values of $x \in [a,b]$ and X is a continuous random variable, then the probability density function of X is

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \quad x \in [a,b] \\ 0 & \text{elsewhere} \end{cases}$$

we usually write $X \sim U[a,b]$

It is also called rectangular distribution.

"a" and "b" are the parametres of the



uniform distribution. It depends on value of "a" and "b"

properties :-

1) Mean

$$Mean = E(x)$$

$$= \int_x x\,p(x)\,dx = \int_x x\,f(x)\,dx$$

$$= \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b$$

$$= \frac{1}{2(b-a)} \left[b^2 - a^2\right]$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$Mean = \frac{1}{2}(b+a)$$

$$E(x) = \frac{a+b}{2}$$

2) Variance

$$variance = E(x^2) - \left[E(x)\right]^2 \rightarrow (1)$$

Take

$$E(x^2) = \int_a^b x^2 f(x)\,dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left.\frac{x^3}{3}\right|_a^b$$

$$= \frac{1}{3(b-a)} \left[b^3 - a^3\right]$$

$$= \frac{(b-a)(a^2+ab+b^2)}{3(b-a)}$$

$$E(x^2) = \frac{a^2+ab+b^2}{3}$$

put in (1)

$$\text{variance} = \frac{a^2+ab+b^2}{3} - \left[\frac{a+b}{2}\right]^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6a}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$\text{var} = \frac{(b-a)^2}{12}$$

3) m.g.f

$$M_x(t) = E(e^{tx})$$

$$= \int_x e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \frac{e^{tx}}{t}\Big|_a^b$$

$$M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$$

Now,

$$\mu_1' = \frac{d}{dt} M_x(t)\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{e^{bt} - e^{at}}{t(b-a)}\right|_{t=0}$$

$$= \frac{1}{b-a} \frac{d}{dt}\left[\frac{e^{bt} - e^{at}}{t}\right]_{t=0}$$

$$= \frac{1}{b-a}\left[\frac{t(be^{bt} - ae^{at}) - (e^{bt} - e^{at})}{t^2}\right|_{t=0}$$

$$= \frac{0}{0}$$ Using Del' Hospital Rule.

$$\mu_1' = \frac{1}{b-a}\left[\frac{(be^{bt} - ae^{at}) + t(b^2e^{bt} - a^2e^{at}) - (be^{bt} - ae^{at})}{2t}\right|_{t=0}$$

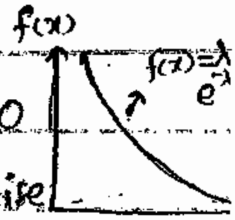
$$= \frac{1}{b-a}\left(\frac{t(b^2e^{bt} - a^2e^{at})}{2t}\right)\Big|_{t=0} = \frac{0}{0}$$

$$= \frac{1}{b-a}\left[\frac{(b^2e^{bt} - a^2e^{at}) + t(b^3e^{bt} - a^3e^{at})}{2}\right|_{t=0}$$

$$\mu_1' = \frac{1}{b-a}\left[\frac{b^2 - a^2 + 0}{2}\right] = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

Exponential Distribution:-

A random variable X is said to have an exponential dist if



$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{Otherwise} \end{cases}$$

where $0 < x < \infty$ or $x > 0$

and $\lambda > 0$ is the parameter of exponential dist.

we usually write it as $X \sim exp(\lambda)$

Properties:-

1) Mean

$$\text{Mean} = E(X)$$

$$= \int_x x f(x) dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x} dx$$

put

$$\lambda x = y$$

$$\Rightarrow \lambda dx = dy \quad , \quad x = \frac{y}{\lambda}$$

Then

$$\text{Mean} = \frac{1}{\lambda} \int_0^\infty y e^{-y} dy$$

Gemma - function is

$$\int_0^\infty x^n e^{-x}\, dx = \Gamma(n+1) = n! \rightarrow (a)$$

By (a)

$$E(x) = \frac{1}{\lambda}\Gamma(2) = \frac{1}{\lambda}(1!) = \frac{1}{\lambda}$$

2) variance

$$\text{variance} = E(x^2) - [E(x)]^2 \rightarrow (1)$$

$$E(x^2) = \int_x x^2 f(x)\, dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x}\, dx$$

put $y = \lambda x$

$dy = \lambda dx$

$x = y/\lambda$

$$E(x^2) = \frac{1}{\lambda^2}\int_0^\infty y^2 e^{-y}\, dy$$

$$= \frac{1}{\lambda^2}\Gamma(2+1)$$

$$E(x^2) = \frac{1}{\lambda^2} 2!$$

$$\text{variance} = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{2-1}{\lambda^2}$$

variance $= \frac{1}{\lambda^2}$

3) m.g.f

$$M_x(t) = E(e^{tx})$$

$$= \int_x e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$$M_x(t) = \lambda \int_0^\infty e^{-(\lambda-t)x} dx$$

Put $(\lambda - t)x = y$

$(\lambda - t)dx = dy$

$$M_x(t) = \frac{\lambda}{(\lambda-t)} \int_0^\infty e^{-y} dy$$

$$= \frac{\lambda}{\lambda-t} \overline{|0+1}$$

$$M_x(t) = \frac{\lambda}{\lambda-t} \quad \text{for } t < \lambda$$

Example:-

The duration of long distance calls is found to be exponentially distributed with mean duration of 3-minutes. What is the prob that a call will lost

(i) more than 3 minutes

(ii) less than 5 minutes

(iii) between 1 and 4 minutes.

Sol:-

Let x be the call duration in minutes.

$$X \sim exp(\lambda)$$

Given that Mean = 3 minutes

we know for exponential dist

$$Mean = \frac{1}{\lambda}$$

$$\therefore \quad 3 = \frac{1}{\lambda}$$

$$\Rightarrow \lambda = \frac{1}{3} = 0.333$$

$\therefore$

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & ; x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(i) $P(X>3) = \int_3^\infty f(x)\,dx$

$$= \int_3^\infty \frac{1}{3} e^{-x/3}\,dx$$

$$= \frac{1}{3} \frac{e^{-x/3}}{-\frac{1}{3}} \Big|_3^\infty$$

$$= -e^{-x/3} \Big|_3^\infty$$

$$= -(e^{-\infty} - e^{-3/3})$$

$$= -(0 - e^{-1})$$

$$= \frac{1}{e} = 0.3679$$

(ii) $P(X<5) = \int_0^5 \frac{1}{3} e^{-x/3}\,dx$

$$= -e^{-x/3} \Big|_0^5$$

$$= -(e^{-5/3} - e^{0})$$

$$= -(0.1889 - 1) = 0.8111$$

$P(X \leq 5) = P(X<5)$
$+ P(X=5$
$= P(X<5) +$
$= P(X<5)$

(iii) $P(1 < x < 4) = \int_1^4 \frac{1}{3} e^{-x/3}\,dx$

$$= -e^{-x/3} \Big|_1^4$$

$$= -\left(e^{-4/3} - e^{-1/3}\right)$$

$$P(1<x<4) = -(0.2636 - 0.7165)$$

$$= -(-0.4529)$$

$$P(1<x<4) = 0.4529$$

Find μ_1':-

$$\mu_1' = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$\mu_1' = \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right)\Big|_{t=0}$$

$$= \lambda \frac{d}{dt}\left((\lambda - t)^{-1}\right)\Big|_{t=0}$$

$$= \lambda\left(-1(\lambda - t)^{-2}(-1)\right)\Big|_{t=0}$$

$$= \lambda(\lambda - t)^{-2}\Big|_{t=0}$$

$$= \frac{\lambda}{(\lambda - t)^2}\Big|_{t=0}$$

$$= \frac{\lambda}{\lambda^2}$$

$$\mu_1' = \frac{1}{\lambda}$$

Similarly, we can find μ_2', μ_3' and μ_4'.

GAMMA DISTRIBUTION:-

A continuous random variable X is said to have a gamma distribution if its p.d.f is defined as

$$f(x) = \begin{cases} \frac{1}{\Gamma(m)} x^{m-1} e^{-x} & \text{for } 0 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

where $m > 0$ is the parametre of gamma distribution.

Properties:-

1) Total area $= \int_x f(x)\,dx$

$$= \int_0^\infty \frac{1}{\Gamma(m)} x^{m-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty x^{m-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma(m)} \Gamma(m-1+1)$$

$$= \frac{1}{\Gamma(m)} \Gamma(m)$$

Total area $= 1$

2) Mean $= E(X)$

$$= \int_x x\, f(x)\,dx$$

$$= \int_0^\infty x \frac{1}{\Gamma m} x^{m-1} e^{-x} dx$$

$$= \frac{1}{\Gamma m} \int_0^\infty x^m e^{-x} dx$$

$$= \frac{1}{\Gamma m} \Gamma m+1$$

$$= \frac{1}{\Gamma m} m \Gamma m$$

Mean $= m$

3)

variance $= E(x^2) - [E(x)]^2 \rightarrow (1)$

Take

$$E(x^2) = \int_x x^2 f(x)\,dx$$

$$= \int_0^\infty x^2 \frac{1}{\Gamma m} x^{m-1} e^{-x} dx$$

$$= \frac{1}{\Gamma m} \int_0^\infty x^{m+1} e^{-x} dx$$

$$= \frac{1}{\Gamma m} \Gamma m+2$$

$$= \frac{1}{\Gamma m} m(m+1)\Gamma m$$

$$E(x^2) = (m+1)m = m^2+m$$

Put in (1)

$$\text{variance} = m^2+m-m^2$$

$$\text{variance} = m$$

4)

$$M_X(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \frac{1}{\Gamma m} x^{m-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma m}\int_0^\infty x^{m-1} e^{(t-1)x}\,dx$$

$$= \frac{1}{\Gamma m}\int_0^\infty x^{m-1} e^{-(1-t)x}\,dx$$

Put $(1-t)x = y$

$(1-t)dx = dy$

$$M_X(t) = \frac{1}{\Gamma m}\int_0^\infty \left(\frac{y}{1-t}\right)^{m-1} e^{-y}\,\frac{dy}{1-t}$$

$$= \frac{1}{\Gamma m\,(1-t)^{m-1+1}}\int_0^\infty y^{m-1} e^{-y}\,dy$$

$$= \frac{1}{\Gamma m}\,\frac{1}{(1-t)^m}\,\Gamma m$$

$$M_X(t) = \frac{1}{(1-t)^m}$$

μ_1' :-

$$\mu_1' = \frac{d}{dt} M_X(t)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\frac{1}{(1-t)^m}\right|_{t=0}$$

$$= \frac{d}{dt}(1-t)^{-m}\Big|_{t=0}$$

$$= -m(1-t)^{-m-1}(-1)\Big|_{t=0}$$

$$= m(1-t)^{-m-1}\Big|_{t=0}$$

$$= \frac{m}{(1-t)^{m+1}}\Big|_{t=0}$$

$$= \frac{m}{(1-0)^{m+1}}$$

$$\mu_1' = m$$

Similarly, $\mu_2' = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$

$$= \frac{d}{dt}(\mu_1')\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\frac{m}{(1-t)^{m+1}}\right|_{t=0}$$

$$\mu_2' = \frac{m(m+1)}{\ldots}\Big| = m(m+1)$$

(5) $K_X(t) = \log_e M_X(t)$

$$= \log_e \frac{1}{(1-t)^m}$$

$$= \log_e (1-t)^{-m}$$

$$= -m \log_e (1-t)$$

$$= -m \left[-\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots\right) \right]$$

$$K_X(t) = m\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots\right)$$

so

$$K_1 = K_2 = K_3 = \cdots = m$$

Beta function:-

The function

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{where } m>0 \text{ and } n>0$$

Beta distribution (Ist kind):-

A random variable x is said to follow a beta dist. of Ist kind if its p.d.f is defined as

$$f(x) = \begin{cases} \frac{1}{B(m,n)} x^{m-1} (1-x)^{n-1} & \text{for } 0<x<1 \\ 0 & \text{elesewhere} \end{cases}$$

where $m>0$ and $n>0$ are the

parameters of beta distribution.

Properties:-

1) Total area $= \int_x f(x)dx$

$$= \int_0^1 \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1} dx$$

$$= \frac{1}{B(m,n)} \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$= \frac{1}{B(m,n)} B(m,n)$$

Total area $= 1$

2)

Mean $= E(X)$

$$= \int_x x\, f(x) dx$$

$$= \int_0^1 x \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1} dx$$

$$= \frac{1}{B(m,n)} \int_0^1 x^{m+1-1}(1-x)^{n-1} dx$$

$$= \frac{1}{B(m,n)} B(m+1,n)$$

$$= \frac{1}{\frac{\Gamma m \, \Gamma n}{\Gamma m+n}} \frac{\Gamma m+1 \, \Gamma n}{\Gamma m+1+n}$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \cdot \frac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$

$$\text{Mean} = \frac{m}{m+n}$$

3) $\text{var} = E(x^2) - [E(x)]^2 \rightarrow (1)$

$$E(x^2) = \int_x x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{B(m,n)} \int_0^1 x^{m+2-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{B(m,n)} B(m+2, n)$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \cdot \frac{\Gamma(m+2)\,\Gamma(n)}{\Gamma(m+2+n)}$$

$$= \frac{\Gamma(m+n)\,(m+1)m\,\Gamma(m)}{\Gamma(m)\,(m+n+1)(m+n)\,\Gamma(m+n)}$$

$$E(x^2) = \frac{m(m+1)}{(m+n)(m+n+1)}$$

put in (1)

$$\text{var} = \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m^2}{(m+n)^2}$$

$$= \frac{m}{m+n}\left[\frac{m+1}{m+n+1} - \frac{m}{m+n}\right]$$

$$= \frac{m}{m+n}\left\{\frac{(m+1)(m+n)-m(m+n+1)}{(m+n)(m+n+1)}\right\}$$

$$= \frac{m}{m+n}\left\{\frac{\cancel{m^2}+\cancel{m}+\cancel{mn}+n-\cancel{m^2}-\cancel{mn}-\cancel{m}}{(m+n)(m+n+1)}\right\}$$

$$= \frac{m}{(m+n)^2(m+n+1)}(n)$$

$$var = \frac{mn}{(m+n)^2(m+n+1)}$$

Beta distribution (Second kind) :-

A r.v X with a p.d.f $f(x)$ is said to follow a beta dist. of second kind if

$$f(x) = \begin{cases} \frac{1}{B(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & \text{for } 0<x<\infty \\ 0 & \text{elsewhere} \end{cases}$$

where $m>0$ and $n>0$ are the parameters.

Properties :-

1) Total area $= \int_x f(x)\,dx$

$$= \int_0^\infty \frac{1}{B(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

put $1+x = \frac{1}{y}$

$$\Rightarrow \quad y = \frac{1}{1+x}$$

$$dy = \frac{-1}{(1+x)^2} dx$$

$$dy = \frac{-1}{y^{-2}} dx$$

when $x = 0$, $y = 1$ $\qquad x = \frac{1-y}{y}$

$x \to \infty$, $y = 0$

Total area becomes

$$\text{Total area} = \frac{1}{B(m,n)} \int_1^0 \left(\frac{1-y}{y}\right)^{m-1} y^{m+n} (-y^{-2}) dy$$

$$= \frac{1}{B(m,n)} \int_0^1 y^{m+n-m+1-2} (1-y)^{m-1} dy$$

$$= \frac{1}{B(m,n)} \int_0^1 y^{n-1} (1-y)^{m-1} dy$$

$$= \frac{1}{B(m,n)} B(n,m)$$

Total area = 1

2)

$$\text{Mean} = E(X)$$

$$= \int_x x f(x) dx$$

$$= \int_0^\infty x \frac{1}{B(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \frac{1}{B(m,n)} \int_0^\infty \frac{x^{m+1-1}}{(1+x)^{m+n}} dx$$

Put $y = \frac{1}{1+x}$ or $1+x = \frac{1}{y}$

$$dy = \frac{-1}{(1+x)^2} dx$$

$$dy = \frac{-1}{(1/y)^2} dx$$

$$dy = -y^2 dx$$

$$\Rightarrow dx = -y^{-2} dy$$

at $x=0$, $y=1$; $x = \frac{1-y}{y}$

$x=\infty$, $y=0$

So,

$$\text{Mean} = \frac{1}{B(m,n)} \int_1^0 \left(\frac{1-y}{y}\right)^{m+1-1} y^{m+n} (-y^{-2} dy)$$

$$= \frac{1}{B(m,n)} \int_0^1 (1-y)^{m+1-1} y^{m+n-2-m+1-1} dy$$

$$= \frac{1}{B(m,n)} \int_0^1 (1-y)^{m+1-1} y^{n-2} dy$$

$$= \frac{1}{B(m,n)} \int_0^1 (1-y)^{m+1-1} y^{n-1-1} dy$$

$$= \frac{1}{B(m,n)} B(m+1, n-1)$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \; \frac{\Gamma(m+1)\,\Gamma(n-1)}{\Gamma(m+1+n-1)}$$

$$= \frac{\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m)}\,\Gamma(n)} \; \frac{m\,\cancel{\Gamma(m)}\,\Gamma(n-1)}{\cancel{\Gamma(m+n)}}$$

$$= \frac{m\,\cancel{\Gamma(n-1)}}{(n-1)\,\cancel{\Gamma(n-1)}} \qquad \because \Gamma(m+1) = m!$$

$$E(X) = \frac{m}{n-1}$$

3)

$$var = E(X^2) - [E(X)]^2 \rightarrow (1)$$

Take

$$E(X^2) = \int_x x^2 f(x)\,dx$$

$$= \int_0^\infty x^2 \frac{1}{B(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{1}{B(m,n)} \int_0^\infty \frac{x^{m+2-1}}{(1+x)^{m+n}}\,dx$$

Put $y = \frac{1}{1+x}$ or $1+x = 1/y$

$(1+x)y = 1$

$y + xy = 1$

$$x = \frac{1-y}{y}$$

$$dy = \frac{-1}{(1+x)^2}\,dx$$

$$dy = \frac{-1}{y^{-2}}\,dx$$

$$-y^{-2}\,dy = dx$$

at $x = 0$, $y = 1$

$x = \infty$, $y = 0$

so

$$E(X^2) = \frac{1}{B(m,n)} \int_1^0 \left(\frac{1-y}{y}\right)^{m+2-1} y^{m+n} \left(-y^{-2}\,dy\right)$$

$$= \frac{1}{B(m,n)} \int_0^1 (1-y)^{m+2-1}\, y^{m+n-2-m-2+1}\, dy$$

$$= \frac{1}{B(m,n)} \int_0^1 y^{n-3} (1-y)^{m+2-1}\, dy$$

$$= \frac{1}{B(m,n)} \int_0^1 (1-y)^{m+2-1}\, y^{n-2-1}\, dy$$

$$= \frac{1}{B(m,n)}\, B(m+2, n-2)$$

$$= \frac{1}{\frac{\Gamma m\,\Gamma n}{\Gamma m+n}} \cdot \frac{\Gamma m+2\;\Gamma n-2}{\Gamma m+n}$$

$$E(X^2) = \frac{\Gamma m+2\;\Gamma n-2}{\Gamma m\;\Gamma n}$$

$$E(X^2) = \frac{(m+1)m\,\cancel{\Gamma m}\;\cancel{\Gamma n-2}}{\cancel{\Gamma m}\,(n-1)(n-2)\,\cancel{\Gamma n-2}}$$

$$E(X^2) = \frac{m(m+1)}{(n-1)(n-2)}$$

Put in (1)

$$Var = \frac{m(m+1)}{(n-1)(n-2)} - \left(\frac{m}{n-1}\right)^2$$

$$= \frac{m(m+1)}{(n-1)(n-2)} - \frac{m^2}{(n-1)^2}$$

$$= \frac{m(m+1)(n-1) - m^2(n-2)}{(n-1)^2(n-2)}$$

$$= \frac{(m^2+m)(n-1) - m^2 n + 2m^2}{(n-1)^2(n-2)}$$

$$= \frac{\cancel{m^2 n} - m^2 + mn - m - \cancel{m^2 n} + 2m^2}{(n-1)^2(n-2)}$$

$$= \frac{m^2 + m(n-1)}{(n-1)^2(n-2)} \qquad \because mn - m = m(n-1)$$

$$Var = \frac{m(m+n-1)}{(n-1)^2(n-2)}$$

Normal Distribution :-

Normal distribution involves error term ϵ_i

$\{\epsilon_i\}_{i=1}^{n}$ are random errors which can be +ve or −ve but average at zero.

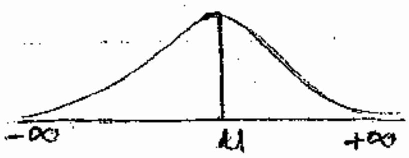
Normal Distribution:- (Gaussian Distribution).

Probability density function of a normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } -\infty < x < \infty$$

where μ and σ^2 are the parameters of normal distribution.

$X \sim N(\mu, \sigma^2)$



Note:-

(i) $\int_0^\infty y^{n-1} e^{-y} dy = \Gamma n$

(ii) $\Gamma \frac{1}{2} = \sqrt{\pi}$

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

μ is location parameter and σ is scale parameter.

Properties :-

1) Total area $= \int_x f(x)\,dx$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$

$$dx = \sigma\, dz$$

Total area $= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma\, dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz$$

is even function

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-\frac{1}{2}z^2} dz$$

put $y = \frac{1}{2}z^2 \Rightarrow z^2 = 2y$

$$2z\,dz = 2dy$$

$$dz = \frac{1}{z}\,dy = \frac{1}{\sqrt{2y}}\,dy$$

Then,

Total area $= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-y} \frac{1}{\sqrt{2y}}\, dy$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{-\frac{1}{2}} e^{-y}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{\frac{1}{2}-1} e^{-y}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\tfrac{1}{2} = \frac{1}{\sqrt{\pi}} \sqrt{\pi}$$

Total area = 1

2)

$$\text{Mean} = E(X)$$
$$= \int_x x f(x) dx$$
$$= \int_{-\infty}^{+\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$

$$dx = \sigma dz$$

$$\text{Mean} = \int_{-\infty}^{\infty} (\mu + \sigma z) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma dz$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} dz$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}z^2} dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma z e^{-\frac{1}{2}z^2} dz$$

Since, $\int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}z^2} dz$ is even function

and $\int_{-\infty}^{\infty} \sigma z e^{-\frac{1}{2}z^2} dz$ is odd function.

So,

$$\text{Mean} = \frac{1}{\sqrt{2\pi}} \left[2 \int_0^{\infty} \mu e^{-\frac{1}{2}z^2} dz \right] + 0$$

$$= 2\mu \int^{\infty} e^{-\frac{1}{2}z^2} dz$$

Properties:-

1) Total area $= \int_x f(x)\,dx$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $Z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma Z$

$dx = \sigma\,dZ$

$$\text{Total area} = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} \sigma\,dZ$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}Z^2} dZ$$

is even function

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-\frac{1}{2}Z^2} dZ$$

put $y = -\frac{1}{2}Z^2 \Rightarrow Z^2 = 2y$

$2Z\,dZ = 2\,dy$

$dZ = \frac{1}{Z}\,dy = \frac{1}{\sqrt{2y}}\,dy$

Then,

$$\text{Total area} = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-y} \frac{1}{\sqrt{2y}}\,dy$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{-1/2} e^{-y}\,dy$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{\frac{1}{2}-1} e^{-y}\,dy$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\tfrac{1}{2} = \frac{1}{\sqrt{\pi}} \sqrt{\pi}$$

Total area = 1

2)

$$\text{Mean} = E(X)$$

$$= \int_x x f(x) dx$$

$$= \int_{-\infty}^{+\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$

$dx = \sigma dz$

$$\text{Mean} = \int_{-\infty}^{\infty} (\mu + \sigma z) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}z^2} dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma z e^{-\frac{1}{2}z^2} dz$$

Since, $\int_{-\infty}^{\infty} \mu e^{-\frac{1}{2}z^2} dz$ is even function

and $\int_{-\infty}^{\infty} \sigma z e^{-\frac{1}{2}z^2} dz$ is odd function.

