Measure Theory: Notes

by

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Partial Contents

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Algebra on X:

Let $X \neq \phi$ be non-empty set, the collection of sub set of X, $\mathcal{A}$ is called algebra on X if $\mathcal{A}$ satisfy the following axioms

(i) $\mathcal{A}$ is closed under complement.
i.e If $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$.

(ii) $\mathcal{A}$ is closed under finite union. i.e
If $E_1, E_2, \dots, E_n \in \mathcal{A}$ then $\bigcup_{i=1}^{n} E_i \in \mathcal{A}$.

Theorem If $\mathcal{A}$ is algebra on X then
Prove that
(i) $\phi, X \in \mathcal{A}$.
(ii) If $E_1, E_2, \dots, E_n \in \mathcal{A}$ then $\bigcap_{i=1}^{n} E_i \in \mathcal{A}$
(iii) If $A, B \in \mathcal{A}$ then $A \setminus B \in \mathcal{A}$.

Proof:

(i) Let $E \subseteq X$ s.t $E \in \mathcal{A}$
then $E^c \in \mathcal{A}$ $\because$ $\mathcal{A}$ is algebra on X.

also

$E \cup E^c \in \mathcal{A}$ $\because$ $\mathcal{A}$ is algebra on X.

$\Rightarrow X \in \mathcal{A}$ $\because$ $X = E \cup E^c$

By complement property of $\mathcal{A}$

$X^c \in \mathcal{A}$

$\Rightarrow \phi \in \mathcal{A}$ $\because$ $X^c = \phi$

Proof (ii) If $E_1, E_2, \ldots, E_n \in \mathcal{A}$ then

$E_1^c, E_2^c, \ldots, E_n^c \in \mathcal{A}$ so that

$\bigcup_{i=1}^{n} E_i^c \in \mathcal{A}$ (by def of $\mathcal{A}$)

Then

$\left(\bigcup_{i=1}^{n} E_i^c\right)^c \in \mathcal{A}$ (by def of $\mathcal{A}$)

Then by De Morgan's Law

$$\left(\bigcup_{i=1}^{n} E_i^c\right)^c = \bigcap_{i=1}^{n} (E_i^c)^c$$

$$= \bigcap_{i=1}^{n} E_i$$

So $\bigcap_{i=1}^{n} E_i \in \mathcal{A}$ $\because$ $\left(\bigcup_{i=1}^{n} E_i^c\right)^c \in \mathcal{A}$

Proof (iii)

Let $A, B \in \mathcal{A}$ then $B^c \in \mathcal{A}$

So

$A \setminus B = A \cap B^c \in \mathcal{A}$ $\because$ $A \setminus B = A \cap B^c$ & intersection of two sets belongs to $\mathcal{A}$.

Sigma Algebra i.e σ-algebra on X:

Let $X \neq \phi$, $\mathcal{A}$ be the collection of subsets of X called σ-algebra on X if it satisfy the following axiomes

(i) If $E \in \mathcal{A}$ Then $E^c \in \mathcal{A}$
(ii) If $E_1, E_2, E_3, --- \in \mathcal{A}$ Then

$\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$ i.e closed undee countable union.

Remark: (i) Every algebra is σ-algebra but not every σ-algebra is algebra on X.

(ii) If X is finite Then algebra and σ-algebra are equal mean that both have same meaning.

Theorem

If $\mathcal{A}$ is σ-algebra on X Then
(i) $\phi, X \in \mathcal{A}$
(ii) If $\{E_i : i \in N\}_{i=1}^{\infty}$ in $\mathcal{A}$ Then

$\bigcap_{i=1}^{\infty} E_i \in \mathcal{A}$

(iii) If $A, B \in \mathcal{A}$ Then $A \cap B \in \mathcal{A}$
(Do yourself).

Trivial σ-algebra:

Let $X \neq \phi$ and $\mathcal{A} = \{\phi, X\}$ form σ-algebra on X called trivial σ-algebra on X.

Largest σ-algebra:

Let $X \neq \phi$ and $\mathcal{A} = P(X)$ is a σ-algebra called largest σ-algebra on X.

Question: Let $X \neq \phi$ be non-empty set and

$$\mathcal{A} = \{ E : E \subseteq X \mid E \text{ is countable or } E^c \text{ is countable} \}$$

is a σ-algebra on X.

Proof: Let $E \in \mathcal{A}$ then E is countable or E^c is countable.

Case-I (i) If E is countable then E^c or $(E^c)^c$ is countable. so $E^c \in \mathcal{A}$

Case-II (ii) If E^c is countable then $(E^c)^c$ or E^c is countable so $E^c \in \mathcal{A}$.

let $\{E_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{A}$ we are to show that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

Case I. If each E_i is countable then $\bigcup_{i=1}^{\infty} E_i$ is countable because countable union of countable sets is countable.

Case II Suppose $E_k \in \{E_i\}_{i=1}^{\infty}$ is not countable for some $k \in N$. Then E_k^c is countable (by def $\mathcal{A}$).

Now

$$E_k \subseteq \bigcup_{i=1}^{\infty} E_i$$

$$\Rightarrow \left(\bigcup_{i=1}^{\infty} E_i\right)^c \subseteq E_k^c$$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i^c$ is countable. $\because E_k^c$ is countable

$$\Rightarrow \bigcup_{i=1}^{\infty} E_i^c \in \mathcal{A}$$

So $\mathcal{A}$ is σ- algebra on X.

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Theorem: Intersection of any no of σ- algebras is a σ- algebra.

Proof let $\{\mathcal{A}_i : i \in N\}$ be the family of σ- algebras. we are to proove that $\bigcap_{i=1}^{\infty} \mathcal{A}_i$ is a σ- algebra.

(i) let $\mathcal{A} = \bigcap_{i=1}^{\infty} \mathcal{A}_i$ and $E \in \mathcal{A}$

$\therefore \quad E \in \bigcap_{i=1}^{\infty} \mathcal{A}_i$

$\Rightarrow E \in \mathcal{A}_i \quad \forall \quad i = 1, 2, \ldots$

$\Rightarrow E^c \in \mathcal{A}_i \ \forall i \because$ each $\mathcal{A}_i$ is σ-algebra

$\Rightarrow \quad E^c \in \bigcap \mathcal{A}_i = \mathcal{A}$

$\Rightarrow \quad E^c \in \mathcal{A}.$

(ii) Let $\{E_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{A}$.

$\Rightarrow \{E_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{A}_i \ \forall i$ $\because \mathcal{A} = \bigcap_{i=1}^{\infty} \mathcal{A}_i.$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}_i \ \forall i \quad \because$ each $\mathcal{A}_i$ is σ-algebra.

$\Rightarrow \quad \bigcup_{i=1}^{\infty} E_i \in \bigcap_{i=1}^{\infty} \mathcal{A}_i = \mathcal{A}$

Hence $\mathcal{A} = \bigcap_{i=1}^{\infty} \mathcal{A}_i$ is a σ- algebra.

Remark: The union of two σ-algebras may or may not be σ-algebra.

For e.g. let $X = \{a, b, c, d\}$ and

$\mathcal{A}_1 = \{\phi, X, \{a\}, \{b, c, d\}\}$, $\mathcal{A}_2 = \{\phi, X, \{b\}, \{a, c, d\}\}$

are σ-algebras on X.

$\mathcal{A}_1 \cup \mathcal{A}_2 = \{\phi, X, \{a\}, \{b\}, \{a, c, d\}, \{b, c, d\}\}$

is not σ-algebra because $\{a\}, \{b\} \in \mathcal{A}_1 \cup \mathcal{A}_2$

But $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$.

Increasing Sequence of Sets:

A sequence of sets $\{A_n\}_{n=1}^{\infty}$ is said to be increasing sequence if

$A_n \subseteq A_{n+1} \quad \forall\ n \in N$

i.e $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. & denoted as

$\{A_n\}_{n=1}^{\infty} \uparrow$ and limit value of the increasing sequence is defined as

$$\boxed{\lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n}$$

Decreasing Sequence of Sets :

A sequence of sets $\{A_n\}_{n=1}^{\infty}$ is said to be decreasing if

$$A_n \supseteq A_{n+1} \quad \forall n \in N$$

and it is denoted as $\{A_n\}_1^{\infty} \downarrow$.

limit value of the decreasing sequence is defined as

$$\boxed{\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n}$$

For Example (i) The sequence $\{A_n\}_{n=1}^{\infty}$ with

$$A_n = \left(0, \frac{1}{n}\right), \quad n = 1, 2, 3, \ldots \text{ is}$$

decreasing sequence so

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$$

$$= \phi.$$

(ii) If $A_n = \left[0, \frac{1}{n}\right)$, so $\{A_n\}_{n=1}^{\infty}$ is decreasing so

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} [0, 1/n)$$

$$\lim_{n\to\infty} A_n = \{0\} \ //.$$

Define $\limsup\limits_{k\to\infty} A_k$ and $\liminf\limits_{k\to\infty} A_k$.

Let $\{A_k\}_{k=1}^{\infty}$ be an arbitrary sequence of sub sets of set X. Define two new sequences

(i) $\underline{A}_k = \bigcap\limits_{n\geq k} A_n$

i.e $\underline{A}_1 = \bigcap\limits_{n\geq 1} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots\cdots$

$\underline{A}_2 = \bigcap\limits_{n\geq 2} A_n = A_2 \cap A_3 \cap A_4 \cap \cdots\cdots$

$\underline{A}_3 = \bigcap\limits_{n\geq 3} A_n = A_3 \cap A_4 \cap A_5 \cap \cdots\cdots$

$\vdots$

obviously $\{\underline{A}_k\}_{k=1}^{\infty}$ is increasing. so

$$\lim_{k\to\infty} \underline{A}_k = \bigcup_{k\geq 1} \underline{A}_k = \bigcup_{k\geq 1} \left(\bigcap_{n\geq k} A_n \right) \quad \text{—(A)}$$

so limit inferior of the origional sequence $\{A_k\}_{k=1}^{\infty}$ is defined as

$$\liminf_{k\to\infty} A_k = \lim_{k\to\infty} \underline{A}_k$$

$$\boxed{\liminf_{k\to\infty} A_k = \bigcup_{k\geq 1} \left(\bigcap_{n\geq k} A_n \right)}$$

To define $\lim\limits_{n \to \infty} \sup A_k$ of the sequence $\{A_k\}_{k=1}^{\infty}$ we define a new sequence $\{\bar{A}_k\}_{k=1}^{\infty}$ s.t

$$\bar{A}_k = \bigcup_{n \geq k} A_n \quad \text{i.e} \quad \bar{A}_1 = \bigcup_{n \geq 1} A_1$$

$$\bar{A}_1 = A_1 \cup A_2 \cup \cdots$$

$$\bar{A}_2 = A_2 \cup A_2 \cup \cdots$$

$$\bar{A}_3 = A_3 \cup A_4 \cup \cdots$$

$$\vdots$$

Clearly $\{\bar{A}_k\}_{k=1}^{\infty}$ is decreasing.

Therefore

$$\lim_{k \to \infty} \bar{A}_k = \bigcap_{k \geq 1} (\bar{A}_k)$$

$$\lim_{k \to \infty} \bar{A}_k = \bigcap_{k \geq 1} \left(\bigcup_{n \geq k} A_n \right) \quad \text{— (B)}$$

So the lim inf A_k of the sequence $\{A_k\}_{k=1}^{\infty}$ is

$$\liminf_{k \to \infty} A_k = \lim_{k \to \infty} \bar{A}_k$$

$$\boxed{\liminf_{k \to \infty} A_k = \bigcap_{k \geq 1} \left(\bigcup_{n \geq k} A_n \right)} \quad \text{by (B)}$$

//.

The limit of the arbitrary sequence $\{A_k\}_{k=1}^{\infty}$ exist if

$$\lim_{k\to\infty} \inf A_k = \lim_{k\to\infty} \sup A_k = \lim_{k\to\infty} A_k.$$

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Question let $\mathcal{A}$ be σ-algebra on X and $\{A_k\}_{k=1}^{\infty}$ be arbitrary sequence in $\mathcal{A}$. Then show that $\lim_{k\to\infty} \inf A_k$ and $\lim_{k\to\infty} \sup A_k$ is in $\mathcal{A}$.

Proof we know that

$$\lim_{k\to\infty} \inf A_k = \bigcup_{k\geq 1} \left(\bigcap_{n\geq k} A_n \right)$$

Since $\{A_k\}_1^{\infty}$ is in $\mathcal{A}$.

$\therefore \bigcap_{n\geq k} A_n \in \mathcal{A}$

also $\bigcup_{k\geq 1} \left(\bigcap_{n\geq k} A_n \right) \in \mathcal{A}$ $\because$ $\mathcal{A}$ σ-algebra on X.

so $\lim_{k\to\infty} \inf A_k \in \mathcal{A}$

Similarly

$$\lim_{k\to\infty} \sup A_k = \bigcap_{k\geq 1} \left(\bigcup_{n\geq k} A_k \right)$$

Since $\{A_k\}_1^{\infty}$ is in $\mathcal{A}$ and $\mathcal{A}$ σ-algebra

$\therefore \bigcap_{k\geq 1} \left(\bigcup_{n\geq k} A_k \right) \in \mathcal{A}$ so $\lim_{k\to\infty} \sup A_k \in \mathcal{A}$. //

Remark: If $\{A_k\}_{k=1}^{\infty}$ is in $\mathcal{A}$, $\mathcal{A}$ = σ-algebra on X. & $\lim_{k \to \infty} A_k$ exists then

$\lim_{k \to \infty} A_k \in \mathcal{A}$.

Smallest σ- Algebra

Let $\mathcal{E}$ be an arbitrary collection of subsets of a set X. The smallest σ- algebra "$\sigma(\mathcal{E})$" is the intersection of σ-algebras containg '$\mathcal{E}$'. i.e

$$\sigma(\mathcal{E}) = \bigcap_{i=1}^{\infty} \mathcal{A}_i \quad , \text{where } \mathcal{E} \subseteq \mathcal{A}_i \ \forall i.$$

Remarks

(1) If $\mathcal{E}_1$, $\mathcal{E}_2$ are arbitrary collections of subsets of X & $\mathcal{E}_1 \subseteq \mathcal{E}_2$ then $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$.

Proof: Let $\{\mathcal{A}_i : i \in N\}$ be family of σ-algebras s.t

$$\mathcal{E}_2 \subseteq \mathcal{A}_i \quad \forall i$$

$\therefore$ $\sigma(\mathcal{E}_2) = \bigcap \mathcal{A}_i$ now since

$\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{A}_i \Rightarrow \mathcal{E}_1 \subseteq \mathcal{A}_i$

so

$$\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$$

//.

(2) If $\mathcal{A}$ is a σ-algebra of subsets of X then

$$\sigma(\mathcal{A}) = \mathcal{A}.$$

Proof:

Since $\mathcal{A}$ is smallest sub collection of $\mathcal{A}$. Therefore by definition

$$\sigma(\mathcal{A}) = \mathcal{A}.$$

(3) $\sigma(\sigma(E)) = \sigma(E)$

Proof:

Since $\sigma(E)$ is σ-algebra on X.

$\therefore$ by Remark (2)

$$\sigma(\sigma(E)) = \sigma(E).$$

Recall If $X \neq \phi$ and $Y \neq \phi$ are two sets & $f: X \longrightarrow Y$ is a function then

(1) $f(X) \subseteq Y$

(2) If $E \subseteq Y$ then E need not to be subset of $f(X)$ &

$f^{-1}(E) = \{x : x \in X \mid f(x) \in E\}$ Thus

If $E \cap f(X) = \phi$ then $f^{-1}(E) = \phi$.

(3) for $E \subseteq Y$, $f(f^{-1}(E)) \subseteq E$.

(4) $f^{-1}(Y) = X$

5. $f^{-1}(E^c) = f^{-1}(Y \setminus E) = f^{-1}(Y) \setminus f^{-1}(E)$
$= X \setminus f^{-1}(E)$
$= (f^{-1}(E))^c$

i.e $f^{-1}(E^c) = (f^{-1}(E))^c$.

6. $f^{-1}\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(E_i)$

7. $f^{-1}\left(\bigcap_{i=1}^{\infty} E_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(E_i)$

8. If $\mathcal{E}$ is an arbitrary collection of subset of Y then

$$f^{-1}(\mathcal{E}) = \{f^{-1}(E) \mid E \in \mathcal{E}\}$$

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Theorem let $f: X \to Y$ be function and β is σ-algebra of subset of Y then Show that $f^{-1}(\beta)$ is σ-algebra on X.

Proof: let $A \in f^{-1}(\beta)$ then $\exists$ $E \in \beta$ such that

$$A = f^{-1}(E).$$

Since $E \in \beta$ then $E^c \in \beta$ $\because$ β is σ-algebra.

$\therefore$ $f^{-1}(E^c) \in f^{-1}(\beta)$

so $\quad A^c \in f^{-1}(\beta) \quad \because \quad f^{-1}(E^c) = [f^{-1}(E)]^c$
$f^{-1}(E^c) = A^c$

Let $\{A_n\}_{1}^{\infty}$ be a sequence in $f^{-1}(\beta)$ Then $\exists$ $\{E_n\}_{1}^{\infty}$ sequence in β s.t

$$A_n = f^{-1}(E_n)$$

Since β is σ-algebra on Y

$$\therefore \bigcup_{i=1}^{\infty} E_n \in \beta$$

$$\Rightarrow f^{-1}\left(\bigcup_{i=1}^{\infty} E_n\right) \in f^{-1}(\beta)$$

$$\Rightarrow \bigcup_{i=1}^{\infty} f^{-1}(E_n) \in f^{-1}(\beta) \quad \because f^{-1}\left(\bigcup_{i=1}^{\infty} E_n\right) = \bigcup_{i=1}^{\infty} f^{-1}(E_n)$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_n \in f^{-1}(\beta) \quad \because f^{-1}(E_n) = A_n.$$

So $f^{-1}(\beta)$ is σ-algebra on set X.

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Theorem Let $f: X \to Y$ be function Then for any arbitrary collection '$\mathcal{E}$' of subsets of Y

$$\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E})).$$

Proof: Since $\mathcal{E}$ collection of subsets of Y

$\therefore$ $\sigma(\mathcal{E})$ is σ-algebra on Y.

Then $f^{-1}(\sigma(\mathcal{E}))$ is σ-algebra on X because " If $f: X \to Y$ is function & β is σ-algebra on Y then $f^{-1}(\beta)$ is σ-algebra on X".

$$\therefore \sigma(f^{-1}(\sigma(\mathcal{E}))) = f^{-1}(\sigma(\mathcal{E})) \quad \text{--- (i)}$$

$\because \sigma(\sigma(\mathcal{E})) = \sigma(\mathcal{E})$.

Now since

$$\mathcal{E} \subseteq \sigma(\mathcal{E}) \quad \text{by def. of } \sigma(\mathcal{E})$$

$$f^{-1}(\mathcal{E}) \subseteq f^{-1}(\sigma(\mathcal{E}))$$

$$\Rightarrow \sigma(f^{-1}(\mathcal{E})) \subseteq \sigma(f^{-1}(\sigma(\mathcal{E}))) \quad \text{--- (ii)}$$

$\because \mathcal{E}_1 \subseteq \mathcal{E}_2 \Rightarrow \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$.

using (i) in (ii) we get

$$\sigma(f^{-1}(\mathcal{E})) \subseteq f^{-1}(\sigma(\mathcal{E})) \quad \text{--- (iii)}$$

To prove the inverse inclusion.

Let $\mathcal{A}_1$ be σ- algebra on X. Then we claim that

$$\mathcal{A}_2 = \{A : A \subseteq Y \mid f^{-1}(A) \in \mathcal{A}_1\}$$

is a σ- algebra on Y.

let $E \in \mathcal{A}_2$ Then $f^{-1}(E) \in \mathcal{A}_1$ so that $(f^{-1}(E))^c = f^{-1}(E^c) \in \mathcal{A}_1 \because \mathcal{A}_1$ σ-algebra.

$$\Rightarrow E^c \in \mathcal{A}_2$$

let $\{E_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{A}_2$ Then

$\{f^{-1}(E_i)\}_{i=1}^{\infty}$ is a sequence in $\mathcal{A}_1$.

Since $\mathcal{A}_1$ - σ algebra on X. Therefore

$$\bigcup_{i=1}^{\infty} f^{-1}(E_i) \in \mathcal{A}_1.$$

i.e $\bigcup_{i=1}^{\infty} f^{-1}(E_i) = f^{-1}\left(\bigcup_{i=1}^{\infty} E_i\right) \in \mathcal{A}_1$

So $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}_2$

Hence $\mathcal{A}_2$ is σ-algebra on X.

Since $\mathcal{A}_1$ is any arbitrary σ-algebra.

So we choose

$$\mathcal{A}_1 = \sigma(f^{-1}(\mathcal{E}))$$

Then

$$\mathcal{A}_2 = \{A : A \subseteq Y \mid f^{-1}(A) \in \sigma(f^{-1}(\mathcal{E}))\}$$

is σ-algebra on Y.

Now

$\mathcal{E} \subseteq \mathcal{A}_2 \quad \because \quad A \in \mathcal{E}$ then

Then $\qquad f^{-1}(A) \in f^{-1}(\mathcal{E}) \subseteq \sigma(f^{-1}(\mathcal{E}))$

$\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{A}_2) = \mathcal{A}_2 \quad \Rightarrow f^{-1}(A) \subseteq \sigma(f^{-1}(\mathcal{E})$

$\Rightarrow f^{-1}(\sigma(\mathcal{E})) \subseteq f^{-1}(\mathcal{A}_2) \qquad \Rightarrow A \in \mathcal{A}_2.$

$$f^{-1}(\sigma(\mathcal{E})) \subseteq f^{-1}(\mathcal{A}_2) \subseteq \sigma(f^{-1}(\mathcal{E}))$$

$$\Rightarrow \quad f^{-1}(\sigma(\mathcal{E})) \subseteq \sigma(f^{-1}(\mathcal{E})) \quad \text{---(iv)}$$

from (iii) & (iv) we get

$$\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$$

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Borel Set & Borel σ-algebra:

let $(X, \mathcal{T})$ be topological space and $\mathcal{D}$ be the collection of all open sets i.e $\mathcal{T} = \mathcal{D}$ then smallest σ-algebra $\sigma(\mathcal{D})$ is called Borel σ-algebra on X or is denoted by $\mathcal{B}(X)$ or $\mathcal{B}_X$. The member of Borel σ-algebra are called Borel set.

Lemma let $\mathcal{C}$ be the collection of all closed sets in topological space $(X, \mathcal{T})$ then $\sigma(\mathcal{C}) = \sigma(\mathcal{D})$.

Proof: let $E \in \mathcal{C}$ then $E^c \in \mathcal{D}$

$\Rightarrow E^c \in \sigma(\mathcal{D})$ $\because \mathcal{D} \subseteq \sigma(\mathcal{D})$

$\Rightarrow E \in \sigma(\mathcal{D})$ $\because \sigma(\mathcal{D})$ is σ-algebra on X.

So $\quad \mathcal{C} \subseteq \sigma(\mathcal{D})$

$\Rightarrow \sigma(\mathcal{C}) \subseteq \sigma(\sigma(\mathcal{D})) \quad \because \mathcal{E}_1 \subseteq \mathcal{E}_2 \Rightarrow \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$

$\Rightarrow \sigma(\mathcal{C}) \subseteq \sigma(\mathcal{D})$ — (1) $\because \sigma(\sigma(\mathcal{E})) = \sigma(\mathcal{E})$

Similarly we can prove that

$\sigma(\mathcal{D}) \subseteq \sigma(\mathcal{C})$ — (2)

from (1) & (2) we get

$$\sigma(\mathcal{C}) = \sigma(\mathcal{D})$$

G_δ - Set

Let $(X, \mathcal{J})$ be topological space. A subset E of X is called G_δ-set if E is the intersection of countabaly many open sets i.e $E = \bigcap_{i=1}^{\infty} G_i$, where $G_i \in \mathcal{J}$.

F_σ- Set

Let $(X, \mathcal{J})$ be Top-space, a subset F of X is called F_σ-set. If F is the union of countabaly many closed sets. i.e

$F = \bigcup_{i=1}^{\infty} F_i$, where F_i are closed subsets of X.