So,

$$\text{Mean} = \frac{1}{\sqrt{2\pi}} \left[2 \int_0^{\infty} \mu e^{-\frac{1}{2}z^2} dz \right] + 0$$

$$= \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2} dz$$

put $y = +\frac{1}{2}z^2 \Rightarrow z^2 = 2y \Rightarrow z = \sqrt{2y}$

$2z\,dz = 2\,dy \Rightarrow dz = \frac{dy}{z}$

$\Rightarrow dz = \frac{1}{\sqrt{2y}}\,dy$ Then

$$\text{Mean} = \frac{2\mu}{\sqrt{2\pi}} \int_0^\infty e^{-y} \frac{1}{\sqrt{2y}}\,dy$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^\infty y^{-1/2} e^{-y}\,dy$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^\infty y^{1/2 - 1} e^{-y}\,dy$$

$$= \frac{\mu}{\sqrt{\pi}} \Gamma\tfrac{1}{2}$$

$$= \frac{\mu}{\sqrt{\pi}} \sqrt{\pi}$$

$$\text{Mean} = \mu$$

3)

$$\text{Variance} = E(x - \text{Mean})^2$$

for normal dist $\text{var} = E(x - \mu)^2 \quad \because E(x) = \mu$

(General) $\text{Variance} = E(X - E(X))^2$

$$= E[X^2 + (E(X))^2 - 2XE(X)]$$

$$= E(X^2) + (E(X))^2 - 2E(X)E(X)$$

$$= E(X^2) + [E(X)]^2 - 2[E(X)]^2$$

$$\text{var} = E(X^2) - [E(X)]^2$$

3) variance $= E(X - \text{Mean})^2$

$$= E(X-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = \sigma z$

$$dx = \sigma\,dz$$

$$\text{variance} = \int_{-\infty}^{\infty} (\sigma z)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma\,dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

$\because \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$ is even function

$$\text{var} = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

put $\frac{1}{2}z^2 = y \Rightarrow z^2 = 2y$

$$2z\,dz = 2dy \Rightarrow z\,dz = dy$$

$$\text{variance} = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2y}\, e^{-y}\, dy$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2}\, y^{1/2} e^{-y} dy$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} y^{3/2 - 1} e^{-y} dy$$

$$\text{variance} = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\tfrac{3}{2}\right)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left(\frac{1}{2} \Gamma\left(\tfrac{1}{2}\right)\right)$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi}$$

$$\text{variance} = \sigma^2$$

4)

Moment generating function

$$M_x(t) = E(e^{tx})$$

$$= \int_x e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$M_x(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$

$dx = \sigma\, dz$

$$M_x(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{1}{2}z^2} \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\mu - 2t\sigma z)} dz$$

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\mu t - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2)} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z + t^2\sigma^2) + \mu t + \frac{1}{2}t^2\sigma^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - t\sigma)^2} dz$$

$$= \frac{2}{\sqrt{2\pi}} e^{\mu t + \frac{1}{2}t^2\sigma^2} \int_{0}^{\infty} e^{-\frac{1}{2}(z - t\sigma)^2} dz$$

Put $y = \frac{1}{2}(z - t\sigma)^2$

$dy = (z - t\sigma)\, dz$

$\because z - t\sigma = \sqrt{2y}$), $dz = \frac{1}{\sqrt{2y}} dy$

$$M_X(t) = \frac{2}{\sqrt{2\pi}} e^{\mu t + \frac{1}{2}t^2\sigma^2} \int_{0}^{\infty} e^{-y} \frac{1}{\sqrt{2y}} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\mu t + \frac{1}{2}t^2\sigma^2} \int_{0}^{\infty} y^{-1/2} e^{-y} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\mu t + \frac{1}{2}t^2\sigma^2} \int_{0}^{\infty} y^{1/2 - 1} e^{-y} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\mu t + \frac{1}{2}t^2\sigma^2} \Gamma(1/2)$$

$$= \frac{1}{\sqrt{\pi}} e^{\mu t + \frac{1}{2}t^2\sigma^2} \sqrt{\pi}$$

$$\Rightarrow M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

Mean about origin :-

$$\mu_1' = \frac{d}{dt}\left(M_x(t)\right|_{t=0}$$

$$= \frac{d}{dt}\left(e^{\mu t + \frac{1}{2}t^2\sigma^2}\right|_{t=0}$$

$$\mu_1' = e^{\mu t + \frac{1}{2}t^2\sigma^2}\left(\mu + \frac{1}{2}2t\sigma^2\right)\Big|_{t=0}$$

$$\mu_1' = e^{\mu t + \frac{1}{2}t^2\sigma^2}\left(\mu + t\cancel{\sigma^2}\right)\Big|_{t=0}$$

$$= e^{0+0}(\mu + 0)$$

$$\mu_1' = \mu$$

and

$$\mu_2' = \frac{d^2}{dt^2}\left(M_x(t)\right|_{t=0}$$

$$= \frac{d}{dt}\left(\mu_1'\right|_{t=0}$$

$$= \frac{d}{dt}\left(e^{\mu t + \frac{1}{2}t^2\sigma^2}(\mu + t\sigma^2)\right|_{t=0}$$

$$= \left(e^{\mu t + \frac{1}{2}t^2\sigma^2}\left(\frac{d\mu}{dt} + \sigma^2\right) + e^{\mu t + \frac{1}{2}t^2\sigma^2}(\mu + t\sigma^2)(\mu + t\sigma^2)\right|_{t=0}$$

$$= \left(e^{\mu t + \frac{1}{2}t^2\sigma^2}(0 + \sigma^2) + e^{\mu t + \frac{1}{2}t^2\sigma^2}(\mu + t\sigma^2)^2\right|_{t=0}$$

$$= \left(\sigma^2 e^{\mu t + \frac{1}{2}t^2\sigma^2} + e^{\mu t + \frac{1}{2}t^2\sigma^2}(\mu + t\sigma^2)^2\right|_{t=0}$$

$$= e^{\mu t + \frac{1}{2}t^2\sigma^2}\left(\sigma^2 + (\mu + t\sigma^2)^2\right|_{t=0}$$

$$\mu_2' = e^{\mu t + \frac{1}{2}t^2\sigma^2}\left(\mu^2 + 2t\sigma^2\mu + t^2\sigma^4 + \sigma^2\right|_{t=0}$$

$$\mu_2' = e^{0+0}(\mu^2 + 0 + 0 + \sigma^2)$$
$$\mu_2' = e^0(\mu^2 + \sigma^2)$$
$$\mu_2' = \mu^2 + \sigma^2$$

5) Median:-

Median is the middle most observation of arranged data

Suppose median $= M$ then

$$P(-\infty < x \le M) = \frac{1}{2} = P(M \le x < \infty)$$

Take

$$P(-\infty < x \le M) = \frac{1}{2}$$

$$\int_{-\infty}^{M} f(x)dx = \frac{1}{2}$$

$$\int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

put $z = \dfrac{x-\mu}{\sigma}$

$$dz = \frac{1}{\sigma}dx$$

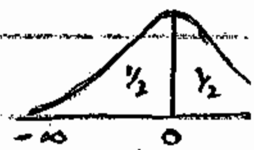
when

$x \to -\infty$, $z \to -\infty$

$x \to M$, $z = \dfrac{M-\mu}{\sigma}$

$$\therefore \int_{-\infty}^{\frac{M-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2} \rightarrow (1)$$

As $\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$



$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2} \rightarrow (2)$$

Comparing (1) and (2)

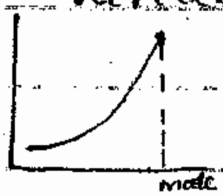
$$\Rightarrow \quad \frac{M-\mu}{\sigma} = 0$$

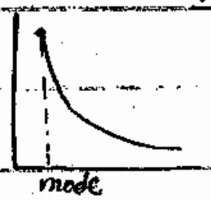
$$\Rightarrow \quad M = \mu$$

$$\Rightarrow \quad \text{Median} = \mu$$

6) **Mode**:-

Mode is the most frequent value of variable.





Mode is the maxima of a function

we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Differentiate w.r.t "x"

$$\frac{d}{dx}(f(x)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} -\frac{1}{2}\left(2\left(\frac{x-\mu}{\sigma^2}\right)\right)$$

$$= -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Put $f'(x) = 0$

$$\Rightarrow \quad -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = 0$$

$$\Rightarrow \quad x-\mu = 0$$

$\Rightarrow \quad x = \mu$ is a stationary pt.

Now take

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\left[(x-\mu)e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left[-\frac{(x-\mu)}{\sigma^2}\right] + e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\cdot 1\right]$$

= Put $x = \mu$

$$f''(\mu) = \frac{-1}{\sigma^3\sqrt{2\pi}}\left[(\mu-\mu)e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2}\left(-\left(\frac{\mu-\mu}{\sigma^2}\right)\right) + e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2}\right]$$

$$= \frac{-1}{\sigma^3\sqrt{2\pi}}\left[0+1\right]$$

$$f''(\mu) = \frac{-1}{\sigma^3\sqrt{2\pi}} < 0$$

$\therefore$ $x = \mu$ is maxima for $f(x)$.

Thus Mode $= \mu$

Note :-

Mean, Median and Mode of normal distribution are same.

i.e. Mean = Median = Mode $= \mu$

7) Points of Inflexion (Inflection) :-

The points at which the curve changes its behave. To find find point of inflexion, we derive $f''(x)$ and equate it to zero.

$$\Rightarrow f''(x) = 0$$

we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \; -\frac{2}{2}\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}$$

$$f'(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} (x-\mu) e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\left[(1-0)e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} + (x-\mu) e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left(-\frac{2}{2}\left(\frac{x-\mu}{\sigma^2}\right)\right)\right]$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} - \left(\frac{x-\mu}{\sigma}\right)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right.$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left[-\left(\frac{x-\mu}{\sigma}\right)^2 + 1\right]$$

Now put $f''(x) = 0$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left[-\left(\frac{x-\mu}{\sigma}\right)^2 + 1\right] = 0$$

$$\Rightarrow \quad -\left(\frac{x-\mu}{\sigma}\right)^2 + 1 = 0$$

$$\Rightarrow \quad \left(\frac{x-\mu}{\sigma}\right)^2 = 1$$

$$\Rightarrow \quad \frac{x-\mu}{\sigma} = \pm 1$$

$$\Rightarrow \quad x - \mu = \pm \sigma$$

$$\therefore \quad f(\quad x = \mu \pm \sigma$$

$$\therefore \quad f(\mu \pm \sigma) = \frac{1}{\sigma\sqrt{2\pi e}} = f(\mu - \sigma)$$

Thus, points of inflection are

$$\left(\mu - \sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right)\left(\mu + \sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right)$$

Area under the normal curve:-

Area always given in table from 0 to Z. If we want to see the area of $Z \leq 1.78$, That is $P(0 \leq Z \leq 1.78) = 0.08183$

Z	0.00	0.01	---	0.08	0.09
0.0					
0.1					
⋮					
0.9					
1					
⋮					
1.7				0.08183	
⋮					

Example:-

2011 Q#04(b)

Find the prob. that a random variable having the standard normal distribution will take on a value.

(i) less than 1.72

(ii) less than −0.88

(iii) between 1.30 and 1.75

(iv) between −0.25 and 0.45

Sol:-

Z is the standard normal variat i.e $Z \sim N(0,1)$

(i) less than 1.72

$P(Z < 1.72) = 0.5 + P(0 < Z < 1.72)$

$= 0.5 + 0.4573$

$P(Z < 1.72) = 0.9573$

(ii) less than −0.88

$P(Z < -0.88)$

$= 0.5 - P(-0.88 < Z < 0)$

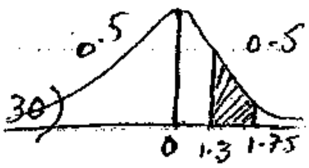
$= 0.5 - 0.3106$

$= 0.2106$

(iii) b/w 1.30 and 1.75

$P(1.30 \le Z \le 1.75) =$

$P(0 \le Z \le 1.75) - P(0 \le Z \le 1.30)$



$= 0.4599 - 0.4032$

$P(1.30 \le Z \le 1.75) = 0.0567$

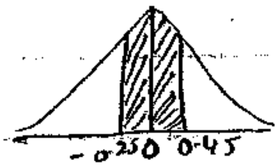
(iv)

b/w -0.25 and 0.45

$P(-0.25 \le Z \le 0.45)$

$= P(-0.25 \le Z \le 0) +$

$P(0 \le Z \le 0.45)$



$\Rightarrow P(-0.25 \le Z \le 0.45) = 0.0987 + 0.1736$

$P(-0.25 \le Z \le 0.45) = 0.2723$

A/2009

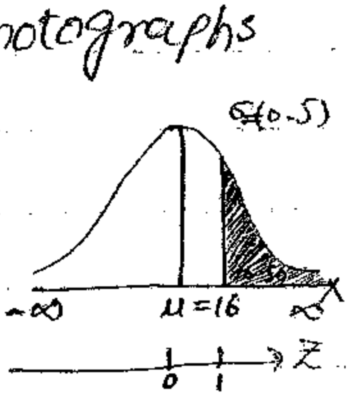
Q#4(b):-

In a photographic process,

Sol:-

X: Developing time of photographs (in seconds)

$X \sim N(16, (0.5)^2)$



i) at least 16.50 seconds

$P(X \geq 1.65)$

$$P(X \geq 1.65) = P\left(Z \geq \frac{1.65-16}{0.5}\right)$$

$$= P(Z \geq 1)$$

$$= 0.5 - P(0 \leq Z \leq 1)$$

$$= 0.5 - 0.3413$$

$$P(X \geq 1.65) = 0.1587$$

(ii) at most 15.40 seconds.

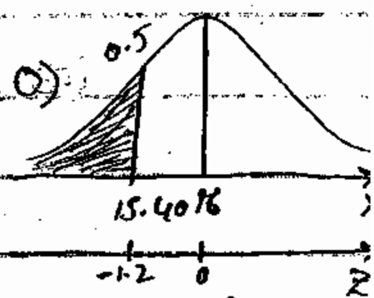
$$P(X \leq 15.40) = P\left(Z \leq \frac{15.40-16}{0.5}\right)$$

$$= P(Z \leq -1.2)$$

$$= 0.5 - P(\cancel{-1.2} \leq Z \leq 0)$$

$$= 0.5 - 0.3849$$

$$P(X \leq 15.40) = 0.1151$$



(iii) Anywhere b/w 15 and 16.80 seconds.

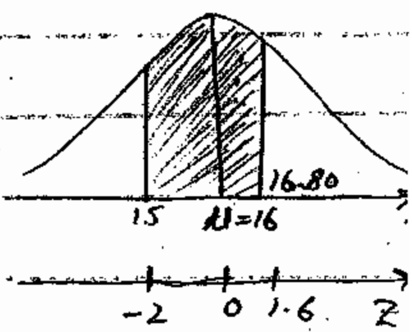
$$P(15 \leq X \leq 16.80) = P\left(\frac{15-16}{0.5} \leq Z \leq \frac{16.80-16}{0.5}\right)$$

$$= P(-2 \leq Z \leq 1.6)$$

$$P(-2 \leq Z \leq 1.6) = P(-2 \leq Z \leq 0) + P(0 \leq Z \leq 1.6)$$

$$= 0.4772 + 0.4452$$

$$P(15 \leq X \leq 16.8) = 0.9224$$



Inverse Normal Distribution:-

Examples:-

Students score in an examination is normally distributed with mean = 55 and $\sigma = 10$. If a student wants to be among

(i) top 10% students,

(ii) top 5% students,

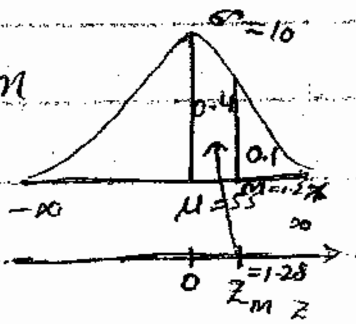
what the minimum marks he/she needs to obtain?

Sol:-

X: Exam marks

$$X \sim N(55, (10)^2)$$

(i)



Suppose, M is the minimum score among top 10% students.

$$P(X \geq M) = 0.1 = 10\%$$

$$Z_M = \frac{M - \mu}{\sigma}$$

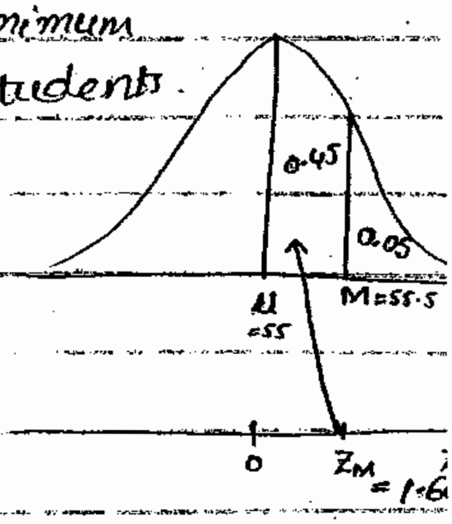
$$Z_M = \frac{M - 55}{10}$$

$$1.28 = \frac{M - 55}{10}$$

$$\Rightarrow M = 67.8$$

(ii) Suppose, M is the minimum score among 5% top students.

$P(X \geq M)$



$$Z_M = \frac{M - \mu}{\sigma}$$

$$1.645 = \frac{M - 55}{10}$$

$$M = 71.45$$

Ex:-

The heights of applicants to the police force are normally distributed with mean 170cm and standard deviation 3.8cm. If 30% of applicants are rejected due to small height, what is the minimum acceptable height for the police force?

Sol:-

X : Height in cm

$\mu = 170$ cm, $\sigma = 3.8$ cm

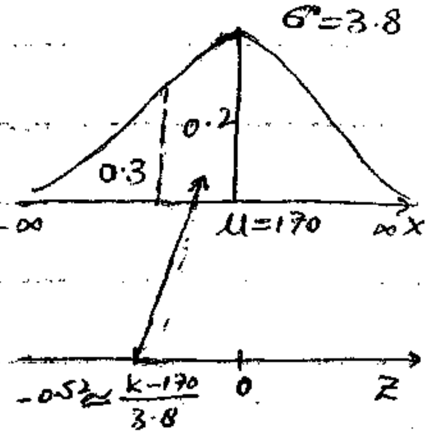
$X \sim N(170, (3.8)^2)$

Suppose minimum acceptable height for police force $= k$ cm.

$P(X < k) = 30\% = 0.3$

$$P\left(Z < \frac{k-170}{3.8}\right) = 0.3$$

$$\frac{k-170}{3.8} = -0.52$$



$$k - 170 = (-0.52)(3.8)$$

$$k = 170 - 1.976$$

$$k = 168.02 \text{ cm}$$

Ex:-

The average life of a certain type of small motors is 10 years with SD of 2 yrs. The manufacturer replaces free all motors that fail under gurantee. If he is willing to replace only 3% of the motors that fails, how long the guarantee he should offer? Assume that the lives of motor are normally distributed?

Sol:-

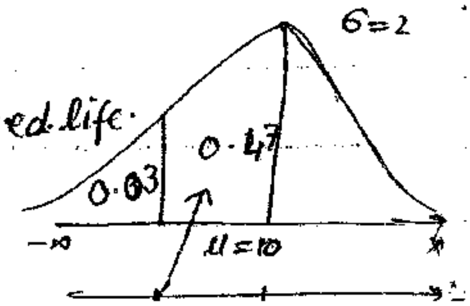
X : life of replacing motors

$\mu = 10$ yrs, $\sigma = 2$ yrs

$$X \sim N(10, (2)^2)$$

Suppose, k is minimum guaranteed life.

$$P(X < k) = 0.03 = 3\%$$



$$P(X < k) = P\left(Z < \frac{k-10}{2}\right)$$

$$\frac{k-10}{2} = -1.89$$

$$k-10 = -3.78$$

$$k = 6.22 \text{ yrs}$$

Ex:-

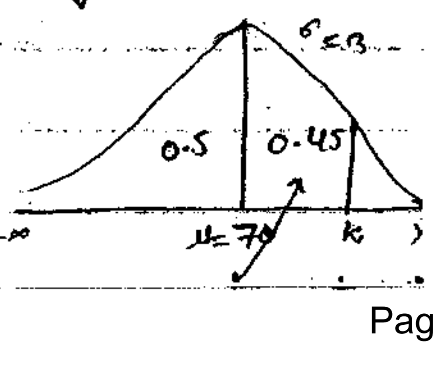
An architect is designing the interior doors, he wants to make them heigh enough so that 95% of the persons using the door will have at least 1-ft clearness. Assume that the heights are normally distributed with mean 70 inches and S.D 3-inches. How, high or what should be the height of door.

Sol:-

X: Height of person (inches)

$$X \sim N(70, (3)^2)$$

k: Tallest person among the 95% using the door.



$P(X < k) = 95\%$

$P(X < k) = 0.95$

$$Z_M = \frac{k - 70}{3}$$

$$1.645 = \frac{k - 70}{3}$$

Sort of inter-
pollation
$$Z_M = \frac{1.64 + 1.65}{2} = 1.645$$

$k = 74.94$ inches

is height of Tallest person

Door height $= 74.94 + 12$

$= 86.935$

$= 87$ inches

Normal Approximation to Binomial Distribution:-

Statement :-

If $X \sim b(x; n, p)$ then

$$\lim_{n \to \infty} X \xrightarrow{\text{Approx}} N(np, npq)$$

Proof :-

By definition

$$M_X(t) = E(e^{tX})$$

Let us define

$$Z = \frac{X - np}{\sqrt{npq}}$$

Take $M_Z(t) = E(e^{tZ})$

$$= E\left[e^{t\left(\frac{X-np}{\sqrt{npq}}\right)}\right]$$

$$= e^{-npt/\sqrt{npq}}\, E\left[e^{tX/\sqrt{npq}}\right]$$

$$M_Z(t) = e^{\frac{-npt}{\sqrt{npq}}} \sum_{x=0}^{n} e^{\frac{tx}{\sqrt{npq}}} p(x)$$

$$= e^{\frac{-npt}{\sqrt{npq}}} \sum_{x=0}^{n} e^{tx/\sqrt{npq}}\ {}^nC_x p^x q^{n-x}$$

$$= e^{\frac{-npt}{\sqrt{npq}}} \sum_{x=0}^{n} {}^nC_x \left(p e^{t/\sqrt{npq}}\right)^x q^{n-x}$$

$$= e^{\frac{-npt}{\sqrt{npq}}} \left(q + p e^{t/\sqrt{npq}}\right)^n$$

By $\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} = (q+p)^n$

$$M_Z(t) = \left(q e^{\frac{-pt}{\sqrt{npq}}} + p e^{\frac{t(1-p)}{\sqrt{npq}}}\right)^n$$

$$M_Z(t) = \left(q e^{-pt/\sqrt{npq}} + p e^{qt/\sqrt{npq}}\right)^n$$

As

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$M_Z(t) = \left(q \sum_{r=0}^{\infty} \left(\frac{-pt}{\sqrt{npq}}\right)^r \frac{1}{r!} + p \sum_{r=0}^{\infty} \left(\frac{qt}{\sqrt{npq}}\right)^r \frac{1}{r!}\right)^n$$

$$= \left[q\left(1 - \frac{pt}{\sqrt{npq}} + \frac{1}{2!}\frac{p^2t^2}{(\sqrt{npq})^2} - \frac{1}{3!}\frac{p^3t^3}{(\sqrt{npq})^3} + \cdots\right)\right.$$

$$\left. + p\left(1 + \frac{qt}{\sqrt{npq}} + \frac{1}{2!}\frac{q^2t^2}{(\sqrt{npq})^2} + \frac{1}{3!}\frac{q^3t^3}{(\sqrt{npq})^3} + \cdots\right)\right]^n$$

$$= \left[(q+p) - \frac{qpt}{\sqrt{npq}} + \frac{qpt}{\sqrt{npq}} + \frac{1}{2!}\frac{qp^2t^2}{(\sqrt{npq})^2} + \frac{1}{2!}\right.$$

$$\frac{pq^2t^2}{(\sqrt{npq})^2} - \frac{1}{3!} q \frac{p^3t^3}{(\sqrt{npq})^3} + \frac{1}{3!}\frac{pq^3t^3}{(\sqrt{npq})^3} + \cdots \Big]^n$$

$$= \Big[(q+p) + \frac{1}{2!}\frac{t^2pq}{(\sqrt{npq})^2}(p+q) + \frac{1}{3!}\frac{pqt^3}{(\sqrt{npq})^3}$$

$$(q^2-p^2) + \cdots \Big]^n \qquad \because q+p=1$$

$$M_z(t) = \Big[1 + \frac{1}{2!}\frac{t^2pq}{(\sqrt{npq})^2} + \frac{1}{3!}\frac{pqt^3}{(\sqrt{npq})^3}(q-p)$$

$$+ \cdots \Big]^n$$

$$\log M_z(t) = n \log\Big[1 + \frac{1}{2!}\frac{t^2pq}{npq} + \frac{1}{3!}\frac{pqt^3}{(\sqrt{npq})^3}(q-p)$$

$$+ \cdots \Big]^n$$

$$= n \log\Big[1 + \frac{1}{2!}\frac{t^2}{n} + \frac{1}{3!}\frac{pqt^3}{(\sqrt{npq})^3}(q-p) + \cdots\Big]^n$$

As $\log(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$\therefore \log M_z(t) = n\Big[\Big(\frac{1}{2!}\frac{t^2}{n} + \frac{1}{3!}\frac{pqt^3}{(\sqrt{npq})^3}(q-p) + \cdots\Big)$$

$$-\frac{1}{2}\Big(\frac{1}{2!}\frac{t^2}{n} + \frac{1}{3!}\frac{pqt^3}{(\sqrt{npq})^3}(q-p) + \cdots\Big)^2 + \frac{1}{3}$$

$$\Big(\frac{1}{2!}\frac{t^2}{n} + \frac{1}{3!}\frac{pqt^3}{(\sqrt{npq})^3}(q-p) + \cdots\Big)^3 + \cdots\Big]$$

$$\log M_z(t) = \frac{1}{2}t^2 + O(n^{-1/2})$$

$$\lim_{n\to\infty} \log M_z(t) = \frac{t^2}{2}$$

$$\lim_{n\to\infty} M_z(t) = e^{t^2/2} \rightarrow (1)$$

As for normal distribution

$$M_x(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

For $Z = \frac{X-\mu}{\sigma}$

Then

$$M_z(t) = e^{(0)t + \frac{1}{2}t^2(1)^2}$$

$$M_z(t) = e^{t^2/2}$$

By (1)

$$\lim_{n\to\infty} M_z(t)_{\text{binomial}} = M_z(t)_{\text{Normal}}$$

$$\Rightarrow \lim_{n\to\infty} b(x; n, p) \simeq N(np, npq)$$

Correction For Continuity:-

If X is a discrete r.v. for which we want to use a continuous p.d.f as an approximation then we need to make a correction for continuity in the limits of interval for probability calculations. purpose.

Discrete	Continuous
$P(X=a)$	$P(a-0.5 \le X \le a+0.5)$

$P(a \leq X \leq b)$ $\quad$ $P(a-0.5 \leq X \leq b+0.5)$

$P(a < X < b)$ $\quad$ $P(a+0.5 \leq X \leq b-0.5)$

$P(X \geq a)$ $\quad$ $P(X \geq a-0.5)$

$P(X \leq a)$ $\quad$ $P(X \leq a+0.5)$

$P(X > a)$ $\quad$ $P(X \geq a+0.5)$

$P(X < a)$ $\quad$ $P(X \leq a-0.5)$

Ex:-

A fair coin is tossed 30 times, what is the prob of obtaining

(i) exactly 10 heads

(ii) less than 25 heads

(iii) at least 2 heads

(iv) between 10 and 20 heads (both inclusive)

Sol:-

X: No. of heads

$n = 30$, $p = P(\text{Head}) = \frac{1}{2}$, $q = \frac{1}{2}$

$$P(X = x) = {}^{30}C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{30-x}$$

$; x = 0, 1, \cdots, 30$

(i) $P(X = 10) = {}^{30}C_{10} \left(\tfrac{1}{2}\right)^{10} \left(\tfrac{1}{2}\right)^{20}$

$= 30045015 \times 9.765625 \times 10^{-4} \times 9.536 \times 10^{-7}$

$P(X = 10) = 0.028$

(ii)

$P(X < 25) = 1 - P(X \geq 25)$

$$= 1 - \left[{}^{30}C_{25}(1/2)^{25}(1/2)^{5} + {}^{30}C_{26}(1/2)^{26}(1/2\right.$$

$$+ {}^{30}C_{27}(1/2)^{27}(1/2)^{3} + {}^{30}C_{28}(1/2)^{28}(1/2)^{2} + {}^{30}C_{29}(1/2)^{29}(1/2)^{1}$$

$$\left. + {}^{30}C_{30}(1/2)^{30}(1/2)^{0}\right]$$

$$= 1 - \left(\tfrac{1}{2}\right)^{30}\left[{}^{30}C_{25} + {}^{30}C_{26} + {}^{30}C_{27} + {}^{30}C_{28} + {}^{30}C_{29}\right.$$

$$\left. + {}^{30}C_{30}\right]$$

$$= 1 - \left(\tfrac{1}{2}\right)^{30}\left[142506 + 27405 + 4060 + 435 +\right.$$

$$\left. 30 + 1\right] = 1 - \left(\tfrac{1}{2}\right)^{30}\left[174437\right]$$

$$= 0.9998$$

(iii)

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - \left[P(X=0) + P(X=1)\right]$$

$$= 1 - \left[{}^{30}C_{0}(1/2)^{0}(1/2)^{30} + {}^{30}C_{1}(1/2)^{1}(1/2)^{29}\right]$$

$$= 1 - (1/2)^{30}\left[{}^{30}C_{0} + {}^{30}C_{1}\right]$$

$$= 1 - (1/2)^{30}\left[1 + 30\right]$$

$$= 1 - 2.8871 \times 10^{-8}$$

$$= 0.999999$$

$$P(X \geq 2) \simeq 1$$

(iv)

$$P(10 \leq X \leq 20) = P(X=10) + P(X=11) + \cdots$$

$$+ P(X=20)$$

$$= {}^{30}C_{10}(1/2)^{10}(1/2)^{20} + {}^{30}C_{11}(1/2)^{11}(1/2)^{19} + {}^{30}C_{12}$$

$$(1/2)^{12}(1/2)^{18} + \cdots + {}^{30}C_{20}(1/2)^{20}(1/2)^{10} = (1/2)^{30}\left[{}^{30}C_{10} + \cdots + {}^{30}C_{20}\right] = 0.9572$$

Normal Approximation :-

$$X \overset{Approx}{\sim} N(np, npq)$$

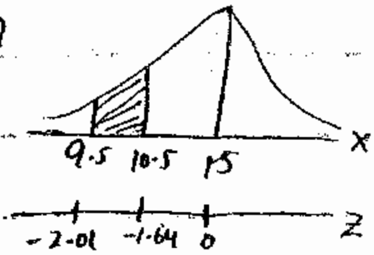
$$X \overset{Approx}{\sim} N(15, 7.5)$$

$$\mu = 15, \quad \sigma^2 = 7.5 \Rightarrow \sigma = 2.7386$$

(i) $P(X = 10)$

After continuity correction

$P(9.5 \le X \le 10.5)$



$$Z = \frac{X - \mu}{\sigma}$$

$$Z_1 = \frac{9.5 - 15}{2.7386} = -2.01$$

$$Z_2 = \frac{10.5 - 15}{2.7386} = -1.64$$

$$\begin{aligned} P(9.5 \le X \le 10.5) &= P(-2.01 \le Z \le -1.64) \\ &= P(-2.01 \le Z \le 0) - P(-1.64 \le Z \le 0) \\ &= 0.4778 - 0.4495 \\ &= 0.0283 \end{aligned}$$

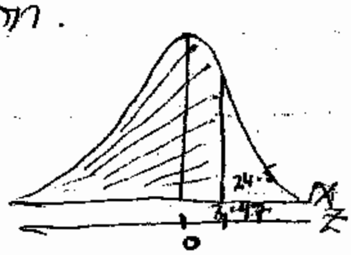
(ii)

$P(X < 25)$

After continuity correction.

$P(X \le 24.5)$

$$Z = \frac{24.5 - 15}{2.7386} = 3.47$$

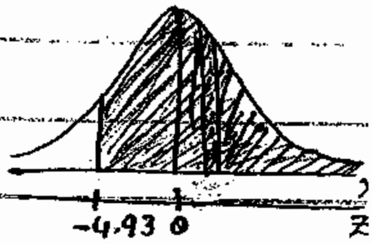


$$P(Z \leq 3.47) = 0.5 + P(0 \leq Z \leq 3.47)$$
$$= 0.5 + 0.4999$$
$$\simeq 0.9999$$

large "n" i.e $n \geqslant 30$
"p" is not small
if $np \geqslant 5$ then we use normal approximation.

(iii) $P(X \geqslant 2)$ After continuity correction
$P(X \geqslant 1.5)$

$$Z = \frac{1.5 - 15}{2.7386} = -4.93$$

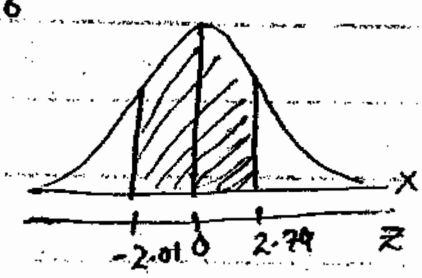


$$P(Z \geqslant -4.93) = 0.5 + P(-4.93 \leq Z \leq 0)$$
$$= 0.5 +$$

(iv) $P(10 \leq X \leq 20)$
After continuity correction
$P(9.5 \leq X \leq 20.5)$

$$Z_1 = \frac{9.5 - 15}{2.7386} = -2.01$$

$$Z_2 = \frac{20.5 - 15}{2.7386} = 2.7386$$



$$P(-2.01 \leq Z \leq 2.74) = P(-2.01 \leq Z \leq 0) + P(0 \leq Z \leq 2.74) = 0.4778 + 0.4969 = 0.9747$$

Normal Approximation to poisson distribution:-

$$\lim_{n\to\infty} P_0(x;\lambda) = N(\lambda,\lambda)$$

Proof:-

If x is a poisson r.v then we can standardize it as $Z = \dfrac{X-\lambda}{\sqrt{\lambda}}$

Now,

$$M_Z(t) = E(e^{tZ})$$

$$= E\left(e^{t\left(\frac{x-\lambda}{\sqrt{\lambda}}\right)}\right)$$

$$= e^{\frac{-\lambda t}{\sqrt{\lambda}}} E\left(e^{\frac{tx}{\sqrt{\lambda}}}\right)$$

$$= e^{-\sqrt{\lambda}t} E\left(e^{tx/\sqrt{\lambda}}\right)$$

$$= e^{-t\sqrt{\lambda}} \sum_{x=0}^{\infty} e^{tx/\sqrt{\lambda}} P_0(x)$$

$$= e^{-t\sqrt{\lambda}} \sum_{x=0}^{\infty} e^{\frac{tx}{\sqrt{\lambda}}} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-t\sqrt{\lambda}-\lambda} \sum_{x=0}^{\infty} \frac{e^{\frac{tx}{\sqrt{\lambda}}}\lambda^x}{x!}$$

$$= e^{-t\sqrt{\lambda}-\lambda} \sum_{x=0}^{\infty} \frac{\left(e^{t/\sqrt{\lambda}}\lambda\right)^x}{x!} \qquad \because \sum_{x=0}^{\infty} \frac{(n)^x}{x!} = e^n$$

$$\text{or } \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$M_z(t) = e^{-t\sqrt{\lambda}-\lambda}\, e^{\lambda e^{t/\sqrt{\lambda}}}$$

$$= e^{-t\sqrt{\lambda}-\lambda+\lambda e^{t/\sqrt{\lambda}}}$$

$$\log M_z(t) = \log_e e^{-t\sqrt{\lambda}-\lambda+\lambda e^{t/\sqrt{\lambda}}}$$

$$= -t\sqrt{\lambda}-\lambda+\lambda e^{t/\sqrt{\lambda}}$$

$$= -t\sqrt{\lambda}-\lambda+\lambda\left[1+\frac{t/\sqrt{\lambda}}{1!}+\frac{(t/\sqrt{\lambda})^2}{2!} + \cdots\right]$$

$$\log M_z(t) = -\cancel{t\sqrt{\lambda}} - \cancel{\lambda} + \cancel{\lambda} + \cancel{t\sqrt{\lambda}} + \frac{1}{2}t^2 + \frac{1}{3!}\frac{t^3}{\sqrt{\lambda}} + O(\lambda^{-1})$$

$$\lim_{\lambda\to\infty} \log M_z(t) = \lim_{\lambda\to\infty}\left[\frac{1}{2}t^2 + \frac{1}{3!}\frac{t^3}{\sqrt{\lambda}} + O(\lambda^{-1})\right]$$

$$\lim_{\lambda\to\infty} \log M_z(t) = \frac{1}{2}t^2$$

$$\lim_{\lambda\to\infty} M_z(t) = e^{\frac{1}{2}t^2} = M_{z_{normal}}(t)$$

Ex:-

A telephone exchange receives, on average, 5 calls per minute. Find the prob that in a 20 minutes period no more than 102 calls are received.

Sol:-

X: No. of calls received $\sim$ poisson.

Average = 5 calls per minute.

$\lambda = 5 \times 20$ calls per 20 minutes.

$\lambda = 100$

$X \sim P_o(100)$

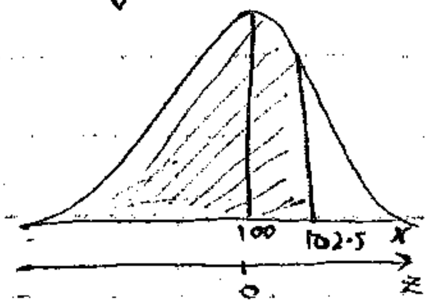
As $\lambda \to \infty$

$X \overset{\text{Approx}}{\sim} N(100, 100)$

$P(X \leq 102)$ After continuity correction

$P(X \leq 102.5)$

$Z = \dfrac{102.5 - 100}{\sqrt{100}}$



$Z = 0.25$

$P(X \leq 102.5) = P(Z \leq 0.25)$

$= 0.5 + P(0 \leq Z \leq 0.25)$

$= 0.5 + 0.0987$

$= 0.5987$

Ex

If X is $b(x; 20, 0.4)$ find $p(6 \leq X \leq 10)$. Also find the approximations to this prob. using (i) Poisson distribution (ii) Normal dist.

Sol

$P(X = x) = {}^{20}C_x \, (0.4)^x (0.6)^{20-x}$; $x = 0, 1, \ldots, 20$

$P(6 \leq X \leq 10) = P(X = 6) + P(X = 7) + \cdots + P(X = 10)$

$$P(6 \le X \le 10) = {}^{20}C_6 (0.4)^6 (0.6)^{14} + {}^{20}C_7 (0.4)^7 (0.6)^{13} + {}^{20}C_8 (0.4)^8 (0.6)^{12}$$
$$+ {}^{20}C_9 (0.4)^9 (0.6)^{11} + {}^{20}C_{10} (0.4)^{10} (0.6)^{10} = 0.124 + 0.166 + 0.1797 + 0.1597 + \ldots$$

$$P(6 \le X \le 10) = 0.7649$$

i)

$$X \overset{\text{Approx}}{\sim} P(x ; \lambda)$$

where $\lambda = np$

$$\lambda = 8 \Rightarrow P(X = x) = e^{-8} 8^x / x!$$

$$P(6 \le X \le 10) = P(X=6) + P(X=7) + \cdots + P(X=10)$$
$$= \frac{e^{-8} 8^6}{6!} + \frac{e^{-8} 8^7}{7!} + \frac{e^{-8} 8^8}{8!} + \frac{e^{-8} 8^9}{9!} + \frac{e^{-8} 8^{10}}{10!}$$
$$= e^{-8}(1862.055)$$

$$P(6 \le X \le 10) = 0.6246$$

ii)

$$X \overset{\text{Approx}}{\sim} N(\mu, \sigma^2)$$

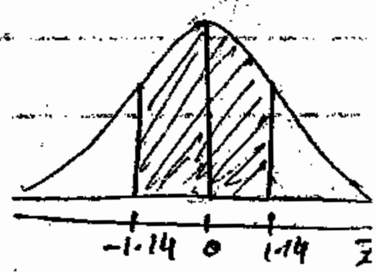
where $\mu = np = 8$

$$\sigma^2 = npq = 4.8$$

$$P(6 \le X \le 10) = P(5.5 \le X \le 10.5)$$

$$Z_1 = \frac{5.5 - 8}{\sqrt{4.8}} = -1.14$$

$$Z_2 = \frac{10.5 - 8}{\sqrt{4.8}} = 1.14$$



$$P(6 \le X \le 10) = P(-1.14 \le Z \le 1.14)$$
$$= P(-1.14 \le Z \le 0) + P(0 \le Z \le 1.14)$$
$$= 0.3729 + 0.3729$$

$P(6 \leq X \leq 10) = 0.7458$

Now $X \sim b(x; 20, 0.1)$

$$P(X = x) = {}^{20}C_x \, (0.1)^x (0.9)^{20-x} \; ; x = 0, 1, \ldots, 20$$

$$P(6 \leq X \leq 10) = P(X=6) + P(X=7) + \cdots + P(X=10)$$

$$= {}^{20}C_6 (0.1)^6 (0.9)^{14} + {}^{20}C_7 (0.1)^7 (0.9)^{13} + {}^{20}C_8$$

$$(0.1)^8 (0.9)^{12} + {}^{20}C_9 (0.1)^9 (0.9)^{11} + {}^{20}C_{10} (0.1)^{10} (0.9)^{10}$$

$P(6 \leq X \leq 10) = 0.0113$

i) Poisson distribution:-

$$X \overset{\text{Approx}}{\sim} P(x; \lambda)$$

where $\lambda = np = 20 \times 0.1 = 2$

$$P(X = x) = e^{-2} 2^x / x!$$

$$P(6 \leq X \leq 10) = P(X=6) + P(X=7) + \cdots + P(X=10)$$

$$= \frac{e^{-2} 2^6}{6!} + \frac{e^{-2} 2^7}{7!} + \frac{e^{-2} 2^8}{8!} + \frac{e^{-2} 2^9}{9!} + \frac{e^{-2} 2^{10}}{10!}$$

$$= e^{-2} \left[\frac{2^6}{6!} + \frac{2^7}{7!} + \frac{2^8}{8!} + \frac{2^9}{9!} + \frac{2^{10}}{10!} \right]$$

$P(6 \leq X \leq 10) = 0.0166$

(ii) Normal distribution:-

$$X \overset{\text{App}}{\sim} N(\mu, \sigma^2) \; ; \; \mu = np = 20 \times 0.1 = 2$$

$$\sigma^2 = npq = 20 \times 0.1 \times 0.9 = 1.8 \Rightarrow \sigma = 1.342$$

$$P(6 \leq X \leq 10) = P\left(\frac{6-2}{1.342} \leq Z \leq \frac{10-2}{1.342}\right)$$

$$P(2.98 \leq Z \leq 5.96) = P(0 \leq Z \leq 5.96) - P(0 \leq Z \leq 2.98)$$

$$= 0.5 - 0.4986 = 0.0014$$

28th Q.No5(a) (12 marks)

Properties of Mathematical Expectation:-

1) $E(c) = c$, where c is a constant.

proof:-

By definition

$$E(c) = \sum_x c f(x)$$
$$= c \sum_x f(x)$$

Since $\sum_x f(x) = 1$ (sum of all probabili

$$E(c) = c$$

2)

$$E(ax+b) = aE(x)+b$$

proof:-

By definition

$$E(ax+b) = \sum_x (ax+b)f(x)$$
$$= \sum_x axf(x) + \sum_x bf(x)$$

Since $\sum_x x f(x) = E(x)$, $\sum_x f(x) = 1$

$$= a\sum_x xf(x) + b\sum_x f(x)$$

$$E(ax+b) = aE(x)+b$$

3)

If X and Y are two variables having a joint prob. dist. $f(x,y)$ then

$$E(X+Y) = E(X) + E(Y)$$

proof:-

$$E(X+Y)=\sum_x \sum_y (x+y) f(x,y)$$

$$=\sum_x \sum_y x f(x,y) + \sum_x \sum_y y f(x,y)$$

$$=\sum_x x \sum_y f(x,y) + \sum_y y \sum_x f(x,y)$$

$\because \sum_y f(x,y) = g(x)$

$$=\sum_x x\, g(x) + \sum_y y\, h(y)$$

$$E(X+Y) = E(X) + E(Y)$$

4) If X and Y are two independent random variable with p.d $f(x,y)$ then $E(XY) = E(X)E(Y)$.

Proof:-

$$E(XY)=\sum_x \sum_y xy f(x,y)$$

If X and Y are independent r.Vs then $f(x,y) = g(x)h(y)$

$$\therefore E(XY) = \sum_x \sum_y xy\, g(x) h(y)$$

$$=\sum_x x\, g(x) \sum_y y\, h(y)$$

$$E(XY) = E(X)E(Y)$$

Ex:-

Let X and Y be two r.vs with joint p.d given as

$y \backslash x$	2	4	$h(y)$
1	0.1	0.15	
3	0.2	0.3	
5	0.1	0.15	

Find $E(X)$, $E(Y)$, $E(2X-3Y)$ and $E(XY)$.

Sol:-

$$E(X) = \sum_x x\, g(x) \rightarrow (1)$$

$$E(Y) = \sum_y y\, h(y) \rightarrow (2)$$

$y \backslash x$	2	4	$h(y)$
1	0.1	0.15	0.25
3	0.2	0.3	0.5
5	0.1	0.15	0.25
$g(x)$	0.4	0.6	1

put values in (1) and (2)

$$E(X) = 2(0.4) + 4(0.6)$$

$$= 0.8 + 2.4$$

$$E(X) = 3.2$$

$$E(Y) = 1(0.25) + 3(0.5) + 5(0.25)$$

$$E(Y) = 3$$

$$E(2X-3Y) = \sum_x \sum_y (2x-3y) f(x,y)$$

$$= [2(2)-3(1)](0.1) + [2(2)-3(3)](0.2) + [2(2)-3(5)](0.1) + [2(4)-3(1)](0.15) + [2(4)-3(3)](0.3) + [2(4)-3(5)](0.15)$$

$$E(2X-3Y) = (4-3)(0.1) + (4-9)(0.2) + (4-15)(0.1) + (8-3)(0.15) + (8-9)(0.3) + (8-15)(0.15)$$

$$= 0.1 + (-1) + (-1.1) + 0.75 + (-0.3) - 1.05$$

$$E(2X-3Y) = -2.6$$

By property

$$E(2X-3Y) = 2E(X) - 3E(Y)$$

$$= 2(3.2) - 3(3)$$

$$= -2.6$$

$$E(XY) = E(X)\,E(Y)$$

$$= 3.2 \times 3$$

$$E(XY) = 9.6$$

By Definition

$$E(XY) = \sum_x \sum_y xy f(x,y)$$

$$= 2(0.1) + 6(0.2) + 10(0.1) + 4(0.15) + 12(0.3) + 20(0.15)$$

$$E(XY) = 0.2 + 1.2 + 1 + 0.6 + 3.6 + 3$$
$$E(XY) = 9.6$$

CHAPTER

CORRELATION:-

The co-efficient of correlation describes the strength of the relationship between two sets of interval-scaled or ratio-scaled variables.

Strength of correlation = Co-efficient of correl.

It is denoted by r and is also called Pearson product moment correlation co-efficient and lies b/w ± 1.

Moments of uni-variate variable:-

A uni-variate variable $x = \{x_1, x_2, \dots, x_n\}$ has moments

Mean $= \bar{x} = \frac{\sum x}{n}$, variance $= S^2 = \frac{\sum (x-\bar{x})^2}{n} \geq 0$

Moments of Bi-variate variable:-

A Bi-variate variable X and Y

$\Rightarrow \{(x_i, y_i) \,;\, i = 1, 2, \dots, n\}$ has moments.

Covariance $= Cov(x, y) = \frac{\sum (X-\bar{X}) \sum (Y-\bar{Y})}{n}$

$Cov(X, Y) \in \mathbb{R}$

For same orientation of x and y

$Cov(x, Y)$ is +ve, otherwise $Cov(X, Y)$ is −ve.

same			opposite			No relation		
dx	dy	dxdy	dx	dy	dxdy	dx	dy	dxdy
+	+	+	+	−	−	+	+	+
		+	−	+	−	+	−	−

Standardized covariance :-

Standardized co-variance is

$$r = \frac{Cov(x, y)}{S_x S_y} = \frac{\Sigma(x-\bar{x})\Sigma(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \Sigma(y-\bar{y})^2}}$$

which is correlation and $r \in [-1, 1]$.