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Lemma Let $\{E_n\}_{n=1}^{\infty}$ be an arbitrary sequence of subsets of X in σ-algebra '$\mathcal{A}$' Then $\exists$ a disjoint sequence $\{F_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ such that

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n .$$

Proof

Define a new sequence $\{F_n\}_{n=1}^{\infty}$ in $\mathcal{A}$. such that

$$F_1 = E_1$$
$$F_2 = E_2 \setminus E_1$$
$$F_3 = E_3 \setminus E_1 \cup E_2$$
$$\vdots$$
$$F_n = E_n \setminus E_1 \cup E_2 \cup \cdots \cup E_{n-1}$$
$$\vdots$$

F_n can be expressed as

$$F_n = E_n \setminus E_1 \cup E_2 \cup \cdots \cup E_{n-1}$$
$$F_n = E_n \cap (E_1 \cup E_2 \cup \cdots \cup E_{n-1})^c$$
$$F_n = E_n \cap (E_1^c \cap E_2^c \cap \cdots \cap E_{n-1}^c)$$

Since $\{E_i\}_{i=1}^{\infty}$ is in $\mathcal{A}$ σ-algebra.

Therefore

$F_n \in \mathcal{A} \;\; \forall\, n \in \mathbb{N}$ $\;\;\because$ by definition of σ-algebra.

So $\{F_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$.

Now we are to show $\{F_n\}_{n=1}^{\infty}$ is disjoint sequence. i.e $F_m \cap F_n = \phi$, where $m \neq n$

let $m < n$ then by definition of F_n we have

$$F_m \subseteq E_m$$

$$F_m \cap F_n \subseteq E_m \cap F_n \quad — (1)$$

consider

$$E_m \cap F_n = E_m \cap \left(E_n \cap \left(\bigcap_{i=1}^{n-1} E_i^c\right)\right)$$

$$E_m \cap F_n = (E_m \cap E_m^c) \cap (E_n \cap E_1^c \cap \cdots \cap E_{m-1}^c \cap \cdots \cap E_{n-1}^c)$$

by distributive law of intersection.

$$= \phi \cap (E_n \cap E_1^c \cap \cdots \cap E_{m-1}^c \cap \cdots \cap E_{n-1}^c)$$

$$E_m \cap F_n = \phi$$

So (1) $\Rightarrow$ $F_m \cap F_n \subseteq \phi$

But $\phi \subseteq F_m \cap F_n$

So $F_m \cap F_n = \phi$

Hence

$\{F_n\}_{n=1}^{\infty}$ is disjoint sequence in A. Now we are to prove that

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

Since $F_n \subseteq E_n$

$$\Rightarrow \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n \quad \text{---(2)}$$

conversely suppose that

$x \in \bigcup_{n=1}^{\infty} E_n$ Then $\exists$ $n \in N$

s.t

$x \in E_n$

let m be the smallest +ve integer s.t

$x \in E_m$ but $x \notin E_1, \cdots, E_{m-1}$

$\Rightarrow$ $x \in E_m \setminus E_1 \cup E_2 \cup \cdots \cup E_{m-1}$

$\Rightarrow$ $x \in F_m$ by def of F_m.

$\Rightarrow x \in \bigcup_{n=1}^{\infty} F_m$. so

$$\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n \longrightarrow (3)$$

from (2) & (3) we get

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

//.

Set of Extended Real number:

If we include the two symboles '$-\infty$' and '∞' in the set of real number 'R' it become set of extended real number i.e

$$\bar{R} = \{-\infty\} \cup R \cup \{\infty\}$$

Set function:

Let $\mathcal{E}$ be an arbitrary collection of sub sets of a set X The the function

$$f : \mathcal{E} \longrightarrow [0, \infty]$$ is called set function.

Properties of set function:

(1) Monoton Property.

A set function $f : \mathcal{E} \longrightarrow [0, \infty]$ is said to be monoton if $E_1, E_2 \in \mathcal{E}$ s.t

$$E_1 \subseteq E_2 \implies f(E_1) \leq f(E_2).$$

(2) Finitely additive:

A set function $f : \mathcal{E} \longrightarrow [0, \infty]$ is said to be finitly additive if for every disjoint sequence $\{E_i\}_{i=1}^{n}$ in $\mathcal{E}$ s.t

$$f\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} f(E_i)$$

(3) Countably additive:

A set function $f: \mathcal{E} \to [0,\infty]$ is said to be countably additive if for every disjoint sequence $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{E}$ s.t

$$f\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} f(E_i).$$

(4) finitly Sub additive:

A set function $f: \mathcal{E} \to [0,\infty]$ is said to finitly sub additive if for every finite sequence $\{E_i\}_{i=1}^{\infty}$ s.t

$$f\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} f(E_i)$$

(5) Countably sub additive:

A set function $f: \mathcal{E} \to [0,\infty]$ is said to be countably sub additive if for every $\{E_i\}_{i=1}^{\infty}$ s.t

$$f\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{n} f(E_i)$$ //.

Measure:

Let $X \neq \phi$ be non-empty set and $\mathcal{A}$ is σ-algebra on X. Then the set function

$\mu : \mathcal{A} \to [0, \infty]$ is called measure if

(i) $\mu(\phi) = 0$

(ii) If $\{E_i\}_{i=1}^{\infty}$ is a disjoint sequence in $\mathcal{A}$

Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

i.e μ is countably additive.

Examples

(1) Let $\mathbb{R}$ be the set of real number and $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on $\mathbb{R}$. Then the set function

$\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ defined as

$\mu(E) = |E|$ = number of elements in E.

is measure.

(2) The set function

$\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ defined as

$$\mu(E) = \begin{cases} 0, & \text{if } 2 \notin E \\ 1, & \text{if } 2 \in E \end{cases}$$

is a measure.
"

Question Given an example of a set function which is not measure.

Solution:

let $X = \mathbb{R}$ be the set real number and $A = P(\mathbb{R})$ Then the set function

$$\mu : P(R) \to [0, \infty] \text{ defined by}$$

$$\mu(E) = \begin{cases} 0 \,, & \text{if } E \text{ is finite} \\ 1 \,, & \text{if } E \text{ is infinite} \end{cases}$$

is not measure because

(i) $\mu(\phi) = 0 \quad \because \phi$ is finite

But

(ii) If we consider the disjoint sequence $\{\{n\}\}_{n=1}^{\infty}$ in $P(\mathbb{R})$. Then

$$\mu\{n\} = 0 \quad \forall \; n \in \mathbb{R} \quad \because \{n\} \text{ is finite}$$

$$\therefore \quad \sum_{n=1}^{\infty} \mu\{n\} = 0$$

But $\mu\left(\bigcup_{n=1}^{\infty} \{n\}\right) = 1 \quad \because \bigcup_{n=1}^{\infty} \{n\}$ is infinite set.

So $\mu\left(\bigcup_{n=1}^{\infty} \{n\}\right) \neq \sum_{i=1}^{\infty} \mu(\{n\})$.

So μ is not-measure //.

Lemma Let $X \neq \phi$ and $\mathcal{A}$ σ-algebra on X and $\mu : \mathcal{A} \to [0, \infty]$ is measure on σ-algebra on $\mathcal{A}$ Then Prove that

(1) μ has finitly additive property.

(2) μ has monotonocity property.

(3) If $E_1, E_2 \in \mathcal{A}$ Then

$\mu(E_1 \setminus E_2) = \mu(E_1) \setminus \mu(E_2)$

(4) μ has countably sub additive property.

(5) μ has finitly sub additive property.

Proof (1)

Let $\{E_i\}_{i=1}^{\infty}$ be a disjoint sequence in $\mathcal{A}$ s.t $E_i = \phi$ $\forall$ $i = n+1, n+2, \ldots$

Since μ is measure

$\therefore$

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \quad \text{— (A)}$$

& $\mu(\phi) = 0$

By the definition of the given sequence

$$\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad \mu(E_i) = 0 \ \forall \ i = n+1, n+2, \ldots$$

$\therefore$ (A) becomes $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$

$\Rightarrow$ μ is finitly additive. //

(2) Proof: let $E_1, E_2 \in \mathcal{A}$ s.t $E_1 \subseteq E_2$

we are to show that $\mu(E_1) \leq \mu(E_2)$.

Since $E_1 \subseteq E_2$

Then $E_2 = E_1 \cup (E_2 \setminus E_1)$

$\therefore \mu(E_2) = \mu(E_1 \cup (E_2 \setminus E_1))$

$\Rightarrow \mu(E_2) = \mu(E_1) + \mu(E_2 \setminus E_1)$ $\because E_1 \cap E_2 \setminus E_1 = \phi$ and μ is finitely additive

$\Rightarrow \mu(E_2) \geq \mu(E_1)$

$\because \mu(E_2 \setminus E_1) \geq 0$. Proved in (1).

So $\mu(E_1) \leq \mu(E_2)$.

(3) Proof: Since $E_1, E_2 \in \mathcal{A}$ with $E_1 \subseteq E_2$

Then

$E_2 = E_1 \cup (E_2 \setminus E_1)$

$\Rightarrow \mu(E_2) = \mu(E_1) + \mu(E_2 \setminus E_1)$

$\Rightarrow \mu(E_2 \setminus E_1) = \mu(E_2) - \mu(E_1)$.

(4) let $\{E_i\}_{i=1}^{\infty}$ be sequence in $\mathcal{A}$. Then

$\bigcup_{i=1}^{\infty} E_i = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_1 \cup E_2) \cup \cdots\cdots$

$\because E_i \cup E_j = E_i \cup (E_j \setminus E_i)$.

$\Rightarrow \mu(\bigcup_{i=1}^{\infty} E_i) = \mu(E_1 \cup (E_2 \setminus E_1) \cup E_3 \setminus E_1 \cup E_2 \cup \cdots)$

$= \mu(E_1) + \mu(E_2 \setminus E_1) + \mu(E_3 \setminus E_1 \cup E_2)$

$+ \cdots$.

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu(E_1) + \mu(E_2 \setminus E_1) + \mu(E_3 \setminus E_1 \cup E_2) + \cdots$$

$$\leq \mu(E_1) + \mu(E_2) + \mu(E_3) + \cdots$$

$\because \quad E_i \setminus E_j \subseteq E_j$

then

$\mu(E_i \setminus E_j) \leq \mu(E_i)$.

$$= \sum_{i=1}^{\infty} \mu(E_i)$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

so μ is countably sub additive.

(5) Let $\{E_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{A}$ such that $E_i = \phi \quad \forall \; i = n+1, n+2, \cdots$.

then

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{n} E_i$$

and

$$\sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{n} \mu(E_i)$$

If μ is measure then countably sub additive property which is already prove in (4) (Previous part) is reduce to finitly sub additive i.e

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mu(E_i)$$

. ——— % ——— % ——— .

Finit Measure:

Let $X \neq \phi$ and $\mathcal{A}$ is σ-algebra on X. A measure $\mu : \mathcal{A} \to [0, \infty]$ is called finite measure if $\mu(X) < \infty$.

σ- finite Measure:

Let $X \neq \phi$, $\mathcal{A}$ is σ-algebra on X, A measure $\mu : \mathcal{A} \to [0, \infty]$ is called σ- finite measure if $\exists$ a sequence $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ such that

$$X = \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad \mu(E_i) < \infty.$$

Question Given an example of a measure which is σ- finite but not finite measure.

Proof.

Let N be the set of natural number and $P(N)$ is σ- algebra on N, Define a measure

$$\mu : P(N) \to [0, \infty] \quad \text{s.t}$$

$\mu(E) = |E|$ is not finite measure because $\mu(N) = \infty$ but μ is σ- finite because $\exists$ a sequence $\{\{n\}\}_{n=1}^{\infty}$ s.t $N = \bigcup_{n=1}^{\infty} \{n\}$ and $\mu(\{n\}) = |\{n\}|$

$$\mu(\{n\}) = 1 < \infty.$$

Theorem (Monoton Convergence Theorem)

(a) If $\{E_n\}_{n=1}^{\infty}$ is increasing sequence then

$$\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n).$$

(b) If $\{E_n\}_{n=1}^{\infty}$ is decreasing sequence then

$$\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n)$$ provided that $\mu(E_1) < \infty$.

Proof:

(a) Suppose that $\{E_n\}_{n=1}^{\infty}$ is increasing sequence then by monotonicity property of measure the sequence $\{\mu(E_n)\}_{n=1}^{\infty}$ in $[0,\infty]$ is increasing.

Here we discusse two cases

Case 1: If $\mu(E_{n_0}) = \infty$ for some $n_0 \in N$ then

$$\lim_{n\to\infty} \mu(E_n) = \infty \quad \text{--- (i)}$$

Now $E_{n_0} \subseteq \bigcup_{n=1}^{\infty} E_n = \lim_{n\to\infty} E_n$ $\because$ $\{E_n\}_{n=1}^{\infty}\uparrow$ then $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$.

$$\Rightarrow \mu(E_{n_0}) \leq \mu(\lim_{n\to\infty} E_n)$$

$$\Rightarrow \mu(\lim_{n\to\infty} E_n) = \infty \text{ --- (ii)} \quad \because \mu(E_{n_0}) = \infty$$

from (i) & (ii)

$$\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n).$$

Case II If $\mu(E_n) < \infty \quad \forall\, n \in N.$
take $E_0 = \phi$ and define
a disjoint sequence $\{F_n\}_{n=1}^{\infty}$ s.t

$$F_n = E_n \setminus E_{n-1} \quad \forall\, n \in N.$$

obviausely

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$$

$$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} F_n \quad \because \{E_n\}_{n=1}^{\infty} \uparrow$$

then $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$.

operate or taking measure on both sides

$$\mu(\lim_{n\to\infty} E_n) = \mu(\bigcup_{n=1}^{\infty} F_n)$$

$$\Rightarrow \mu(\lim_{n\to\infty} E_n) = \sum_{n=1}^{\infty} \mu(F_n) \quad \because \mu \text{ is measure.}$$

$$= \sum_{n=1}^{\infty} \mu(E_n \setminus E_{n-1})$$

$$= \sum_{n=1}^{\infty} [\mu(E_n) - \mu(E_{n-1})] \quad \because \mu(A\setminus B) = \mu(A) - \mu(B)$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} [\mu(E_n) - \mu(E_{n-1})]$$

$$\mu(\lim_{n\to\infty} E_n) = \lim_{k\to\infty} \sum_{n=1}^{k} \left[\mu(E_n) - \mu(E_{n-1})\right]$$

$$= \lim_{k\to\infty} \left[(\mu(E_1)-\mu(E_0)) + (\mu(E_2)-\mu(E_1)) + \cdots + (\mu(E_k)-\mu(E_{k-1})) \right].$$

$$= \lim_{k\to\infty} \mu(E_k)$$

$$= \lim_{n\to\infty} \mu(E_n).$$ considering 'k' as a dummy variable.

Hence

$$\mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(E_n).$$

%———%———%———%———%.

(b) Proof Suppose that $\{E_n\}_{n=1}^{\infty}$ is decreasing sequence with $\mu(E_1) < \infty$.

$\therefore \lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$.

Consider $E_1 \setminus \bigcap_{n=1}^{\infty} E_n = E_1 \cap (\bigcap_{n=1}^{\infty} E_n)^c$ $\because A\setminus B = A\cap B^c$

$= E_1 \cap (\bigcup_{n=1}^{\infty} E_n^c)$ $\because$ of De-Morgen Law.

$= \bigcup_{n=1}^{\infty} (E_1 \cap E_n^c)$ by Distributive law.

$= \bigcup_{n=1}^{\infty} (E_1 \setminus E_n)$

$= \lim_{n\to\infty} (E_1 \setminus E_n)$ $\because$ $\{E_1\setminus E_n\}_{n=1}^{\infty} \uparrow$

Since $\{E_1\setminus E_n\}_{n=1}^{\infty}$ is $\uparrow$ then by (a) part of the theorem

Therefore

$$\mu\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right) = \mu\left(\lim_{n\to\infty} E_1 \setminus E_n\right)$$

$$\mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_1 \setminus E_n)$$ by (a) part of theorem.

$$\mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_1) - \lim_{n\to\infty} \mu(E_n)$$

$$\cancel{\mu(E_1)} - \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \cancel{\mu(E_1)} - \lim_{n\to\infty} \mu(E_n)$$

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

$\because \{E_n\}_{n=1}^{\infty}$ is $\downarrow$

then

$\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$

Notes by: Mr. Anwar Khan

Theorem: Let $X \neq \phi$ be non-empty set, $\mathcal{A}$ is σ-algebra on and $\mu : \mathcal{A} \to [0, \infty]$ is measure Then

(a) for an arbitrary sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{A}$, such that

$$\mu(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n).$$

(b) If there exist a set $A \in \mathcal{A}$ with finite measure i.e $\mu(A) < \infty$ and $E_n \subseteq A \ \forall n \in \mathbb{N}$ Then

$$\mu(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} \mu(E_n).$$

Proof (a) By definition of $\liminf_{n\to\infty} E_n$ we have

$$\liminf_{n\to\infty} E_n := \bigcup_{n\geq 1}\left(\bigcap_{k\geq n} E_k\right) \quad \text{where } \left\{\bigcap_{k\geq n} E_k\right\}_{k=1}^{\infty} \uparrow$$

$$\liminf_{n\to\infty} E_n = \lim_{n\to\infty}\left(\bigcap_{k\geq n} E_k\right) \because$$

Then $\lim_{k\to\infty}\left(\bigcap_{k\geq n} E_k\right) = \bigcup_{k=1}^{\infty}\left(\bigcap_{k\geq n} E_k\right).$

operating 'μ' on both sides

$$\mu(\liminf_{n\to\infty} E_n) = \mu\left(\lim_{n\to\infty} \bigcap_{k\geq n} E_k\right)$$

$$= \lim_{n\to\infty} \mu\left(\bigcap_{k\geq n} E_k\right) \quad \text{by M.C.T \& } \left\{\bigcap_{k\geq n} E_k\right\}_{k=1}^{\infty} \uparrow$$

$$= \liminf_{n\to\infty} \mu\left(\bigcap_{k\geq n} E_k\right).$$

$\because \{E_n\}_{n=1}^{\infty}$ exist.

Note: M.C.T stand for Monoton convergence Theorem

$$\mu(\liminf_{n\to\infty} E_n) = \liminf_{n\to\infty} \mu(\bigcap_{k\geq n} E_k)$$

$$\leq \liminf_{n\to\infty} \mu(E_n).$$

$\because \bigcap_{k\geq n} E_k \subseteq E_n$ so

$\mu(\bigcap_{k\geq n} E_k) \leq \mu(E_n)$.

Note: If limit of $\{E_n\}_{n=1}^{\infty}$ exist then $\{\mu(E_n)\}_{n=1}^{\infty}$ exist in $[0,\infty]$ and

(i) $\liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = \lim_{n\to\infty} E_n$

(ii) $\liminf_{n\to\infty} \mu(E_n) = \limsup_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(E_n)$.

So

$$\mu(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n).$$

(b) Proof:

By definition of $\limsup_{n\to\infty} E_n$ we have

$$\limsup_{n\to\infty} E_n = \bigcap_{n\geq 1} (\bigcup_{k\geq n} E_k)$$

$$= \lim_{n\to\infty} (\bigcup_{k\geq n} E_k) \quad \text{--- (1)}$$

$\because \{\bigcup_{k\geq n} E_k\}_{n=1}^{\infty} \downarrow$ then $\lim_{n\to\infty} (\bigcup_{k\geq n} E_k) = \bigcap_{n\geq 1} (\bigcup_{k\geq n} E_k)$,

operating measure 'μ' on both sides we get

$$\mu(\limsup_{n\to\infty} E_n) = \mu(\lim_{n\to\infty} (\bigcup_{k\geq n} E_k))$$

$$= \lim_{n\to\infty} \mu(\bigcup_{k\geq n} E_k)$$

$\{\bigcup_{k\geq n} E_k\} \downarrow$

" $\lim_{n\to\infty} \mu\left(\bigcup_{k\geq n} E_k\right)$ exist because

$E_n \subseteq A \quad \forall\, n \in N$

$\therefore$

$$\bigcup_{k\geq n} E_k \subseteq A \Rightarrow \mu\left(\bigcup_{k\geq n} E_k\right) \leq \mu(A) < \infty$$

$$\Rightarrow \mu\left(\bigcup_{k\geq} E_k\right) < \infty \text{."}$$

So

$$\mu\left(\lim_{n\to\infty} \sup E_n\right) = \mu\left(\lim_{n\to\infty}\left(\bigcup_{k\geq n} E_k\right)\right)$$

$$= \lim_{n\to\infty} \mu\left(\bigcup_{k\geq n} E_k\right)$$

$\because$ M.C.T (a) part.

$$= \lim_{n\to\infty} \sup \mu\left(\bigcup_{k\geq n} E_n\right)$$

$\because$ limit of $\left\{\mu\left\{\bigcup_{k\geq n} E_n\right\}\right\}$ exist.

$$\geq \lim_{n\to\infty} \sup \mu(E_n) \quad \because \bigcup_{k\geq n} E_n \supseteq E_n$$

$$\Rightarrow \mu\left(\bigcup_{k\geq n} E_n\right) \geq \mu(E_n)$$

Hence

$$\mu\left(\lim_{n\to\infty} \sup E_n\right) \geq \lim_{n\to\infty} \sup \mu(E_n).$$

Measurable space & Measure Space:

Let $X \neq \phi$, $\mathcal{A}$ is σ-algebra on X and $\mu : \mathcal{A} \to [0, \infty]$ is measure on $\mathcal{A}$ Then the pair $(X, \mathcal{A})$ is called measurable Space and the triplet $(X, \mathcal{A}, \mu)$ is called measure space.

Finite Measure Space A measure space $(X, \mathcal{A}, \mu)$ is called finite Measure Space if $\mu : \mathcal{A} \to [0, \infty]$ is finite measure i.e $\mu(X) < \infty$.

σ-Finit Measure space: A measure space $(X, \mathcal{A}, \mu)$ is called σ-finite Measure space if $\mu : \mathcal{A} \to [0, \infty]$ is σ-finite measure i.e There exist a sequence $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ s.t $X = \bigcup_{i=1}^{\infty} E_i$ with $\mu(E_i) < \infty \ \forall i$.

$\mathcal{A}$-Measurable Set: Let $(X, \mathcal{A})$ be a measurable space Then members of $\mathcal{A}$ are called $\mathcal{A}$-Measurable set.

σ- Finite Set

Let $(X, \mathcal{A}, \mu)$ is a measure space, a set $D \in \mathcal{A}$ is called σ- Finite set if ∃ a sequence $\{D_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ s.t

$$D = \bigcup_{n=1}^{\infty} D_n \quad \text{with} \quad \mu(D_n) < \infty \quad \forall\ n \in N.$$

Lemma :

(1) Let $(X, \mathcal{A}, \mu)$ be a measurable space, $D \in \mathcal{A}$ is σ- finite set Then show that ∃ an increasing sequence $\{F_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ s.t $\lim_{n\to\infty} F_n = D$ and $\mu(F_n) < \infty$. Also ∃ a disjoint sequence $\{G_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ s.t

$$\bigcup_{n=1}^{\infty} G_n = D \quad \text{and} \quad \mu(G_n) < \infty \quad \forall\ n \in N.$$

Proof: Suppose that $D \in \mathcal{A}$ is σ- finite set Then ∃ a sequence $\{D_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ s.t

$$D = \bigcup_{n=1}^{\infty} D_n \quad \text{and} \quad \mu(D_n) < \infty.$$

Define a sequence $\{F_n\}_{n=1}^{\infty}$ s.t

$F_n = \bigcup_{i=1}^{n} D_i$ Then clearly the sequence $\{F_n\}_{n=1}^{\infty}$ is increasing sequence. Now we are to show that $\lim_{n \to \infty} F_n = D$.

Since $F_n = \bigcup_{i=1}^{n} D_i$ therefore

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} D_i \right)$$

$$= \bigcup_{n=1}^{\infty} D_n$$

$$\bigcup_{n=1}^{\infty} F_n = D \quad \text{—} \quad (1)$$

Now since $\{F_n\}_{n=1}^{\infty}$ is increasing sequence

$\therefore \lim_{n \to \infty} F_n = \bigcup_{n=1}^{\infty} F_n$.

So

eqn (1) $\Rightarrow$ $\boxed{D = \lim_{n \to \infty} F_n.}$

Now we are to show that $\mu(F_n) < \infty \ \forall n$.

Since

$$F_n = \bigcup_{i=1}^{n} D_i$$

operating measure 'μ' on both sides

$$\mu(F_n) = \mu\left(\bigcup_{i=1}^{n} D_i\right)$$

$$\leq \sum_{i=1}^{n} \mu(D_i) < \infty \quad \because \mu(D_i) < \infty \ \forall i$$

$\Rightarrow \mu(F_n) < \infty$ which required.