Example :-

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$	$(y-\bar{y})^2$
1	2	-2	-3	6	4	9
2	5	-1	0	0	1	0
3	3	0	-2	0	0	4
4	8	1	3	3	1	9
5	7	2	2	4	4	4
15	25	0	0	13	10	26

$$\bar{x} = \frac{\Sigma x}{n} = \frac{15}{5} = 3$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{25}{5} = 5$$

$$Cov(x, y) = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n}$$

$$Cov(x, y) = \frac{13}{5} = 2.6$$

$$r = \frac{Cov(x, y)}{S_x S_y} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \Sigma(y-\bar{y})^2}}$$

$$\gamma = \frac{13}{\sqrt{10 \times 26}}$$

$$\gamma = \frac{13}{\sqrt{260}} = 0.81$$

$\Rightarrow$ x and y have strong +ve correlation.

Property 1:-

Correlation is independent of change of origin and scale i.e $\gamma_{uv} = \gamma_{xy}$

Proof:-

we have to prove $\gamma_{xy} = \gamma_{uv}$

where $u = \frac{x-a}{b}$, $v = \frac{y-c}{d}$

$\Rightarrow x = a + ub$, $y = c + vd$

and $\bar{x} = a + \bar{u}b$, $\bar{y} = c + \bar{v}d$

Now, $\gamma_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 \sum(y-\bar{y})^2}}$

$$\gamma_{xy} = \frac{\sum(a+ub-a-\bar{u}b)(c+vd-c-\bar{v}d)}{\sqrt{\sum(a+ub-a-\bar{u}b)^2 \sum(c+vd-c-\bar{v}d)^2}}$$

$$\gamma_{xy} = \frac{bc\sum(u-\bar{u})(v-\bar{v})}{bc\sqrt{\sum(u-\bar{u})^2 \sum(v-\bar{v})^2}}$$

$\gamma_{xy} = \gamma_{uv}$

Ex:-

X	Y	$U=X-3$	$V=Y-3$	$U-\bar{U}$	$V-\bar{V}$	$(U-\bar{U})^2$	$(V-$
1	2	-2	-1	-2	-3	4	9
2	5	-1	2	-1	0	1	0
3	3	0	0	0	-2	0	4
4	8	1	5	1	3	1	9
5	7	2	4	2	+2	4	4
		0	10	0	0	10	26

$$r_{uv} = \frac{\sum(U-\bar{U})(V-\bar{V})}{\sqrt{\sum(U-\bar{U})^2 \sum(V-\bar{V})^2}} \rightarrow (1)$$

$$\bar{U} = \frac{\sum U}{n} = \frac{0}{5} = 0$$

$$\bar{V} = \frac{\sum V}{n} = \frac{10}{5} = 2$$

put all values in (1)

$$r_{uv} = \frac{13}{\sqrt{(10)(26)}}$$

$$r_{uv} = \frac{13}{\sqrt{260}} = \frac{13}{16.125}$$

$$r_{uv} = 0.806$$

Note:- Numeric value shows the relation b/w no. of students at pu and gold price but by common sense or theoratical they have no relation. This type is called con-founding effect.

REGRESSION:-

The technique used to develop the equation and provide the estimates is called regression analysis. An equation that defines linear relationship between two variables.

$$\text{Regression} \begin{cases} \text{Linear} \\ \text{Non-linear} \end{cases}$$

Response = Signal + Noise
= 99% + 1%

As Noise is of (0.1) 1%. So, we have strong response.

Linear Regression:-

Let we have n-observations $\{(x_i, y_i);\ i=1,2,\dots,n\}$, where Y is the response (dependent) variable and X is the predictor (independent) variable, then a simple linear regression model is defined as,

$$Y_i = \underbrace{\alpha + \beta X_i}_{\text{signal}} + \underbrace{\epsilon_i}_{\text{Noise}} \quad ; \ i=1,2,\dots,n$$

where α : intercept

β : regression co-efficient (Slope)

ϵ_i : Error term ; $\Sigma \epsilon_i = 0$

"α" and "β" are the parameters which need to be estimated for fitting the regression model which can later be used for prediction purpose.

Method of least squares:-

we define sum of square of error as an objective function and the parameters 'α' and "β" are estimated by minimizing this objective function.

Let us define

$$S = \sum_{i=1}^{n} \epsilon_i^2$$

$$S = \sum_{i=1}^{n} (Y_i - \alpha - \beta X_i)^2 \rightarrow (1)$$

partially differentiate (1) w.r.t "α".

$$\frac{\partial S}{\partial \alpha} = -2 \sum_{i=1}^{n} (Y_i - \alpha - \beta X_i)$$

$$\frac{\partial^2 S}{\partial^2 \alpha} = 2 > 0$$

put $\frac{\partial S}{\partial \alpha} = 0$

$$0 = -2 \sum_{i=1}^{n} (y_i - \hat{\alpha} - \hat{\beta} x_i)$$

$$\Rightarrow \sum_{i=1}^{n} y_i - n\hat{\alpha} - \hat{\beta} \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i = n\hat{\alpha} + \hat{\beta} \sum_{i=1}^{n} x_i \rightarrow (2)$$

Partially Differentiate (1) w.r.t "β"

$$\frac{\partial S}{\partial \beta} = -2 \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)\, x_i$$

$$= -2 \sum_{i=1}^{n} x_i (y_i - \alpha - \beta x_i)$$

$$\frac{\partial^2 S}{\partial^2 \beta} = 2 \sum_{i=1}^{n} x_i^2 > 0$$

put $\frac{\partial S}{\partial \beta} = 0$

$$0 = -2 \sum_{i=1}^{n} x_i (y_i - \hat{\alpha} - \hat{\beta} x_i)$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i - \hat{\alpha} \sum_{i=1}^{n} x_i - \hat{\beta} \sum_{i=1}^{n} x_i^2$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i = \hat{\alpha} \sum_{i=1}^{n} x_i + \hat{\beta} \sum_{i=1}^{n} x_i^2 \rightarrow (3)$$

Equation (2) and (3) are simultaneous equations involving unknown quantities $\hat{\alpha}$ and $\hat{\beta}$. These equations are called "Normal equations" and unique solution can easily be obtained by solving them simultaneously.

Take eq (2)

$$\Sigma Y = n\hat{\alpha} + \hat{\beta}\Sigma X$$

Divide both sides by "n"

$$\frac{\Sigma Y}{n} = \frac{n\hat{\alpha}}{n} + \hat{\beta}\frac{\Sigma X}{n}$$

$$\Rightarrow \quad \bar{Y} = \hat{\alpha} + \hat{\beta}\bar{X}$$

$$\Rightarrow \quad \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X} \rightarrow (4)$$

put in (3)

$$\Sigma XY = (\bar{Y} - \hat{\beta}\bar{X})\Sigma X + \hat{\beta}\Sigma X^2$$

$$\Sigma XY = \bar{Y}\Sigma X - \hat{\beta}\bar{X}\Sigma X + \hat{\beta}\Sigma X^2$$

$$= \bar{Y}\Sigma X + \hat{\beta}(\Sigma X^2 - \bar{X}\Sigma X)$$

$$\Rightarrow \quad \hat{\beta} = \frac{\Sigma XY - \bar{Y}\Sigma X}{\Sigma X^2 - \bar{X}\Sigma X}$$

As $\bar{X} = \frac{\Sigma X}{n}$, $\bar{Y} = \frac{\Sigma Y}{n}$

$$\Rightarrow \quad \Sigma X = n\bar{X} \quad , \quad \Sigma Y = n\bar{Y}$$

Then,

$$\therefore \quad \hat{\beta} = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma X^2 - n\bar{X}^2} \rightarrow (5)$$

or

$$\hat{\beta} = \frac{\Sigma XY - \frac{\Sigma Y \Sigma X}{n}}{\sqrt{\Sigma X^2 - \frac{\Sigma X \Sigma X}{n}}}$$

$$\hat{\beta} = \frac{\frac{n\Sigma XY - \Sigma X \Sigma Y}{n}}{\frac{n\Sigma X^2 - (\Sigma X)^2}{n}}$$

$$\Rightarrow \hat{\beta} = \frac{n\Sigma XY - \Sigma X \Sigma Y}{n\Sigma X^2 - (\Sigma X)^2}$$

Put value of $\hat{\beta}$ from (5) in (4)

$$\hat{\alpha} = \bar{Y} - \left(\frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma X^2 - n\bar{X}^2}\right)\bar{X}$$

$$= \frac{\bar{Y}(\Sigma X^2 - n\bar{X}^2) - (\Sigma XY - n\bar{X}\bar{Y})\bar{X}}{\Sigma X^2 - n\bar{X}^2}$$

$$\hat{\alpha} = \frac{\bar{Y}\Sigma X^2 - n\bar{X}^2\bar{Y} - \bar{X}\Sigma XY + n\bar{X}^2\bar{Y}}{\Sigma X^2 - n\bar{X}^2}$$

$$\hat{\alpha} = \frac{\bar{Y}\Sigma X^2 - \bar{X}\Sigma XY}{\Sigma X^2 - n\bar{X}^2} \rightarrow (6)$$

Ex :-

Fit a least square regression line for x and y

x	1	2	3	4	5
y	2	5	3	8	7

Sol :-

1 Simple linear regression model is

$$Y_i = \alpha + \beta X_i + \epsilon_i \; ; \; i = 1, 2, \ldots, n$$

Let $\hat{\alpha}$ and $\hat{\beta}$ are the least squares estimates of α and β respectively, then

$$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$$

where

$$\hat{\beta} = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma X^2 - n\bar{X}^2}$$

Now

	X	Y	XY	X^2	$\hat{Y}$	$\hat{\epsilon}$
	1	2	2	1	2.4	-0.4
	2	5	10	4	3.7	1.3
	3	3	9	9	5	-2
	4	8	32	16	6.3	1.7
	5	7	35	25	7.6	-0.6
Σ	15	25	88	55		0

$$\bar{X} = \frac{\Sigma X}{n} = \frac{15}{5} = 3$$

$$\bar{y} = \frac{\sum y}{n} = \frac{25}{5} = 5$$

Now $\hat{\beta} = \frac{88 - 5(3)(5)}{55 - 5(3)^2}$

$$\hat{\beta} = \frac{88 - 75}{55 - 45} = \frac{13}{10} = 1.3$$

$$\hat{\beta} = 1.3$$

$$\hat{\alpha} = 5 - (1.3)(3)$$

$$= 5 - 3.9$$

$$\hat{\alpha} = 1.1$$

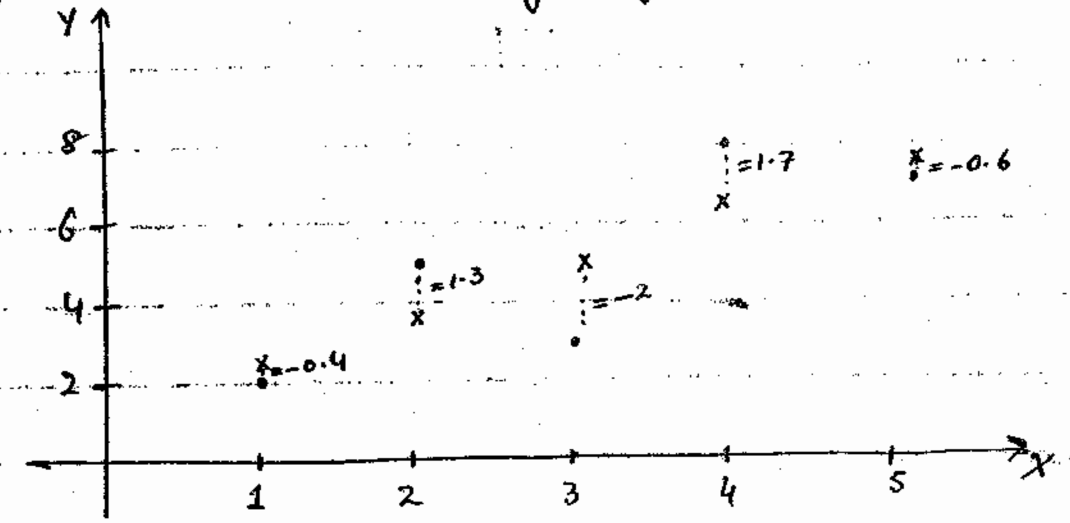
$\therefore$ fitted (estimated) model is

$$\hat{y}_i = \hat{\alpha} + \hat{\beta} x_i \quad ; \; i = 1, 2, \cdots, n$$

$$\Rightarrow \hat{y}_i = 1.1 + 1.3 x_i$$

Residual $= \hat{\epsilon} = y - \hat{y}$

and sum of residual $\hat{\epsilon}$ of least squares is always zero.



Sum of squares of residuals $= \sum \hat{\epsilon}_i^2$

$$\sum \hat{\epsilon}_i^2 = 0.16 + 1.69 + 4 + 2.89 + 0.36 = 9.1$$

NOTE:-

Sum of residual $\hat{\epsilon} = y - \hat{y}$ is always zero for only least square meth

2 To predict X on the basis of Y. So, the simple linear regression model is

$$x_i = \alpha_0 + \beta_0 y_i + \gamma_i \quad ; i = 1, 2, \dots, n$$

Let $\hat{\alpha}_0$ and $\hat{\beta}_0$ are least squares estimates of α_0 and β_0 respectively.

$$\hat{\alpha}_0 = \bar{X} - \hat{\beta}_0 \bar{Y}$$

$$\hat{\beta}_0 = \frac{\sum XY - n \bar{X} \bar{Y}}{\sum Y^2 - n \bar{Y}^2}$$

Y^2
4
25
9
64
49
151

$$\hat{\beta}_0 = \frac{88 - 5(3)(5)}{151 - 5(5)^2}$$

$$= \frac{88 - 75}{151 - 125} = \frac{13}{26}$$

$$\hat{\beta}_0 = 0.5$$

$$\hat{\alpha}_0 = 3 - 0.5(5)$$

$$\hat{\alpha}_0 = 0.5$$

$\therefore$ fitted (estimated) model of X on Y is

$$\hat{X}_i = 0.5 + 0.5 Y_i$$

Ex:- N/2009 Q8(a)

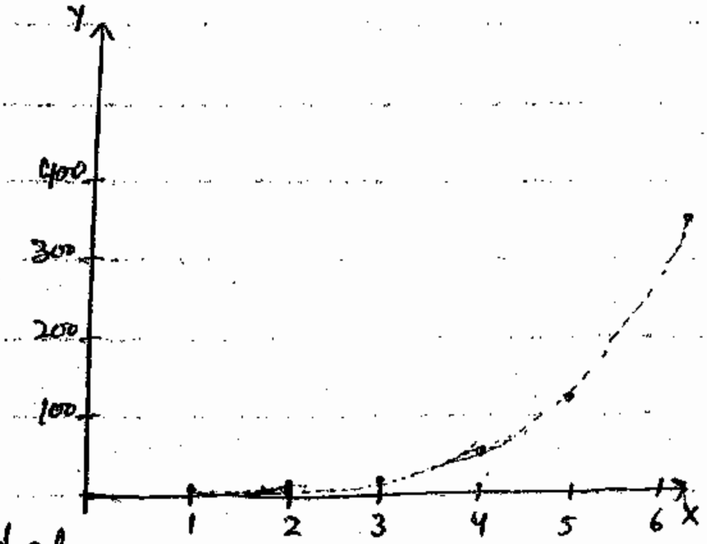
Fit an exponential curve $y = ae^{bx}$ to the following data.

x	1	2	3	4	5	6
y	1.6	4.5	11.8	40.2	125	363

Sol:-

$y = ae^{bx}$ is a non-linear model.

First, we transform the given exponential model to a linear model



by taking log on both sides.

$$\log_e y = \log_e (ae^{bx})$$

$$\log_e y = \log_e a + bx \log_e e$$

$$\log_e y = \log_e a + bx$$

$$\Rightarrow \quad y' = a' + bx$$

where $y' = \log_e y$, $a' = \log_e a$.

$$\hat{b} = \frac{\sum xy' - n\bar{x}\bar{y}'}{\sum x^2 - n\bar{x}^2}$$

$$\hat{a}' = \bar{y}' - \hat{b}\bar{x}$$

x	y	y'	xy'	x^2
1	1.6	0.2041	0.2041	1
2	4.5	0.65321	1.3064	4
3	11.8	1.07188	3.2156	9
4	40.2	1.6042	6.4168	16
5	125	2.0969	10.4845	25
6	363	2.5599	15.3594	36
21		8.19019	36.9868	91

$\bar{x} = \sum x/n = 21/6 = 3.5$, $\bar{y}' = \frac{\sum y'}{n} = 1.365$

$$\hat{b} = \frac{36.9868 - 6(3.5)(1.365)}{91 - 6(3.5)^2}$$

$$\hat{b} = \frac{36.9868 - 28.665}{91 - 6(12.25)}$$

$$\hat{b} = \frac{8.3218}{17.5}$$

$$\hat{b} = 0.47555 = 0.476$$

$$\hat{a}' = 1.365 - 0.476(3.5) = -0.301$$

$a = \text{antilog}(-0.301) = 0.5$

The fitted curve is $y = 0.5\, e^{0.476x}$

Regression lines:-

$$y = a + bx \quad ; \; b = b_{yx}$$

is regression line of Y on X.

$$x = a_0 + b_{xy}\, x$$

is regression line of X on Y.

1) b_{yx}, b_{xy} and r have same signs.
2) r is the geo-metric mean of b_{yx} and b_{xy}

i.e

$$r = \sqrt{b_{yx} \times b_{xy}}$$

Ex:-

Y on X : $\hat{Y} = 1.1 + 1.3\hat{X} \Rightarrow b_{yx} = 1.3$

X on Y : $\hat{X} = 0.5 + 0.5\hat{X} \Rightarrow b_{xy} = 0.5$

by

$$r = \sqrt{1.3 \times 0.5} = 0.8$$

Notes:-

In Quantitative variable, zero is arbitrary value. Somewhere zero has importance and some-where not. e.g "0-weight" means nothing but 0°c temp has value.

A/2009 Q9(C):-

Co-variance $(X, Y) = 8$

$\sigma_x = 5, \quad \sigma_y = 5$

Find Corr(x, y)

Sol:-

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_x \sigma_y}$$

$$= \frac{8}{5.5} = \frac{8}{25}$$

$Corr(X, Y) = 0.32$

Scales:-

Variable

Qualitative (categorical) $\downarrow$ | Quantitative $\downarrow$

1 Nominal (Distinction)
- Gender (1=F, 2=M)
- Nationality

1 Interval (Arbitrary Zero (i.e.) Temp 0°C)

2 Ordinal (Natural Ranking) (Designation)
- Income group
- Educational Qualification
(M=1, I=2, B=3, M=4
$\Rightarrow$ (1<2<3<4)

2 Ratio (Weight)

Rank Correlation:- (variables are on ordinal scales)

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

where $d = |X - Y|$ and X and Y are ranks, r_s: Spearman's rank correlation.

Ex:-

Stdnts	Statistic	Math	d=X-Y	d^2
1	1	2	-1	1
2	2	4	-2	4
3	3	3	0	0
4	4	1	3	9
5	5	7	-2	4
6	6	5	1	1
7	7	8	-1	1
8	8	10	-2	4
9	9	6	3	9
10	10	9	1	1
			0	34

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$= 1 - \frac{6(34)}{10(100-1)}$$

$$= 1 - \frac{204}{990}$$

$$r_s = 0.794$$

The correlation will be 1 when $X=Y$.

Pearson Product Correlation formulas:-

$$r = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}$$

$$r = \frac{\sum(X-\bar{X})(Y-\bar{Y})}{\sqrt{\sum(X-\bar{X})^2\sum(Y-\bar{Y})^2}}$$

$$r = \frac{\sum XY - n\bar{X}\bar{Y}}{\sqrt{(\sum X^2 - n\bar{X}^2)(\sum Y^2 - n\bar{Y}^2)}}$$

$$r = \frac{\sum XY - (\sum X)(\sum Y)/n}{\sqrt{\left(\sum X^2 - \frac{(\sum X)^2}{n}\right)\left(\sum Y^2 - \frac{(\sum Y)^2}{n}\right)}}$$

$$r = \frac{n\sum XY - (\sum X)(\sum Y)}{\sqrt{(n\sum X^2 - (\sum X)^2)(n\sum Y^2 - (\sum Y)^2)}}$$

Derivation of rank correlation-co-efficient

Suppose, we have "n" pairs of observations $\{(x_i, y_i);\ i=1,2,\cdots,n\}$ then correlation co-efficient between X and Y is given as

$$r = \frac{\Sigma XY - (\Sigma X)(\Sigma Y)/n}{\sqrt{\left(\Sigma X^2 - \frac{(\Sigma X)^2}{n}\right)\left(\Sigma Y^2 - \frac{(\Sigma Y)^2}{n}\right)}} \rightarrow (1);$$

If X and Y are ranked data then both of X and Y consists of numbers ranging '1' to 'n'.

Thus, $\sum_{i=1}^{n} X_i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$$\Sigma Y = \Sigma X = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} X_i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Sigma X^2 = \Sigma Y^2$$

To calculate ΣXY, Take $d = X-Y$

$$\Rightarrow d^2 = (X-Y)^2$$

$$\Sigma d^2 = \Sigma (X-Y)^2$$

$$= \Sigma (X^2 + Y^2 - 2XY)$$

$$\Sigma d^2 = \Sigma X^2 + \Sigma Y^2 - 2\Sigma XY$$

$$= \Sigma X^2 + \Sigma X^2 - 2\Sigma XY$$

$$\Sigma d^2 = 2\Sigma X^2 - 2\Sigma XY$$

$$\frac{\Sigma d^2}{2} = \Sigma X^2 - \Sigma XY$$

$$\Rightarrow \Sigma XY = \Sigma X^2 - \frac{\Sigma d^2}{2}$$

$$\Rightarrow \Sigma XY = \left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{\Sigma d^2}{2}$$

put all values in (1)

$$\gamma = \frac{\left[\frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\Sigma d^2\right] - \left[\frac{n(n+1)}{2}\right]^2/n}{\sqrt{\left[\frac{n(n+1)(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2/n\right]\left[\frac{n(n+1)(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2/n\right]}}$$

$$\gamma = \frac{\frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\Sigma d^2 - \frac{n^2(n+1)^2}{4n}}{\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4n}}$$

$$\Rightarrow \gamma = \frac{\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4n} - \frac{1}{2}\Sigma d^2}{\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4n}}$$

$$\Rightarrow \gamma = 1 - \frac{\frac{1}{2}\Sigma d^2}{\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4n}} \rightarrow (2)$$

$$\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4n} = \frac{n(n+1)}{2}\left[\frac{2n+1}{3} - \right.$$

$$\frac{n(n+1)}{2n}\Big]$$

$$\Rightarrow \frac{n(n+1)}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$\Rightarrow \frac{n(n+1)}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$\Rightarrow \frac{n(n+1)}{2}\left[\frac{n-1}{6}\right]$$

$$= +\frac{1}{6}\,\frac{n(n+1)(n-1)}{2} = \frac{1}{12}n(n^2-1)$$

So, $\dfrac{n(n+1)(2n+1)}{6} - \dfrac{n^2(n+1)^2}{4n} = \dfrac{1}{12}n(n^2-1)$

put in (2)

$$r_s = 1 - \frac{\frac{1}{2}\Sigma d^2}{\frac{1}{12}n(n^2-1)}$$

$$r_s = 1 - \frac{12}{2}\frac{\Sigma d^2}{n(n^2-1)}$$

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

is rank correlation co-efficient.

Ex :-

Find the pearson product correlation co-efficient of the following data.