Define a sequence $\{G_n\}_{n=1}^{\infty}$ s.t $G_1 = F_1$ and $G_n = F_n \setminus F_{n-1} \ \forall n \geq 2$. Then $\{G_n\}_{n=1}^{\infty}$ is a disjoint sequence s.t

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} G_n$$

$$\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} F_n$$

$$= D \qquad \because \bigcup_{n=1}^{\infty} F_n = D.$$

Since

$$G_1 = F_1$$

$$\therefore \mu(G_1) = \mu(F_1) < \infty$$

$$\Rightarrow \mu(G_1) < \infty$$

and

$$G_n = F_n \setminus F_{n-1}$$

$$\therefore \mu(G_n) = \mu(F_n \setminus F_{n-1})$$

$$= \mu(F_n) - \mu(F_{n-1})$$

$$\leq \mu(F_n) < \infty$$

$$\Rightarrow \mu(G_n) < \infty \quad \forall\ n \geq 2.$$

which is the required result.

(2) If $(X, \mathcal{A}, \mu)$ is σ-finite measurable space then every $D \in \mathcal{A}$ is a σ-finite set.

Proof Since $(X, \mathcal{A}, \mu)$ is σ-finite space.

$\therefore \exists$ a sequence $\{E_n\}_{n\geq 1}^{\infty}$ in $\mathcal{A}$ such that

$$X = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad \mu(E_n) < \infty \ \forall\, n \in N.$$

Let $D \in \mathcal{A}$. Define a sequence $\{D_n\}_{n=1}^{\infty}$ s.t

$D_n = D \cap E_n$ Then

$$D = \bigcup_{n=1}^{\infty} D_n$$

Now we are to show that $\mu(D_n) < \infty \ \forall n \in N$.

Since $D_n \subseteq E_n \because D_n = D \cap E_n$

$\therefore \mu(D_n) \leq \mu(E_n) < \infty \quad \forall n \in N$

$\Rightarrow \mu(D_n) < \infty$.

Hence $D \in \mathcal{A}$ is a σ-finite set.

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<u>Null Set:</u> Let $(X, \mathcal{A}, \mu)$ be a measure space. A subset E of X is called null set iff $\mu(E) = 0$.

for e.g ϕ is a null set because $\mu(\phi) = 0$.

<u>Note:</u> ϕ is null in every measure space but a null set need not to be ϕ.

<u>Lemma:</u>

Show that countable union of null set is null set.

<u>Proof:</u>

let $\{G_i : i \in N\}$ be collection of null set. we are to prove that

$\mu\left(\bigcup_{i=1}^{\infty} G_i\right) = 0$.

Since

$$\mu\left(\bigcup_{i=1}^{\infty} G_i\right) \leq \sum_{i=1}^{\infty} \mu(G_i) = 0 \quad \because \mu(G_i) = 0 \ \forall i \in N$$

$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} G_i\right) = 0 \because \mu$ is always +ve.

so $\bigcup_{i=1}^{\infty} G_i$ is null set. //

Complet σ- Algebra:

Let $(X, \mathcal{A}, \mu)$ be measure space. The σ-algebra '$\mathcal{A}$' is said to be complet if every sub set of E_0 of a null set E is a member of $\mathcal{A}$. In otherword

$E_0 \subseteq E$

$\Rightarrow \mu(E_0) \leq \mu(E) = 0 \Rightarrow \mu(E_0) = 0, E_0 \in \mathcal{A}.$

Complete Measure space:

A measure space $(X, \mathcal{A}, \mu)$ is called complete measure space if σ-algebra '$\mathcal{A}$' is complet σ-algebra.

Outer Measure:

Let $X \neq \phi$, A set function $\mu^*: P(X) \longrightarrow [0, \infty]$ is called outer measure on σ-algebra $P(X)$ If it satisfies the following axioms;

(1) $\mu^*(\phi) = 0$

(2) If $E_1, E_2 \in P(X)$ s.t $E_1 \subseteq E_2$

$\Rightarrow \mu^*(E_1) \leq \mu^*(E_2)$ i.e μ^* has monotonicity Property.

(3) μ^* has countably sub additive i.e For a sequence $\{E_n\}_{n\geq 1}^{\infty}$ in $P(X)$ s.t. $\mu^*\left(\bigcup_{n=1}^{\infty} E_i\right) \leq \sum_{n=1}^{\infty} \mu^*(E_i)$.

Note: Let $X \neq \phi$ be non-set, $P(X)$ is Power set of X. Let $E \in P(X)$ Then for any set $A \in P(X)$ we have

(i) $(A \cap E) \cap (A \cap E^c) = \phi$

(ii) $(A \cap E) \cup (A \cap E^c) = A$.

μ^*- Measurable Set:

Let $\mu^* : P(X) \to [0, \infty]$ be an outer measure on $P(X)$, A set $E \in P(X)$ is called μ^*- measurable set if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall A \in P(X).$$

where set A is called Testing set.

Remark

Since $A = (A \cap E) \cup (A \cap E^c) \quad \forall A \in P(X)$ and μ^* is sub-additive Therefore

$$\mu^*(A) = \mu^*((A \cap E) \cup (A \cap E^c))$$
$$\leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

so

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

So in order to show that E is μ^*- measurable set we only need to show that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Q// Prove that ϕ and X are μ^*-measurable sets.

Proof

for $A \in P(X)$

consider

$$\mu^*(A \cap \phi) + \mu^*(A \cap \phi^c)$$

$$= \mu^*(\phi) + \mu^*(A \cap X)$$

$$= \mu^*(A) \quad \because \quad \mu^*(\phi) = 0 \text{ and } A \cap X = A.$$

so

$$\mu^*(A) = \mu^*(A \cap \phi) + \mu^*(A \cap \phi^c)$$

$\Rightarrow$ ϕ is μ^*-measurable.

Now we are to prove that X is μ^*-measurable.

for $A \in P(X)$.

consider

$$\mu^*(A \cap X) + \mu^*(A \cap X^c)$$

$$= \mu^*(A \cap X) + \mu^*(A \cap \phi)$$

$$= \mu^*(A) + \mu^*(\phi)$$

$$= \mu^*(A) \quad \because \quad \mu^*(\phi) = 0$$

Hence

$$\mu^*(A) = \mu^*(A \cap X) + \mu^*(A \cap X^c)$$

$\Rightarrow$ X is μ^*-measurable set.

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Question: If E is μ^* - measurable set then E^c is μ^* - measurable set.

Proof: Since E is μ^*-measurable set then $\forall \; A \in P(X)$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

$$= \mu^*(A \cap (E^c)^c) + \mu^*(A \cap E^c)$$

$$\mu^*(A) = \mu^*(A \cap E^c) + \mu^*(A \cap (E^c)^c)$$

by interchaing the terms of R.H.S.

Hence E^c is μ^*- measurable.

Remark: If E is not μ^*- measurable set Then E^c is also not μ^*- measurable set.

Note : The collection all μ^*- measurable sets is denoted by

$$m(\mu^*).$$

Lemma:

Let $X \neq \phi$, $\mu^*: P(X) \to [0,\infty]$ be an outer measure on $P(X)$. If $E_1, E_2 \in P(X)$ are μ^*-measurable then prove that $E_1 \cup E_2$ is μ^*-measurable.

OR

If $E_1, E_2 \in m(\mu^*)$ then $E_1 \cup E_2 \in m(\mu^*)$.

Proof To show that $E_1 \cup E_2$ is μ^*-measurable we are to prove that $\forall\ A \in P(X)$

$$\mu^*(A) = \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c).$$

Since E_1 is μ^*-measurable. Therefore $\forall\ A \in P(X)$ we have

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c). \quad \text{--- (1)}$$

Since E_2 is μ^*-measurable.

$$\therefore\ \mu^*(A \cap E_1^c) = \mu^*((A \cap E_1^c) \cap E_2) + \mu^*((A \cap E_1^c) \cap E_2^c) \quad (2)$$

by considering $(A \cap E_1^c)$ as a testing set.

using eqn (2) in (1) we get

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*((A \cap E_1^c) \cap E_2) + \mu^*((A \cap E_1^c) \cap E_2^c)$$

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*((A \cap E_1^c)) + \mu^*(A \cap (E_1 \cup E_2)^c)$$

by Demorgan Law

$$\mu^*(A) = \mu^*(A\cap E_1) + \mu^*(A\cap E_1^c \cap E_2) + \mu^*(A\cap(E_1\cup E_2)^c)$$

$$\geq \mu^*((A\cap E_1)\cup(A\cap(E_1^c\cap E_2)) + \mu^*(A\cap(E_1\cup E_2)^c)$$

$\because \mu^*$ is finitely sub additive

$$= \mu^*(A\cap(E_1\cup(E_1^c\cap E_2)) + \mu^*(A\cap(E_1\cup E_2)^c)$$

By Distributive Law

$$= \mu^*(A\cap(E_1\cup(E_2\setminus E_1)) + \mu^*(A\cap(E_1\cup E_2)^c)$$

$$= \mu^*(A\cap(E_1\cup E_2)) + \mu^*(A\cap(E_1\cap E_2)^c)$$

$$\Rightarrow \mu^*(A) \geq \mu^*(A\cap(E_1\cup E_2)) + \mu^*(A\cap(E_1\cup E_2)^c)$$

obviousely

$$\mu^*(A) \leq \mu^*(A\cap(E_1\cup E_2)) + \mu^*(A\cap(E_1\cup E_2)^c)$$

so

$$\mu^*(A) = \mu^*(A\cap(E_1\cup E_2)) + \mu^*(A\cap(E_1\cup E_2)^c).$$

Hence $E_1\cup E_2$ is μ^*- measurable set

i.e $E_1\cup E_2 \in m(\mu^*)$.

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Question Prove that intersection of the two μ^*-measurable sets is μ^*-measurable set.

OR

Proof. If $E_1, E_2 \in m(\mu^*)$ then $E_1 \cap E_2 \in m(\mu^*)$.

Sol:// Let $E_1, E_2 \in m(\mu^*)$ Then $E_1^c, E_2^c \in m(\mu^*)$ $\because$ If E is μ^*-measurable then E^c is μ^*-measurable.

So. $E_1 \cup E_2 \in m(\mu^*)$

$\Rightarrow (E_1 \cup E_2)^c \in m(\mu^*)$ $\because$ If $E_1, E_2 \in m(\mu^*)$ then $E_1 \cup E_2 \in m(\mu^*)$

$\Rightarrow E_1^c \cap E_2^c \in m(\mu^*)$ by De-Morgen Law

$\therefore$ $(E_1 \cap E_2)^c = E_1^c \cup E_2^c \in m(\mu^*)$

$\Rightarrow (E_1 \cap E_2)^c \in m(\mu^*)$

So $[(E_1 \cap E_2)^c]^c \in m(\mu^*)$ $\because$ If $E \in m(\mu^*)$ then $E^c \in m(\mu^*)$

Hence

$E_1 \cap E_2 \in m(\mu^*)$.

which is required.

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Lemma:

Let $X \neq \phi$, $\mu^* : P(X) \to [0, \infty]$ be an outer measure and $E \in P(X)$ s.t $\mu^*(E) = 0$, Then every sub set of E is μ^*-measurable. In particular E itself is μ^*-measurable.

OR

Prove that every sub set of a null set is μ^*-measurable. In particular a null set is μ^*-measurable.

Proof let E be null set i.e

$$\mu^*(E) = 0 \quad \text{and} \quad E_0 \subseteq E$$

we are to show that E_0 is μ^*-measurable.

Since $E_0 \subseteq E$

$\therefore \mu^*(E_0) \leq \mu^*(E)$ by Monotonicity property of outer measure μ^*.

$\Rightarrow \mu^*(E_0) = 0 \quad \because \mu^*(E) = 0$

Now for $A \in P(X)$

we have

$$A \cap E_0 \subseteq E_0 \Rightarrow \mu^*(A \cap E_0) \leq \mu^*(E_0) \text{ --- (i)}$$

& $$A \cap E_0^c \subseteq A \Rightarrow \mu^*(A \cap E_0^c) \leq \mu^*(A) \text{ --- (ii)}$$

from the inequalities (i) & (ii) we have

$$\mu^*(A \cap E_0) + \mu^*(A \cap E_0^c) \leq \mu^*(E_0) + \mu^*(A) \quad \because \mu^* \text{ is +ve.}$$

$$\Rightarrow \mu^*(A \cap E_0) + \mu^*(A \cap E_0^c) \leq \mu^*(A) \text{ --- (iii)} \quad \because \mu^*(E_0) = 0$$

obviously

$$\mu^*(A \cap E_0) + \mu^*(A \cap E_0^c) \geq \mu^*(A) \text{ --- (iv)}$$

from (iii) & (iv)

$$\mu^*(A) = \mu^*(A \cap E_0) + \mu^*(A \cap E_0^c)$$

$\Rightarrow E_0$ is μ^*-measurable.

If we apply the same arguments on a set $A \in P(X)$ and E we can show that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \le \mu^*(E) + \mu^*(A)$$

$$\Rightarrow \mu^*(A \cap E) + \mu^*(A \cap E^c) \le \mu^*(A) \quad \because \mu^*(E) = 0$$

i.e $\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$

$\Rightarrow$ E is μ^*- measurable.

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Lemma Let $X \neq \phi$, $\mu^* : P(X) \to [0, \infty]$ be an outer-measure and $E_1, E_2 \in m(\mu^*)$ s.t $E_1 \cap E_2 = \phi$ Then

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2).$$

Proof Since $E_1 \in m(\mu^*)$

$$\therefore \quad \mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) \quad \forall A \in P(X).$$

Take $A = E_1 \cup E_2$ we have

$$\mu^*(E_1 \cup E_2) = \mu^*((E_1 \cup E_2) \cap E_1) + \mu^*((E_1 \cup E_2) \cap E_1^c)$$

$$= \mu^*(E_1) + \mu^*((E_1 \cup E_2) \setminus E_1)$$

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2) \quad \because E_1 \cap E_2 = \phi.$$

which is the required result.

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Lemma: Let $X \neq \phi$ and $\mu^*: P(X) \rightarrow [0, \infty]$ be an outer measure. If $E, F \in P(X)$ such that $\mu^*(F) = 0$ Then $\mu^*(E \cup F) = \mu^*(E)$.

Proof: By the definition of outer measure μ^* we have

$$\mu^*(E \cup F) \leq \mu^*(E) + \mu^*(F)$$

$$\Rightarrow \quad \mu^*(E \cup F) \leq \mu^*(E) \text{ ---(1)} \quad \because \mu^*(F) = 0$$

also

$$E \subseteq E \cup F$$

$$\therefore \quad \mu^*(E) \leq \mu^*(E \cup F) \text{ ---(2)} \quad \because \mu^* \text{ has monotonicity Property.}$$

from (1) & (2) we have

$$\mu^*(E \cup F) = \mu^*(E).$$

x ——— x ——— x ——— x ——— x

Lemma: If $A, B \in m(\mu^*)$ Then show that $A \setminus B \in m(\mu^*)$.

OR

If A, B are μ^* - measurable sets then $A \setminus B$ is also μ^* - measurable set.

Proof Since $A, B \in m(\mu^*)$ $\therefore$ $B^c \in m(\mu^*)$

so $A \cap B^c \in m(\mu^*)$ $\because$ Intersection of two measurable sets is measurable.

Hence $A \setminus B \in m(\mu^*)$ $\because$ $A \setminus B = A \cap B^c$

$\Rightarrow$ $A \setminus B$ is μ^* - measurable sets. $\square$.

Theorem: let $X \neq \phi$, $\mu^*: P(X) \rightarrow [0,\infty]$ be an outer measure. let $\{E_i\}_{i=1}^{n}$ be a disjoint sequence in $m(\mu^*)$ then $\forall A \in P(X)$ we have

$$\mu^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$$

Proof we prove the result by mathematical induction on 'n'.

for $n = 1$ we

$$\mu^*(A \cap E_1) = \mu^*(A \cap E_1) \quad \text{--- (1)}$$

the result is true for $n=1$

Suppose that the result is true for $n=k$

i.e

$$\mu^*(A \cap (\bigcup_{i=1}^{k} E_i)) = \sum_{i=1}^{k} \mu^*(A \cap E_i). \quad \text{--- (2)}$$

we are to prove that the result is true for

Since E_{k+1} is μ^*-measurable. Therefore considering the testing $A \cap (\bigcup_{i=1}^{k+1} E_i)$ we have

$$\mu^*(A \cap (\bigcup_{i=1}^{k+1} E_i)) = \mu^*(A \cap (\bigcup_{i=1}^{k+1} E_i) \cap E_{k+1}) + \mu^*(A \cap (\bigcup_{i=1}^{k+1} E_i) \cap E_{k+1}^c)$$

by definition μ^*-measurable set.

$$= \mu^*(A \cap (E_{k+1} \cap (\bigcup_{i=1}^{k+1} E_i))) + \mu^*(A \cap (\bigcup_{i=1}^{k} E_i))$$

$\because \{E_i\}_{i=1}^{n}$ is disjoint.

$$= \mu^*(A \cap E_{k+1}) + \sum_{i=1}^{k} \mu^*(A \cap E_i) \quad \text{using eqn (2).}$$

$$\mu^*(A \cap \bigcup_{i=1}^{k+1} E_i) = \sum_{i=1}^{k+1} \mu^*(A \cap E_i)$$ (54)

Result is true for $n = k+1$. So induction is complete. Hence

$$\mu^*(A \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu^*(A \cap E_i). \square$$

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Theorem: Let $X \neq \phi$ be non-empty set and $\mu^* : P(X) \longrightarrow [0,\infty]$ be an outer measure on $P(X)$. Show that $m(\mu^*)$ is σ-algebra on X. Where $m(\mu^*)$ is the collection of all μ^*-measurable subsets of X.

Proof: To show that $m(\mu^*)$ is σ-algebra on X. We are to show that $m(\mu^*)$ is (i) closed under complement.
(ii) closed under countable union.

Let $E \in m(\mu^*)$ Then E is μ^*-measurable i.e $\forall A \in P(X)$ we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$$\mu^*(A) = \mu^*(A \cap (E^c)^c) + \mu^*(A \cap E^c) \quad \forall A \in P(X)$$

$\Rightarrow E^c$ is μ^*-measurable. So $E^c \in m(\mu^*)$

So $m(\mu^*)$ is closed under complement.

let $\{E_i\}_{i=1}^{\infty}$ be a sequence in $m(\mu^*)$
we are to show $\bigcup_{i=1}^{\infty} E_i \in m(\mu^*)$.

Since $m(\mu^*)$ is closed under finit union because " If E_1, E_2 are μ^*-measurable then $E_1 \cup E_2$ is μ^*-measurabe. Generaly if $E_1, E_2, \ldots, E_n$ are μ^*-measurable then $E_1 \cup E_2 \cup \cdots \cup E_n$ is μ^*-measurable"

Therefore $\forall \ A \in P(X)$ we have

$$\mu^*(A) = \mu^*(A \cap (\bigcup_{i=1}^{n} E_i)) + \mu^*(A \cap (\bigcup_{i=1}^{n} E_i)^c) \quad \text{---(1)}$$

Since L.H.S of eqn (1) is independent of 'n' therefore R.H.S of (1) must be independent of 'n' so eqn (1) becomes

$$\mu^*(A) = \mu^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) + \mu^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c)$$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i$ is μ^*- measurable set.

Hence $\bigcup_{i=1}^{\infty} E_i \in m(\mu^*)$. which shows that $m(\mu^*)$ is closed under countable union.

Hence $m(\mu^*)$ is σ- algebra on X.

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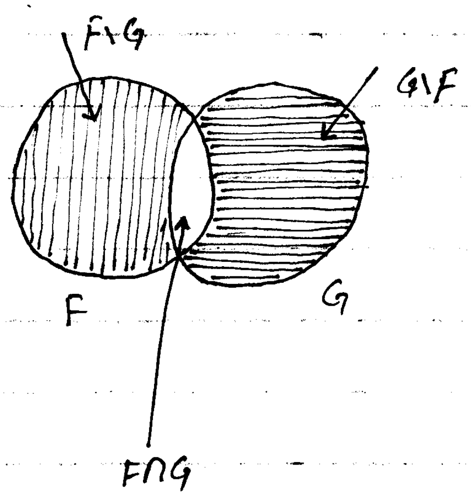
Question: If F is μ^*-measurable set and $F \Delta G$ is symmetric difference of F and G s.t $\mu^*(F \Delta G) = 0$ Then show that G is μ^*-measurable.



Solution:

Since

$F \setminus G \subseteq F \Delta G$ and $G \setminus F \subseteq F \Delta G$

$\therefore$ $F \setminus G$ and $G \setminus F$ are μ^*-measurable. "because every subset of a null set is μ^*-measurable".

so $(F \setminus G)^c$ is μ^*-measurable.

and Symmetric difference of F and G is defined as

$$F \Delta G = (F \setminus G) \cup (G \setminus F).$$

Now $F \cap G = F \cap (F \setminus G)^c$

being Intersection of two μ^*-measurable set μ^*-measurable. Then

From Fig

$G = (F \cap G) \cup (G \setminus F)$ being union of two μ^*-measurable sets is μ^*-measurable.

Hence G is μ^*-measurable.

———— % ———— % ———— % ————

Theorem: Let $X \neq \phi$ be non-empty set and $\mathcal{E} \subseteq P(X)$ such that $\phi, X \in \mathcal{E}$

a set function

$$f : \mathcal{E} \longrightarrow [0, \infty] \quad s.t$$

(i) $f(\phi) = 0$ (ii) $f\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} f(E_i)$ Then show that the set function $\mu^* : P(X) \longrightarrow [0, \infty]$ defined as

$$\mu^*(E) = \inf\left\{ \sum_{i=1}^{\infty} f(E_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{E} \right\} \quad \text{—(A)}$$

is an outer measure.

Proof: To show that $\mu^* : P(X) \longrightarrow [0, \infty]$ is an outer measure we will prove

(i) $\underline{\mu^*(\phi) = 0}$

Since $\phi \subseteq \phi \cup \phi \cup \phi \cup \cdots$ and $f(\phi) = 0$

$$\therefore \quad \sum_{i=1}^{\infty} f(\phi) = 0 \quad \text{so that}$$

$$\inf\left\{ \sum_{i=1}^{\infty} f(E_i) \;\middle|\; \phi \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{E} \right\} = 0$$

$\Rightarrow \mu^*(\phi) = 0$ by definition of μ^* in (A)

(ii) Now we are to prove that $\mu^* : P(X) \to [0, \infty]$ has monotonicity property. Let $A, B \in P(X)$ such that $A \subseteq B$. Then every sequence which is cover of B also cover of A.

But cover of a set A need not be cover of B. Therefore

$$\left\{ \sum_{i=1}^{\infty} \rho(E_i) \;\middle|\; B \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{E} \right\} \subseteq \left\{ \sum_{i=1}^{\infty} \rho(F_i) \;\middle|\; A \subseteq \bigcup_{i=1}^{\infty} F_i,\ F_i \in \mathcal{E} \right\}$$

$$\Rightarrow \operatorname{Inf} \left\{ \sum_{i=1}^{\infty} \rho(E_i) \;\middle|\; B \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{E} \right\} \geq \operatorname{Inf} \left\{ \sum_{i=1}^{\infty} \rho(F_i) \;\middle|\; A \subseteq \bigcup_{i=1}^{\infty} F_i,\ F_i \in \mathcal{E} \right\}$$

$\because$ $A \subseteq B$

$\Rightarrow \operatorname{Inf} A \geq \operatorname{Inf} B.$

$\Rightarrow \mu^*(B) \geq \mu^*(A)$ by definition of μ^* in (A)

i.e $\mu^*(A) \leq \mu^*(B)$.

(iii) Now we are to show that $\mu^* : P(X) \longrightarrow [0, \infty]$ is countablity sub additive. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $P(X)$. Let $\{E_i^1\}_{i=1}^{\infty}$ in $\mathcal{E}$ is cover of A_1 i.e

$$A_1 \subseteq \bigcup_{i=1}^{\infty} E_i^1$$

Then by hypothesis

$$\mu^*(A_1) \leq \sum_{i=1}^{\infty} \rho(E_i^1).$$

Let $\varepsilon > 0$ be +ve real number such that

$$\sum_{i=1}^{\infty} \rho(E_i^1) \leq \mu^*(A_1) + \frac{\varepsilon}{2^1}.$$

For A_2 $\exists$ $\{E_i^2\}$ in $\mathcal{E}$ s.t

$A_2 \subseteq \bigcup_{i=1} E_i^2$ Then by hypothesis

$\mu^*(A_2) \leq \sum_{i=1}^{\infty} \rho(E_i^2)$. and for $\varepsilon > 0$

we have

$$\sum_{i=1}^{\infty} \rho(E_i^2) \leq \mu^*(A_2) + \frac{\varepsilon}{2^2}$$

Similarly for each $A_K \in \{A_i\}_{i=1}^{\infty}$ we have $\{E_i^K\}_1^{\infty}$ in $\mathcal{E}$ s.t

$A_K \subseteq \bigcup_{i=1}^{\infty} E_i^k$ and for $\varepsilon > 0$

$$\sum_{i=1}^{\infty} \rho(E_i^k) \leq \mu^*(A_K) + \frac{\varepsilon}{2^K} \quad \forall\ K = 1, 2, 3, \cdots$$

Then countable union of $\{A_i\}_{i=1}^{\infty}$ is cover by the sequence $\left\{\left(\bigcup_{i=1}^{\infty} E_i^n\right)\right\}_{n=1}^{\infty}$ in $\mathcal{E}$ s.t

$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty}\left(\bigcup_{i=1}^{\infty} E_i\right)$ Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \inf\left\{\sum_{n=1}^{\infty} \rho\left(\bigcup_{i=1}^{\infty} E_i^n\right) \,\middle|\, \bigcup_{n=1}^{\infty}\left(\bigcup_{i=1}^{\infty} E_i^n\right) \supseteq \bigcup_{n=1}^{\infty} A_n\right\}$$

$$\leq \sum_{n=1}^{\infty} \rho\left(\bigcup_{i=1}^{\infty} E_i^n\right)$$

$$\leq \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} \rho(E_i^n)\right) \quad \because \rho \text{ is countabily sub additive.}$$

$$\leq \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right)$$

$$= \sum_{n=1}^{\infty} \mu^*(A_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$

$$= \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon$$

$\because \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$ is Geometric series with common ratio $|r| = |1/2| < 1$ is convergent. s.t $\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$ (say).

Hence $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ because 'ε' is any arbitrary +ve real numbers.

So μ^* is countably sub additive

Hence $\mu^* : P(X) \to [0,\infty]$ defined as

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \;\middle|\; \bigcup_{i=1}^{\infty} E_i \supseteq E,\; E_i \in \mathcal{E} \right\}$$

is an outer measure.

Theorem: Let $X \neq \phi$, $\mu^* : P(X) \to [0,\infty]$ be an outer measure on $P(X)$ & $m(\mu^*)$ is σ-algebra of measurable sets on X. Show the restriction of μ^* to $m(\mu^*)$ is become measure i.e $\mu^*|_{m(\mu^*)} : m(\mu^*) \to [0,\infty]$ is measure i.e $\mu^*|_{m(\mu^*)} = \mu$. Furthermore $(X, m(\mu^*), \mu)$ is complete measure space.

Proof) Since $\mu^* : P(X) \to [0,\infty]$ is countably sub additive therefore

Therefore $\mu^*\big|_{m(\mu^*)} : m(\mu^*) \longrightarrow [0,\infty]$ is countably sub additive.

we are to show $\mu^*\big|_{m(\mu^*)} : m(\mu^*) \longrightarrow [0,\infty]$ is measure on $m(\mu^*)$.

(i) Since $\mu^*(\phi) = 0$

$\therefore$

$$\mu^*\big|_{m(\mu^*)} (\phi) = 0$$

(ii) To show that $\mu^*\big|_{m(\mu^*)}$ is countably additive.

Let $\{E_i\}_{i=1}^{\infty}$ be sequence in $m(\mu^*)$. Therefore

$\{E_i\}_{i=1}^{\infty}$ is disjoint sequence in $m(\mu^*)$ and

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i) \quad \because \mu^* \text{ is outer measure.}$$

$$\Rightarrow \mu^*\big|_{m(\mu^*)} \left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*\big|_{m(\mu^*)} (E_i) \quad \text{--- (1)}$$

~~Since~~ Now for $n \in N$, we have

$$\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i=1}^{\infty} E_i$$

$\therefore$

$$\mu^*\big|_{m(\mu^*)}\left(\bigcup_{i=1}^{n} E_i\right) \le \mu^*\big|_{m(\mu^*)}\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$\because$ μ^* has monotonicity property.

OR

$$\mu^*\big|_{m(\mu^*)}\left(\bigcup_{i=1}^{\infty} E_i\right) \ge \mu^*\big|_{m(\mu^*)}\left(\bigcup_{i=1}^{n} E_i\right)$$

$$= \sum_{i=1}^{n} \mu^*\big|_{m(\mu^*)}(E_i)$$

$\because$ If $E_1, E_2 \in m(\mu^*)$ & $E_1 \cap E_2 = \phi$ then $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$.

Since this is true $\forall\, n \in N$ therefore

$$\mu^*\big|_{m(\mu^*)}\left(\bigcup_{i=1}^{\infty} E_i\right) \ge \sum_{i=1}^{\infty} \mu^*\big|_{m(\mu^*)}(E_i) \quad \text{---(2)}$$

From (1) & (2)

$$\mu^*\big|_{m(\mu^*)}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*\big|_{m(\mu^*)}(E_i)$$

So $\mu^*\big|_{m(\mu^*)} : m(\mu^*) \to [0, \infty]$ is measure on $m(\mu^*)$.

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Notations: $\mathbb{R}$: (Set of real numbers).

$\mathcal{J}_o$ = Collection of ϕ and all open intervals in $\mathbb{R}$.
$\mathcal{J}_{oc}$ = Collection of ϕ and all open-closed intervals in $\mathbb{R}$.
$\mathcal{J}_{co}$ = Collection of ϕ and all closed-open intervals in $\mathbb{R}$.
$\mathcal{J}_c$ = collection of ϕ and all closed intervals in $\mathbb{R}$.

$$\mathcal{J} = \mathcal{J}_o \cup \mathcal{J}_{oc} \cup \mathcal{J}_{co} \cup \mathcal{J}_c .$$

$\because$ Let $l : \mathcal{J} \longrightarrow [0,\infty]$ be non-negative real value function s.t

(i) $\forall\ I \in \mathcal{J}$ with end point $a, b \in \mathbb{R}$, $a<b$
$l(I) = b-a$ and $l(\phi) = 0$

(ii) If I is an infinite interval then
$l(I) = \infty$

(iii) for an arbitrary disjoint sequence $\{I_n\}_{n=1}^{\infty}$ in $\mathcal{J}$,

$$l\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} l(I_n).$$

Lebesgue Outer Measure:

let $\mathbb{R}$ be the set of real number and $l : \mathcal{J} \longrightarrow [0,\infty]$ s.t $l(\phi) = 0$, $l(I) = b-a$ then the set function $\mu_L^* : P(R) \longrightarrow [0,\infty]$ defined by

$$\mu_L^*(E) = \inf\left\{ \sum_{n=1}^{\infty} l(I_n) \,/\, E \subseteq \bigcup_{n=1}^{\infty} I_n ,\ I_n \in \mathcal{J}_o \right\} \quad \forall\ E \in P(\mathbb{R})$$

is called Lebesgue outer measure.

Lebesgue measurable set :
OR
μ_L^*- measurable set :

Let $\mu_L^* : P(R) \longrightarrow [0,\infty]$ is Lebesgue outer measure, A set $E \in P(R)$ is called μ_L^*- measurable set OR Lebesgue measurable set if

$$\mu_L^*(A) = \mu_L^*(A \cap E) + \mu_L^*(A \cap E^c) \quad \forall A \in P(R).$$

Remark: The condition
$$\mu_L^*(A) = \mu_L^*(A \cap E) + \mu_L^*(A \cap E^c) \quad \forall A \in P(R)$$
is equivalent to
$$\mu_L^*(I) = \mu_L^*(I \cap E) + \mu^*(I \cap E^c) \quad \forall I \in \mathcal{I}_0.$$

Lebesgue σ- algebra :

The collection of all μ_L^*-measurable sets form σ-algebra on R called Lebesgue σ-algebra and it is denoted by m_L.

Lebesgue Measurable space

The pair (R, m_L) is Lebesgue measure space, where R is the set of real numbers and m_L is Lebesgue σ-algebra.

Lebesgue Measure space:

The Triplet $(\mathbb{R}, m_L, \mu_L)$ is called Lebesgue Measure Space, where $\mathbb{R}$ is the set of real number and μ_L is measure on m_L.

Lemma

(1) Prove that Lebesgue outer measure of singleton set is zero i.e $\mu_L^*(\{x\}) = 0 \;\forall\, x \in \mathbb{R}$. and $\{x\} \in m_L$.

Proof:

Let $x \in \mathbb{R}$, Then $\forall\; \varepsilon > 0$ we have $(x-\varepsilon, x+\varepsilon) \in \mathcal{J}_0$, so that

$(x-\varepsilon, x+\varepsilon), \phi, \phi, \phi, \ldots$ is an open cover of $\{x\}$

Then by definition of Lebesgue outer measure

$$\mu_L^*(\{x\}) = \operatorname{Inf}\left\{ \sum_{i=1}^{\infty} l(I_i) \;\middle|\; \{x\} \subseteq \bigcup_{i=1}^{\infty} I_i,\; I_i \in \mathcal{J}_0 \right\}$$

$$\therefore \quad \mu_L^*(\{x\}) \leq l(x-\varepsilon, x+\varepsilon) + l(\phi) + l(\phi) + \cdots$$

$$= 2\varepsilon \quad \forall\; \varepsilon > 0.$$

Since ε is any arbitrary +ve real number

$\therefore \quad \mu_L^*(\{x\}) = 0$

& $\quad \because$ If $\mu^*(E) = 0$ then $E \in m(\mu^*)$.

$\{x\} \in m_L$

(2) Prove that Every countable sub set of $\mathbb{R}$ is null set in $(\mathbb{R}, m_L, \mu_L)$.

Proof let E be countable subset of $\mathbb{R}$. Then E is countable union of singleton.

i.e $E = \bigcup_{x \in E} \{x\}$.

operating μ_L - on both sides

$$\mu_L(E) = \mu_L\left(\bigcup_{x \in E} \{x\}\right)$$

$$= \sum_{x \in E} \mu_L(\{x\}) \quad \because \mu_L \text{ is measure.}$$

$$= 0 \quad \because \mu_L(\{x\}) = 0 \ \forall x \in \mathbb{R}$$

$\Rightarrow E$ is a null set.

Question Prove that set of rational number (Q) is null set. & $Q \in m_L$.

Proof:

let $(\mathbb{R}, m_L, m_L)$ be lebesgue measure space and $Q \subseteq R$. Since set of rational number is countable, Therefore set of rational number is null set. $\because$ Every countable subset of $\mathbb{R}$ in (R, m_L, μ_L) is null set.

so

$Q \in m_L$ $\because$ Every null set belonges to m_L.

Question let (R, μ_L, m_L) is measurable space and Q is the set of rational number and Q^c is the set of irrational number Then Prove that $\mu_L(Q^c) = \infty$ and $Q^c \in m_L$.

Proof:

Since $R = Q \cup Q^c$

$\therefore$

$$Q^c = R \setminus Q$$

Then

$$\mu_L(Q^c) = \mu_L(R \setminus Q)$$
$$= \mu_L(R) - \mu_L(Q)$$
$$= \infty - 0 \quad \because Q \text{ is null set.}$$

$$\mu_L(Q^c) = \infty$$

Since $R \in m_L$ and $Q \in m_L$

$\therefore$ $R \setminus Q \in m_L$ $\because$ m_L is σ-algebra.

$\Rightarrow$ $Q^c \in m_L$ $\because$ $Q^c = R \setminus Q$.

Dense Sub Set of X:

let $(X, \mathcal{J})$ be Topological space, A subset E of X is called dense in X if for all open set O in X s.t

$$O \cap E \neq \phi.$$

OR

$\bar{E} = X$. where $\bar{E}$ (closure of E).

Proposition:

If E is null set in (R, m_L, μ_L) then E^c is dense in $\mathbb{R}$.

Proof: let $I \in J_0$ s.t

$I \subseteq E$

Then by montonoticity property of μ_L we have

$\mu_L(I) \leq \mu_L(E) = 0 \quad \because E$ is null set

$\Rightarrow \mu_L(I) = 0$

which is controdiction to the fact that

$\mu_L(I) > 0 \quad \forall \; I \in J_0$.

Hence $I \nsubseteq E$ then $I \cap E^c \neq \phi \quad \forall \; I \in J_0$

so E^c is dense in (X, m_L, μ_L).

* ———————— *

Lemma:

Prove that Lebesgue outer measure of an interval is its length. i.e $\mu_L^*(I) = \ell(I) \quad \forall \; I$ in R. where I is an interval in $\mathbb{R}$.

Proof Case I

First we considered the case when I is finite closed interval i.e $I = [a, b]$ where $a, b \in \mathbb{R}$ s.t $a < b$. For every $\varepsilon > 0$

$[a, b] \subseteq (a - \varepsilon, a + \varepsilon)$.

Then $\{(a-\varepsilon, b+\varepsilon), \phi, \phi, \phi, \cdots\}$ is sequence in J_0 s.t it cover the interval $I=[a,b]$.
Then by definition of μ_L^* we have

$$\mu_L^*([a,b]) \le l(a-\varepsilon, b+\varepsilon) + l(\phi) + l(\phi) + \cdots,$$
$$= b-a+2\varepsilon.$$

Since this true for all $\varepsilon > 0$ Therefore

$$\mu_L^*[a,b] \le b-a = l(I)$$

i.e
$$\mu^*[a,b] \le l(I) \quad \text{--- (1)}$$

Now we prove the reverse inequality $\mu_L^*(I) \ge l(I)$. But it is equivalent to show

$$\sum_{n=1}^{\infty} l(I_n) \ge b-a \text{ --- (2)}$$ for any countable cover $\{I_n\}_{n=1}^{\infty}$ in J_0 of the interval I.

"By Heien-Borel Theorem Every countable cover of closed interval can be reduced to finite sub cover".

So it is sufficent to prove inequality (2) for a finite sub cover. i.e If $\{I_n\}_{n=1}^{N}$ is finit sub cover of the interval $I=[a,b]$ Then we are to prove that

$$\sum_{n=1}^{N} l(I_n) \ge b-a = l(I) \quad \text{--- (3)}$$

Since $I \subseteq \bigcup_{n=1}^{N} I_n$.

$\therefore \quad a \in \bigcup_{n=1}^{N} I_n$

$\Rightarrow \exists$ an open interval $(a_1, b_1) \in \{I_n\}_{n=1}^{N}$ s.t

$a \in (a_1, b_1)$ Then $a_1 < a < b_1$.

If $b_1 \leq b$ Then $b_1 \in [a, b]$ and $b_1 \notin (a_1, b_1)$

Then there is an open interval $(a_2, b_2) \in \{I_n\}_{n=1}^{N}$ s.t

$b_1 \in (a_2, b_2)$ Then $a_2 < b_1 < b_2$.

Proceeding in the same way, we reach an open interval $(a_k, b_k) \in \{I_n\}_{n=1}^{N}$ s.t

$a_k < b < b_k$ i.e $b \in (a_k, b_k)$. So we obtain a sub-sequence of $\{I_n\}_{n=1}^{N}$ i.e

$\{(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)\} \subseteq \{I_n\}_{n=1}^{N}$.

Therefore

$$\sum_{n=1}^{N} l(I_n) \geq \sum_{i=1}^{k} l(a_i, b_i)$$

$$= l(a_1, b_1) + l(a_2, b_2) + \dots + l(a_k, b_k).$$

OR

$$= l(a_k, b_k) + l(a_{k-1}, b_{k-1}) + \dots + l(a_2, b_2) + l(a_1, b_1)$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \dots + (b_2 - a_2) + (b_1 - a_1)$$

$$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \dots - (a_2 - b_1) - a_1.$$

$> b_k - a_1 \quad \because \quad a_i < b_{i-1} \ \forall\, i = 1, 2, 3, \dots, k$

$\Rightarrow a_i - b_{i-1} < 0.$

$> b - a \quad \because \quad b_k > b$ & $a_1 < a \Rightarrow -a > -a_1$

$\Rightarrow b_k - a_1 > b - a.$

i.e $\sum_{n=1}^{N} l(I_n) > b-a = l(I)$

so $\sum_{n=1}^{\infty} l(I_n) > l(I)$

$\Rightarrow \mu_L^*(I) \geq l(I)$ —— (4)

from (3) & (4) we have

$\mu_L^*(I) = l(I)$ for $I = [a,b]$

Case II :- If $I = (a,b)$ then

$(a,b) \subseteq [a,b]$

$\therefore \mu_L^*(a,b) \leq \mu_L^*[a,b]$ by monotonicity property

$= l([a,b])$

$= b-a$

i.e $\mu^*(a,b) \leq b-a$ —— (i)

If for $\varepsilon > 0$ we have

$\mu^*(a,b) \geq b-a-\varepsilon$

Since ε is an arbitrary +ve real number

$\therefore \mu^*(a,b) \geq b-a$ —— (ii)

from (i) & (ii)

$\mu_L^*(a,b) = b-a = l(I)$

Case III

If $I = (a,b]$ then since

$(a,b] = (a,b) \cup \{b\}$ & $\mu_L^*(\{b\}) = 0$

$\therefore$

$$\mu_L^*((a,b]) = \mu_L^*(a,b)$$
$$= b-a = l((a,b]).$$

<u>case IV</u> If $I=[a,b)$ Then since

$$[a,b) = \{a\} \cup (a,b) \quad \text{and} \quad \mu_L^*(\{a\})=0$$

$\therefore$

$$\mu_L^*([a,b)] = \mu_L^*(a,b)$$
$$= b-a \quad \text{by case II}$$
$$= l([a,b))$$

so $\mu_L^*([a,b)] = l([a,b))$.

<u>case V:</u> let $I=(a,\infty)$, Then $\forall\, n \in N$.

$$(a,\infty) \supseteq (a,n) \quad \text{so that}$$
$$\mu_L^*((a,\infty)) \geq \mu_L^*((a,n)) = n-a$$

Since this holds $\forall\, n \in N$, we must have

$$\mu_L^*((a,\infty)) = \infty = l((a,\infty)).$$

<u>case VI</u>

let $I=(-\infty,b)$ Then $\forall\, n \in N$

$$(-n,b) \subseteq (-\infty,b)$$

i.e $\mu_L^*((-n,b)) \leq \mu_L^*((-\infty,b))$

$$b-(-n) \leq \mu_L^*((-\infty,b))$$

Since this hold $\forall\, n$ $\therefore$ $\mu_L^*(-\infty,b) = \infty = l(-\infty,b)$ //.

Lemma

Prove that every interval in R is lebesgue measural or μ_L- measurable. OR Prove that $\mathcal{I} \subseteq m_L$.

Proof A subset E of R is μ_L^*- measurable if $\forall\ I \in \mathcal{I}_0$ s.t

$$\mu_L^*(I) = \mu_L^*(I \cap E) + \mu_L^*(I \cap E^c).$$

Case-I If $I = (a, \infty) \in \mathcal{I}_0$, $a \in R$ we have

$$I = I \cap R$$

$$I = I \cap \left((a,\infty) \cup (a,\infty)^c \right)$$

$$I = I \cap (a,\infty) \cup I \cap (a,\infty)^c$$

Since $I \cap (a,\infty)$ and $I \cap (a,\infty)^c$ are disjoint so that

$$l(I) = l(I \cap (a,\infty)) + l(I \cap (a,\infty)^c)$$

$$\Rightarrow \mu_L^*(I) = \mu_L^*(I \cap (a,\infty)) + \mu_L^*(I \cap (a,\infty)^c)$$

$$\because \mu_L^*(I) = l(I)$$

$\Rightarrow (a,\infty)$ is μ_L^*- measurable.

$\Rightarrow (a,\infty) \in m_L$

By the similar argument we can prove that $(-\infty, b) \in m_L$.

Case-II If $I = (a, b)$.

Since $I = (-\infty, b) \cup (a, \infty) \in m_L$ being the union of two μ_L^*- measurable interval is μ_L^*- measurable.

Case III when $I = [a,b]$

Since $I = \{a\} \cup (a,b) \cup \{b\} \in m_L$

$\Rightarrow I = [a,b] \in m_L.$ $\because (a,b), \{a\}, \{b\} \in m_L.$

Case IV If $I = (a,b]$ or $I = [a,b)$

then $I = (a,b) \cup \{b\}$ or $I = (a,b) \cup \{b\}$

$\Rightarrow I \in m_L$ $\because (a,b), \{a\}, \{b\} \in m_L.$

Hence every interval in $\mathbb{R}$ is μ_L^*- measured

OR $\mathcal{J} \subseteq m_L$

Question Show that (R, m_L, μ) is σ- finite but not finit space.

Sol:

Since $\mathbb{R} = (-\infty, \infty)$

$\therefore$ $\mu_L(-\infty, \infty) = l(-\infty, \infty) = \infty$

So (R, m_L, μ) is not finite.

Now consider the sequence $\{(-n,n)\}_{n=1}^{\infty}$ in m_L

Then

$$\bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}$$

and

$$\mu_L(-n, n) = l(-n, n) = 2n < \infty \quad \forall n \in N.$$

Hence (R, m_L, μ_L) is σ- finite space.

Theorem :

Prove that every Borel set is a lebesgue measurable set. i.e $\mathcal{B}_R \subseteq m_L$.

Proof :

Since every open interval in $\mathbb{R}$ is μ_L^*-measurable and every open set in $\mathbb{R}$ is countable union of open sets (intervals) in $\mathbb{R}$. Therefore it is member of m_L. if we $\mathcal{D}$ be the collection of all open sets in $\mathbb{R}$ Then

$\mathcal{D} \subseteq m_L$ so that

$$\sigma(\mathcal{D}) \subseteq \sigma(m_L) = m_L$$

i.e $\mathcal{B}_L \subseteq m_L$.

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Translation of a Set :

Let X be a vector (Linear) space over the field of scalars $\mathbb{R}$ Then for $E \subseteq X$ and $x_0 \in X$ we define translation of E by x_0 as $E + x_0 = \{ x + x_0 \mid x \in E \}$

Dielation of a Set :

Let $X(R)$ be vector space over a field $\mathbb{R}$, for $E \subseteq X$, $\alpha \in \mathbb{R}$ The dielation of E by α is defined as

$$\alpha E = \{ \alpha x : x \in E \}$$

Note: For a collection $\mathcal{E}$ of subsets of X we have $\forall\, x_0 \in X$, $\alpha \in \mathbb{R}$

$$\mathcal{E} + x_0 = \{ E + x \mid E \in \mathcal{E} \},\ \alpha\mathcal{E} = \{ \alpha E : E \in \mathcal{E} \}.$$

NOTE: Properties of Transation & Dilation of a Set.

(1) $(E + x_1) + x_2 = E + (x_1 + x_2)$

(2) $(E + x_1)^c = E^c + x$

(3) $E_1 \subseteq E_2 \Rightarrow E_1 + x \subseteq E_2 + x$

(4) $\left(\bigcup_{i=1}^{\infty} E_i\right) + x = \bigcup_{i=1}^{\infty} (E_i + x)$

(5) $\left(\bigcap_{i=1}^{\infty} E_i\right) + x = \bigcap_{i=1}^{\infty} (E_i + x)$

(6) $\alpha(\beta E) = (\alpha\beta) E.$

(7) $(\alpha E)^c = \alpha E^c.$

Translation Invarient:-

Let $(X, \mathcal{A}, \mu)$ be a measurable space as well as vector space over a field F then

(1) The σ-algebra $\mathcal{A}$ is translation invarient if $\forall E \in \mathcal{A}$ and $x \in X$ implies that $E + x \in \mathcal{A}$.

(2) The measure μ is said to be translation invarient if $\forall E \in \mathcal{A} \Rightarrow E + x \in \mathcal{A}$ and
$$\mu(E) = \mu(E + x), \quad \forall x \in X, E \in \mathcal{A}.$$

(3) The measure space $(X, \mathcal{A}, \mu)$ is translation invarient if $\mathcal{A}$ and μ both are translation invarient.

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Lemma:

Prove that Lebesgue outer measure is Translation invarient. OR for every $E \in P(R)$, $x \in R$ show that
$\mu_L^*(E+x) = \mu_L^*(E)$.