Sol:-

As $\quad r = \dfrac{\sum(X-\bar{X})(Y-\bar{Y})}{\sqrt{\sum(X-\bar{X})^2 \sum(Y-\bar{Y})^2}} \rightarrow (1)$

Stdnts	Math X	Stats Y	$X-\bar{X}$	$Y-\bar{Y}$	$(X-\bar{X})^2$	$(Y-\bar{Y})^2$
1	1	2	-4.5	-3.5	20.25	12.25
2	2	4	-3.5	-1.5	12.25	2.25
3	3	3	-2.5	-2.5	6.25	6.25
4	4	1	-1.5	-4.5	2.25	20.25
5	5	7	-0.5	1.5	0.25	2.25
6	6	5	0.5	-0.5	0.25	0.25
7	7	8	1.5	2.5	2.25	6.25
8	8	10	2.5	4.5	6.25	20.25
9	9	6	3.5	0.5	12.25	0.25
10	10	9	4.5	3.5	20.25	12.25
Σ	55	55			82.5	82.5

$\bar{X} = \dfrac{\sum X}{n} = \dfrac{55}{10} = 5.5$

$\bar{Y} = \dfrac{\sum Y}{n} = \dfrac{55}{10} = 5.5$

$\sum(X-\bar{X})(Y-\bar{Y})$	
15.75	-0.25
5.25	3.75
6.25	11.25
6.75	1.75
-0.75	15.75
65.5	

put all values in (1)

$r = \dfrac{65.5}{\sqrt{(82.5)(82.5)}} = \dfrac{65.5}{82.5}$

$r = 0.794$

is same as $r_s = 0.794$

Ex:-

Stdnts	Eco x	phy y	Rank x (x')	Rank y (y')	$x'-y'=d$	d^2
1	36	62	8	2	6	36
2	56	42	4	9	-5	25
3	41	60	7	3	4	16
4	46	53	5	4	1	1
5	59	46	3	6	-3	9
6	45	50	6	5	1	1
7	65	43	2	8	-6	36
8	31	66	9	1	8	64
9	68	44	1	7	-6	36
						224

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2-1)} \rightarrow (1)$$

$n = 9$

put all values

$$r_s = 1 - \frac{6(224)}{9(9^2-1)}$$

$$= 1 - \frac{1344}{9(81-1)}$$

$$= 1 - \frac{1344}{9(80)}$$

$$= 1 - \frac{1344}{720}$$

$\Rightarrow r_s = -0.8667$

Now, use simple correlation co-efficient

$$r = \frac{\sum(X-\bar{X})(Y-\bar{Y})}{\sqrt{\sum(X-\bar{X})^2 \sum(Y-\bar{Y})^2}} \rightarrow (1)$$

X	Y	$X-\bar{X}=I$	$Y-\bar{Y}=J$	$(X-\bar{X})^2$	$(Y-\bar{Y})^2$	IJ
36	62	-13.67	10.22	186.8689	104.4484	-139.71
56	42	6.33	-9.78	40.0689	95.6484	-61.91
41	60	-8.67	8.22	75.1689	67.5684	-71.26
46	53	-3.67	1.22	13.4689	1.4884	-4.47
59	46	9.33	-5.78	87.0489	33.4084	-53.93
45	50	-4.67	-1.78	21.8089	3.1684	8.3126
65	43	15.33	-8.78	235.0089	77.0884	-134.59
31	66	-18.67	14.22	348.5689	202.2084	-265.49
68	44	18.33	-7.78	335.9889	60.5284	-142.61
447	466			1344.0001	645.56	-865.6784

$$\bar{X} = \frac{\sum X}{n} = \frac{447}{9} = 49.67, \quad \bar{Y} = \frac{\sum Y}{n} = \frac{466}{9} = 51.78$$

Put all values in (1)

$$r = \frac{-865.6784}{\sqrt{(1344.0)(645.56)}}$$

$$r = \frac{-865.6784}{931.46800}$$

$$r = -0.9294$$

Rank Correlation for tied ranks:-

Suppose we have "n" pairs of ranks $\{(x_i, y_i); i=1,2,\dots,n\}$ and their exist "m" ties in the data, the rank correlation can be calculated as

$$r_s = 1 - \frac{6(\Sigma d^2 + T)}{n(n^2-1)}$$

where $T = \frac{1}{12} \sum_{j=1}^{m} (t_j^3 - t_j)$

and t_j = No. of observations having tie at a specific rank

Ex:-

Given the following ranked data, calculate the rank correlation co-efficient

Sol:-

x	y	d=x-y	d^2
8	8	0	0
3✓	9	-6	36
6.5*	6.5*	0	0
3✓	2.5✓	0.5	0.25
6.5*	4	2.5	6.25
9	5	4	16
3✓	6.5*	-3.5	12.25

1	1	0	0
5	2.5 ✓	2.5	6.25
			77

m = No. of ties = 4

t_1 = No. of observations in X tied at 3 = 3

t_2 = No. of observations in X tied at 6.5 = 2

t_3 = No. of observations in y tied at 2.5 = 2

t_4 = No. of observations in Y tied at 6.5 = 2

$$T = \frac{1}{12}\sum_{j=1}^{m}(t_j^3 - t_j)$$

$$= \frac{1}{12}\sum_{j=1}^{4}(t_j^3 - t_j)$$

$$= \frac{1}{12}\left[(t_1^3 - t_1) + (t_2^3 - t_2) + (t_3^3 - t_3) + (t_4^3 - t_4)\right]$$

$$= \frac{1}{12}\left[(3^3 - 3) + (2^3 - 2) + (2^3 - 2) + (2^3 - 2)\right]$$

$$= \frac{1}{12}((27 - 3) + (8 - 2) + (8 - 2) + (8 - 2))$$

$T = 3.5$

$$r_s = 1 - \frac{6(\Sigma d^2 + T)}{n(n^2 - 1)}$$

$$= 1 - \frac{6(77 + 3.5)}{9(81 - 1)} = 1 - \frac{483}{720}$$

$$r_s = 1 - 0.671 = 0.329$$

Ex8-

Entry	J.X	J.Y	J.Z	$d_{xy}=x-y$	$d_{xz}=x-z$
A	5	1	6	4	-1
B	2	7	4	-5	-2
C	6	6	9	0	-3
D	8	10	8	-2	0
E	1	4	1	-3	0
F	7	5	2	2	5
G	4	3	3	1	1
H	9	8	10	1	-1
K	3	2	5	1	-2
L	10	9	7	1	3

$d_{yz}=y-z$	d^2_{xy}	d^2_{xz}	d^2_{yz}
-5	16	1	25
3	25	4	9
-3	0	9	9
2	4	0	4
3	9	0	9
3	4	25	9
0	1	1	0
-2	1	1	4
-3	1	4	9
2	1	9	4
	62	54	82

Now

$$r_{xy} = 1 - \frac{6\Sigma d^2_{xy}}{n(n^2-1)}$$

$$= 1 - \frac{6(62)}{10(100-1)}$$

$$r_{xy} = 1 - \frac{372}{990}$$

$$r_{xy} = 0.624$$

$$r_{xz} = 1 - \frac{6\Sigma d^2_{xz}}{n(n^2-1)}$$

$$= 1 - \frac{6(54)}{10(99)}$$

$$r_{xz} = 1 - \frac{324}{990} = 0.6727$$

$$r_{yz} = 1 - \frac{6\Sigma d_{yz}^2}{n(n^2-1)} = 1 - \frac{6(82)}{990} = 1 - \frac{492}{990}$$

$r_{yz} = 0.503$

Co-efficient of determination:-

Suppose we have a linear model $Y = \alpha + \beta X + \epsilon$

Total variation $= \sum_{i=1}^{n} (Y_i - \bar{Y})^2$

Also, assume the fitted model is

$$\hat{Y} = a + bX$$

where a and b are the estimates of α and β

Explained variation $= \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2$

Un-explained variation = Total variation − Explained variation

we can define co-efficient of determination as

$$r^2 = \frac{\text{Explained variation}}{\text{Total variation}}$$

$$r^2 = \frac{\Sigma(\hat{Y} - \bar{Y})^2}{\Sigma(Y - \bar{Y})^2}$$

"The proportion of the total variation in the dependent variable Y that is explained, or accounted for, by the variation in the independent variable X.

Unexplained variation $= \Sigma(y-\bar{y})^2 - \Sigma(\hat{y}-\bar{y})^2$

$= \Sigma(y-\hat{y})^2$

As $\quad r^2 = \dfrac{\Sigma(\hat{y}-\bar{y})^2}{\Sigma(y-\bar{y})^2}$

$$= \frac{\Sigma(\hat{y}-\bar{y})^2}{\Sigma(\hat{y}-\bar{y})^2+\Sigma(y-\hat{y})^2}$$

If the model is a good fit, ideally, $\hat{y}=y$ then $\Sigma(y-\hat{y})^2=0$

then

$$r^2 = \frac{\Sigma(\hat{y}-\bar{y})^2}{\Sigma(\hat{y}-\bar{y})^2}$$

$$\Rightarrow r^2 = 1$$

$$\Rightarrow r = \pm 1$$

If the fitted model is a simple average of observed values of y (i.e.) the model is not utilizing the information of predictor X, then $\hat{y}=\bar{y}$.

$\Rightarrow \Sigma(\hat{y}-\bar{y})^2=0$

$$\therefore r^2 = \frac{0}{\Sigma(y-\hat{y})^2} = 0$$

$\Rightarrow \quad r = 0$

Thus $\quad 0 \leq r^2 \leq 1$ and $-1 \leq r \leq 1$

Standard Error of Estimate:-

Suppose, we have a regression model

$$Y = \alpha + \beta X + \epsilon$$

Also assume the fitted model is

$$\hat{y} = a + bx$$

where a and b are the estimates of α and β.

Then, standard error of estimate is defined as

$$s_{y \cdot x} = \sqrt{\frac{\Sigma(y-\hat{y})^2}{n-2}}$$

Take $\quad \Sigma(y-\hat{y})^2 = \Sigma(y-a-bx)^2$

$$= \Sigma(y-a-bx)(y-a-bx)$$

$$= \Sigma[y^2 - ay - bxy - ay + a^2 - abx - bxy + abx + b^2xy]$$

$$= [\Sigma y^2 - a\Sigma y - b\Sigma xy] - a[\Sigma y - na - b\Sigma x] - b[\Sigma xy - a\Sigma x - b\Sigma xy] \rightarrow (1)$$

Due to the result of normal eqs for model $Y = \alpha + \beta x + \epsilon$

$$\Sigma y = na + b\Sigma x$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2$$
$$\Rightarrow \Sigma y - na - b\Sigma x = 0$$
$$\Rightarrow \Sigma xy - a\Sigma x - b\Sigma x^2 = 0$$

put in (i)

$$\Rightarrow \Sigma(y-\hat{y})^2 = (\Sigma y^2 - a\Sigma y - b\Sigma xy) + 0 + 0$$
$$= \Sigma y^2 - a\Sigma y - b\Sigma xy$$

Then,

$$S_{y \cdot x} = \sqrt{\frac{\Sigma y^2 - a\Sigma y - b\Sigma xy}{n-2}}$$

(i)

when true values of α and β are given then standard error of estimate is

$$S_{y \cdot x} = \sqrt{\frac{\Sigma y^2 - a\Sigma y - b\Sigma xy}{n}}$$

(ii)

when true values of one of α and β are given and one of them we estimate α or β then standard error of estimate is

$$S_{y \cdot x} = \sqrt{\frac{\Sigma y^2 - a\Sigma y - b\Sigma xy}{n-1}}$$

Ex:-

X	Y	$\hat{Y}$	$(Y-\bar{Y})^2$	$(\hat{Y}-\bar{Y})^2$	$(Y-\hat{Y})^2$
1	2	2.4	9	6.76	0.16
2	5	3.7	0	1.69	1.69
3	3	5	4	0	4
4	8	6.3	9	1.69	2.89
5	7	7.6	4	6.76	0.36
			26	16.86	9.1

The fitted regression model is

$$\hat{y} = 1.1 + 1.3X$$

$$\Sigma Y = \frac{25}{5} = 5$$

$$\text{Total variation} = \Sigma(Y-\bar{Y})^2 = 26$$

$$\text{Explained variation} = \Sigma(\hat{Y}-\bar{Y})^2 = 16.9$$

$$\text{Unexplained variation} = \Sigma(Y-\hat{Y})^2 = 9.1$$

Total variation = Explained + Unexplained

$$26 = 16.9 + 9.1$$

r^2 = co-efficient of determination

$$r^2 = \frac{\text{Explained variation}}{\text{Total variation}} = \frac{16.9}{26} = 0.65$$

$$s_{y.x} = \sqrt{\frac{\Sigma(Y-\hat{Y})^2}{n-2}} = \sqrt{\frac{9.1}{5-2}} = \sqrt{\frac{9.1}{3}}$$

$$s_{y.x} = 1.74$$

If $x = 6$ Then, $\hat{y} = 1.1 + 1.3(6) = 8.9$
and $s_{y.x} = 1.74$

Then, we have an interval in which true value of y is contained, The interval is

$$(8.9 - 1.74,\ 8.9 + 1.74) = (7.16,\ 10.64)$$

in which true value of y contained.

Multiple Regression :-

y : Response variable

$$X \longrightarrow \boxed{f} \longrightarrow Y$$

$$\begin{matrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{matrix} \longrightarrow \boxed{f} \longrightarrow Y$$

(i.e) Age $\longrightarrow \boxed{f} \longrightarrow$ Height

$$\begin{matrix} \text{Age} \\ \text{Parent's Height} \\ \text{Nutrition} \end{matrix} \longrightarrow \boxed{f} \longrightarrow \text{Height}$$

$X_1, X_2, \ldots, X_k$: Predictors

so, $$y = \underbrace{\alpha + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k}_{\text{signal}} + \underbrace{\varepsilon}_{\text{Noise}}$$

Suppose $k=2$

$$y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \epsilon$$

we want to estimate α, β_1 and β_2.

If we have a sample of "n" observations $\{(\underline{x}_i^T, y_i) ; i = 1, 2, \ldots, n\}$

where $\underline{x} = (X_1, X_2)$

we can define error as

$$\epsilon_i = y_i - \alpha - \beta_1 X_{1i} - \beta_2 X_{2i} \quad ; \quad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - \alpha - \beta_1 X_{1i} - \beta_2 X_{2i})^2 = S \rightarrow (1)$$

we can obtain least square estimates of α, β_1, and β_2 by minimizing $\sum \epsilon_i^2$ w.r.t α, β_1 and β_2

Partially differentiate (1) w.r.t "α"

$$\frac{\partial S}{\partial \alpha} = -2 \sum_i (y_i - \alpha - \beta_1 X_{1i} - \beta_2 X_{2i})$$

$$\frac{\partial^2 S}{\partial \alpha^2} = 2 > 0$$

$\therefore$ put $\frac{\partial S}{\partial \alpha} = 0$

$$0 = -2 \sum_i (y_i - \hat{\alpha} - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})$$

$$\Rightarrow \sum_i (y_i - \hat{\alpha} - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}) = 0$$

$$\Rightarrow \sum_i y_i = n\hat{\alpha} + \hat{\beta}_1 \sum_i X_{1i} + \hat{\beta}_2 \sum_i X_{2i} \rightarrow (2)$$

Partially differentiate (1) w.r.t "β_1"

$$\frac{\partial S}{\partial \beta_1} = -2 \sum_i (y_i - \alpha - \beta_1 X_{1i} - \beta_2 X_{2i}) X_{1i}$$

$$\frac{\partial S}{\partial \beta_1} = -2 \sum_i X_{1i} (y_i - \alpha - \beta_1 X_{1i} - \beta_2 X_{2i})$$

$$\frac{\partial^2 S}{\partial \beta_1^2} = 2 \sum_i X_{1i}^2 > 0$$

put $\frac{\partial S}{\partial \beta_1} = 0$

$$\Rightarrow -2 \sum_i X_{1i} (y_i - \alpha - \beta_1 X_{1i} - \beta_2 X_{2i}) = 0$$

$$\Rightarrow \sum_i X_{1i} Y_i - \alpha \sum_i X_{1i} - \beta_1 \sum_i X_{1i}^2 - \beta_2 \sum_i X_{1i} X_{2i} = 0$$

$$\Rightarrow \sum_i X_{1i} Y_i = \alpha \sum_i X_{1i} + \beta_1 \sum_i X_{1i}^2 + \beta_2 \sum_i X_{1i} X_{2i} \rightarrow (3)$$

partially differentiate (1) w.r.t β_2

$$\frac{\partial S}{\partial \beta_2} = -2 \sum_i X_{2i} (Y_i - \alpha - \beta_1 X_{1i} - \beta_2 X_{2i})$$

$$\frac{\partial^2 S}{\partial \beta_2^2} = 2 \sum_i X_{2i}^2 > 0$$

Put $\dfrac{\partial S}{\partial \beta_2} = 0$

$$\Rightarrow -2 \sum_i X_{2i} (Y_i - \alpha - \beta_1 X_{1i} - \beta_2 X_{2i}) = 0$$

$$\Rightarrow \sum_i X_{2i} Y_i - \alpha \sum_i X_{2i} - \beta_1 \sum_i X_{1i} X_{2i} - \beta_2 \sum X_{2i}^2 = 0$$

$$\Rightarrow \sum_i X_{2i} Y_i = \alpha \sum_i X_{2i} + \beta_1 \sum_i X_{1i} X_{2i} + \beta_2 \sum X_{2i}^2 \rightarrow (4)$$

Normal eqs (2), (3) and (4) can also be written as

$$\begin{bmatrix} \Sigma Y \\ \Sigma X_1 Y \\ \Sigma X_2 Y \end{bmatrix} = \begin{bmatrix} n & \Sigma X_1 & \Sigma X_2 \\ \Sigma X_1 & \Sigma X_1^2 & \Sigma X_1 X_2 \\ \Sigma X_2 & \Sigma X_1 X_2 & \Sigma X_2^2 \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

Let $\hat{A} = \hat{C} \hat{B}$

$$\Rightarrow \underline{B} = \underline{C}^{-1} \underline{A}$$

where

$$\underline{B} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} n & \Sigma X_1 & \Sigma X_2 \\ \Sigma X_1 & \Sigma X_1^2 & \Sigma X_1 X_2 \\ \Sigma X_2 & \Sigma X_1 X_2 & \Sigma X_2^2 \end{bmatrix}$$

Ex:-

X_1	X_2	Y	X_1X_2	X_1Y	X_2Y	X_1^2	X_2^2
8	0	2	0	16	0	64	0
8	1	5	8	40	5	64	1
6	1	7	6	42	7	36	1
5	3	8	15	40	24	25	9
3	4	5	12	15	20	9	16
30	9	27	41	153	56	198	27

The multiple regression line is

$$Y = \hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 \rightarrow (1)$$

1) Normal equations are

$$\Sigma Y = n\hat{\alpha} + \hat{\beta}_1 \Sigma X_1 + \hat{\beta}_2 \Sigma X_2 \rightarrow (a)$$

$$\Sigma X_1 Y = \hat{\alpha} \Sigma X_1 + \hat{\beta}_1 \Sigma X_1^2 + \hat{\beta}_2 \Sigma X_1 X_2 \rightarrow (b)$$

$$\Sigma X_2 Y = \hat{\alpha} \Sigma X_2 + \hat{\beta}_1 \Sigma X_1 X_2 + \hat{\beta}_2 \Sigma X_2^2 \rightarrow (c)$$

putting all values in (a), (b) and (c).

$$27 = 5\hat{\alpha} + 30\hat{\beta}_1 + 9\hat{\beta}_2 \rightarrow (2)$$

$$153 = 30\hat{\alpha} + 198\hat{\beta}_1 + 41\hat{\beta}_2 \rightarrow (3)$$

$$56 = 9\hat{\alpha} + 41\hat{\beta}_1 + 27\hat{\beta}_2 \rightarrow (4)$$

6 Eq (2) − Eq (3)

$$162 = \cancel{30\hat{\alpha}} + 180\hat{\beta}_1 + 54\hat{\beta}_2$$

$$\underline{\pm 153 = \pm \cancel{30\hat{\alpha}} \pm 198\hat{\beta}_1 \pm 41\hat{\beta}_2}$$

$$9 = -18\hat{\beta}_1 + 13\hat{\beta}_2 \rightarrow (5)$$

$9Eq(2) - 5Eq(4)$

$$243 = 45\hat{\alpha} + 270\hat{\beta}_1 + 81\hat{\beta}_2$$

$$\pm 280 = \pm 45\hat{\alpha} \pm 205\hat{\beta}_1 \pm 135\hat{\beta}_2$$

$$-37 = 65\hat{\beta}_1 - 54\hat{\beta}_2 \rightarrow (6)$$

$65Eq(5) + 18Eq(6)$

$$585 = -1170\hat{\beta}_1 + 845\hat{\beta}_2$$

$$-666 = 1170\hat{\beta}_1 - 972\hat{\beta}_2$$

$$-81 = -127\hat{\beta}_2$$

$$\hat{\beta}_2 = \frac{-81}{-127}$$

$$\hat{\beta}_2 = 0.638$$

put this in (5)

$$9 = -18\hat{\beta}_1 + 13(0.638)$$

$$9 = -18\hat{\beta}_1 + 8.294$$

$$9 - 8.294 = -18\hat{\beta}_1$$

$$0.706 = -18\hat{\beta}_1$$

$$\Rightarrow \hat{\beta}_1 = \frac{0.706}{-18}$$

$$\Rightarrow \hat{\beta}_1 = -0.039$$

Put value of $\hat{\beta}_1$ and $\hat{\beta}_2$ in (2)

$$27 = 5\hat{\alpha} + 30(-0.039) + 9(0.638)$$

$$27 = 5\hat{\alpha} - 1.17 + 5.742$$

$$27 = 5\hat{\alpha} + 4.572$$

$$27 - 4.572 = 5\hat{\alpha}$$

$$22.428 = 5\hat{\alpha}$$

$$\Rightarrow \hat{\alpha} = \frac{22.428}{5}$$

$$\Rightarrow \hat{\alpha} = 4.486$$

The fitted multiple regression line is

$$\hat{y} = 4.486 + (-0.039)X_1 + 0.638X_2$$

$$\Rightarrow \hat{y} = 4.486 - 0.039X_1 + 0.638X_2$$

2)

As Normal eq's can be written as

$$\begin{bmatrix} \Sigma y \\ \Sigma X_1 y \\ \Sigma X_2 y \end{bmatrix} = \begin{bmatrix} n & \Sigma X_1 & \Sigma X_2 \\ \Sigma X_1 & \Sigma X_1^2 & \Sigma X_1 X_2 \\ \Sigma X_2 & \Sigma X_1 X_2 & \Sigma X_2^2 \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

$$\Rightarrow \bar{A} = \bar{C}\bar{B}$$

$$\Rightarrow \underline{B} = \underline{C}^{-1}\underline{A} \rightarrow (1)$$

and

$$\underline{C}^{-1} = \frac{\operatorname{adj} C}{|C|} \quad ; \quad |C| \neq 0$$

$$|C| = \begin{vmatrix} n & \Sigma X_1 & \Sigma X_2 \\ \Sigma X_1 & \Sigma X_1^2 & \Sigma X_1 X_2 \\ \Sigma X_2 & \Sigma X_1 X_2 & \Sigma X_2^2 \end{vmatrix}$$

put all values.

$$|C| = \begin{vmatrix} 5 & 30 & 9 \\ 30 & 198 & 41 \\ 9 & 41 & 27 \end{vmatrix}$$

$$\begin{aligned}
|C| &= 5(5346-1681) - 30(810-369) + 9(1230-1782) \\
&= 5(3665) - 30(441) + 9(-552) \\
&= 18325 - 13230 - 4968 \\
&= 18325 - 18198 \\
|C| &= 127 \neq 0 .
\end{aligned}$$

$$\text{adj}\, C = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\text{adj}C = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix}198 & 41\\ 41 & 27\end{vmatrix} & (-1)^{2+1}\begin{vmatrix}30 & 9\\ 41 & 27\end{vmatrix} & (-1)^{3+1}\begin{vmatrix}30 & \\ 198 & \end{vmatrix} \\ (-1)^{1+2}\begin{vmatrix}30 & 41\\ 9 & 27\end{vmatrix} & (-1)^{2+2}\begin{vmatrix}5 & 9\\ 9 & 27\end{vmatrix} & (-1)^{3+2}\begin{vmatrix}5 & \\ 30 & \end{vmatrix} \\ (-1)^{1+3}\begin{vmatrix}30 & 198\\ 9 & 41\end{vmatrix} & (-1)^{2+3}\begin{vmatrix}5 & 30\\ 9 & 41\end{vmatrix} & (-1)^{3+3}\begin{vmatrix}5 & 3\\ 30 & 19\end{vmatrix} \end{bmatrix}$$

$$\text{adj}C = \begin{bmatrix} (5346-1681) & -(810-369) & (1230-178 \\ -(810-369) & (135-81) & -(205-270) \\ +(1230-1782) & -(205-270) & (990-900) \end{bmatrix}$$

$$\operatorname{adj} C = \begin{bmatrix} 3665 & -441 & -552 \\ -441 & 54 & 65 \\ -552 & 65 & 90 \end{bmatrix}$$

so

$$\underline{C}^{-1} = \frac{\operatorname{adj} C}{|C|} \quad ; \quad |C| \neq 0$$

$$\underline{C}^{-1} = \begin{bmatrix} 3665 & -441 & -552 \\ -441 & 54 & 65 \\ -552 & 65 & 90 \end{bmatrix} \cdot \frac{1}{127}$$

$$\underline{C}^{-1} = \begin{bmatrix} 28.858 & -3.472 & -4.346 \\ -3.472 & 0.425 & 0.512 \\ -4.346 & 0.512 & 0.709 \end{bmatrix}$$

Now by eq.(1)

$$\underline{B} = \underline{C}^{-1}\,\underline{A}$$

$$\Rightarrow \begin{bmatrix} \hat{\alpha} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 28.858 & -3.472 & -4.346 \\ -3.472 & 0.425 & 0.512 \\ -4.346 & 0.512 & 0.709 \end{bmatrix} \begin{bmatrix} 27 \\ 153 \\ 56 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{\alpha} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} (28.858)(27) + (-3.472)(153) - (4.346)(56) \\ (-3.472)(27) + (0.425)(153) + (0.512)(56) \\ (-4.346)(27) + (0.512)(153) + (0.709)(56) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{\alpha} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 4.574 \\ -0.047 \\ 0.698 \end{bmatrix}$$

$\Rightarrow \hat{\alpha} = 4.574, \ \hat{\beta}_1 = -0.047, \ \hat{\beta}_2 = 0.698$

The fitted multiple regression line is

$$\hat{Y} = 4.574 - 0.047X_1 + 0.698X_2$$

Ex:-

Given $\Sigma Y = 89$, $\Sigma X_1 = 30$, $\Sigma X_2 = 52$, $\Sigma X_1^2 = 238$, $\Sigma X_2^2 = 582$, $\Sigma X_1X_2 = 351$, $\Sigma X_1Y = 619$, $\Sigma X_2Y = 1007$, $n = 5$. Find the estimated regression model for Y regressed on X_1 and X_2.