Proof: First we show that $l : \mathcal{J}_0 \to [0,\infty]$ s.t
for $I = (a,b)$
$l(I) = b-a$ is translation invarient. If $I = (a,b)$ then $I+x = (a+x, b+x) \in \mathcal{J}_0$.

$$l(I+x) = b+x-(a+x)$$
$$= b-a = l(I)$$

If $I = (a,\infty)$ or $I = (-\infty, b)$ or $I = (-\infty,\infty)$ Then
$I+x = (a+x, \infty)$, $I+x = (-\infty, b+x)$, $I+x = (-\infty,\infty) \in \mathcal{J}_0$
and $l(I+x) = +\infty = l(I)$ in each case. Hence
$\forall$ $I \in \mathcal{J}_0$ & $x \in R$ we have $I+x \in \mathcal{J}_0$ and
$l(I+x) = l(I)$

So $l : \mathcal{J}_0 \to [0,\infty]$ is Translation invarient.

Let $\{I_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $\mathcal{J}_0$ s.t
$E \subseteq \bigcup_{n=1}^{\infty} I_n$ Then for an arbitrary $x \in R$, $\{I_n+x\}_{n=1}^{\infty}$
is sequence in $\mathcal{J}_0$ with $l(I_n+x) = l(I_n)$, $\forall n \in N$.

Now $\bigcup_{n=1}^{\infty}(I_n+x) = \left(\bigcup_{n=1}^{\infty} I_n\right) + x \supseteq E+x$

$$\Rightarrow \bigcup_{n=1}^{\infty}(I_n+x) \supseteq E+x$$

So that
$$\sum_{n=1}^{\infty} l(I_n) = \sum_{i=1}^{\infty} l(I_n+x) \geq \mu_L^*(E+x)$$
by def of μ_L^*.

Since this hold for an arbitrary sequence $\{I_n\}_{n=1}^{\infty}$ s.t $E \subseteq \bigcup_{n=1}^{\infty} I_n$ &

$$\mu_L^*(E) = \inf\left\{ \sum_{n=1}^{\infty} l(I_n) \;\middle|\; E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

Therefore

$$\mu_L^*(E) \geq \mu_L^*(E+x) \quad \text{—— (1)}$$

Applying (1) to $E+x$ and its translation by $-x$ i.e $E+x+(-x)$ we obtain $E+x$.

Therefore from (1) we have

$$\begin{aligned} \mu_L^*(E+x) &\geq \mu_L^*(E+x+(-x)) \\ &= \mu_L^*(E+(x-x)) \\ &= \mu_L^*(E+0) \\ &= \mu_L^*(E) \end{aligned}$$

i.e $\mu_L^*(E+x) \geq \mu_L^*(E)$ —— (2)

from (1) & (2) we have

$$\mu_L^*(E+x) = \mu_L^*(E).$$

Hence Lebesgue Outer measure is translation invariant.

Theorem:

The Lebesgue measure space (R, m_L, μ_L) is translation invarient i.e $\forall\ E \in m_L$ and $x \in \mathbb{R}$, $E + x \in m_L$ and

$\mu_L(E+x) = \mu_L(E)$ furthermore

$m_L + x = m_L$.

Proof:

Let $E \in m_L$ and $x \in \mathbb{R}$ we are to show that $E + x \in m_L$. Let A be an aribtrary sub set of R i.e $A \in P(R)$.

Consider

$$\mu_L^*(A \cap (E+x)) + \mu_L^*(A \cap (E+x)^c)$$

$$= \mu_L^*((A \cap (E+x)) - x) + \mu_L^*((A \cap (E+x)^c) - x)$$

$\because \mu_L^*$ is Translation invarient.

$$= \mu_L^*((A-x) \cap (E+x) - x) + \mu_L^*((A-x) \cap (E^c + x - x))$$

$$= \mu_L^*((A-x) \cap E) + \mu_L^*((A-x) \cap E^c)$$

$$= \mu_L^*(A-x) \quad \because E \in m_L \text{ \& considering } A-x \text{ is a Testing set}$$

$$= \mu_L^*(A) \quad \because \mu_L^* \text{ is Translation invarient.}$$

So

$$\mu^*(A \cap (E+x)) + \mu_L^*(A \cap (E+x)^c) = \mu^*(A) \ \forall A \in P(R)$$

Hence $E + x \in m_L$.

Since restriction of μ_L^* to m_L become measure mean outer measure become measure

Therefore $\mu_L^* = \mu_L$ so

$$\mu_L(E+x) = \mu_L^*(E+x) = \mu_L^*(E) = \mu_L(E)$$

i.e $\mu_L(E+x) = \mu_L(E)$.

so (R, m_L, μ_L) is translation invarient.

for $E \in m_L$ & $x \in R$ we have

$E + x \in m_L$ But actual $E+x \in m_L + x$.

$\therefore$

$$m_L + x \subseteq m_L \text{ --- (i)}$$

let $\underset{E-x \in m_L}{E \in m_L}$, $x \in R$ we have

$\Rightarrow$ $E + \cancel{x-x} \in m_L + x$ $\because$ m_L is Translation invarient.

$\Rightarrow$ $E + 0 \in m_L + x$

$\Rightarrow$ $E \in m_L + x$

so $m_L \subseteq m_L + x$ --- (ii)

from (i) & (ii) we get

$$m_L + x = m_L.$$

Addition Modulo 1 :

let $I = [0,1)$ be an interval in R. For $x, y \in I = [0,1)$ we defined addition modulo 1 by

$$x \dot{+} y = \begin{cases} x+y, & \text{if } x+y < 1 \\ x+y-1, & \text{if } x+y \geq 1 \end{cases}$$

Note: $x \dot{+} y = y \dot{+} x$. $\forall\, x, y \in I = [0,1)$.

Translation of E mod 1:

Let $E \subseteq I = [0,1)$ and $y \in I = [0,1)$ we define

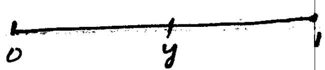
$E \dot{+} y = \{ x \dot{+} y \mid x \in E \}$ which is called translation of E mod 1.

Lemma

Prove that Lebesgue measure is translation invariant mod 1. OR Let $E \subseteq [0,1)$, If $E \in \mathcal{M}_L$ then for every $y \in (0,1)$, $E \dot{+} y \in \mathcal{M}_L$ and $\mu_L(E \dot{+} y) = \mu_L(E)$.

Proof Let $E \subseteq [0,1)$ & $y \in (0,1)$.

Define the intervals



$[0, 1-y)$ and $[1-y, 1)$. clearly

$[0,1-y) \cap [1-y, 1) = \phi$. Now we define two subsets of E s.t

$E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$

Then

$E_1 \cap E_2 = \phi$ and $E_1 \cup E_2 = E$

Since $E \in \mathcal{M}_L$ and $\mathcal{J} \subseteq \mathcal{M}_L$ also $\mathcal{M}_L$ is a σ-algebra.

$\therefore$ $E_1 \in \mathcal{M}_L$ and $E_2 \in \mathcal{M}_L$.

Note: $\mathcal{J}$ is the collection of all interval in $\mathbb{R}$

Since $E_1 \stackrel{\circ}{+} y = \{ x \stackrel{\circ}{+} y \mid x \in E_1 \}$

$= \{ x+y \mid \forall\, x \in E_1,\ \underset{\text{i.e } x<y-1}{x+y<1} \}$

$= E_1 + y \in \mathcal{M}_L \quad \because$ $\mathcal{M}_L$ is Translation invarient i.e $\forall\ E \in \mathcal{M}_L,\ x \in \mathbb{R}$ $\Rightarrow E+x \in \mathcal{M}_L$.

&

$E_2 \stackrel{\circ}{+} y = \{ x \stackrel{\circ}{+} y \mid x \in E_2 \}$

$= \{ x+y-1 \mid x \in E_2,$ if $x+y \geq 1$ i.e $x \geq y-1 \}$

$= E_2 + (y-1) \in \mathcal{M}_L \quad \because$ $\mathcal{M}_L$ is Translation invarient.

Now

$E \stackrel{\circ}{+} y = (E_1 \cup E_2) \stackrel{\circ}{+} y \qquad \because E = E_1 \cup E_2$

$= (E_1 \stackrel{\circ}{+} y) \cup (E_2 \stackrel{\circ}{+} y) \in \mathcal{M}_L \quad \because$ $\mathcal{M}_L$ is σ-algebra.

So $E \stackrel{\circ}{+} y \in \mathcal{M}_L$.

Now we are to show that $\mu_L(E \stackrel{\circ}{+} y) = \mu_L(E)$

for this First we are to show that

$\mu_L(E_1 \stackrel{\circ}{+} y) = \mu_L(E_1)$ and $\mu_L(E_2 \stackrel{\circ}{+} y) = \mu_L(E_2)$.

Since $E_1 \stackrel{\circ}{+} y = E_1 + y \quad \because x \in E_1$ Then $x+y<1$ i.e $x<y-1$.

operating μ_L on both sides

$\mu_L(E_1 \stackrel{\circ}{+} y) = \mu_L(E_1 + y)$

$\mu_L(E_1 \overset{\circ}{+} y) = \mu_L(E_1)$.

Now

Since $E_2 \overset{\circ}{+} y = E_2 + (y-1)$ $\because x \geq y-1$ in this case.

operating 'μ_L' on both sides we get

$$\mu_L(E_2 \overset{\circ}{+} y) = \mu_L(E_2 + (y-1))$$
$$= \mu_L(E_2) \quad \because \mu_L \text{ is translation invarient.}$$

Since

$$E \overset{\circ}{+} y = (E_1 \overset{\circ}{+} y) \cup (E_2 \overset{\circ}{+} y)$$

operating 'μ_L' on both sides

$$\mu_L(E \overset{\circ}{+} y) = \mu_L((E_1 \overset{\circ}{+} y) \cup (E_2 \overset{\circ}{+} y)) \quad \because m_L \text{ } \sigma\text{-algebra}$$
$$= \mu_L(E_1 \overset{\circ}{+} y) + \mu_L(E_2 \overset{\circ}{+} y)$$

& $E_1, E_2, E_1 \overset{\circ}{+} y, E_2 \overset{\circ}{+} y$ belong's m_L

$$= \mu_L(E_1) + \mu_L(E_2)$$

& $E_1 \overset{\circ}{+} y \cap E_2 \overset{\circ}{+} y = \phi$, $E_1 \cap E_2 = \phi$

$$= \mu_L(E_1 \cup E_2)$$

$E_1 \overset{\circ}{+} y \cup E_2 \overset{\circ}{+} y = E \overset{\circ}{+} y$

$$\mu_L(E \overset{\circ}{+} y) = \mu_L(E) \quad \because E_1 \cup E_2 = E$$

$\Rightarrow \mu_L$ is Translation invarient modulo 1.

Hence proved.

Theorem:

Prove that the interval $[0,1) \in \mathcal{M}_L$ contains a non-Lebesgue measurable set.

Proof:

Step 1: First we define a relation '$\sim$' on $[0,1)$ s.t for $x, y \in [0,1)$

$x \sim y \iff x - y$ is a rational. clearly the relation '$\sim$' is an equivalence relation. The relation '$\sim$' partioned $[0,1)$ into equivalence classes $\{E_n\}$. Any two elements $x, y \in [0,1)$ s.t $x, y \in E_k$ for some k if $x - y$ is rational & $x \in E_i$ & $y \in E_j$ for some i, j if $x - y$ is irrational.

Step II

By axiom of choice, construct a set $P \subseteq [0,1)$ by Picking exactly on element from each equivalence class.

let $\{r_n : n \in \mathbb{Z}_+'\}$ be rationals in $[0,1)$. where $r_0 = 0$.

Define a collection $\Omega = \{P_n \mid P_n = P \dotplus r_n ; n \in \mathbb{Z}_+'\}$.

we claim that the collection "Ω" is a disjoint collection.

let P_m and $P_n \in \Omega$ for $m \neq n$ and suppose that $P_m \cap P_n \neq \phi$.

Then $x \in P_m \cap P_n$

$\Rightarrow x \in P_m$ & $x \in P_n$. Therefore $\exists\, p_n, p_m \in P$

s.t $x = p_m \dot{+} r_m$ and $x = p_n \dot{+} r_n$

$\Rightarrow$

$$p_m \dot{+} r_m = p_n \dot{+} r_n$$

Since $p_m \dot{+} r_m$ is either $p_m + r_m$ or $p_m + r_m - 1$.

& $p_n \dot{+} r_n$ is either $p_n + r_n$ or $p_n + r_n - 1$.

Therefore in either case $p_m - p_n$ is a rational.

Note:

$$x \dot{+} y = \begin{cases} x+y, & x+y<1 \\ x+y-1, & x+y \geq 1 \end{cases}$$

If $p_m \dot{+} r_m = p_m + r_m$ & $p_n \dot{+} r_n = p_n + r_n$ then $p_m + r_m = p_n + r_n$

$p_m - p_n = r_n - r_m$ (rational)

Similarly

$p_m - p_n = r_n - r_m$ (rational in other case.

$\therefore$ $p_m, p_n \in E_\alpha$, for som α.

Since P contains exactly one element from each class $\therefore$ $p_m = p_n$

$\Rightarrow m = n$ a controduction.

Hence

$$P_m \cap P_n = \phi \quad m \neq n.$$

Step III Now we claim that

$$\bigcup_{n \in \mathbb{Z}_+} P_n = [0,1).$$

Since $P_n \subset [0,1)$ $\forall\, n \in \mathbb{Z}_+$

so that

$$\bigcup_{n \in \mathbb{Z}_+} P_n \subseteq [0,1) \quad \text{—— (1)}$$

let $x \in [0,1)$ Then $x \in E_\alpha$ for some α, Since $P \subseteq [0,1)$ contains exactly one element from each equivalence ~~class~~, therefore $\exists\ p \in E_\alpha$, where $p \in P$. Since $x, p \in E_\alpha$

$\therefore$ $x - p$ is a rational number. So

$$x - p \in \{ r_n \mid \forall n \in \mathbb{Z}_+ \}$$

$\therefore$ $x - p = r_n$ for some $n \in \mathbb{Z}_+$

Here we discussed two cases.

(i). If $x \geq p$ then $x - p \geq 0 \in [0,1)$ so $x = p + r_n \in P_n$

(ii) If $x < p$ Then $x - p < 0$ Then $p - x = r_n'$

let $r_m = 1 - r_n'$

So that $x = p - r_n'$

$x = p - (1 - r_m)$

$x = p - r_m + 1 \in P_m$ $\because$ $r_m + 1$ is rational.

$$\Rightarrow x \in \bigcup_{n \in \mathbb{Z}_+} P_n$$

So $[0,1) \subseteq \bigcup_{n \in \mathbb{Z}_+} P_n$ ——— (2)

from (1) & (2)

$$[0,1) = \bigcup_{n \in \mathbb{Z}_+} P_n$$

Suppose that $P \in m_L$ and

taking μ_L (lebesgue measure) on both sides

$$\mu_L([0,1)) = \mu_L\left(\bigcup_{n \in \mathbb{Z}_+} P_n \right)$$

$$\mu_L([0,1)) = \mu_L\left(\bigcup_{n\in Z_+} P_n\right)$$

$$1 = \sum_{n\in Z_+} \mu_L(P_n) \quad \because \ \mu_L[a,b] = b-a$$

and μ_L is countably additive.

$$= \sum_{n\in Z_+} \mu_L(P) \quad \text{—(3)}$$

$P_n = P \dot{+} r_n$

$\mu_L(P \dot{+} r_n) = \mu_L(P)$

$\because \mu_L$ is mod 1 Translation invarient.

Since μ_L is always positive.

$\therefore \quad \mu_L(P) \geq 0$

If $\mu_L(P) = 0$ then (3) reduce to

$1 = 0$ which is contraduction.

If $\mu_L(P) > 0$ then (3) reduce to

$1 = \infty$ which is contraduction.

Hence $P \notin \mathfrak{M}_L$.

$\Rightarrow$ The interval $[0,1) \in \mathfrak{M}_L$ containg a non-Lebesgue measurable set.

—— x —— x —— x ——

Measurable function:

Let $(X, \mathcal{A})$ be a measurable space, $D \in \mathcal{A}$. A function $f: D \to \bar{\mathbb{R}}$ is said to be $\mathcal{A}$-measurable function on D if the set $\{x \in D \mid f(x) < \alpha\} \in \mathcal{A}$ for every real number 'α'.

Equivalently if

$$\{x \in D \mid f(x) \in [-\infty, \alpha)\} \in \mathcal{A}$$

OR

$$f^{-1}([-\infty, \alpha)) \in \mathcal{A}.$$

Lemma Let $(X, \mathcal{A})$ be a measurable space, & $f: D \to \bar{\mathbb{R}}$ be a function defined on set $D \in \mathcal{A}$. Then the following conditions are equivalent

(a) $\{x \in D \mid f(x) \leq \alpha\} = f^{-1}([-\infty, -\alpha]) \in \mathcal{A}, \quad \forall \alpha \in \mathbb{R}$

(b) $\{x \in D \mid f(x) > \alpha\} = f^{-1}((\alpha, \infty]) \in \mathcal{A}, \quad \forall \alpha \in \mathbb{R}$

(c) $\{x \in D \mid f(x) \geq \alpha\} = f^{-1}([\alpha, \infty]) \in \mathcal{A}, \quad \forall \alpha \in \mathbb{R}$

(d) $\{x \in D \mid f(x) < \alpha\} = f^{-1}((\alpha, \infty]) \in \mathcal{A}, \quad \forall \alpha \in \mathbb{R}.$

Proof: (1) (a) $\iff$ (b)

Let $\alpha \in \mathbb{R}$ and $D_1 = \{x \in D \mid f(x) \leq \alpha\}$, $D_2 = \{x \in D \mid f(x) > \alpha\}$ Then clearly $D_1 \cup D_2 = D$ and $D_1 \cap D_2 = \phi$.

If we let $D_1 \in \mathcal{A}$ Then $D_2 \in \mathcal{A}$ $\because$ $D_2 = D \setminus D_1$ & $\mathcal{A}$ is σ-algebra.

& If we let $D_2 \in \mathcal{A}$ Then $D_1 \in \mathcal{A}$.

So (a) $\iff$ (b).

Now we are to show that

(2) (c) $\Leftrightarrow$ (d)

let $\alpha \in \mathbb{R}$ and let $D_1 = \{x \in D \mid f(x) \geq \alpha\}$ and $D_2 = \{x \in D \mid f(x) < \alpha\}$ Then clearly $D_1 \cup D_2 = D$ and $D_1 \cap D_2 = \phi$.

. If we suppose (c) is hold i.e $D_1 \in \mathcal{A}$ Then $D_2 \in \mathcal{A}$ $\because$ $D_2 = D \setminus D_1$ and $\mathcal{A}$ - σ-algebra.

$\Rightarrow$ (d) hold.

Now suppose that (d) hold i.e $D_2 \in \mathcal{A}$ Then $D_1 \in \mathcal{A}$ because $D_1 = D \setminus D_2 \in \mathcal{A}$.

so (c) hold. Hence $c \Leftrightarrow d$.

(3) To show that (d) $\Rightarrow$ (a). Suppose that (d) is true i.e $\{x \in D \mid f(x) < \alpha\} \in \mathcal{A}$ $\forall \alpha \in \mathbb{R}$.

for every $x \in D$ and $\alpha \in \mathbb{R}$ we have

$$f(x) \leq \alpha \Leftrightarrow f(x) < \alpha + \frac{1}{n}, \ \forall n \in N.$$

So $\{x \in D \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in D \mid f(x) < \alpha + \frac{1}{n}\} \in \mathcal{A}$

$\because$ (d) is True and $\mathcal{A}$ is σ-algebra.

i.e $\{x \in D \mid f(x) \leq \alpha\} \in \mathcal{A}$ $\forall \alpha \in \mathbb{R}$

which is (a).

(4) To show that (b) $\Rightarrow$ (c).

Suppose that (b) is true i.e $\{x \in D \mid f(x) > \alpha\} \in \mathcal{A}$.
Then for $x \in D$ and $\alpha \in \mathbb{R}$ we have

$$f(x) \geq \alpha \iff f(x) > \alpha - \frac{1}{n} \quad \forall \ n \in N.$$

so

$$\{x \in D \mid f(x) \geq \alpha\} = \bigcap \{x \in D \mid f(x) > \alpha - \tfrac{1}{n}\} \in \mathcal{A}$$

$\because$ by (b) & $\mathcal{A}$ is σ-algebra.

i.e $\{x \in D \mid f(x) \geq \alpha\} \in \mathcal{A} \quad \forall \ \alpha \in \mathbb{R}$.

$\Rightarrow$ (c) is hold.

So all the conditions (a), (b), (c) and (d) are equivalent.

<u>Question</u> Let $(X, \mathcal{A})$ be a measurable space and a set $D \in \mathcal{A}$. A function $f : D \to \overline{\mathbb{R}}$ is measurable function on D. Then show that

(i) $f^{-1}([c, d)) = \{x \in D \mid c \leq f(x) < d\} \in \mathcal{A}$.

(ii) $f^{-1}((c, d]) = \{x \in D \mid c < f(x) \leq d\} \in \mathcal{A}$.

(iii) $f^{-1}((c, d)) = \{x \in D \mid c < f(x) < d\} \in \mathcal{A}$.

(iv) $f^{-1}([c, d]) = \{x \in D \mid c \leq f(x) \leq d\} \in \mathcal{A}$.

(v) $f^{-1}(\{\infty\}) \in \mathcal{A}$.

(vi) $f^{-1}(\{-\infty\}) \in \mathcal{A}$.

(vii) $f^{-1}(\{c\}) \in \mathcal{A}$.

Proof

(i) Since $f : D \longrightarrow \bar{\mathbb{R}}$ therefore

$$f^{-1}([c,d)) = f^{-1}([c,\infty] \cap [-\infty, d))$$
$$= f^{-1}([c,\infty]) \cap f^{-1}([-\infty,d)) \in \mathcal{A}$$

$\because$ $f^{-1}([c,\infty]) \in \mathcal{A}$

" $f^{-1}([-\infty,d)) \in \mathcal{A}$

by Lemma (previous)

& $\mathcal{A}$ is σ-algebra.

$\Rightarrow f^{-1}([c,d)) \in \mathcal{A}$.

(ii) Since $f : D \rightarrow \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function on $D \in \mathcal{A}$

and $f^{-1}((c,d]) = f^{-1}((c,\infty] \cap [-\infty, d])$

$$= f^{-1}(c,\infty] \cap f^{-1}[-\infty,d] \in \mathcal{A}$$

$\because$ f is $\mathcal{A}$ measureable

& $\mathcal{A}$ is σ-algebra on X.

$\Rightarrow f^{-1}((c,d]) \in \mathcal{A}$.

(iii) Since $f : D \rightarrow \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function on $D \in \mathcal{A}$

and $f^{-1}((c,d)) = f^{-1}((c,\infty] \cap [-\infty, d))$

$$= f^{-1}((c,\infty]) \cap f^{-1}([-\infty,d)) \in \mathcal{A}$$

i.e $f^{-1}((c,d)) \in \mathcal{A}$ $\because$ f is $\mathcal{A}$-measureable function on D and $\mathcal{A}$ σ-algebra on X.

(iv) Since $f : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measural function on $D \in \mathcal{A}$

and $f^{-1}([c, d]) = f^{-1}([c, \infty] \cap [-\infty, d])$

$= f^{-1}([c, \infty]) \cap f^{-1}([-\infty, d]) \in \mathcal{A}$

$\because$ f is $\mathcal{A}$-measureable function

& $\mathcal{A}$ is σ-algebra on X.

$\Rightarrow f^{-1}([c, d]) \in \mathcal{A}$.

(v) $f^{-1}(\{\infty\}) = \{x \in D \mid f(x) = \infty\}$

$= \bigcap_{k=1}^{\infty} \{x \in D \mid f(x) > k,\ k \in \mathbb{R}\} \in \mathcal{A}$ $\because$ f is $\mathcal{A}$-measureble & $\mathcal{A}$-σ-algebra.

$\Rightarrow f^{-1}(\{\infty\}) \in \mathcal{A}$.

(vi) Since $f^{-1}(\{-\infty\}) = \{x \in D \mid f(x) = -\infty\}$

$= \bigcap_{k=1}^{\infty} \{x \in D \mid f(x) < -k,\ k \in \mathbb{R}\} \in \mathcal{A}$

$\because$ f is $\mathcal{A}$-measurable & $\mathcal{A}$-σ-algebra.

(vii) Since $f : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function

$\therefore$ $f^{-1}(\{c\}) = \{x \in D \mid f(x) = c\}$

$= \{x \in D \mid f(x) \geq c\} \cap \{x \in D \mid f(x) \leq c\} \in \mathcal{A}$

$\because$ f is $\mathcal{A}$-measurable function & $\mathcal{A}$-σ-algebra on X

$\Rightarrow f^{-1}(\{c\}) \in \mathcal{A}$.

Question

(1) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras on X s.t $\mathcal{A}_1 \subseteq \mathcal{A}_2$ then every $\mathcal{A}_1$-measureable function is $\mathcal{A}_2$-measureable.

Proof let $D \in \mathcal{A}_1$ then $D \in \mathcal{A}_2$ $\because \mathcal{A}_1 \subseteq \mathcal{A}_2$.

let $f : D \to \overline{\mathbb{R}}$ is $\mathcal{A}_1$-measureable.

The by def of measureable function

$$\{x \in D \mid f(x) < \alpha, \forall \alpha \in \mathbb{R}\} \in \mathcal{A}_1$$

$$\Rightarrow \{x \in D \mid f(x) < \alpha, \forall \alpha \in \mathbb{R}\} \in \mathcal{A}_2 \quad \because \mathcal{A}_1 \subseteq \mathcal{A}_2.$$

so f is $\mathcal{A}_2$-measureable. which required result.

---

(2) If $\mathcal{A} = \{\phi, X\}$ is the smallest σ-algebra on X. Then $f : X \to \overline{\mathbb{R}}$ is $\mathcal{A}$-measureable $\Leftrightarrow$ f is constant function.

Proof Suppose that $f : X \to \overline{\mathbb{R}}$ is $\mathcal{A}$-measureable. Then by definition of $\mathcal{A}$-measureable function the set $\{x \in X \mid f(x) < \alpha, \alpha \in \mathbb{R}\} \in \mathcal{A}$.

Case I If

$$\{x \in X \mid f(x) < \alpha\} = \phi \text{ then}$$

$$f(x) = c \geq \alpha \quad \forall x \in X$$

$\Rightarrow f$ is constant.

Case II

$$\text{If } \{x \in X \mid f(x) < \alpha\} = X.$$

$$\Rightarrow f(x) = d < \alpha \quad \forall x \in X$$

$\Rightarrow f$ is constant.

Conversely suppose that f is constant we are we are to show that f is $\mathcal{A}$- measureable. when f is constant then $f(x) = c \ \forall \ x \in X$. let $\alpha \in \mathbb{R}$ then

$$\{x \in X \mid f(x) < \alpha\} = \begin{cases} X, & \text{if } c < \alpha \\ \phi, & \text{if } c \geq \alpha \end{cases}$$

In each case $\{x \in X \mid f(x) < \alpha\} \in \mathcal{A} \ \because \mathcal{A} = \{\phi, X\}$
so f is $\mathcal{A}$- measureable function.

(3) Prove that every $f : X \to \overline{\mathbb{R}}$ is $\mathcal{A}$ - measurable function on X if $\mathcal{A} = P(X)$,

<u>Proof</u>: for $\alpha \in \mathbb{R}$, every subset of X i.e

$$\{x \in D \mid f(x) < \alpha\} \in P(X)$$

so f is $\mathcal{A}$ - measureable function.

<u>Characterestic function</u>:

let $X \neq \phi$ be non-empty set and $E \subseteq X$ A function $\chi_E : X \to \{0, 1\}$ defined as

$$\chi_E(x) = \begin{cases} 0 ; & \text{if } x \notin E \\ 1 ; & \text{if } x \in E. \end{cases}$$

<u>Note</u>: In Measure Theory χ_E is replaced by 1_E.

Question Let $(X, \mathcal{A})$ be a measureable space and $E \subseteq X$. Then characteristic function 1_E is $\mathcal{A}$-measureable function iff $E \in \mathcal{A}$.

Proof:

Suppose $E \in \mathcal{A}$ we are to show that 1_E is $\mathcal{A}$-measureable function. Let $\alpha \in \mathbb{R}$ be fixed. Then

$$\{x \in X \mid 1_E(x) \leq \alpha\} = \begin{cases} \phi & ; \text{ if } \alpha < 0 \\ E^c & ; \ 0 \leq \alpha < 1 \\ X & ; \ \alpha \geq 1 \end{cases}$$

In each case the set

$$\{x \in X \mid 1_E(x) \leq \alpha\} \in \mathcal{A}.$$

So

1_E is $\mathcal{A}$-measureable function.

Question Let $(\mathbb{R}, m_L)$ be Lebesgue measureable space and G be an open subset of $\mathbb{R}$ and $f : D \to \overline{\mathbb{R}}$ is m_L-measureable function on $D \in m_L$. Then show that $f^{-1}(G) \in m_L$.

Solution: Since G is open subset of $\mathbb{R}$ Therefore $\exists$ disjoint collection of open interval in $\mathbb{R}$ s.t

$G = \bigcup_{n=1}^{\infty} I_n$

$$\Rightarrow \quad f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$= \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in m_L \quad \because \ f \text{ is } m_L\text{-measurable and } m_L \ \sigma\text{-algebra on } \mathbb{R}.$$

So $f^{-1}(G) \in m_L$.

Proposition:

Let $(X, \mathcal{A})$ be a measureable space &
$f: D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function on $D \in \mathcal{A}$
Then $\{x \in D \mid f(x) = \alpha\} \in \mathcal{A} \quad \forall \alpha \in \bar{\mathbb{R}}$.

Proof for $\alpha \in \mathbb{R}$ the

set $\{x \in D \mid f(x) = \alpha\} = f^{-1}(\{\alpha\}) \in \mathcal{A}$

$\because f^{-1}(\{c\}) \in \mathcal{A}$
$\forall c \in \mathbb{R}$

So $\{x \in D \mid f(x) = \alpha\} \in \mathcal{A}$.

If $\alpha = \infty$ then the set

$\{x \in D \mid f(x) = \infty\} = f^{-1}(\{\infty\}) \in \mathcal{A}$

$\Rightarrow \{x \in D \mid f(x) = \infty\} \in \mathcal{A}$

If $\alpha = -\infty$ then the set $\{x \in D \mid f(x) = -\infty\} = f^{-1}(\{-\infty\}) \in \mathcal{A}$

so

$\{x \in D \mid f(x) = -\infty\} \in \mathcal{A}$.

Hence

$\{x \in D \mid f(x) = \alpha\} \in \mathcal{A} \quad \forall \alpha \in \bar{\mathbb{R}}$.

Result: Prove that a function $f: D \to \bar{\mathbb{R}}$ on $D \in \mathcal{A}$ satisfying $\{x \in D \mid f(x) = \alpha\} \in \mathcal{A} \ \forall \alpha \in \bar{\mathbb{R}}$ need not be $\mathcal{A}$-measureable.

Proof:

Consider $(\mathbb{R}, m_L)$ Lebesgue measure space.
Since the interval $[0,1)$ containing non-Lebesgue measureable sub set call it $P \subseteq [0,1)$.

let $f:[0,1) \longrightarrow \{x, -x\}$ defined by

$$f(x) = \begin{cases} x & ; \text{ If } x \in P \\ -x & ; \text{ If } x \in [0,1) \setminus P. \end{cases}$$

Then $\forall\ \alpha \in \bar{\mathbb{R}}$ the set $\{x \in [0,1) \mid f(x) = \alpha\}$ is either singleton or empty set. In each case it is member of m_L. But If we choose $\alpha = 0$ then $\{x \in [0,1) \mid f(x) \geq 0\} = P \notin m_L$. so that f is not m_L-measureable.

Theorem:

let $(X, \mathcal{A})$ be measureable space

(1) If $f: D \to \bar{\mathbb{R}}$ is e.r.v measureable function define on set $D \in \mathcal{A}$ Then for every $D_0 \subseteq D$ s.t $D_0 \in \mathcal{A}$, the restriction of f on D_0 is $\mathcal{A}$-measureable.

Proof:

let $f: D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function and $\alpha \in \mathbb{R}$ Then $\{x \in D \mid f(x) \leq \alpha\} \in \mathcal{A}$.

Consider the set

$$\{x \in D_0 \mid f(x) \leq \alpha\} = \{x \in D \mid f(x) \leq \alpha\} \cap D_0 \in \mathcal{A}$$

$\because$ f is $\mathcal{A}$-measureable & $\mathcal{A}$ is σ-algebra.

(2) Let $(X, \mathcal{A})$ be measureable space and $f: D_i \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function on $D_i \in \mathcal{A}$. If $\{D_i\}_{i=1}^{\infty}$ is sequence in $\mathcal{A}$ s.t $\bigcup_{i=1}^{\infty} D_i = D$ Then $f: D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function.

Proof

for $\alpha \in \mathbb{R}$ consider the set

$$\{x \in D \mid f(x) \leq \alpha\} = \{x \in \bigcup_{i=1}^{\infty} D_i \mid f(x) \leq \alpha\}$$

$$= \bigcup_{i=1}^{\infty} \{x \in D_i \mid f(x) \leq \alpha\} \in \mathcal{A}$$

$\because$ f is $\mathcal{A}$-measureable on D_i $\forall i=1,2,\ldots$ and $\mathcal{A}$ is σ-algebra on X.

Proposition:

Let $(X, \mathcal{A})$ be measureable space then prove that constant function $f: D \to \bar{\mathbb{R}}$ defined on $D \in \mathcal{A}$ is $\mathcal{A}$-measureable.

Proof:

Let $f: D \to \bar{\mathbb{R}}$ is defined $f(x) = c$ $\forall x \in D$.

Let $\alpha \in \mathbb{R}$, consider the set

$$\{x \in D \mid f(x) \leq \alpha\} = \begin{cases} D & ; \text{ if } c \leq \alpha \\ \phi & ; \text{ if } c > \alpha \end{cases}$$

In both cases the set

$\{x \in D \mid f(x) \leq \alpha\} \in \mathcal{A}$. $\Rightarrow$ f is $\mathcal{A}$-measureable function.

Theorem: let $(X, \mathcal{A})$ be measureable space and $f: D \to \bar{\mathbb{R}}$ and $g: D \to \bar{\mathbb{R}}$, $D \in \mathcal{A}$ are $\mathcal{A}$-measureable functions on D Then Prove that

(i) $f+c : D \to \bar{\mathbb{R}}$ defined as $f+c\,(x) = f(x)+c$ where c is any real number is $\mathcal{A}$-measureable function on D.

(ii) $cf : D \to \bar{\mathbb{R}}$ defined as $cf\,(x) = c \cdot f(x)$ is $\mathcal{A}$-measureable function on D.

(iii) $f+g : D \to \bar{\mathbb{R}}$ defined as $f+g\,(x) = f(x) + g(x)$ is $\mathcal{A}$-measureable function on D.

(iv) $f-g : D \to \bar{\mathbb{R}}$ defined as $f-g\,(x) = f(x) - g(x)$ is $\mathcal{A}$-measureable function on D.

(v) $fg : D \to \bar{\mathbb{R}}$ defined as $fg\,(x) = f(g(x))$ is $\mathcal{A}$-measureable function on D.

(vi) $f^2 : D \to \bar{\mathbb{R}}$ defined as $f^2(x) = f(f(x))$ is $\mathcal{A}$-measureable function on D.

(vii) $f/g : D \to \bar{\mathbb{R}}$ defined as $f/g\,(x) = \frac{f(x)}{g(x)}$ $(g \neq 0)$ is $\mathcal{A}$-measureable function.

Proof: (i) let $\alpha \in \mathbb{R}$ Then

$$\{x \in D \mid f+c\,(x) \leq \alpha\} = \{x \in D \mid f(x)+c \leq \alpha\}$$

$$= \{x \in D \mid f(x) \leq \alpha - c\}$$

$$= \{x \in D \mid f(x) \leq \beta\} \in \mathcal{A}$$ where $\alpha - c = \beta \in \mathbb{R}$

$\because f$ is $\mathcal{A}$-measureable function.

So $\{x \in D \mid f+c\,(x) \leq \alpha\} \in \mathcal{A}$

$\Rightarrow f+c$ is $\mathcal{A}$-measureable function.

(2) Now we are to show that $cf : D \to \bar{\mathbb{R}}$ defined as $cf(x) = c \cdot f(x) \quad \forall x \in D$. is $\mathcal{A}$-measureable function. Here we discusse the following cases of 'c' $\in \mathbb{R}$ i.e

If $\underline{\underline{c = 0}}$ Then $cf(x) = 0 \quad \forall \ x \in D$

$\Rightarrow$ cf is constant function.

So cf is $\mathcal{A}$-measureable function.

because " Every constant function is $\mathcal{A}$-measureable function ".

If $\underline{\underline{c > 0}}$ Then for $\alpha \in \mathbb{R}$ we have

$$\{x \in D \mid cf(x) \geq \alpha\}$$
$$= \{x \in D \mid c \cdot f(x) \geq \alpha\}$$
$$= \{x \in D \mid f(x) \leq \tfrac{\alpha}{c}\}$$
$$= \{x \in D \mid f(x) \leq \beta\} \in \mathcal{A} \quad \text{where } \tfrac{\alpha}{c} = \beta \in \mathbb{R}$$

$\because f$ is $\mathcal{A}$-measureable

If $c < 0$ Then $\alpha \in \mathbb{R}$, The set

$$\{x \in D / cf(x) \leq \alpha\} = \{x \in D / c \cdot f(x) \leq \alpha\}$$
$$= \{x \in D / f(x) \geq \tfrac{\alpha}{c}\}$$
$$= \{x \in D / f(x) \geq \beta\} \in \mathcal{A}$$

$\because f$ is $\mathcal{A}$-measureable function and $\frac{\alpha}{c} = \beta$.

Hence $cf : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function. and particularly '$-f$' is $\mathcal{A}$-measureable function on D. for $c = -1 \in \mathbb{R}$.

(3) Proof:- Now we are to show that $f+g : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function equivalently we are to show that the set $\{x \in D / f+g\,(x) > \alpha\} \in \mathcal{A}$.

Consider the set

$$\{x \in D \mid (f+g)x > \alpha\} = \{x \in D \mid f(x) + g(x) > \alpha\}$$
$$= \{x \in D \mid f(x) > \alpha - g(x)\}$$

Since $f(x)$ & $\alpha - g(x) \in \mathbb{R}$ & set of rational number $\mathbb{Q}$ is dense in $\mathbb{R}$.

$\therefore \quad f(x) > r > \alpha - g(x)$ where $r \in \mathbb{Q}$.

we claim that

$$\{x \in D \mid (f+g)(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} \left[\{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) < r\}\right]$$

let $y \in \{x \in D \mid (f+g)(x) > \alpha\}$

then $(f+g)(y) > \alpha \Rightarrow f(y) + g(y) > \alpha$

$\Rightarrow f(y) > \alpha - g(y)$

$\therefore \qquad \Rightarrow f(y) > r > \alpha - g(y), \ r \in \mathbb{Q}$

$y \in [\{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) < r\}$

$\Rightarrow y \in \bigcup_{r \in \mathbb{Q}} [\{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) < r\}]$

so $\{x \in D \mid (f+g)(x) > \alpha\} \subseteq \bigcup_{r \in \mathbb{Q}} [\{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) > r\}]$ ①

conversely suppose that

$y \in \bigcup_{r \in \mathbb{Q}} [\{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) < r\}]$

$\Rightarrow y \in \{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) < r\}$, for some $r \in \mathbb{Q}$.

$\Rightarrow \quad f(y) > r > \alpha - g(y)$

$f(y) > \alpha - g(y)$

$f(y) > \alpha - g(y)$

$\Rightarrow f(y) + g(y) > \alpha$

$\Rightarrow (f+g)(y) > \alpha$

so $y \in \{x \in D \mid (f+g)(x) > \alpha\}$

&

$$\bigcup_{r \in Q} \left[\{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) < r\} \right] = \{x \in D \mid f+g(x) > \alpha\}$$

Since f and g are $\mathcal{A}$-measureable functions on D

these $\{x \in D \mid f(x) > r\} \in \mathcal{A}$ and $\{x \in D \mid \alpha - g(x) < r\} \in \mathcal{A}$

$\Rightarrow \{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) < r\} \in \mathcal{A}$

$\Rightarrow \bigcup_{r \in Q} \left[\{x \in D \mid f(x) > r\} \cap \{x \in D \mid \alpha - g(x) < r\} \right] \in \mathcal{A}$ $\because$ $\mathcal{A}$ is σ-algebra on X.

Hence $\{x \in D \mid f+g(x) > \alpha\} \in \mathcal{A}$

So $f+g$ is $\mathcal{A}$-measureable function on $D \in \mathcal{A}$.

(4) Proof: Since g is $\mathcal{A}$-measureable function on D.

$\therefore$ $-g$ is $\mathcal{A}$-measurable function on D.

also f is $\mathcal{A}$-measureable function on D. So by part (3) $f + (-g) = f - g$ is $\mathcal{A}$-measureable function.

(5) Let $f^2 : D \to \bar{\mathbb{R}}$ is e.r.v function defined on D

s.t $f^2(x) = [f(x)]^2 \quad \forall x \in D$.

We are to show that $f^2 : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function. Consider the set

$\{x \in D \mid f^2(x) > \alpha\}$.

If $\alpha \in \mathbb{R}$ s.t $\alpha < 0$ Then

$\{x \in D \mid f^2(x) > \alpha\} = D \in \mathcal{A}$. $\because f$ is $\mathcal{A}$-measureable.

Now if $\alpha > 0$ Then

$$\{x \in D \mid f^2(x) \le \alpha\} = \{x \in D \mid [f(x)]^2 \le \alpha\}$$
$$= \{x \in D \mid f(x) \le \pm\sqrt{\alpha}\}$$
$$= \{x \in D \mid f(x) \le \sqrt{\alpha}\} \cup \{x \in D \mid f(x) \ge -\sqrt{\alpha}\} \in \mathcal{A}$$

$\because \mathcal{A}$ is σ-algebra & f is $\mathcal{A}$-measureable function.

So $\{x \in D \mid f^2(x) \le \alpha\} \in \mathcal{A}$.

Hence $f^2 : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function.

(6) Since

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right).$$

also $f, g, f^2, g^2, f+g, f-g$ are $\mathcal{A}$-measureable functions. Therefore

fg is $\mathcal{A}$-measureable function on D.

(7) Proof: First we show If $g : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function on D Then $1/g : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measurable function.

consider the set $\{x \in D \mid (\frac{1}{g})(x) > \alpha\}$ and discusse it under the following assumptions on $\alpha \in \mathbb{R}$.

First If $\alpha = 0$ Then the set

$$\{x \mid x \in D : (\tfrac{1}{g})(x) > 0\} = \{x \in D \mid \tfrac{1}{g(x)} > 0\}$$
$$= \{x \in D \mid g(x) > 0\} \in \mathcal{A}$$

$\because g$ is $\mathcal{A}$-measurable function.

2nd If $\alpha > 0$ Then the set

$$\{x \in D \mid \tfrac{1}{g}(x) > \alpha\} = \{x \in D \,\tfrac{1}{g(x)} > \alpha\}$$
$$= \{x \in D \mid g(x) < 1/\alpha\}$$

$$= \{x \in D \mid g(x) < \beta\} \in \mathcal{A}$$ $\because g$ is $\mathcal{A}$-measureable function and we take $\frac{1}{\alpha} = \beta \in \mathbb{R}$.

3rd if $\alpha < 0$ Then the set

$$\{x \in D \mid \tfrac{1}{g}(x) > \alpha\} = \{x \in D \mid \tfrac{1}{g(x)} > \alpha\}$$
$$= \{x \in D \mid \tfrac{1}{g(x)} > \alpha,\ g(x) > 0\} \cup \{x \in D \mid \tfrac{1}{g(x)} > \alpha,\ g(x) < 0\}$$

Since g is $\mathcal{A}$-measureable function and $\mathcal{A}$-is σ-algebra on X. Therefore

$\{x \in D \mid \frac{1}{g}(x) > \alpha\} \in \mathcal{A} \Rightarrow \frac{1}{g}$ is $\mathcal{A}$-measureable functions.

So in each case $\frac{1}{g} : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function on D.

Now we are to show $\frac{f}{g} : D \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measureable function on D. By (6) Part of theorem i.e " If f & g are measureable function then $f \cdot g$ is $\mathcal{A}$-measureable function"

Therefore

$\frac{f}{g}$ is $\mathcal{A}$-measureable function on D. $\because$ f and $\frac{1}{g}$ are $\mathcal{A}$-measureable function.

Almost every where Property:

Let $(X, \mathcal{A}, \mu)$ be a measured Space. A property 'P' holds almost every where in X iff $\exists$ a set $N \in \mathcal{A}$ s.t $\mu(N) = 0$ (null set) and 'P' is hold for all $x \in X \setminus N$.

Equal almost every where ($f = g$ a.e)

Let $(X, \mathcal{A}, \mu)$ be measure space and $f : D \to \bar{\mathbb{R}}$ and $g : D \to \bar{\mathbb{R}}$, are $\mathcal{A}$-measureable function on $D \in \mathcal{A}$ are said to be equal almost every where on D i.e

$f = g$ a.e on D if $f(x) = g(x)$ $\forall x \in D \setminus N$ where $\mu(N) = 0$ i.e N is null set.

Observation:

Let $(X, \mathcal{A}, \mu)$ be complete measure space

Then

(1) Every function $f : N \to \overline{\mathbb{R}}$ where $\mu(N) = 0$ i.e N is null set is $\mathcal{A}$-measureable function on N.

Proof: Let $\alpha \in \mathbb{R}$ and consider the set
$\{x \in N \mid f(x) \leq \alpha\} \subseteq N$.

Since $(X, \mathcal{A}, \mu)$ is complete measure space & N is null set. Therefore the set $\{x \in N \mid f(x) \leq \alpha\} \in \mathcal{A}$.

Hence f is $\mathcal{A}$-measureable function.

(2) If $f : D \to \overline{\mathbb{R}}$ and $g : D \to \overline{\mathbb{R}}$, $D \in \mathcal{A}$ s.t $f = g$ a.e on D and if f is $\mathcal{A}$-measureable function on D Then g is also $\mathcal{A}$-measureable function on D.

Proof:- Since $f = g$ a.e on D
$\therefore \exists$ a null set N s.t
$f(x) = g(x) \quad \forall x \in D \setminus N$.

Since f is $\mathcal{A}$-measureable function D Then f is f is $\mathcal{A}$-measureable function on $D \setminus N$ $\because D \setminus N \subseteq D$.

As $f = g$ on $D \setminus N$ & f is $\mathcal{A}$-measureable on $D \setminus N$.

So g is $\mathcal{A}$-measureable. By First part g is $\mathcal{A}$-measureable function on $N \subseteq D$. So g is $\mathcal{A}$-measureable function on $D \setminus N \cup N = D$. $\because$ "If f is $\mathcal{A}$-measureable on $D_1, D_2, \ldots, D_n$ Then f is $\mathcal{A}$-measureable on $\bigcup_{i=1}^{n} D_i$".

Limit inferior & Limit superior of real values sequence.

Let (x_n) be real values sequence we define two new sequences $\{\underline{x}_n\}$ and $\{\overline{x}_n\}$ s.t

$\underline{x}_n = \inf\{x_1, x_2, \cdots\}$ and $\overline{x}_n = \sup\{x_1, x_2, x_3 \cdots\}$

$\underline{x}_n = \inf\limits_{k \geq n} \{x_k\}$ and $\overline{x}_n = \sup\limits_{k \geq n} \{x_k\}$

clearly $(\underline{x}_n)$ is increasing sequence i.e $\underline{x}_n \leq \underline{x}_{n+1}\ \forall n \in \mathbb{N}$ and $(\overline{x}_n)$ is decreasing sequence i.e $\overline{x}_n \geq \overline{x}_{n+1}\ \forall n \in \mathbb{N}$

$\therefore$ $\lim\limits_{n \to \infty} \underline{x}_n$ and $\lim\limits_{n \to \infty} \overline{x}_n$ exist in $\overline{\mathbb{R}}$. Then we define lim inf x_n as

$$\boxed{\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \underline{x}_n = \lim_{n \to \infty} \inf_{k \geq n} \{x_k\}.}$$

Similarly lim sup x_n is defined as

$$\boxed{\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \overline{x}_n = \lim_{n \to \infty} \sup_{k \geq n} \{x_k\}}$$

If $\lim\limits_{n \to \infty} \inf x_n = \lim\limits_{n \to \infty} \sup x_n$ then limit of the sequence (x_n) i.e $\lim\limits_{n \to \infty} x_n$ exist &

$$\boxed{\lim_{n \to \infty} \inf x_n = \lim_{n \to \infty} \sup x_n = \lim_{n \to \infty} x_n}.$$

Sequence of $\mathcal{A}$-measureable functions & its limits & their properties.

let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{A}$-measureable functions defime on set $D \in \mathcal{A}$. & its limit is denoted as $\lim_{n\to\infty} f_n$.