Sol:-

The regression model for Y regressed on X_1 and X_2 is

$$\hat{Y} = \hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 \rightarrow (A)$$

Normal equations of (A) are

$$\Sigma \hat{Y} = n\hat{\alpha} + \hat{\beta}_1 \Sigma X_1 + \hat{\beta}_2 \Sigma X_2 \rightarrow (a)$$

$$\Sigma X_1 \hat{Y} = \hat{\alpha} \Sigma X_1 + \hat{\beta}_1 \Sigma X_1^2 + \hat{\beta}_2 \Sigma X_1 X_2 \rightarrow (b)$$

$$\Sigma X_2 \hat{Y} = \hat{\alpha} \Sigma X_2 + \hat{\beta}_1 \Sigma X_1 X_2 + \hat{\beta}_2 \Sigma X_2^2 \rightarrow (c)$$

$$89 = 5\hat{\alpha} + 30\hat{\beta}_1 + 52\hat{\beta}_2 \rightarrow (1)$$

$$619 = 30\hat{\alpha} + 238\hat{\beta}_1 + 351\hat{\beta}_2 \rightarrow (2)$$

$$1007 = 52\hat{\alpha} + 351\hat{\beta}_1 + 582\hat{\beta}_2 \rightarrow (3)$$

Eq (1), (2) and (3) are obtained by put values

Eq(2) - 6 Eq(1)

$$619 = 30\hat{\alpha} + 238\hat{\beta}_1 + 351\hat{\beta}_2$$

$$\pm 534 = \pm 30\hat{\alpha} \pm 180\hat{\beta}_1 \pm 312\hat{\beta}_2$$

$$85 = 58\hat{\beta}_1 + 39\hat{\beta}_2 \rightarrow (4)$$

Eq(3) - 10.4 Eq(1)

$$1007 = 52\hat{\alpha} + 351\hat{\beta}_1 + 582\hat{\beta}_2$$

$$\pm 925.6 = \pm 52\hat{\alpha} \pm 312\hat{\beta}_1 \pm 540.8\hat{\beta}_2$$

$$81.4 = 39\hat{\beta}_1 + 41.2\hat{\beta}_2 \rightarrow (5)$$

39 Eq(4) - 58 Eq(5)

$$3315 = 2262\hat{\beta}_1 + 1521\hat{\beta}_2$$

$$\pm 4721.2 = \pm 2262\hat{\beta}_1 \pm 2389.6\hat{\beta}_2$$

$$-1406.2 = -868.6\hat{\beta}_2$$

$$\hat{\beta}_2 = 1.6189 \simeq 1.62$$

Put in (4)

$$85 = 58\hat{\beta}_1 + 39(1.62)$$

$$85 - 63.18 = 58\hat{\beta}_1$$

$$21.82 = 58\hat{\beta}_1$$

$$\Rightarrow \hat{\beta}_1 = 0.376 \simeq 0.38$$

Put $\hat{\beta}_1$ and $\hat{\beta}_2$ in (1)

$$89 = 5\hat{\alpha} + 30(0.38) + 52(1.62)$$

$$-5\hat{\alpha} = -89 + 11.4 + 84.24 = 6.64$$

$$\hat{\alpha} = -1.328 \simeq -1.33$$

$\therefore$ fitted model is $\hat{Y} = -1.33 + 0.38X_1 + 1.62X_2$

Deviation Form:-

The estimated regression model is

$$\hat{Y} = \hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 \quad \text{(sample form)}$$

$$Y = \alpha + \beta_1 X_1 + \beta_2 X_2 \quad \text{(pop form)}$$

Take average of regression model of pop form

$$\Rightarrow \quad \bar{Y} = \alpha + \beta_1 \bar{X}_1 + \beta_2 \bar{X}_2$$

Take $Y - \bar{Y} = \beta_1 (X_1 - \bar{X}_1) + \beta_2 (X_2 - \bar{X}_2)$ is deviation form and can be written as

$$y = \beta_1 x_1 + \beta_2 x_2$$

where $y = Y - \bar{Y}$, $x_1 = X_1 - \bar{X}_1$, $x_2 = X_2 - \bar{X}_2$

Method of least square provides 2 equations

$$\Sigma x_1 y = \hat{\beta}_1 \Sigma x_1^2 + \hat{\beta}_2 \Sigma x_1 x_2 \rightarrow (a)$$

$$\Sigma x_2 y = \hat{\beta}_1 \Sigma x_1 x_2 + \hat{\beta}_2 \Sigma x_2^2 \rightarrow (b)$$

Now $\Sigma x_1 y = \Sigma (X_1 - \bar{X}_1)(Y - \bar{Y}) = \Sigma [X_1 Y - X_1 \bar{Y} - \bar{X}_1 Y + \bar{X}_1 \bar{Y}_1] = \Sigma X_1 Y - \bar{Y} \Sigma X_1 - \bar{X}_1 \Sigma Y + n \frac{\Sigma X_1}{n} \frac{\Sigma Y}{n}$

$$\Sigma x_1 y = \Sigma X_1 Y - \frac{(\Sigma X_1)(\Sigma Y)}{n}$$

$$\Sigma x_1 y = 619 - \frac{(30)(89)}{5}$$

$$\Sigma x_1 y = 85$$

Similarly,

$$\Sigma x_1^2 = \Sigma X_1^2 - \frac{(\Sigma X_1)^2}{n}$$

$$\Sigma x_1^2 = 238 - \frac{(30)^2}{5} = 58$$

$\Sigma X_1 Y - \frac{\Sigma Y}{n} \Sigma X_1 - \frac{\Sigma x}{r}$

$\Sigma Y + \frac{\Sigma X_1 \Sigma Y}{n}$

$= \Sigma X_1 Y - \frac{\Sigma X_1 \Sigma Y}{n} - \frac{\Sigma X}{n}$

$\Sigma Y + \frac{\Sigma X_1 \Sigma Y}{n}$

$= \Sigma X_1 Y - 2 \frac{\Sigma X_1 \Sigma Y}{n}$

$+ \frac{\Sigma X_1 \Sigma Y}{n}$

$= \Sigma X_1 Y - \frac{\Sigma X_1 \Sigma Y}{n}$

$$\sum x_1 x_2 = \sum (X_1 - \bar{X}_1)(X_2 - \bar{X}_2)$$

$$\Rightarrow \sum x_1 x_2 = \sum X_1 X_2 - \frac{(\sum X_1)(\sum X_2)}{n}$$

$$= 351 - \frac{(30)(52)}{5}$$

$$\sum x_1 x_2 = 39$$

$$\sum x_2 y = \sum (X_2 - \bar{X}_2)(Y - \bar{Y})$$

$$\sum x_2 y = \sum X_2 Y - \frac{(\sum X_2)(\sum Y)}{n}$$

$$= 1007 - \frac{(52)(89)}{5}$$

$$\sum x_2 y = 81.4$$

$$\sum x_2 x_2 = \sum (X_2 - \bar{X}_2)(X_2 - \bar{X}_2)$$

$$\sum x_2^2 = \sum X_2^2 - \frac{(\sum X_2)^2}{n}$$

$$= 582 - \frac{(52)^2}{5}$$

$$\sum x_2^2 = 41.2$$

put these values in (a) and (b)

$$85 = 58\hat{\beta}_1 + 39\hat{\beta}_2 \rightarrow (c)$$

$$81.4 = 39\hat{\beta}_1 + 41.2\hat{\beta}_2 \rightarrow (d)$$

39 Eq (c) − 58 Eq (d)

$$3315 = 2262\hat{\beta}_1 + 1521\hat{\beta}_2$$

$$\underline{\pm 4721.2 = \pm 2262\hat{\beta}_1 \pm 2389.6\hat{\beta}_2}$$

$$-1406.2 = \quad -868.6\hat{\beta}_2$$

$$\Rightarrow \hat{\beta}_2 = +1.619 \Rightarrow \hat{\beta}_2 = 1.62$$

put in (c) $\quad 85 = 58\hat{\beta}_1 + 39(1.62)$

$$\Rightarrow \hat{\beta}_1 = 0.376 = 0.38$$

$$y = 0.38x_1 + 1.62x_2$$

$$Y - \bar{Y} = 0.38(X_1 - \bar{X}_1) + (1.62)(X_2 - \bar{X}_2)$$

As $\bar{Y} = \frac{\sum Y}{n} = \frac{81}{5} = 17.8$

$$\bar{X}_1 = \frac{\sum X_1}{n} = \frac{30}{5} = 6, \quad \bar{X}_2 = \frac{\sum X_2}{n} = \frac{52}{5} = 10.4$$

So

$$Y - 17.8 = 0.38(X_1 - 6) + 1.62(X_2 - 10.4)$$

$$Y = 17.8 + 0.38X_1 - 2.28 + 1.62X_2 - 16.848$$

$$Y = 17.8 - 2.28 - 16.848 + 0.38X_1 + 1.62X_2$$

$$Y = -1.328 + 0.38X_1 + 1.62X_2$$

$$Y = -1.33 + 0.38X_1 + 1.62X_2$$

(or)

As $\hat{\alpha} = \bar{Y} - \hat{\beta}_1 \bar{X}_1 - \hat{\beta}_2 \bar{X}_2$

After calculating $\hat{\beta}_1$ and $\hat{\beta}_2$

$$\hat{\alpha} = 17.8 - (0.38)(6) - (1.62)(10.4)$$

$$\hat{\alpha} = 17.8 - 2.28 - 16.848$$

$$\hat{\alpha} = -1.328 \simeq -1.33$$

So, the estimated fitted regression model is $Y = -1.33 + 0.38X_1 + 1.62X_2$

Standard error of estimate:-

$$Y = \alpha + \beta_1 X_1 + \beta_2 X_2$$

The fitted estimated model is

$$\hat{Y} = \hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2$$

The standard error of estimate can be defined as $s_{y\cdot x_1x_2}$

$$s_{y\cdot x_1x_2} = \sqrt{\frac{\Sigma(y-\hat{y})^2}{n-3}} \rightarrow (A)$$

Take $\Sigma(y-\hat{y})^2 = \Sigma(y-(\hat{\alpha}+\hat{\beta}_1 x_1+\hat{\beta}_2 x_2))^2$

$$= \Sigma(y-\hat{\alpha}-\hat{\beta}_1 x_1-\hat{\beta}_2 x_2)(y-\hat{\alpha}-\hat{\beta}_1 x_1-\hat{\beta}_2 x_2)$$

$$\Sigma(y-\hat{y})^2 = \Sigma y(y-\hat{\alpha}-\hat{\beta}_1 x_1-\hat{\beta}_2 x_2) - \Sigma(\hat{\alpha}+\hat{\beta}_1 x_1+\hat{\beta}_2 x_2)(y-\hat{\alpha}-\hat{\beta}_1 x_1-\hat{\beta}_2 x_2) \rightarrow (B)$$

Take

$$\Sigma(\hat{\alpha}+\hat{\beta}_1 x_1+\hat{\beta}_2 x_2)(y-\hat{\alpha}-\hat{\beta}_1 x_1-\hat{\beta}_2 x_2)$$

$$= \Sigma\hat{\alpha}(y-\hat{\alpha}-\hat{\beta}_1 x_1-\hat{\beta}_2 x_2)+\hat{\beta}_1\Sigma x_1(y-\hat{\alpha}-\hat{\beta}_1 x_1-\hat{\beta}_2 x_2)+\hat{\beta}_2\Sigma x_2(y-\hat{\alpha}-\hat{\beta}_1 x_1-\hat{\beta}_2 x_2)$$

$$= \hat{\alpha}(\Sigma y-n\hat{\alpha}-\hat{\beta}_1\Sigma x_1-\hat{\beta}_2\Sigma x_2)+\hat{\beta}_1(\Sigma x_1 y-\hat{\alpha}\Sigma x_1-\hat{\beta}_1\Sigma x_1^2-\hat{\beta}_2\Sigma x_1x_2)+\hat{\beta}_2(\Sigma x_2 y-\hat{\alpha}\Sigma x_2-\hat{\beta}_1\Sigma x_1x_2-\hat{\beta}_2\Sigma x_2^2)$$

As we know, normal eqn of $\hat{y}=\hat{\alpha}+\hat{\beta}_1 x_1+\hat{\beta}_2 x_2$ are

$$\Sigma y = n\hat{\alpha}+\hat{\beta}_1\Sigma x_1+\hat{\beta}_2\Sigma x_2$$

$$\Rightarrow \Sigma y-n\hat{\alpha}-\hat{\beta}_1\Sigma x_1-\hat{\beta}_2\Sigma x_2 = 0$$

$$\Sigma x_1 y = \hat{\alpha}\Sigma x_1+\hat{\beta}_1\Sigma x_1^2+\hat{\beta}_2\Sigma x_1x_2$$

$$\Rightarrow \Sigma x_1 y-\hat{\alpha}\Sigma x_1-\hat{\beta}_1\Sigma x_1^2-\hat{\beta}_2\Sigma x_1x_2 = 0$$

and $\sum X_2 Y = \hat{\alpha}\sum X_2 + \hat{\beta}_1 \sum X_1 X_2 + \hat{\beta}_2 \sum X_2^2$

$$\Rightarrow \sum X_2 Y - \hat{\alpha}\sum X_2 - \hat{\beta}_1 \sum X_1 X_2 - \hat{\beta}_2 \sum X_2^2 = 0$$

So

$$\sum(\hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2)(Y - \hat{\alpha} - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2) = \hat{\alpha}(0) + \hat{\beta}_1(0) + \hat{\beta}_2(0)$$

$$\Rightarrow \sum(\hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2)(Y - \hat{\alpha} - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2) = 0$$

Thus, eq.(B) becomes

$$\sum(Y - \hat{Y})^2 = \sum Y(Y - \hat{\alpha} - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2) - 0$$

$$\Rightarrow \sum(Y - \hat{Y})^2 = \sum Y^2 - \hat{\alpha}\sum Y - \hat{\beta}_1 \sum X_1 Y - \hat{\beta}_2 \sum X_2 Y$$

So, the standard error of estimate is

$$s_{Y.X_1X_2} = \sqrt{\frac{\sum Y^2 - \hat{\alpha}\sum Y - \hat{\beta}_1 \sum X_1 Y - \hat{\beta}_2 \sum X_2 Y}{n-3}}$$

Matrix Representation:-

Suppose, we have a multiple regression model

$$Y_i = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + \epsilon_i$$

We can write $\quad ; i = 1, 2, \cdots, n$

$$Y_1 = \alpha + \beta_1 X_{11} + \beta_2 X_{21} + \cdots + \beta_k X_{k1} + \epsilon_1$$

$$Y_2 = \alpha + \beta_1 X_{12} + \beta_2 X_{22} + \cdots + \beta_k X_{k2} + \epsilon_2$$

$$\vdots$$

$$Y_n = \alpha + \beta_1 X_{1n} + \beta_2 X_{2n} + \cdots + \beta_k X_{kn} + \epsilon_n$$

In matrix form

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{(n\times 1)} = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{k1} \\ 1 & X_{12} & X_{22} & \cdots & X_{k2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{1n} & X_{2n} & \cdots & X_{kn} \end{bmatrix}_{(n\times(k+1))} \begin{bmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}_{((k+1)\times 1)} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}_{(n\times 1)}$$

$$\underline{Y} = \underline{X}\,\underline{\beta} + \underline{\epsilon}$$

Suppose $k=2$, then

$$\underline{X} = \begin{bmatrix} 1 & X_{11} & X_{21} \\ 1 & X_{12} & X_{22} \\ \vdots & \vdots & \vdots \\ 1 & X_{1n} & X_{2n} \end{bmatrix}_{(n\times 3)}, \quad \underline{\beta} = \begin{bmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{bmatrix},$$

$$\Rightarrow \underline{X}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \end{bmatrix}_{(3\times n)}$$

We have least squares estimates

$$\hat{\underline{\beta}} = (\underline{X}^T \underline{X})^{-1} \underline{X}^T \underline{Y}$$

$$\hat{\underline{\beta}} = (\underline{A})^{-1} \underline{C}$$

$$\underline{X}^T \underline{X} = \begin{bmatrix} n & \sum_{i=1}^n X_{1i} & \sum_{i=1}^n X_{2i} \\ \sum_{i=1}^n X_{1i} & \sum_{i=1}^n X_{1i}^2 & \sum_{i=1}^n X_{1i}X_{2i} \\ \sum_{i=1}^n X_{2i} & \sum_{i=1}^n X_{1i}X_{2i} & \sum_{i=1}^n X_{2i}^2 \end{bmatrix} = \underline{A} \text{ is symmetric matrix}$$

Note:-

For solution of system of non-linear eqs, we use matrix form, because in matrix form, we can generalised our results.

and $X^T\underline{Y}$

$$X^T\underline{Y} = \begin{bmatrix} \sum_{i=1}^{n} Y_i \\ \sum_{i=1}^{n} X_{1i}Y_i \\ \sum_{i=1}^{n} X_{2i}Y_i \end{bmatrix} = \underline{C}$$

Ex:-

Y	X_1	X_2	X_1X_2	X_1Y	X_2Y	X_1^2	X_2^2
12	2	1	2	24	12	4	1
10	2	1	2	20	10	4	1
9	3	0	0	27	0	9	0
13	4	0	0	52	0	16	0
20	4	3	12	80	60	16	9
64	15	5	16	203	82	49	11

We want to fit multiple regression model

$$Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \epsilon$$

$$\hat{\underline{\beta}} = (\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\underline{Y}$$

$$\underline{Y} = \begin{bmatrix} 12 \\ 10 \\ 9 \\ 13 \\ 20 \end{bmatrix}, \quad \underline{X} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 3 \end{bmatrix}, \quad \underline{X}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \\ 1 & 1 & 0 & 0 & 3 \end{bmatrix}$$

Now

First, we find $\underline{X}^T\underline{X}$ and its inverse

$$\underline{X}^T\underline{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \\ 1 & 1 & 0 & 0 & 3 \end{bmatrix}_{3\times5} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 3 \end{bmatrix}_{5\times3}$$

$$\underline{X}^T\underline{X} = \begin{bmatrix} 1+1+1+1+1 & 2+2+3+4+4 & 1+1+0+0+3 \\ 2+2+3+4+4 & 4+4+9+16+16 & 2+2+0+0+12 \\ 1+1+0+0+3 & 2+2+0+0+12 & 1+1+0+0+9 \end{bmatrix}$$

$$\underline{X}^T\underline{X} = \begin{bmatrix} 5 & 15 & 5 \\ 15 & 49 & 16 \\ 5 & 16 & 11 \end{bmatrix} = \underline{A}$$

Now $\underline{A}^{-1} = \dfrac{\operatorname{adj}\underline{A}}{|\underline{A}|}$; $|\underline{A}| \neq 0$.

; $\underline{A} = \underline{X}^T\underline{X}$

$$|\underline{A}| = \begin{vmatrix} 5 & 15 & 5 \\ 15 & 49 & 16 \\ 5 & 16 & 11 \end{vmatrix} = 5(539-256) - 15(165 - 80) + 5(240-245)$$

$$|\underline{A}| = 1415 - 1275 - 25 = 115 \neq 0$$

$$\operatorname{adj}\underline{A} = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 49 & 16 \\ 16 & 11 \end{vmatrix} & (-1)^{2+1}\begin{vmatrix} 15 & 5 \\ 16 & 11 \end{vmatrix} & (-1)^{3+1}\begin{vmatrix} 15 & 5 \\ 49 & 16 \end{vmatrix} \\ (-1)^{1+2}\begin{vmatrix} 15 & 16 \\ 5 & 11 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 5 & 5 \\ 5 & 11 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 5 & 5 \\ 15 & 16 \end{vmatrix} \\ (-1)^{1+3}\begin{vmatrix} 15 & 49 \\ 5 & 16 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 5 & 15 \\ 5 & 16 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 5 & 15 \\ 15 & 49 \end{vmatrix} \end{bmatrix}$$

$$\operatorname{adj} \underline{A} = \begin{bmatrix} 283 & -85 & -5 \\ -85 & 30 & -5 \\ -5 & -5 & 20 \end{bmatrix}$$

So

$$(\underline{X}^T \underline{X})^{-1} = A^{-1} = \begin{bmatrix} \frac{283}{115} & \frac{-85}{115} & -5/115 \\ \frac{-85}{115} & \frac{30}{115} & -5/115 \\ \frac{-5}{115} & \frac{-5}{115} & 20/115 \end{bmatrix}$$

$$(\underline{X}^T \underline{X})^{-1} = \begin{bmatrix} 2.4608 & -0.7391 & -0.0435 \\ -0.7391 & 0.2609 & -0.0435 \\ -0.0435 & -0.0435 & 0.1739 \end{bmatrix}$$

Now, $\underline{X}^T \underline{Y} = \underline{C}$ (Let)

$$\underline{X}^T \underline{Y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \\ 1 & 1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ 10 \\ 9 \\ 13 \\ 20 \end{bmatrix} = \begin{bmatrix} 12+10+9+13+20 \\ 24+20+27+52+80 \\ 12+10+0+0+60 \end{bmatrix}$$

$$\underline{X}^T \underline{Y} = \begin{bmatrix} 64 \\ 203 \\ 82 \end{bmatrix}$$ Now, finally.

$$\hat{\underline{\beta}} = (\underline{X}^T \underline{X})^{-1} \underline{X}^T \underline{Y} = \begin{bmatrix} 2.4608 & -0.7391 & -0.0435 \\ -0.7391 & 0.2609 & -0.0435 \\ -0.0435 & -0.0435 & 0.1739 \end{bmatrix} \begin{bmatrix} 64 \\ 203 \\ 82 \end{bmatrix}$$

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 3.8869 \\ 2.0933 \\ 2.6453 \end{bmatrix}$$

So, the estimated multiple regression model is

$$\hat{Y} = 3.8869 + 2.0933X_1 + 2.6453X_2$$

Co-efficient of multiple correlation and determination:-

If we have a multiple regression model $Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \epsilon$

Total variation $= \Sigma (Y - \bar{Y})^2$

Explained variation $= \Sigma (\hat{Y} - \bar{Y})^2$

Un-explained variation $=$ Total $-$ Explained

$= \Sigma (Y - \bar{Y})^2 - \Sigma (\hat{Y} - \bar{Y})^2$

Un-explained variation $= \Sigma (Y - \hat{Y})^2$

Co-efficient of determination $= R^2_{Y.X_1X_2}$

$$= \frac{\text{Explained variation}}{\text{Total variation}}$$

$$R^2_{Y.X_1X_2} = \frac{\Sigma (\hat{Y} - \bar{Y})^2}{\Sigma (Y - \bar{Y})^2} \in [0, 1]$$

Co-efficient of multiple correlation $= R_{Y.X_1X_2}$

$= \sqrt{\text{Co-eff of determination}}$

$\in [0, 1]$

Ex:-

Y	12	10	9	13	20
X_1	2	2	3	4	4
X_2	1	1	0	0	3

Find Co-efficient of determination

and co-efficient of correlation of given data.