The functions " $\min_{n=1,2,\dots,N} f_n$, $\max_{n=1,2,\dots,N} f_n$, $\lim_{n\to\infty} \operatorname{Inf} f_n$

$\lim_{n\to\infty} \operatorname{Sup} f_n$, $\lim_{n\to\infty} f_n$, $\inf_{n\in N} f_n$ and $\sup_{n\in N} f_n$"

have the following properties

(1) $\left(\min_{n=1,2,\dots,N} f_n\right)(x) = \min_{n=1,2,\dots,N} \left(f_n(x)\right)$

(2) $\left(\max_{n=1,2,\dots,N} f_n\right)(x) = \max_{n=1,2,\dots,N} \left(f_n(x)\right)$

(3) $\left(\lim_{n\to\infty} \operatorname{Inf} f_n\right)(x) = \lim_{n\to\infty} \operatorname{Inf} \left(f_n(x)\right)$

(4) $\left(\lim_{n\to\infty} \operatorname{Sup} f_n\right)(x) = \lim_{n\to\infty} \operatorname{Sup} \left(f_n(x)\right)$

(5) $\left(\lim_{n\to\infty} f_n\right)(x) = \lim_{n\to\infty} f_n(x)$.

(6) $\left(\operatorname{Inf}_{n\in N} f_n\right)(x) = \operatorname{Inf}_{n\in N} f_n(x)$

(7) $\left(\sup_{n\in N} f_n\right)(x) = \sup_{n\in N} \left(f_n(x)\right)$.

Theorem: Let $(X, \mathcal{A})$ be a measureable space & $\{f_n\}_{n=1}^{\infty}$ be a monotone sequence of e.r.v $\mathcal{A}$-measureable functions defined $D \in \mathcal{A}$. Then $\lim_{n\to\infty} f_n$ exists on D & $\lim_{n\to\infty} f_n$ is $\mathcal{A}$-measureable function on D.

Proof: Since $\{f_n\}$ is monotone sequence on D. Therefore $\{f_n(x)\}$ is monoton sequence in $\bar{\mathbb{R}}$. So that $\lim_{n\to\infty} f_n(x)$ exist in $\bar{\mathbb{R}}$ $\forall\, x \in D$. Hence $\lim_{n\to\infty} f_n$ exist on D.

Now we are to show that $\lim_{n\to\infty} f_n = f$ (say) is $\mathcal{A}$-measureable function on D. If $\{f_n\}\uparrow$ then for $\alpha \in \mathbb{R}$, consider the set

$$\{x \in D \mid (\lim_{n\to\infty} f_n)(x) > \alpha\} = \{x \in D \mid \lim_{n\to\infty} f_n(x) > \alpha\}$$

$$\lim_{n\to\infty} f_n(x) > \alpha \iff f_n(x) > \alpha, \text{ for some } n.$$

So

$$\{x \in D \mid (\lim_{n\to\infty} f_n)(x) > \alpha\} = \bigcup_{n \in \mathbb{N}} \{x \in D \mid f_n(x) > \alpha\} \in \mathcal{A}$$

$\because$ $\mathcal{A}$ - σ-algebra on X and $\{E_n\}\uparrow$ then $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$

If $f_n \downarrow$ then $-f_n \uparrow$ so that $\lim_{n\to\infty}(-f_n)$ is $\mathcal{A}$-measureable function on D then $-\lim_{n\to\infty} f_n$ is $\mathcal{A}$-measureable so that $\lim_{n\to\infty} f_n$ is $\mathcal{A}$-measureable function. //

Theorem: Let $(X, \mathcal{A})$ be a measureable space and Let $\{f_n\}_{n=1}^{\infty}$ be sequence of e.r.v $\mathcal{A}$-measureable functions on a set $D \in \mathcal{A}$, then the functions

(1) $\min_{n=1,2,\dots,N} f_n$ (2) $\max_{n=1,2,\dots,N} f_n$ (3) $\inf_{n\in N} f_n$

(4) $\sup_{n\in N} f_n$ (5) $\lim_{n\to\infty} \inf f_n$ (6) $\lim_{n\to\infty} \sup f_n$

are $\mathcal{A}$-measureable function.

Proof: (1)

Let $\alpha \in \mathbb{R}$ and $x \in D$ Then

$$\min_{n=1,2,\dots,N} \{f_n(x)\} < \alpha \iff f_n(x) < \alpha \text{ for some } n = 1,2,\dots,N.$$

So we have

$$\{x\in D \mid (\min_{n=1,2,\dots,N} f_n)(x) < \alpha\} = \{x\in D \mid \min_{n=1,\dots,N} f_n(x) < \alpha\}$$

$$= \bigcup_{n=1}^{N} \{x\in D \mid f_n(x) < \alpha\} \in \mathcal{A} \quad \because$$

"$\mathcal{A}$ - σ - algebra & each f_n is $\mathcal{A}$-measureable function"

$\Rightarrow \min_{n=1,2,\dots,N} f_n$ is $\mathcal{A}$ - measureable function.

(2) Let $\alpha \in \mathbb{R}$ and $x \in D$ Then

$$\max_{n=1,2,\cdots,N} f_n(x) > \alpha \iff f_n(x) > \alpha \text{ for some } n = 1,2,\cdots,N.$$

$$\Rightarrow \{x \in D \mid (\max_{n=1,2,\cdots,N} f_n)(x) > \alpha\} = \bigcup_{n=1}^{N} \{x \in D \mid f_n(x) > \alpha\} \in \mathcal{A}$$

$\because$ " $\mathcal{A}$ is σ-algebra & each f_n is $\mathcal{A}$-measureable function "

$\Rightarrow \max_{n=1,2,\cdots,N} f_n$ is $\mathcal{A}$-measureable function

(3) Let $\alpha \in \mathbb{R}$ and $x \in D$ we have

$$\inf_{n \in \mathbb{N}} f_n(x) < \alpha \iff f_n(x) < \alpha \text{ for some } n \in \mathbb{N}.$$

$$\Rightarrow \{x \in D \mid (\inf_{n \in \mathbb{N}} f_n)(x) < \alpha\} = \bigcup_{n=1}^{\infty} \{x \in D \mid f_n(x) < \alpha\} \in \mathcal{A}.$$

$\because$ $\mathcal{A}$ is σ-algebra on X & each f_n is $\mathcal{A}$-measureable.

so

$$\{x \in D \mid (\inf_{n \in \mathbb{N}} f_n)(x) < \alpha\} \in \mathcal{A}$$

$\Rightarrow \inf_{n \in \mathbb{N}} f_n$ is $\mathcal{A}$-measureable function on D.

(4) let $\alpha \in \mathbb{R}$, $x \in D$. Then

$$\sup_{n\in N} f_n(x) > \alpha \iff f_n(x) > \alpha, \text{ for some } n \in N.$$

$$\Rightarrow \left\{x \in D \,\middle|\, \left(\sup_{n\in N} f_n\right)(x) > \alpha\right\} = \bigcup_{n=1}^{N} \left\{x \in D \,\middle|\, f_n(x) > \alpha\right\} \in \mathcal{A}$$

$\because$ $\mathcal{A}$ - σ-algebra on X & f_n is $\mathcal{A}$-measureable

$$\Rightarrow \left\{x \in D \,\middle|\, \left(\sup_{n\in N} f_n\right)(x) > \alpha\right\} \in \mathcal{A}$$

Hence $\sup\limits_{n\in N} f_n$ is $\mathcal{A}$-measureable function.

(5) we know that

$$\lim_{n\to\infty} \inf f_n = \lim_{n\to\infty} \inf_{k\geq n} \{f_n\}$$

where $\{\inf\limits_{k\geq n} \{f_n\}\}$ is increasing. By result (4)

$\inf\limits_{k\geq n} \{f_k\}$ is $\mathcal{A}$-measureable function $\forall\, n \in N$.

so $\lim\limits_{n\to\infty} \inf\limits_{k\geq n}\{f_n\} = \lim\limits_{n\to\infty} \inf f_n$ is $\mathcal{A}$-measureable function.

Larger & Smaller of two function :

let $(X, \mathcal{A})$ be measure space and $f: D \to \bar{\mathbb{R}}$ and $g: D \to \bar{\mathbb{R}}$ be two e.r.v functions.

Smaller of "f" & "g" is defined as

$$f \wedge g = \min[f, g] \quad \text{i.e} \quad f \wedge g\,(x) = \min[f(x), g(x)].$$

larger of 'f' & 'g' is defined as

$$f \vee g = \max[f, g] \quad \text{i.e} \quad f \vee g\,(x) = \max[f(x), g(x)].$$

+ve Part of f i.e f^+ :

let $f: D \to \bar{\mathbb{R}}$ is e.r.v function its +ve part f^+ is defined as

$$f^+(x) = (f \vee 0)(x) = \max\{f(x), 0\}$$

-ve Part of f i.e f^- :

let $f: D \to \bar{\mathbb{R}}$ is e.r.v function its -ve part (f^-) is defined as

$$f^-(x) = -f \wedge 0\,(x) = -\min\{f(x), 0\}$$

Absolut function of f i.e $|f|$:

let $f: D \to \bar{\mathbb{R}}$ is e.r.v function on D its absolut function "$|f|$" is defined as $|f| = |f(x)| \geq 0$

Proposition:

Let $f: D \to \bar{\mathbb{R}}$ be e.r.v function, $D \in \mathcal{A}$ is $\mathcal{A}$-measureable function Then f^+, $\bar{f}$ and $|f|$ are $\mathcal{A}$-measureable functions.

Proof:

Since $f^+ = f \vee 0 = \max[f, 0]$ and f and 0 are $\mathcal{A}$-measureable functions on D.

$\therefore$ f^+ is $\mathcal{A}$-measureable on D.

Now we are to show that '$\bar{f}$' is $\mathcal{A}$-measureable.

Since $\bar{f} = -f \wedge 0 = -\min[f, 0]$

$\Rightarrow \bar{f}$ is $\mathcal{A}$-measureable function on D $\because$ f & 0 are $\mathcal{A}$-measureable function. & $\min[f, 0]$ is $\mathcal{A}$-measureable.

Now $|f| = f^+ + \bar{f}$

Since f^+ and $\bar{f}$ are $\mathcal{A}$-measureable functions there $|f|$ is $\mathcal{A}$-measureable

$\because$ f, g are $\mathcal{A}$-measureable Then "$f + g$" is $\mathcal{A}$-measureable

Limit existence almost everywhere:

Let $(X, \mathcal{A})$ be a measure space and $\{f_n\}_{n=1}^{\infty}$ be sequence of e.r.v $\mathcal{A}$-measureable functions defined on a set D. $\lim_{n\to\infty} f_n$ exists a.e on D if $\exists$ a null set N s.t $\lim_{n\to\infty} f_n$ exist on $D\setminus N$..

Equivalently the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges a.e on D if $\{f_n(x)\}$ converges on $D\setminus N$ where $\mu(N) = 0$.

Note:

The convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ depends on the convergence $\{f_n(x)\}_{n=1}^{\infty}$ for $x \in D$.

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measureable space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of e.r.v $\mathcal{A}$-measureable functions on D. If for every $\eta > 0$ $\exists$ an $\mathcal{A}$-measurable sub set E of D with $\mu(E) < \eta$ s.t $\lim_{n\to\infty} f_n(x)$ exists $\forall\ x \in D\setminus E$ then $\lim_{n\to\infty} f_n$ exist a.e on D.

Proof Let $\forall\ n \in \mathbb{N}$ $\exists$ an $\mathcal{A}$-measurable subset $E_n \subseteq D$ s.t $\mu(E_n) < \frac{1}{n}$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x)$ exist $\forall\ x \in D\setminus E_n$. We are to prove that $\lim_{n\to\infty} f_n$ exists a.e on D.

Define $N = \bigcap_{n=1}^{\infty} E_n$ Then $N \subseteq D$

so that

$$\mu(N) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \leq \mu(E_n) < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

i.e $\mu(N) = 0$ so N is null set.

Now

$$\begin{aligned} D \setminus N &= D \cap N^c \\ &= D \cap \left(\bigcap_{n=1}^{\infty} E_n\right)^c \\ &= D \cap \left(\bigcup_{n=1}^{\infty} E_n^c\right) \\ &= \bigcup_{n=1}^{\infty} (D \cap E_n^c) \\ &= \bigcup_{n=1}^{\infty} D \setminus E_n \end{aligned}$$

$\Rightarrow \quad x \in D \setminus N \iff x \in D \setminus E_k$ for $k \in \mathbb{N}$.

Hence

$\lim_{n \to \infty} f_n(x)$ exist $\forall\, x \in D \setminus E_n$.

$\Rightarrow \lim_{n \to \infty} f_n(x)$ exists $\forall\, x \in D \setminus N$.

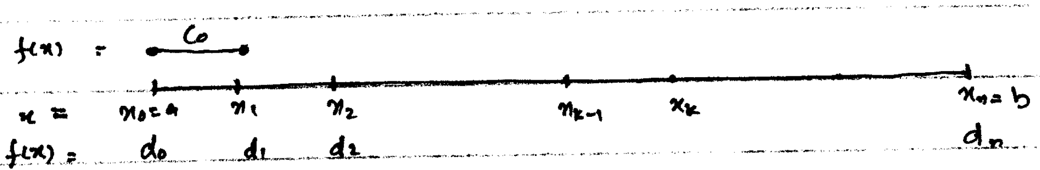
i.e $\lim_{n \to \infty} f_n$ exist a.e on D.

Step function

Let $I = [a,b]$ be an interval in $\mathbb{R}$. and $P = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_n = b\}$ is the partition of I s.t $I = \bigcup_{k=1}^{n} I_k$ where $I_k = (x_{k-1}, x_k)$ then real value function $\phi : I \longrightarrow \mathbb{R}$ defined as

$$\phi(x) = \begin{cases} c_k & \text{if } x \in I_k, \quad k = 1,2,\dots,n \\ d_k & \text{if } x = x_k, \quad k = 0,1,2,\dots,n \end{cases}$$

is called step function.



Riemann Integral

Let $\phi : [a,b] \longrightarrow \mathbb{R}$ be real valued funcion be a step function s.t

$$\phi(x) = \begin{cases} c_k \; ; & x \in (x_{k-1}, x_k), \quad k = 1,2,\dots,n \\ d_k \; ; & x = x_k, \quad k = 0,1,2,\dots,n. \end{cases}$$

The Riemann Integral of ϕ on $[a,b]$ is defined as

$$\int_a^b \phi(x)\,dx = \sum_{k=1}^{n} c_k \,\Delta x_k \quad \text{where } \Delta x_k = |x_k - x_{k-1}|$$

Note

* Step function $\phi : I = [a,b] \longrightarrow R$ defined as

$$\Phi(x) = \begin{cases} c_k , & x \in I_k , \; k = 1,2,\dots,n \; , \; I_k = (x_{k-1}, x_k) \\ d_k , & x = x_k , \; k = 0,1,2,3,\dots,n \end{cases}$$

can be expressed as

$$\phi(x) = \sum_{k=1}^{n} c_k \, 1_{(x_{k-1},x_k)}(x) + \sum_{k=1}^{n} d_k \, 1_{\{x_k\}}(x)$$

* The value of Rimann Integral of step function is independent of the choice of partion of the interval $[a,b]$ as long as step function is constant on the open sub interval of the partitions.

- $1_{(x_{k-1}, x_k)}$ is characteristic function of the open inter (x_{k-1}, x_k) on Interval $[a,b]$

i.e

$$1_{(x_{k-1}, x_k)}(x) = \begin{cases} 1, & \text{if } x \in (x_{k-1}, x_k) \\ 0, & \text{if } x \notin (x_{k-1}, x_k) \end{cases}$$

Similarly

$$1_{\{x_k\}}(x) = \begin{cases} 1, & \text{if } x = x_k \\ 0, & \text{if } x \neq x_k \end{cases}$$

$\dot{x}$ ——— $\dot{x}$ ——— $\dot{x}$ ——— $\dot{x}$

Simple function

Let $(X, \mathcal{A}, \mu)$ be measureable space. A r.v function $\phi : D \to R$, $D \in \mathcal{A}$ is said to be simple function if it satisfies the following conditions

(i) Domain of ϕ i.e $D(\phi) \in \mathcal{A}$.

(ii) Rang of ϕ i.e $R(\phi)$ is finite i.e ϕ assumes only finitely many values of real numbers.

(iii) ϕ is $\mathcal{A}$-measureable function on D.

Question: Prove that every step function is simple function but a simple function need be a step function.

Proof: Let (R, m_L, μ_L) be measureable space. Consider the real value function $\phi : (a,b) \to R$ s.t

$$\phi(x) = \begin{cases} 1 ; & \text{if } x \text{ is rational} \\ 0 ; & \text{if } x \text{ is irrational} \end{cases}$$

is simple function but not a step function.

Canonical Representation of Simple function:

Let $(X, \mathcal{A}, \mu)$ be a measureable space and $\phi : D \to R$, $D \in \mathcal{A}$ is simple function such that 'ϕ' assumes the values $c_1, c_2, \cdots, c_n$.

let $D_i = \{x \in D \mid \phi(x) = c_i\}$ Then clearly the collection $\{D_i\}_{i=1}^{n}$ partitioned the set $D \in \mathcal{A}$

i.e $D = \bigcup_{i=1}^{n} D_i$ and $D_i \cap D_j = \phi \quad \forall\ i,j = 1,2,\cdots,n$.

The expression

$$\phi(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{D_i}(x) \quad \forall\ x \in D$$

is called canonical representation of ϕ on D.

Remark: A simple function is a linear combination of characteristic function.

Lebesgue Integral of Simple function :-

let $(X, \mathcal{A}, \mu)$ be measure space and $\phi : D \to \mathbb{R}$, $D \in \mathcal{A}$ is simple function such that its canonical representation is $\phi(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{D_i}(x)$. The lebesgue integral of ϕ is defined as

$$\int_D \phi \, d\mu = \sum_{i=1}^{n} c_i \mu(D_i).$$

Provided that the sum exist in $\overline{\mathbb{R}}$ Then ϕ is said to semi lebesgue integrable on D. If the sum exist in $\mathbb{R}$ then ϕ is said to lebesgue integrable on D.

Question Prove that lebesgue integral of step function agree with its Reimann Integral.

Proof

Let $\phi : [a,b] \to R$ is a step function. Then ϕ is a simple function on $[a,b]$ $\because$ "Every step ftn is simple ftn"

Then $\phi(x) = \sum_{k=1}^{n} c_k 1_{(x_{k-1}, x_k)}(x) + \sum_{k=0}^{n} d_k 1_{\{x_k\}}(x)$

So lebesgue Integral of 'ϕ' is

$$\int_{D=[a,b]} \phi \, d\mu_L = \sum_{k=1}^{n} c_k \mu_L((x_{k-1}, x_k)) + \sum_{k=0}^{n} d_k \mu_L(\{x_k\})$$

$$= \sum_{k=1}^{n} c_k \Delta x_k + 0 \quad \because \mu_L(\{x_k\}) = 0 \ \forall k = 0,1,\dots,n$$

$\quad$ & $\mu_L[a,b] = b-a$

$$= \sum_{k=1}^{n} c_k \Delta x_k .$$

$$\int_{[a,b]} \phi \, d\mu_L = \int_a^b \phi(x) dx$$

which is required result.

Question Give an example of simple function which is Lebesgue Integrable.

Sol: Consider $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$ Borel measureable space.
Define a simple function
$\phi : [0,1] \longrightarrow \mathbb{R}$ s.t

$$\phi(x) = \begin{cases} 0 ; & x \in \mathbb{Q} \cap [0,1] \\ 1 ; & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$

$\therefore$ Canonical representation is

$$\phi(x) = 0 \cdot 1_{\mathbb{Q} \cap [0,1]}(x) + 1 \cdot 1_{\mathbb{Q}^c \cap [0,1]}(x)$$

so its Lebesgue integral is

$$\int_{[0,1]} \phi(x)\, d\mu_L = 0 \cdot \mu_L[\mathbb{Q} \cap [0,1]) + 1 \cdot \mu_L[[0,1] \cap \mathbb{Q}^c]$$

$$= 0 + \mu_L[[0,1] \cap \mathbb{Q}^c]$$

$$= \mu_L([0,1] \setminus \mathbb{Q}]$$

$$= \mu_L([0,1]) - \mu_L(\mathbb{Q}) \quad \because \text{"Set of rational number is countable union of singletons."}$$

$$= 1 - 0$$

$$= 1 \in \mathbb{R}.$$

Hence ϕ is Lebesgue integrable on $[0,1]$.

Question Give an example of Simple function which is Semi Lebesgue integrable.

Sol: Let $(\mathbb{R}, B_{\mathbb{R}}, \mu_L)$ be Borel measureable space and simple $\phi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\phi(x) = \begin{cases} 0 ; & x \in \mathbb{Q} \\ 1 ; & x \in \mathbb{Q}^c \end{cases}$$

$\therefore$ canonical representation of 'ϕ' is

$$\phi(x) = 0 \cdot 1_{\mathbb{Q}}(x) + 1 \cdot 1_{\mathbb{Q}^c}(x)$$

so its Lebesgue integral is

$$\int_{\mathbb{R}} \phi(x)\, d\mu_L = 0 \cdot \mu_L(\mathbb{Q}) + 1 \cdot \mu_L(\mathbb{Q}^c)$$
$$= 0 + \infty$$
$$= \infty \in \bar{\mathbb{R}} \quad \because \mu_L(\mathbb{Q}^c) = \infty$$

so ϕ is Semi-Lebesgue Integrable.

Question Given an example of simple function which is not Lebesgue integrable.

Solution

Let $(\mathbb{R}, B_{\mathbb{R}}, \mu_L)$ is Lebegue measureable space & $\phi : [0, \infty] \to \mathbb{R}$ is simple function defined as

$$\phi(x) = \begin{cases} -1 ; & \text{If } x \in \bigcup_{x \in \mathbb{Z}_+} [2k+1, 2k+2) \\ 1 ; & \text{If } x \in \bigcup_{x \in \mathbb{Z}_+} [2k, 2k+1) \end{cases}$$

Therefore its canonical representation is

$$\phi(x) = (-1)\, 1_{\bigcup_{k\in Z_+}[2k+1,2k+2)}(x) + (1)\, 1_{\bigcup_{k\in Z_+}[2k\ ,2k+2)}(x)$$

So its lebesgue integral is

$$\int_{[0,\infty)} \phi(x)\, d\mu_L = (-1)\,\mu_L\left(\bigcup_{k\in Z_+}([2k+1,2k+2))\right) + (1)\,\mu_L\left(\bigcup_{k\in Z_{+1}}([2k,2k+2))\right)$$

$$= (-1)\sum_{k\in Z_{+1}} \mu_L[2k+1,\ 2k+2) + (1)\sum_{k\in Z_+} \mu_L\left([2k,2k+2)\right)$$

$$= -\infty + \infty$$

$$= \text{undefined}$$

Hence Lebesgue Integral of simple function ϕ does not exist i.e ϕ is not integrable over $[0,\infty)$.

Proposition :

If ϕ and ψ are simple functions defined on a set D with $\mu(D) < \infty$ and $k \in \mathbb{R}$ Then

(1) $k\phi$ is simple function on D and

$$\int_D k\phi \, d\mu = k \int_D \phi \, d\mu.$$

(2) $\phi + \psi$ is simple function on D and

$$\int_D (\phi + \psi) \, d\mu = \int_D \phi \, d\mu + \int_D \psi \, d\mu.$$

(3) If $\phi \leq \psi$ on D i.e $\phi(x) \leq \psi(x) \ \forall x \in D$ Then

$$\int_D \phi \, d\mu \leq \int \psi \, d\mu.$$

(4) If D_1 & D_2 are dijoint measureable subsets of D with $D = D_1 \cup D_2$ Then

$$\int_D \psi \, d\mu = \int_{D_1} \psi \, d\mu + \int_{D_2} \psi \, d\mu.$$

Proof (1) Since ϕ is simple function on D $\therefore$ $\exists$ a disjoint sequence $\{E_i\}_{i=1}^{n}$ s.t $D = \bigcup_{i=1}^{n} E_i$. So canonical representation of 'ϕ' is

$$\phi(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(x)$$

where $c_1, c_2, \dots, c_n$ are distinct numbers assume by simple function ϕ.

$\therefore$

$$k\phi = k \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(x)$$

$$k\phi(x) = \sum_{i=1}^{n} (kc_i) \mathbb{1}_{E_i}(x)$$

is the canonical representation '$k\phi$'.