Sol:-

Y	X_1	X_2	$\hat{Y}$	$\hat{Y}-\bar{Y}$	$(\hat{Y}-\bar{Y})^2$	$Y-\bar{Y}$	$(Y-\bar{Y})^2$
12	2	1	10.7188	−2.0812	4.3314	−0.8	0.64
10	2	1	10.7188	−2.0812	4.3314	−2.8	7.84
9	3	0	10.1668	−2.6332	6.9337	−3.8	14.4
13	4	0	12.2601	−0.5399	0.2915	0.2	0.0
20	4	3	20.196	7.396	54.701	7.2	51.8
64	15	5			70.589		74.

The fitted estimated multiple regression model is $\hat{Y} = 3.8869 + 2.0933X_1 + 2.6453X_2$

we know

$$\text{Co-efficient of determination} = \frac{\sum(\hat{Y}-\bar{Y})^2}{\sum(Y-\bar{Y})^2} \rightarrow (1)$$

$$\bar{Y} = \frac{\sum Y}{n} = \frac{64}{5} = 12.8$$

put values in (1)

$$R^2_{Y.X_1X_2} = \frac{70.589}{74.8} = 0.9437 \simeq 0.944$$

$$R^2_{Y.X_1X_2} = 0.944$$

$$\text{co-efficient of correlation} = \sqrt{R^2_{Y.X_1X_2}}$$

$$= \sqrt{0.944}$$

$$\Rightarrow R_{Y.X_1X_2} = 0.972$$

Note:-

1) Co-efficient of determination always lies b/w "0" and "1" in both regression models (linear, multiple).

2) Co-efficient of correlation in linear regression model lies between "-1" and "1" but in multiple regression model it lies b/w "0" and "1". Because, in multiple regression model, the variable Y is jointly correlated with x_1, x_2 or $x_1, x_2, \ldots x_n$. when Y is jointly correlated with 2 or more independent variables $(x_1, x_2, x_3, \ldots, x_n)$ then there is no difference of signs. Thats why $R^2_{Y \cdot x_1 x_2}$ or $R^2_{Y \cdot x_1 x_2 x_3}$ always $\in [0, 1]$.

Multiple correlation:- and Partial correlation:-

Suppose we have three variables X_1, X_2, X_3 such that each of X_i; $i=1,2,3$ can be modeled on the basis of remaining two variables X_j, X_k; $j,k = 1,2,3$ $(j \neq k \neq i)$. Then, we can define three models.

$$X_1 = \alpha_{1.23} + \beta_{12.3} X_2 + \beta_{13.2} X_3 + \epsilon_1$$

$$X_2 = \alpha_{2.13} + \beta_{21.3} X_1 + \beta_{23.1} X_3 + \epsilon_2$$

$$X_3 = \alpha_{3.12} + \beta_{31.2} X_1 + \beta_{32.1} X_3 + \epsilon_3$$

We can also define following simple linear models (Bi-variate

$$X_1 = \alpha_{1.2} + \beta_{12} X_2 + \epsilon_1^{*}$$

$$X_1 = \alpha_{1.3} + \beta_{13} X_3 + \epsilon_1^{**} \qquad \because r_{ij} = r_{ji}$$

$$X_2 = \alpha_{2.1} + \beta_{21} X_1 + \epsilon_2^{*} \qquad \beta_{ij} \neq \beta_{ji}$$

$$X_2 = \alpha_{2.3} + \beta_{23} X_3 + \epsilon_2^{**}$$

$$X_3 = \alpha_{3.1} + \beta_{31} X_1 + \epsilon_3^{*}$$

$$X_3 = \alpha_{3.2} + \beta_{32} X_2 + \epsilon_3^{**}$$

We can define bivariate correlations

$r_{12}, r_{13}, r_{23},$

Now, we can find the multiple correlations using simple correlations.

$$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1-r_{23}^2}}$$

$$R_{2.13} = \sqrt{\frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13}}{1-r_{13}^2}}$$

$$R_{3.12} = \sqrt{\frac{r_{13}^2 + r_{23}^2 - 2r_{13}r_{23}r_{12}}{1-r_{12}^2}}$$

Multiple correlation defines the inter-dependence of single variable with joint effect of several variables.

So, $R_{1.23}$ is multiple correlation of X_1 with joint effect of two variables (X_2, X_3).

Partial Correlation is the correlation between two variables when effect of the third variable is remover (or kept constant).

$$r_{12.3} = \sqrt{\beta_{12.3} \cdot \beta_{21.3}} \in [-1, 1]$$

By regression model

$$r_{13.2} = \sqrt{\beta_{13.2} \cdot \beta_{31.2}}$$

$$r_{23.1} = \sqrt{\beta_{23.1} \cdot \beta_{32.1}}$$

$$r_{12\cdot3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$r_{23\cdot1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}}$$

$$r_{13\cdot2} = \frac{r_{13} - r_{12}r_{23}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}}$$

$$r_{21\cdot3} = \frac{r_{21} - r_{23}r_{13}}{\sqrt{(1-r_{23}^2)(1-r_{13}^2)}} = r_{12\cdot3}$$

So, partial correlation is symmetric.

$\Rightarrow r_{21\cdot3} = r_{12\cdot3}$, $r_{13\cdot2} = r_{31\cdot2}$, $r_{23\cdot1} = r_{32\cdot1}$

Ex:-

X_1	X_2	X_3	I $X_1-\bar{X}_1$	J $X_2-\bar{X}_2$	K $X_3-\bar{X}_3$	IJ	JK	IK
12	2	1	-0.8	-1	0	0.8	0	0
10	2	1	-2.8	-1	0	2.8	0	0
9	3	0	-3.8	0	-1	0	0	3.8
13	4	0	0.2	1	-1	0.2	-1	-0.2
20	4	3	7.2	1	2	7.2	2	14.4
64	15	5				11	1	18

we have to find multiple and partial correlations.

First, multiple correlations which are

$$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1-r_{23}^2}} \rightarrow (a)$$

$$R_{2.13} = \sqrt{\frac{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}r_{13}}{1-r_{13}^2}} \rightarrow (b)$$

$$R_{3.12} = \sqrt{\frac{r_{13}^2 + r_{23}^2 - 2r_{13}r_{23}r_{12}}{1-r_{12}^2}} \rightarrow (c)$$

For these, we have to calculate r_{12}, r_{13} and r_{23}.

So $$r_{12} = \frac{\Sigma(X_1-\bar{X}_1)(X_2-\bar{X}_2)}{\sqrt{\Sigma(X_1-\bar{X}_1)^2\,\Sigma(X_2-\bar{X}_2)^2}} \rightarrow (1)$$

As $\frac{\Sigma X_1}{n} = \bar{X}_1$, $\bar{X}_2 = \frac{\Sigma X_2}{n}$, $\bar{X}_3 = \frac{\Sigma X_3}{n}$

$\bar{X}_1 = \frac{64}{5} = 12.8$, $\bar{X}_2 = \frac{15}{5} = 3$, $\bar{X}_3 = \frac{5}{5} = 1$

$(X_1-\bar{X}_1)^2$	$(X_2-\bar{X}_2)^2$	$(X_3-\bar{X}_3)^2$
0.64	1	0
7.84	1	0
14.44	0	1
0.04	1	1
51.84	1	4
74.8	4	6

Put all values in (1)

$$r_{12} = \frac{11}{\sqrt{74.8 \times 4}} = \frac{11}{\sqrt{299.2}}$$

$r_{12} = 0.64 = r_{21}$

Now $$r_{13} = \frac{\Sigma(X_1-\bar{X}_1)(X_3-\bar{X}_3)}{\sqrt{\Sigma(X_1-\bar{X}_1)^2\,\Sigma(X_3-\bar{X}_3)^2}}$$

$$r_{13} = \frac{18}{\sqrt{74.8 \times 6}} = \frac{18}{\sqrt{448.8}} = 0.85 = r_{31}$$

$$r_{23} = \frac{\Sigma(X_2-\bar{X}_2)(X_3-\bar{X}_3)}{\sqrt{\Sigma(X_2-\bar{X}_2)^2\,\Sigma(X_3-\bar{X}_3)^2}} = \frac{1}{\sqrt{4 \times 6}}$$

$$r_{23} = \frac{1}{\sqrt{24}} = 0.204 = r_{32}$$

put these in (a), (b) and (c)

$$R_{1.23} = \sqrt{\frac{(0.64)^2 + (0.85)^2 - 2(0.64)(0.85)(0.204)}{1-(0.204)^2}}$$

$$R_{1.23} = \sqrt{\frac{0.910148}{0.958384}} = 0.97$$

$$R_{2.13} = \sqrt{\frac{(0.64)^2 + (0.204)^2 - 2(0.64)(0.204)(0.85)}{1-(0.85)^2}}$$

$$R_{2.13} = \sqrt{\frac{0.229264}{0.2775}} = 0.91$$

$$R_{3.12} = \sqrt{\frac{(0.85)^2 + (0.204)^2 - 2(0.64)(0.204)(0.85)}{1-(0.64)^2}}$$

$$R_{3.12} = \sqrt{\frac{0.542164}{0.5904}} = 0.96$$

Now, partial correlations are

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}} = \frac{0.64 - (0.85)(0.204)}{\sqrt{(1-(0.85)^2)(1-(0.204)^2)}}$$

$$r_{12.3} = \frac{0.4666}{\sqrt{(0.2775)(0.958384)}} = 0.90$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{23}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}} = \frac{0.85 - (0.64)(0.204)}{\sqrt{(1-(0.64)^2)(1-(0.204)^2)}} = \frac{0.7194}{0.752} = 0.96$$

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}} = \frac{0.204 - (0.64)(0.85)}{\sqrt{(1-(0.64)^2)(1-(0.85)^2)}} = \frac{-0.34}{0.405} = -0.84$$

Notes:-

Suppose we have two variables x and y, such that $\bar{x}=0$; $\bar{y}=0$

$var(x)=\sigma^2$, $var(y)=\sigma^2$, $r_{xy}=0$

If $\bar{x}=0 \Rightarrow \frac{\sum x}{n}=0 \Rightarrow \sum x=0$

Similarly

$\bar{y}=0 \Rightarrow \frac{\sum y}{n}=0 \Rightarrow \sum y=0$

Now

$var(x)=\sigma^2 \Rightarrow \frac{\sum(x-\bar{x})^2}{n}=\sigma^2$

If $\bar{x}=0$

$\Rightarrow \frac{\sum x^2}{n}=\sigma^2 \Rightarrow \sum x^2=n\sigma^2$

Similarly, $\sum y^2=n\sigma^2$

Now, if $r_{xy}=0$

$$\Rightarrow \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2\sum(y-\bar{y})^2}}=0$$

$\Rightarrow \sum(x-\bar{x})(y-\bar{y})=0$

$\Rightarrow \sum xy=0$ because $\bar{x}=0, \bar{y}=0$

A|2007 Q#6(a):-

If x and y have zero mean and same variance σ^2 and zero correlation, then show that $(x\cos\alpha+y\sin\alpha)$ and $(x\sin\alpha-y\cos\alpha)$ have same variance σ^2 and zero correlation.

Sol:-

Let $U = X\cos\alpha + Y\sin\alpha$

$V = X\sin\alpha - Y\cos\alpha$

we have to prove

$var(U) = \sigma^2 = var(V)$ and $r_{uv} = 0$

Since, $\bar{x} = \bar{y} = 0$ and $r_{xy} = 0$

$var(U) = var(X\cos\alpha + Y\sin\alpha)$

As $r_{xy} = 0 \Rightarrow var(X) = \sigma^2$, $var(Y) = \sigma^2$

So $var(U) = var(X\cos\alpha) + var(Y\sin\alpha)$

($\because var(aX) = a^2 var(X)$; $var(X+Y) = var(X) + var(Y)$)

$= \cos^2\alpha\, var(X) + \sin^2\alpha\, (var(Y))$

$= \cos^2\alpha\,(\sigma^2) + \sin^2\alpha\,(\sigma^2)$

$= (\cos^2\alpha + \sin^2\alpha)\,\sigma^2$

$var(U) = \sigma^2$

$\Rightarrow var(X\cos\alpha + Y\sin\alpha) = \sigma^2$

Now $var(V) = var(X\sin\alpha - Y\cos\alpha)$

As $r_{xy} = 0$ and $var(X) = \sigma^2 = var(Y)$

we know $var(X-Y) = var(X) + var(Y)$

So $var(V) = var(X\sin\alpha) + var(Y\cos\alpha)$

Also $var(aX) = a^2 var(X)$

So, $var(V) = \sin^2\alpha\, var(X) + \cos^2\alpha\, var(Y)$

Since, $var(X) = \sigma^2 = var(Y)$

So, $var(V) = \sin^2\alpha(\sigma^2) + \cos^2\alpha(\sigma^2)$

$= (\sin^2\alpha + \cos^2\alpha)\sigma^2$

$var(x\sin\alpha - y\cos\alpha) = \sigma^2$

Now $r_{uv} = \dfrac{\Sigma(u-\bar{u})(v-\bar{v})}{\sqrt{\Sigma(u-\bar{u})^2\Sigma(v-\bar{v})^2}}$

As

$u = x\cos\alpha + y\sin\alpha$

$\bar{u} = \bar{x}\cos\alpha + \bar{y}\sin\alpha = 0 \quad \because \bar{x} = \bar{y} = 0$

$v = x\sin\alpha - y\cos\alpha$

$\bar{v} = \bar{x}\sin\alpha - \bar{y}\cos\alpha = 0 \quad \because \bar{x} = \bar{y} = 0$

$\therefore \quad r_{uv} = \dfrac{\Sigma uv}{\sqrt{\Sigma u^2 \Sigma v^2}}$

$\because \bar{u} = 0$ and $var(u) = \sigma^2$

$\Rightarrow \Sigma u^2 = n\sigma^2$

and $\bar{v} = 0$, $var(v) = \sigma^2$

$\Rightarrow \Sigma v^2 = n\sigma^2$

$r_{uv} = \dfrac{\Sigma uv}{\sqrt{(n\sigma^2)(n\sigma^2)}}$

$= \dfrac{\Sigma uv}{n\sigma^2}$

$= \dfrac{1}{n\sigma^2}\Sigma(x\cos\alpha + y\sin\alpha)(x\sin\alpha - y\cos\alpha)$

$= \dfrac{1}{n\sigma^2}\Sigma(x^2\cos\alpha\sin\alpha - xy\cos^2\alpha$

$+ xy\sin^2\alpha - y^2\sin\alpha\cos\alpha)$

$$= \frac{1}{n\sigma^2}\left(\cos\alpha\sin\alpha\Sigma x^2 - \cos^2\alpha\Sigma xy + \sin^2\alpha\Sigma xy - \sin\alpha\cos\alpha\Sigma y^2\right) \rightarrow (1)$$

$\because$ $r_{xy} = 0$ and $\bar{x} = \bar{y} = 0 \Rightarrow \Sigma xy = 0$

Also $var(x) = \sigma^2$, $var(y) = \sigma^2$

and $\bar{x} = \bar{y} = 0$

$\Rightarrow \Sigma x^2 = n\sigma^2$, $\Sigma y^2 = n\sigma^2$

put in (1)

$$r_{uv} = \frac{1}{n\sigma^2}\left(\cos\alpha\sin\alpha(n\sigma^2) + 0 + 0 - \cos\alpha\sin\alpha(n\sigma^2)\right)$$

$r_{uv} = 0$

Ex:-

X_1	X_2	X_3	X_1^2	X_2^2	X_3^2	X_1X_2	X_2X_3	X_1X_3
1	1	2	1	1	4	1	2	2
4	8	8	16	64	64	32	64	32
1	3	1	1	9	1	3	3	1
3	5	7	9	25	49	15	35	21
2	6	4	4	36	16	12	24	8
4	10	6	16	100	36	40	60	24
15	33	28	47	235	170	103	188	88

we want to calculate all multiple and partial correlations by using fitted regression model, which are

$$X_1 = \hat{\alpha}_{1.23} + \hat{\beta}_{12.3}X_2 + \hat{\beta}_{13.2}X_3 \rightarrow (A)$$

$$X_2 = \hat{\alpha}_{2\cdot13} + \hat{\beta}_{21\cdot3} X_1 + \hat{\beta}_{23\cdot1} X_3 \rightarrow (B)$$

$$X_3 = \hat{\alpha}_{3\cdot12} + \hat{\beta}_{31\cdot2} X_1 + \hat{\beta}_{32\cdot1} X_2 \rightarrow (C)$$

First, we fit (A). For this, normal equations are

$$\Sigma X_1 = n\hat{\alpha}_{1\cdot23} + \hat{\beta}_{12\cdot3}\Sigma X_2 + \hat{\beta}_{13\cdot2}\Sigma X_3 \rightarrow (1)$$

$$\Sigma X_1 X_2 = \hat{\alpha}_{1\cdot23}\Sigma X_2 + \hat{\beta}_{12\cdot3}\Sigma X_2^2 + \hat{\beta}_{13\cdot2}\Sigma X_2 X_3 \rightarrow (2)$$

$$\Sigma X_1 X_3 = \hat{\alpha}_{1\cdot23}\Sigma X_3 + \hat{\beta}_{12\cdot3}\Sigma X_2 X_3 + \hat{\beta}_{13\cdot2}\Sigma X_3^2 \rightarrow (3)$$

put all values in (1), (2) and (3)

$$15 = 6\hat{\alpha}_{1\cdot23} + 33\hat{\beta}_{12\cdot3} + 28\hat{\beta}_{13\cdot2} \rightarrow (a)$$

$$103 = 33\hat{\alpha}_{1\cdot23} + 235\hat{\beta}_{12\cdot3} + 188\hat{\beta}_{13\cdot2} \rightarrow (b)$$

$$88 = 28\hat{\alpha}_{1\cdot23} + 188\hat{\beta}_{12\cdot3} + 170\hat{\beta}_{13\cdot2} \rightarrow (c)$$

33 eq (a) − 6 eq (b)

$$495 - 618 = (198 - 198)\hat{\alpha}_{1\cdot23} + (1089 - 1410)\hat{\beta}_{12\cdot3} + (924 - 1128)\hat{\beta}_{13\cdot2}$$

$$-123 = -321\hat{\beta}_{12\cdot3} - 204\hat{\beta}_{13\cdot2}$$

$$\Rightarrow 123 = 321\hat{\beta}_{12\cdot3} + 204\hat{\beta}_{13\cdot2} \rightarrow (d)$$

28 eq (a) − 6 eq (c)

$$420 - 528 = (168 - 168)\hat{\alpha}_{1\cdot23} + (924 - 1128)\hat{\beta}_{12\cdot3} + (784 - 1020)\hat{\beta}_{13\cdot2}$$

$$-108 = -204\hat{\beta}_{12\cdot3} - 236\hat{\beta}_{13\cdot2}$$

$$\Rightarrow 108 = 204\hat{\beta}_{12\cdot3} + 236\hat{\beta}_{13\cdot2} \rightarrow (e)$$

Now from (d) and (e)

$$41 = 107\hat{\beta}_{12\cdot3} + 68\hat{\beta}_{13\cdot2} \rightarrow (4)$$

$$54 = 102\hat{\beta}_{12\cdot3} + 118\hat{\beta}_{13\cdot2} \rightarrow (5)$$

$$eq(4) - 1.049\, eq(5)$$

$$41 - 56.646 = (107 - 106.998)\hat{\beta}_{12\cdot3} + (68 - 123.782)\hat{\beta}_{13\cdot2}$$

$$\Rightarrow -15.646 = -55.782\hat{\beta}_{13\cdot2}$$

$$\Rightarrow \hat{\beta}_{13\cdot2} = 0.28 \quad \text{put in (5)}$$

$$54 = 102\hat{\beta}_{12\cdot3} + 118(0.28)$$

$$54 - 33.04 = 102\hat{\beta}_{12\cdot3}$$

$$\Rightarrow \hat{\beta}_{12\cdot3} = 0.21 \quad \text{put in (a)}$$

$$15 = 6\hat{\alpha}_{1\cdot23} + 33(0.21) + 28(0.28)$$

$$15 = 6\hat{\alpha}_{1\cdot23} + 6.93 + 7.84$$

$$15 - 14.77 = 6\hat{\alpha}_{1\cdot23}$$

$$\Rightarrow \hat{\alpha}_{1\cdot23} = 0.04$$

So, fitted estimated regression model is

$$X_1 = 0.04 + 0.21X_2 + 0.28X_3 \rightarrow (A^*)$$

Now,

$$X_2 = \hat{\alpha}_{2\cdot13} + \hat{\beta}_{21\cdot3}X_1 + \hat{\beta}_{23\cdot1}X_3$$

Normal equations are

$$\sum X_2 = n\hat{\alpha}_{2\cdot13} + \hat{\beta}_{21\cdot3}\sum X_1 + \hat{\beta}_{23\cdot1}\sum X_3 \rightarrow (a)$$

$$\sum X_1X_2 = \hat{\alpha}_{2\cdot13}\sum X_1 + \hat{\beta}_{21\cdot3}\sum X_1^2 + \hat{\beta}_{23\cdot1}\sum X_1X_3 \rightarrow (b)$$

$$\sum X_2X_3 = \hat{\alpha}_{2\cdot13}\sum X_3 + \hat{\beta}_{21\cdot3}\sum X_1X_3 + \hat{\beta}_{23\cdot1}\sum X_3^2 \rightarrow (c)$$

put all values in (a), (b) and (c).

$$33 = 6\hat{\alpha}_{2\cdot13} + 15\hat{\beta}_{21\cdot3} + 28\hat{\beta}_{23\cdot1} \rightarrow (d)$$

$$103 = 15\hat{\alpha}_{2.13} + 47\hat{\beta}_{21.3} + 88\hat{\beta}_{23.1} \rightarrow (e)$$
$$188 = 28\hat{\alpha}_{2.13} + 88\hat{\beta}_{21.3} + 170\hat{\beta}_{23.1} \rightarrow (f)$$

5 eq (d) − 2 eq (e)

$$(165 - 206) = (30 - 30)\hat{\alpha}_{2.13} + (75 - 94)\hat{\beta}_{21.3} + (140 - 176)\hat{\beta}_{23.1}$$
$$\Rightarrow -41 = -19\hat{\beta}_{21.3} - 36\hat{\beta}_{23.1}$$
$$\Rightarrow 41 = 19\hat{\beta}_{21.3} + 36\hat{\beta}_{23.1} \rightarrow (a')$$

28 eq (d) − 6 eq (f)

$$(924 - 1128) = (168 - 168)\hat{\alpha}_{2.13} + (420 - 528)\hat{\beta}_{21.3} + (784 - 1020)\hat{\beta}_{23.1}$$
$$-204 = -108\hat{\beta}_{21.3} - 236\hat{\beta}_{23.1}$$
$$\Rightarrow 102 = 54\hat{\beta}_{21.3} + 118\hat{\beta}_{23.1} \rightarrow (b')$$

54 eq (a′) − 19 eq (b′)

$$(2214 - 1938) = (1026 - 1026)\hat{\beta}_{21.3} + (1944 - 2242)\hat{\beta}_{23.1}$$
$$\Rightarrow 276 = -298\hat{\beta}_{23.1}$$
$$\Rightarrow \hat{\beta}_{23.1} = -0.93 \quad \text{put in } (a')$$
$$41 = 19\hat{\beta}_{21.3} + 36(-0.93)$$
$$41 = 19\hat{\beta}_{21.3} - 33.48$$
$$\Rightarrow 74.48 = 19\hat{\beta}_{21.3}$$
$$\Rightarrow \hat{\beta}_{21.3} = 3.92 \quad \text{put in } (d)$$
$$33 = 6\hat{\alpha}_{2.13} + 15(3.92) + 28(-0.93)$$
$$33 = 6\hat{\alpha}_{2.13} + 58.8 - 26.04$$
$$33 = 6\hat{\alpha}_{2.13} + 32.76$$

$0.24 = 6\hat{\alpha}_{2.13} \Rightarrow \hat{\alpha}_{2.13} = 0.04$

So, fitted regression model is

$X_2 = 0.04 + 3.92 X_1 - 0.93 X_3$

Now, $X_3 = \hat{\alpha}_{3.12} + \hat{\beta}_{31.2} X_1 + \hat{\beta}_{32.1} X_2 \rightarrow (B^*)$

Normal equations are

$\sum X_3 = n\hat{\alpha}_{3.12} + \hat{\beta}_{31.2} \sum X_1 + \hat{\beta}_{32.1} \sum X_2 \rightarrow (a)$

$\sum X_1 X_3 = \hat{\alpha}_{3.12} \sum X_1 + \hat{\beta}_{31.2} \sum X_1^2 + \hat{\beta}_{32.1} \sum X_1 X_2 \rightarrow (b)$

$\sum X_2 X_3 = \hat{\alpha}_{3.12} \sum X_2 + \hat{\beta}_{31.2} \sum X_1 X_2 + \hat{\beta}_{32.1} \sum X_2^2 \rightarrow (c)$

put all values in (a), (b), (c)

$28 = 6\hat{\alpha}_{3.12} + 15\hat{\beta}_{31.2} + 33\hat{\beta}_{32.1} \rightarrow (a')$

$88 = 15\hat{\alpha}_{3.12} + 47\hat{\beta}_{31.2} + 103\hat{\beta}_{32.1} \rightarrow (b')$

$188 = 33\hat{\alpha}_{3.12} + 103\hat{\beta}_{31.2} + 235\hat{\beta}_{32.1} \rightarrow (c')$

5 eq(a') − 2 eq(b')

$(140-176) = (30-30)\hat{\alpha}_{3.12} + (75-94)\hat{\beta}_{31.2} + (165-206)\hat{\beta}_{32.1}$

$\Rightarrow -36 = -19\hat{\beta}_{31.2} - 41\hat{\beta}_{32.1}$

$\Rightarrow 36 = 19\hat{\beta}_{31.2} + 41\hat{\beta}_{32.1} \rightarrow (e)$

33 eq(a') − 6 eq(c')

$(924-1128) = (198-198)\hat{\alpha}_{3.12} + (495-618)\hat{\beta}_{31.2} + (1089-1410)\hat{\beta}_{32.1}$

$\Rightarrow -204 = -123\hat{\beta}_{31.2} + (-321)\hat{\beta}_{32.1}$

$68 = 41\hat{\beta}_{31.2} + 107\hat{\beta}_{32.1} \rightarrow (f)$

from (e) and (f)

Mathematical models are exact in nature while statistical models are probabilistic in nature (they based on laws of probability).

$$36 = 19\hat{\beta}_{31\cdot2} + 41\hat{\beta}_{32\cdot1} \rightarrow (e)$$
$$68 = 41\hat{\beta}_{31\cdot2} + 107\hat{\beta}_{32\cdot1} \rightarrow (f)$$

41 Eq (e) − 19 Eq (f)

$$1476 = 779\hat{\beta}_{31\cdot2} + 1681\hat{\beta}_{32\cdot1}$$
$$\pm 1292 = \pm 779\hat{\beta}_{31\cdot2} \pm 2033\hat{\beta}_{32\cdot1}$$

$$184 = -352\hat{\beta}_{32\cdot1}$$
$$\Rightarrow \hat{\beta}_{32\cdot1} = \frac{-184}{352}$$
$$\Rightarrow \hat{\beta}_{32\cdot1} = -0.523$$

put in (e)

$$36 = 19\hat{\beta}_{31\cdot2} + 41(-0.523)$$
$$36 + 21.443 = 19\hat{\beta}_{31\cdot2}$$
$$57.443 = 19\hat{\beta}_{31\cdot2}$$
$$\Rightarrow \hat{\beta}_{31\cdot2} = 3.023$$

put these in (a')

$$28 = 6\hat{\alpha}_{3\cdot12} + 15(3.023) + 33(-0.523)$$
$$28 = 6\hat{\alpha}_{3\cdot12} + 45.345 - 17.259$$
$$28 = 6\hat{\alpha}_{3\cdot12} + 28.086$$
$$\Rightarrow 28 - 28.086 = 6\hat{\alpha}_{3\cdot12}$$
$$\Rightarrow -0.086 = 6\hat{\alpha}_{3\cdot12}$$
$$\Rightarrow \hat{\alpha}_{3\cdot12} = -0.014$$

The fitted regression model is

$$X_3 = -0.014 + 3.023X_1 - 0.523X_2 \Rightarrow (C^*)$$

From (A^*), (B^*) and (C^*)

$$\hat{\beta}_{13\cdot2} = 0.28, \quad \hat{\beta}_{12\cdot3} = 0.21$$

$$\hat{\beta}_{23\cdot1} = -0.93, \quad \hat{\beta}_{21\cdot3} = 3.92$$

$$\hat{\beta}_{32\cdot1} = -0.523, \quad \hat{\beta}_{31\cdot2} = 3.023$$

Partial correlations are

$$r_{12\cdot3} = \sqrt{\hat{\beta}_{12\cdot3} \times \hat{\beta}_{21\cdot3}}$$

$$r_{12\cdot3} = \sqrt{0.21 \times 3.92}$$

$$r_{12\cdot3} = \sqrt{0.8232}$$

$$r_{12\cdot3} = 0.91$$

$$r_{13\cdot2} = \sqrt{\hat{\beta}_{13\cdot2} \times \hat{\beta}_{31\cdot2}}$$

$$= \sqrt{0.28 \times 3.023}$$

$$= \sqrt{0.84644}$$

$$r_{13\cdot2} = 0.92$$

$$r_{23\cdot1} = \sqrt{\hat{\beta}_{23\cdot1} \times \hat{\beta}_{32\cdot1}}$$

$$= \sqrt{-0.93 \times -0.523}$$

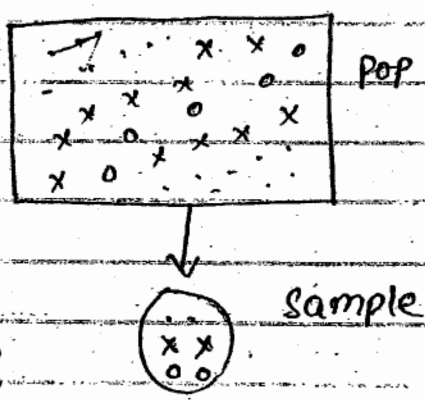
$$r_{23\cdot1} = -\sqrt{0.48639}$$

$$r_{23\cdot1} = -0.68$$

Sampling And Sampling Distributions

In Research question, we have entire collection: population (universal set). we have some restrictions i-e

Natural
Time
Cost
Administrative



Question:-

Average income per month in Pakistan?

Census

$\downarrow$

Sample survey

Population → Summarize (Mean, var)
→ Prob. distribution
→ correlational

$\downarrow$

Sample → Summarize (Mean, var)
→ determine prob dist
→ Correlational

Notations:-

Population size $= N$

Sample size $= n \quad (n \leq N)$

Random or study variable $= X$

Population Mean $= \mu = \Sigma X$

$$\text{Population variance} = \sigma^2 = \frac{\Sigma(x-\mu)^2}{N}$$

$$\text{Sample Mean} = \bar{x} = \frac{\Sigma x}{n}$$

$$\text{Sample variance} = s^2 = \frac{\Sigma(x-\bar{x})^2}{n}$$

Mean of sample means $= \mu_{\bar{x}}$

variance of sample means $= \sigma^2_{\bar{x}}$

Parametre :-

Population summary measure is parametre : μ (population mean)

parametre : μ is a constant value.

Statistic :-

Sample summary measure is statistic : $\bar{x}$ (sample mean)

statistic : $\bar{x}$ is variable quantity, it varies as the sample changes.

Sampling error:-

sampling error is amount of error between a sample statistic and its corresponding population parametre.

sampling error $= \bar{x} - \mu$

As sample size increase, sampling error decreases.

so sampling error $\propto \dfrac{1}{\text{sample size} = n}$

Sampling Techniques:-

(i) Probability sampling (Random sampling)

(ii) Non-Probability sampling (Non-Random Sampling)

(Population parameter and sample statics are Min, Max, var, Proportion)

Example:

Population = 2, 4, 6 $\Rightarrow N = 3$

$X = 2, 4, 6$

pop mean = $\mu = \frac{\Sigma X}{n} = \frac{12}{3}$

$\mu = 4$

pop. variance = $\frac{\Sigma (x-\mu)^2}{N}$

$\sigma^2 = \frac{(2-4)^2 + (4-4)^2 + (6-4)^2}{3}$

$\sigma^2 = \frac{4+4}{3} = \frac{8}{3}$

Sampling:

$n = 2$

we want to select sample of size "2". So,

Samples	$\bar{x}$	$\bar{x}-\mu$	$(\bar{x}-\mu)^2$	X_{max}	X_{min}
2, 2	2	-2	4	2	2
2, 4	3	-1	1	4	2
2, 6	4	0	0	6	2
4, 2	3	-1	1	4	2
4, 4	4	0	0	4	4
4, 6	5	1	1	6	4
6, 2	4	0	0	6	2
6, 4	5	1	1	6	4
6, 6	6	2	4	6	6
			12		

Let M: no of possible samples.
$\bar{X}$ is mean of these samples, so we can find sampling distribution of sample mean. Similarly, if we have X_{min} or X_{max} then we can also make distributions of minimum or maximum sample

$$\mu_{\bar{x}} = \frac{\sum \bar{x}}{n}$$

$$\mu_{\bar{x}} = \frac{36}{9} = 4$$

$$\Rightarrow \mu_{\bar{x}} = \mu \quad \text{(unbiasdness)}$$

Standard error:-

(Estimate of sampling error)

$$\sigma_{\bar{x}}^2 = \frac{\sum(\bar{x} - \mu_{\bar{x}})^2}{M}$$

$$= \frac{\sum(\bar{x} - \mu)^2}{M}$$

$$\sigma_{\bar{x}}^2 = \frac{12}{9} = \frac{4}{3} = 1.333$$

$\sigma_{\bar{x}} = 1.155$ is standard error.

As $\bar{x} = 2$ and $S.E = 1.155$

Then 2 ± 1.155 , $2 \pm 2(1.155)$

$\mu \in (0.845, 3.155)$ $\mu \in (-0.31, 4.31)$

So, we have a confidence level that population mean lies in interval $(-0.31, 4.31)$

Note:-

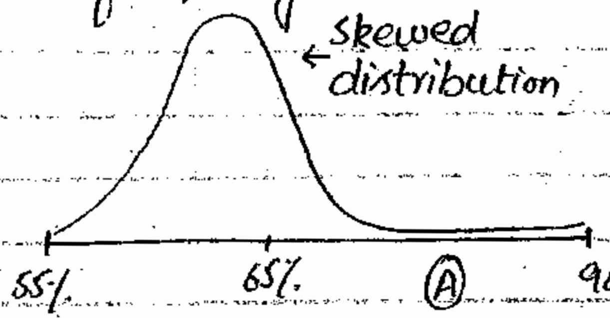
If error is small, then result will be precise or near to the accurate value.

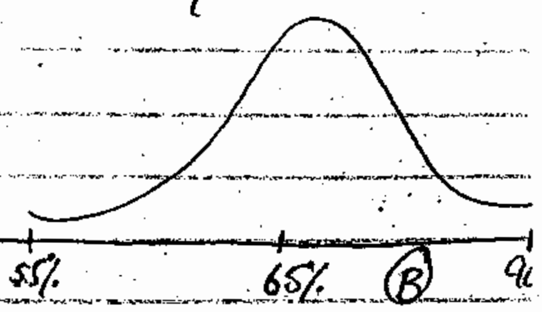
But in above example, sampling error is 1.155 and we have interval $(0.845, 3.155)$ in which $\mu = 4$ does not lie. So, we increase sampling error to get a confidence level in which μ lies and sampling

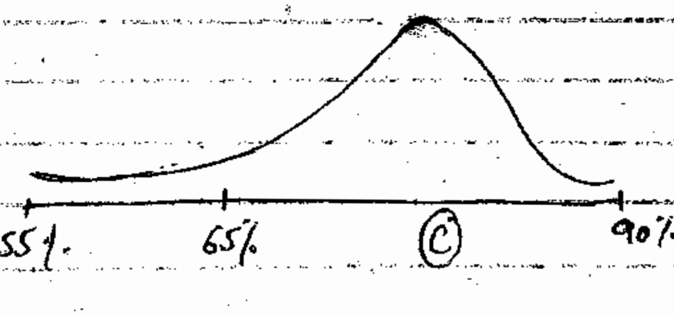
error can be increased by multiplying it by 2, 3, 4, ...
As we multiply S.E = 1.155 by "2" we have a confidence level in which μ lies $\Rightarrow \mu = 4 \in (-0.31, 4.31)$.
But when we multiply S.E = 1.155 by "3" Then we have a confidence level $(3-1.155, 3+1.155) = (1.845, 4.155)$ in which μ lies but this interval is wide. So, it is useless for us.
So, for obtaining confidence level in which μ lies, we multiply S.E by "2" usually.

Examples:-

Average marks in B.A/B.Sc./B.Com of Punjab university is $65\% = \mu$







starting percentage of marks is 55% and ending percentage of marks is 90%.

The curve (A) is skewed distribution (below the mean value) shows that most students obtain marks below 65% and so → This behaviour is called sampling distri-

Sampling distribution:-

Following are sampling distributions.

(1) Chi-square distribution (χ^2-distribution)

(2) Student's t-distribution

(3) F-distribution.

These three distributions approch the normal distribution under certain conditions.

These distributions are limiting form of normal distribution.

Chi-Square distribution (χ^2-distribution):-

If $X_1, X_2, \ldots, X_k$ are "k" independent and identically distributed having normal distribution with zero mean and unit variance, then we can define a statistic χ^2

$$\chi^2 = \sum_{i=1}^{k} X_i^2 \quad \text{such that}$$

$$f(\chi^2)\,d\chi^2 = \frac{1}{2^{k/2}\,\Gamma\frac{k}{2}} (\chi^2)^{k/2 - 1} e^{-\frac{1}{2}\chi^2} d\chi^2$$

Derivation of p.d.f of χ^2-distribution:-

If $X_i \overset{iid}{\sim} N(0,1)$; $i = 1, 2, \ldots, k$

then

$$f(X_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2} \quad ; i = 1, 2, \ldots k$$

$\because X \sim N(\mu, \sigma^2)$ then

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

Let us define a statistic

$$\chi^2 = \sum_{1}^{k} X_i^2, \text{ then}$$

$$f(\chi^2)\,d\chi^2 = f(x_1, x_2, \ldots, x_k)\,dx_1 . dx_2 \ldots dx_k$$

To obtain probability distribution function, integrand prob. density function.

$$f(\chi^2)\,d\chi^2 = \int_S f(x_1, x_2, \ldots, x_k)\,dx_1 . dx_2 \ldots dx_k$$

$$f(\chi^2)\,d\chi^2 = \int_S f(x_1) . f(x_2) . f(x_3) \ldots f(x_k) . dx_1 dx_2 dx_3 \ldots dx_k$$

$$= \int_S \prod_{i=1}^{k} f(x_i)\,dx_i$$

put value of $f(x_i)$

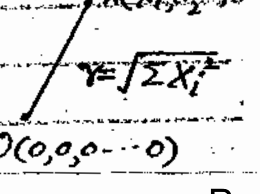
$$= \int_S \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}\,dx_i$$

$$= \int_S \frac{1}{(\sqrt{2\pi})^k} \left(e^{-\frac{x_1^2}{2}} . e^{-\frac{x_2^2}{2}} \ldots e^{-\frac{x_k^2}{2}}\right) dx_1 . dx_2 \ldots dx_k$$

$$= \int_S \frac{1}{(\sqrt{2\pi})^k} e^{-\frac{1}{2}\sum x_i^2}\,dx_1 . dx_2 \ldots dx_k$$

$$f(\chi^2)\,d\chi^2 = \frac{1}{(\sqrt{2\pi})^k} e^{-\frac{1}{2}\sum x_i^2} \int_S dx_1 . dx_2 \ldots dx_k$$

(The variables $x_1, x_2, \ldots, x_k$ could be individually interchange or change but they have constant sum of squares).
The above integral is a $(k-1)$ sphere such that radius is $r = \sqrt{\sum x_i^2}$.
Using this result, we can simplify the above integral as the surface area of



(k-1) sphere multiplied by the infinitesimal thickness of the sphere.

We can write surface area of (k-1) sphere as

$$A = \frac{k \gamma^{k-1} \pi^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)} \qquad \text{put } \gamma = \sqrt{\Sigma x_i^2}$$

$$\gamma = \sqrt{\chi^2}$$

So

$$A = \frac{k\left(\sqrt{\chi^2}\right)^{k-1} \pi^{k/2}}{\frac{k}{2}\Gamma\frac{k}{2}} \qquad \because \Gamma\alpha+1 = \alpha\Gamma\alpha$$

$$A = \frac{2\left(\chi^2\right)^{\frac{k-1}{2}} \pi^{k/2}}{\Gamma\frac{k}{2}}$$

Take $\gamma = \sqrt{\Sigma x_i^2}$; $-\infty < x_i < \infty$

$$\gamma = \sqrt{\chi^2}$$

$$\frac{d\gamma}{d\chi^2} = \frac{1}{2}\left(\chi^2\right)^{-1/2}$$

$$d\gamma = \frac{1}{2\sqrt{\chi^2}} d\chi^2$$

So

$$f(\chi^2)d\chi^2 = \frac{1}{\left(\sqrt{2\pi}\right)^k} e^{-\frac{1}{2}\chi^2} A.d\gamma$$

$\because A.d\gamma = \int_s dx_1 . dx_2 \ldots dx_k$

$$= \frac{1}{(2\pi)^{k/2}} e^{-\frac{\chi^2}{2}} . \frac{2\left(\chi^2\right)^{\frac{k-1}{2}} \pi^{k/2}}{\Gamma\frac{k}{2}} \frac{1}{2\sqrt{\chi^2}} d\chi^2$$

$$f(\chi^2)\,d\chi^2 = \frac{1}{2^{k/2}\,\Gamma\!\left(\frac{k}{2}\right)}\, e^{-\frac{1}{2}\chi^2}\,(\chi^2)^{\frac{k}{2}-1}\,d\chi^2$$

$$\Rightarrow f(\chi^2) = \frac{1}{2^{k/2}\,\Gamma\!\left(\frac{k}{2}\right)}\, e^{-\frac{\chi^2}{2}}\,(\chi^2)^{\frac{k}{2}-1} \quad ; 0 < \chi^2 < \infty$$

is probability distribution function of χ^2-distribution.

Note:-

(i) If $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

then

$$\chi^2 = \sum Z_i^2 \sim \chi^2_{\text{Parametre}}$$

where

$$Z_i = \frac{X_i - \mu}{\sigma^2}$$

(ii) If $X_i \sim \exp(\lambda)$

then

$$\chi^2 = \sum X_i^2 \nsim \chi^2$$

(iii) If $X_i \nsim$ Normal

$$\log X_i \sim N(\mu^*, \sigma^{2*})$$

then $Z^* = \dfrac{\log X_i - \mu^*}{\sigma^*} \sim N(0,1)$

Examples:-

If $n = 5$, $\bar{X} = 10$

then $\bar{X} = \dfrac{\sum X}{n} \Rightarrow 10 = \dfrac{\sum X}{5}$

$\Rightarrow \sum X = 50$ then degree of freedom $= 4$

X	
8	→ freedom
15	
7	
6	
14	
50	

Degrees Of Freedom

If $X_i \xrightarrow{iid} N(0,1)_n$; $i=1,2,\dots,n$
then statistic $\sum_{i=1}^{n} X_i^2 = \chi^2$ follows a χ^2-distribution with probability density function.

$$f(\chi^2) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} e^{-\frac{1}{2}\chi^2} (\chi^2)^{\frac{n}{2}-1}$$

$$; 0 \leq \chi^2 \leq \infty$$

We can say statistic $\sum_{i=1}^{n} X_i^2$ follows a Chi-square distribution with 'n' degrees of freedom.

$$\sum_{i=1}^{n} X_i^2 \sim \chi_n^2$$

where χ^2 is pivotal statistic.

Properties of Chi-square distribution:-

1) Moments Generating function (m.g.f):

Let we have a sample of n-observations $X_1, X_2, \ldots, X_n$ such that $X_i \overset{iid}{\sim} N(0,1)$ then we can define the chi-square statistic as

$$\chi^2 = \sum_{i=1}^{n} X_i^2$$

We can define the m.g.f as

$$M(t) = E(e^{t\chi^2})$$

$$= E(e^{t\Sigma x_i^2})$$

$$= E(e^{t(x_1^2 + x_2^2 + \cdots + x_n^2)})$$

$$= E(e^{tx_1^2} \cdot e^{tx_2^2} \cdot \ldots \cdot e^{tx_n^2})$$

$$M(t) = E(e^{tx_1^2}) \cdot E(e^{tx_2^2}) \cdots E(e^{tx_n^2})$$

$\because x_1, x_2, \ldots, x_n$ are independent

$(E(XY) = E(X)E(Y)$ if X, Y are independent$)$.

$$M(t) = \prod_{i=1}^{n} E(e^{tx_i^2})$$

$$= \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{tx_i^2} f(x_i)\, dx_i$$

$\because x_i$'s are identic x_i

$$= \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{tx_i^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$\because X \sim N(\mu, \sigma)$ then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

and $X \sim N(0, 1)$ then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

So,

$$M(t) = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2 + t x_i^2} \, dx_i$$

$$= \prod_{i=1}^{n} 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i^2 - 2t x_i^2)} \, dx_i$$

$$= \prod_{i=1}^{n} \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}(1-2t)x_i^2} \, dx_i$$

Put $\frac{1}{2}(1-2t)x_i^2 = v$ $\Rightarrow x_i = \sqrt{\frac{2v}{1-2t}}$

$$\frac{2}{2} x_i (1-2t) \, dx_i = dv$$

$$\Rightarrow (1-2t) x_i \, dx_i = dv$$

$$\Rightarrow dx_i = \frac{1}{(1-2t)x_i} dv$$

put x_i - value

$$\Rightarrow dx_i = \frac{1}{(1-2t)\sqrt{\frac{2v}{1-2t}}} dv$$

$$dX_i = \frac{1}{\sqrt{1-2t}\sqrt{2v}}\,dv$$

$$\therefore \quad M(t) = \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-v} \frac{1}{\sqrt{1-2t}\sqrt{2v}}\,dv$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \int_0^\infty \frac{e^{-v}}{\sqrt{v}}\,dv$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \int_0^\infty e^{-v} v^{-1/2}\,dv$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \Gamma\left(\tfrac{1}{2}\right)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \sqrt{\pi}$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{1-2t}}.$$

$$= \left(\frac{1}{\sqrt{1-2t}}\right)^n$$

$$= \frac{1}{(1-2t)^{n/2}}$$

$$M(t) = (1-2t)^{-n/2}$$

for $t < \frac{1}{2}$

2. Mean :-

$$\mu = \frac{d}{dt}(M(t))\Big|_{t=0}$$

$$= \frac{d}{dt}\left((1-2t)^{-n/2}\right)\Big|_{t=0}$$

$$= \left(-\frac{n}{2}(1-2t)^{-\frac{n}{2}-1}(0-2)\right)\Big|_{t=0}$$

$$= \left(n(1-2t)^{-\frac{n}{2}-1}\right)\Big|_{t=0}$$

$$= n(1-0)$$

$$\text{Mean} = n$$