$\therefore$

$$\int_D k\phi \, d\mu = \sum_{i=1}^{n} (kc_i)\mu(E_i)$$

$$= k \sum_{i=1}^{n} c_i \mu(E_i)$$

$$= k \int_D \phi \, d\mu.$$

(2) Since ϕ and ψ are simple function therefore $\exists$ disjoint sequences $\{E_i\}_{i=1}^{n}$ and $\{F_j\}_{j=1}^{m}$ and distinct numbers $\{c_i\}_{i=1}^{n}$ and $\{d_j\}_{j=1}^{m}$ such that the canonical representation of 'ϕ' & 'ψ' are given by

$$\phi(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(x) \quad \text{and} \quad \psi(x) = \sum_{j=1}^{m} \mathbb{1}_{F_j}(x)$$

respectively.

Define $G_{ij} = E_i \cap F_j$ then the collection $\{G_{ij} : i = 1, 2, \dots, n, \, j = 1, 2, \dots, m\}$ is a disjoint collection s.t.

$$\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} G_{ij} = D \qquad \text{Then}$$

$$(\phi + \psi)(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} (c_i + d_j) \mathbb{1}_{G_{ij}}(x) \quad \text{is canonical}$$

representation, so $\phi + \psi$ is simple function on D.

Then Lebesgue integral of '$\phi+\psi$' is given by

$$\int_D (\phi+\psi)\,d\mu = \sum_{i=1}^{n}\sum_{j=1}^{m} (c_i + d_j)\mu(G_{ij})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} c_i\,\mu(G_{ij}) + \sum_{i=1}^{n}\sum_{j=1}^{m} d_j\,\mu(G_{ij})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} c_i\,\mu(E_i\cap F_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} d_j\,\mu(E_i\cap F_j)$$

$$= \sum_{i=1}^{n} c_i\left[\sum_{j=1}^{m}\mu(E_i\cap F_j)\right] + \sum_{j=1}^{m} d_j\left[\sum_{i=1}^{n}\mu(E_i\cap F_j)\right]$$

$$= \sum_{i=1}^{n} c_i\,\mu\left(\bigcup_{j=1}^{m}(E_i\cap F_j)\right) + \sum_{j=1}^{m} d_j\,\mu\left(\bigcup_{i=1}^{n}(E_i\cap F_j)\right)$$

$$= \sum_{i=1}^{n} c_i\,\mu\left(E_i\cap\left(\bigcup_{j=1}^{m}F_j\right)\right) + \sum_{j=1}^{m} d_j\,\mu\left(\bigcup_{i=1}^{n}E_i\cap F_j\right)$$

$$= \sum_{i=1}^{n} c_i\,\mu(E_i\cap D) + \sum_{j=1}^{m} d_j\,\mu(D\cap F_j)$$

$$= \sum_{i=1}^{n} c_i\,\mu(E_i) + \sum_{j=1}^{m} d_j\,\mu(F_j)$$

$$= \int_D \phi(x)\,d\mu + \int_D \psi(x)\,d\mu.$$

Hence $\displaystyle\int_D (\phi+\psi)\,d\mu = \int_D \phi\,d\mu + \int \psi\,d\mu.$

(3) Proof

If $\phi \leq \psi$ then $\psi - \phi \geq 0$

so that

$$\int_D (\psi - \phi)\, d\mu \geq 0$$

$$\Rightarrow \int_D (\psi + (-\phi))\, d\mu \geq 0$$

$$\Rightarrow \int_D \psi\, d\mu + \int -\phi\, d\mu \geq 0 \quad \text{by (2) part of theorem}$$

$$\Rightarrow \int_D \psi\, d\mu - \int \phi\, d\mu \geq 0 \quad \text{by (1) part of theorem}$$

$$\Rightarrow \int_D \psi\, d\mu \geq \int \phi\, d\mu$$

or $$\int \phi\, d\mu \leq \int \psi\, d\mu.$$

(4) Proof

Let ψ be simple function on s.t $\psi(x) = \sum_{j=1}^{m} d_j 1_{F_j}(x)$ is canonical representation of 'ψ'. If $D = D_1 \cup D_2$ with $D_1 \cap D_2 = \phi$. Then $1_D = 1_{D_1} + 1_{D_2}$. The Lebesgue Integral of ψ is given

$$\int_D \phi\, d\mu = \sum_{j=1}^{m} d_j\, \mu(F_j)$$

$$= \sum_{j=1}^{m} d_j\, \mu(F_j \cap D)$$

$$= \sum_{j=1}^{m} d_j\, \mu(F_j \cap (D_1 \cup D_2))$$

$$\int_D \varphi \, d\mu = \sum_{j=1}^{m} d_j \, \mu\left(F_j \cap (D_1 \cup D_2) \right)$$

$$= \sum_{j=1}^{m} d_j \, \mu\left((F_j \cap D_1) \cup (F_j \cap D_2) \right)$$

$$= \sum_{j=1}^{m} d_j \left\{ \mu(F_j \cap D_1) + \mu(F_j \cap D_2) \right)$$

$$= \sum_{j=1}^{m} d_j \, \mu(F_j \cap D_1) + \sum_{j=1}^{m} d_j \, \mu(F_j \cap D_2) \quad \text{---(1)}$$

Now $\{F_j \cap D_1\}_{j=1}^{m}$ and $\{F_j \cap D_2\}_{j=1}^{m}$ are disjoint with

$$\bigcup_{j=1}^{m} (F_j \cap D_1) = \left(\bigcup_{j=1}^{m} F_j \right) \cap D_1 = D \cap D_1 = D_1$$

and

$$\bigcup_{j=1}^{m} (F_j \cap D_2) = \left(\bigcup_{j=1}^{m} F_j \right) \cap D_2 = D \cap D_2 = D_2$$

So from (1) we have

$$\int_D \varphi \, d\mu = \int_{D_1} \varphi \, d\mu + \int_{D_2} \varphi \, d\mu.$$

which is the required result.

Notes by Mr. Anwar Khan

Question:

Let $(X, \mathcal{A}, \mu)$ be a measureable space and $\phi : D \to R$ is a simple function, $D \in \mathcal{A}$ then

(1) If $\mu(D) = 0$ then $\int_D \phi \, d\mu = 0$

Proof: Since

$$\int_D \phi \, d\mu = \sum_{i=1}^{n} c_i \mu(D_i)$$

$$= 0 \quad \because \quad D = \bigcup_{i=1}^{n} D_i \ \& \ D_i \cap D_j = \phi$$

$\&$ $\mu(D) = 0$

$\Rightarrow \mu(D_i) = 0 \ \forall i = 1, 2, \ldots, n.$

(2) If $\phi = 0$ on D then $\int_D \phi \, d\mu = 0$

Proof

Since $\int_D \phi \, d\mu = \sum_{i=1}^{n} c_i \cdot \mu(D_i)$

$\&$ Since

$$\phi = 0 \quad \because \quad \phi(x) = 0 \ \forall \ x \in D$$

i.e. $c_i = 0 \quad \forall \ i = 1, 2, \ldots, n$

So $\int_D \phi \, d\mu = 0$

(3) If $\phi \geq 0$ on then $\int_D \phi \, d\mu \geq 0$.

Proof

Since $\int_D \phi \, d\mu = \sum_{i=1}^{n} c_i \mu(D_i) \geq 0 \quad \because \ c_i \geq 0 \ \forall i = 1, 2, 3, \ldots, n$

$\Rightarrow \int_D \phi \, d\mu \geq 0$

$\because \phi \geq 0$ and μ is always greater or equal to zero.

(4) If $\phi \le 0$ on D then $\int_D \phi d\mu \le 0$.

Proof.

Since $\phi \le 0$ $\therefore$ $-\phi \ge 0$ So by

(3) Part's

$$\int -\phi d\mu \ge 0$$

$$\Rightarrow -\int \phi d\mu \ge 0$$

$$\Rightarrow \int \phi d\mu \le 0.$$

(5) ϕ is μ-integrable on D iff $\mu(\{x \in D \mid \phi(x) \ne 0\}) < \infty$

Proof:

Suppose that ϕ is μ-integrable on D then

$$\int_D \phi d\mu = \sum_{i=1}^{n} c_i \mu(D_i) < \infty$$

NOW

$\mu(\{x \in D \mid \phi(x) \ne 0\}) = \mu(\{x \in \bigcup D_i \mid \phi(x) \ne 0\})$

This implies $\exists$ at least one Set s.t

$\{x \in D_i \mid \phi(x) \ne 0\}$ since $\mu(D_i) < \infty$ $\forall i = 1, 2, 3, \ldots, n$.

$$\Rightarrow \mu\{x \in D_i \mid \phi(x) \ne 0\}) < \infty$$

$$\Rightarrow \mu\{x \in \bigcup_{i=1}^{m} D_i \mid \phi(x) \ne 0\} < \infty$$

$$\Rightarrow \mu\{x \in D \mid \phi(x) \ne 0\} < \infty$$

Conversely let $\mu(\{x \in D \mid \phi(x) \ne 0\}) < \infty$

$\Rightarrow \mu(D) < \infty$ $\forall x \in D$.

$\Rightarrow \sum_{i=1}^{n} c_i \mu(D_i) < \infty$ $\because$ $D_i \subseteq D$ and each c_i is finite.

$\Rightarrow \int_D \phi d\mu < \infty \Rightarrow \phi$ is μ-integrable. //

Theorem : Let $(X, \mathcal{A}, \mu)$ be measureable space and $\phi : D \to \mathbb{R}$, $D \in \mathcal{A}$ is simple function. Let $\{E_1, E_2, E_3, \cdots, E_n\}$ be disjoint collection in $\mathcal{A}$ s.t $\bigcup_{i=1}^{n} E_i = D$ Then Prove that ϕ is simple function on E_i, $i = 1, 2, \cdots, n$ and

$$\int_D \phi \, d\mu = \sum_{i=1}^{n} \int_{E_i} \phi \, d\mu .$$

Proof: Since ϕ is simple function D.

$\therefore \exists$ a disjoint sequence $\{D_j\}_{j=1}^{n}$ s.t $D = \bigcup_{j=1}^{n} D_j$ and cononomical representation of ϕ is

$$\phi(x) = \sum_{j=1}^{n} c_j 1_{D_j}(x)$$ where c_j, $j = 1, 2, \cdots, n$ are distinct numbers assigned by simple function ϕ.

$\therefore$ Lebesgue Integral of 'ϕ' on D is

$$\int_D \mu \, d\mu = \sum_{j=1}^{n} c_j \mu(D_j) \text{ --- (1)}$$

Since ϕ assumes finitly many values on D.

$\therefore$ its restriction to E_i, $i = 1, 2, \cdots, n$ assumes only finitely many values. Hence ϕ is simple function on E_i, $\forall\, i = 1, 2, 3, \cdots, n$. Then we have disjoint sequence $\{D \cap E_i\}_{i=1}^{n}$ s.t $\bigcup_{i=1}^{n} (D_j \cap E_i) = E_i$

$\forall\, i = 1, 2, 3, \cdots, n$
$j = 1, 2, 3, \cdots, n$

and cononomical representation of ϕ on E_i is

$$\phi(x) = \sum_{j=1}^{n} c_j 1_{D_j \cap E_i}(x)$$

from eqn ①

$$\int_D \phi \, d\mu = \sum_{j=1}^{n} c_j \, \mu(D_j)$$

$$= \sum_{j=1}^{n} c_j \, \mu(D_j \cap D)$$

$$= \sum_{j=1}^{n} c_j \, \mu\left(D_j \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) \quad \because D = \bigcup_{i=1}^{n} E_i$$

$$= \sum_{j=1}^{n} c_j \, \mu\left(\bigcup_{i=1}^{n} (D_j \cap E_i)\right) \quad \text{by Distributive Property.}$$

$$= \sum_{j=1}^{n} c_j \sum_{i=1}^{n} \mu(D_j \cap E_i) \quad \text{by definition of measure.}$$

$$= \sum_{i=1}^{m} \left[\sum_{j=1}^{n} c_j \, \mu(D_j \cap E_i) \right]$$

$$\int_D \phi \, d\mu = \sum_{i=1}^{n} \int_{E_i} \phi \, d\mu. \quad \text{As required.}$$

Theorem:

Let $(X, \mathcal{A}, \mu)$ be measureable space and ϕ_1 & ϕ_2 are simple function defined on a set $D \in \mathcal{A}$. Assume that ϕ_1 & ϕ_2 are integrable on D. If $\phi_1 = \phi_2$ a.e on D then prove that

$$\int_D \phi_1 \, d\mu = \int_D \phi_2 \, d\mu.$$

Proof Given that $\phi_1 = \phi_2$ a.e on D.

$\therefore \exists$ a null set N s.t

$$\phi_1(x) = \phi_2(x) \quad \forall \; x \in D \setminus N$$

Since $D = (D \setminus N) \cup N$ and $(D \setminus N) \cap N = \phi$

Therefore

$$\int_D \phi_1 d\mu = \int_{D\setminus N} \phi_1 d\mu + \int_N \phi_1 d\mu$$

$$= \int_{D\setminus N} \phi_1 d\mu + 0 \quad \because N \text{ is null set. Therefore } \int_N \phi_1 d\mu = 0$$

$$= \int_{D\setminus N} \phi_1 d\mu$$

$$= \int_{D\setminus N} \phi_2 d\mu \quad \because \phi_1 = \phi_2 \text{ on } D\setminus N.$$

$$= \int_{D\setminus N} \phi_2 d\mu + 0$$

$$= \int_{D\setminus N} \phi_2 d\mu + \int_N \phi_2 d\mu \quad \because \int_N \phi_2 d\mu = 0$$

$$= \int_D \phi_2 d\mu$$

Hence $\int_D \phi_1 d\mu = \int_D \phi_2 d\mu$. $\square$.

Bounded function:

Let $(X, \mathcal{A}, \mu)$ be a measureable space. A function $f: D \to \mathbb{R}$, $D \in \mathcal{A}$ is said to bounded if for $M > 0$, $M \in \mathbb{R}$ s.t

$$|f(x)| \leq M, \quad \forall x \in D.$$

Note: (i) Every simple function 'ϕ' defined on a set D with $\mu(D) < \infty$ then ϕ is Lebesgue integrable.

(ii) If ϕ and ψ are simple function defined on D, with $\mu(D) <$ also f is bounded function s.t

$$\phi(x) \leq f(x) \leq \psi(x)$$

(Such pair of simple function always exist).

Lower Lebesgue Integral:

Let $f: D \to \mathbb{R}$ be bounded function, $D \in \mathcal{A}$ with $\mu(D) < \infty$ in $(X, \mathcal{A}, \mu)$ measureable space then lower Lebesgue integral of 'f' is defined as

$$\underline{\int_D} f \, d\mu = \sup_{\phi \leq f} \int_D \phi \, d\mu.$$

Upper Lebesgue Integral: Let $(X, \mathcal{A}, \mu)$ be measureable space and 'f' is bounded function define on set $D \in \mathcal{A}$ with $\mu(D) < \infty$ Then upper Lebesgue Integral is defined as

$$\overline{\int_D} f\,d\mu = \inf_{f \le \psi} \int_D \psi\,d\mu \quad \text{where } \psi \text{ is simple function.}$$

Lebesgue Integral of Bounded function :

Let $(X, \mu, \mathcal{A})$ be measure space and f is bounded function define on set $D \in \mathcal{A}$ with $\mu(D) < \infty$, 'f' is said to be Lebesgue integrable if

$$\underline{\int_D} f\,d\mu = \overline{\int_D} f\,d\mu .$$

& Lebesgue Integral of bounded function is written as

$$\int_D f\,d\mu.$$

Lemma: Let $(X, \mathcal{A}, \mu)$ be a measureable space and f_1 and f_2 be bounded real value $\mathcal{A}$-measureable functions on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$ Then

(1) $\int_D cf\,d\mu = c\int_D f\,d\mu, \ \forall c \in \mathbb{R}.$

(2) $\int_D (f_1 + f_2)\,d\mu = \int_D f_1\,d\mu + \int_D f_2\,d\mu.$

Proof: (1) Here we discusse the following cases.

If $c = 0$ Then $cf = 0$ (zero function) on D. So $\int_D cf\,d\mu = 0$. Also since $\int_D f\,d\mu \in \mathbb{R}$ and $c = 0$ Therefore $c \cdot \int_D f\,d\mu = 0$. So

$$\int_D cf\,d\mu = c\int_D f\,d\mu.$$

If $c > 0$ Then

$$\int_D cf\,d\mu = \sup_{\phi \le cf} \int_D \phi\,d\mu$$

$$= \sup_{\frac{1}{c} \le f} \int_D \phi\,d\mu$$

$$= \sup_{\frac{1}{c} \le f} c\int_D \frac{1}{c}\phi\,d\mu$$

$$= c \sup_{\frac{\phi}{c} \le f} \int \frac{1}{c}\phi\,d\mu$$

$$\int_D cf\,d\mu = c\int_D f\,d\mu.$$

If $\underline{\underline{c<0}}$ Then $-c>0$ so

$$\int_D cf\,d\mu = \int_D -|c|f\,d\mu \quad \text{—— (1)}$$

where $|c| = \begin{cases} -c, & c<0 \\ c, & c>0 \end{cases}$

If $c=-1$ Then

$$\int_D -f\,d\mu = \sup_{\phi \le -f} \int_D \phi\,d\mu$$

$$= -\inf_{\phi \le -f} -\int_D \phi\,d\mu$$

$$= -\inf_{f \le -\phi} \int_D -\phi\,d\mu \qquad \because \int_D c\phi\,d\mu = c\int \phi\,d\mu$$

$$\int_D -f\,d\mu = -\int_D f\,d\mu. \quad \text{—— (2)}$$

$\therefore$ eqn (1) $\Rightarrow$ $\displaystyle\int_D cf\,d\mu = \int_D -|c|f\,d\mu$

$$= -\int_D |c|f\,d\mu \quad \text{by using eqn (2)}$$

$$= -|c|\int_D f\,d\mu \quad \because |c|>0$$

$$= -(-c)\int_D f\,d\mu$$

$$= c\int_D f\,d\mu.$$

Hence $\displaystyle\int_D cf\,d\mu = c\int_D f\,d\mu.$

Proof (2) let ϕ_1 and ϕ_2 be simple functions defined on $D \in \mathcal{A}$ with $\mu(D) < \infty$ such that $\phi_1 \leq f_1$ and $\phi_2 \leq f_2$. Since ϕ_1 and ϕ_2 are simple function therefore $\phi_1 + \phi_2$ is simple function and

$$\int_D \phi_1 d\mu + \int_D \phi_2 d\mu = \int_D (\phi_1 + \phi_2) d\mu$$

$$\int_D \phi_1 d\mu + \int_D \phi_2 d\mu = \int_D \phi \, d\mu \quad \text{by letting } \phi_1 + \phi_2 = \phi.$$

also f_1 and f_2 are bounded therefore their sum function $f_1 + f_2 = f$ (say) is bounded and

$$\phi_1 + \phi_2 \leq f_1 + f_2 \quad \text{i.e} \quad \phi \leq f.$$

$$\Rightarrow \sup_{\phi_1 \leq f_1} \int_D \phi_1 d\mu + \int_D \phi_2 d\mu \leq \sup_{\phi \leq f} \int_D f d\mu$$

$$\Rightarrow \int_D f_1 d\mu + \sup_{\phi_2 \leq f_2} \int_D \phi_2 d\mu \leq \int_D f d\mu \quad \because f_1, f_2, f \text{ are Lebesgue integrable.}$$

$$\Rightarrow \int_D f_1 d\mu + \int_D f_2 d\mu \leq \int_D f d\mu \quad \text{—— (1)}$$

Similarly for simple function ψ_1 & ψ_2 we have $\psi_1 + \psi_2$ is simple function and

$$\int_D (\psi_1 + \psi_2) d\mu = \int_D \psi_1 d\mu + \int_D \psi_2 d\mu.$$

Let $f_1 \leq \varphi_1$, $f_2 \leq \varphi_2$ therefore

$$f_1 + f_2 \leq \varphi_1 + \varphi_2 \quad \text{i.e} \quad f \leq \varphi.$$

Then

$$\int_D f\,d\mu \leq \inf_{f_1 \leq \varphi_1} \int_D \varphi_1\,d\mu + \int_D \varphi_2\,d\mu$$

$$\Rightarrow \int_D f\,d\mu \leq \int_D f_1\,d\mu + \inf_{f_2 \leq \varphi_2} \int_D \varphi_2\,d\mu$$

$$\Rightarrow \int_D f\,d\mu \leq \int_D f_1\,d\mu + \int_D f_2\,d\mu \quad \text{---(2)}$$

$\because$ f, f_1, f_2 are lebesgue integrable on set D.

from (1) & (2) we have

$$\int_D (f_1 + f_2)\,d\mu = \int_D f_1\,d\mu + \int_D f_2\,d\mu$$

where $f_1 + f_2 = f$.

×

Theorem Let $(X, \mathcal{A}, \mu)$ be measureable space, 'f' be bounded real valued $\mathcal{A}$-measureable function on D with $\mu(D) < \infty$. Let $\{D_n\}_1^\infty$ be disjoint sequence in $\mathcal{A}$ s.t. $\bigcup_{n=1}^{\infty} D_n = D$ Then Prove that

$$\int_D f d\mu = \sum_{n=1}^{\infty} \int_{D_n} f d\mu.$$

Proof: Let ϕ be an arbitrary simple function defined on set D s.t $\phi \leq f$ on D. Let

$\phi(x) = \sum_{i=1}^{p} a_i 1_{E_i}(x)$ be canonical representation of simple function ϕ. Let ϕ_n be the restriction of simple function ϕ to D_n Then

$$\phi_n(x) = \sum_{i=1}^{p} a_i 1_{E_i \cap D_n}(x)$$

Note that

$$\bigcup_{i=1}^{p} (E_i \cap D_n) = D_n \quad \text{Then}$$

Lebesgue integral of ϕ on D is given by

$$\int_D \phi d\mu = \sum_{i=1}^{p} a_i \mu(E_i)$$

$$= \sum_{i=1}^{p} a_i \mu(E_i \cap D)$$

$$= \sum_{i=1}^{p} a_i \mu(E_i \cap (\bigcup_{n=1}^{\infty} D_n))$$

$$= \sum_{i=1}^{p} a_i \mu(\bigcup_{n=1}^{\infty} (E_i \cap D_n)) \text{ by Dist. property}$$

$$= \sum_{i=1}^{p} a_i \sum_{n=1}^{\infty} \mu(E_i \cap D_n)$$

$$\int_D \phi \, d\mu = \sum_{n=1}^{\infty} \left[\sum_{i=1}^{p} a_i \, \mu(D_n \cap E_i) \right]$$

$$= \sum_{n=1}^{\infty} \int_{D_n} \phi_n \, d\mu$$

$$\leq \sum_{n=1}^{\infty} \int_{D_n} f \, d\mu . \quad \because \begin{array}{l} \phi \leq f \\ \phi_n \leq f \\ \int_{D_n} \phi_n \, d\mu \leq \sup\limits_{\phi_n \leq f} \int_{D_n} \phi_n \, d\mu \\ = \int_{D_n} f \, d\mu . \end{array}$$

So

$$\int_D \phi \, d\mu \leq \sum_{n=1}^{\infty} \int_{D_n} f \, d\mu$$

where the last inequality is from the fact that ϕ_n is simple function on D and $\phi_n \leq f$ on D.

so that

$$\int_{D_n} \phi_n \, d\mu \leq \sup_{\phi_n \leq f} \int_{D_n} \phi_n \, d\mu$$

$$= \int_{D_n} f \, d\mu$$

So $$\int_D \phi \, d\mu \leq \sum_{n=1}^{\infty} \int_{D_n} f \, d\mu$$

$$\Rightarrow \sup_{\phi \leq f} \int_D \phi \, d\mu \leq \sum_{n=1}^{\infty} \int_{D_n} f \, d\mu \quad \because \phi \text{ is arbitrary}$$

$$\Rightarrow \int_D f \, d\mu \leq \sum_{n=1}^{\infty} \int_{D_n} f \, d\mu \quad \text{——} \ (1)$$

Similarly starting with simple function ψ s.t $f \leq \psi$ on $D \in \mathcal{A}$ we obtain

$$\inf_{f \leq \psi} \int_D \psi \, d\mu \geq \sum_{n=1}^{\infty} \int_{D_n} f \, d\mu$$

$$\int_D f\,d\mu \geq \sum_{n=1}^{\infty} \int_{D_n} f\,d\mu \quad \text{—(2)}$$

from (1) & (2) we have

$$\int_D f\,d\mu = \sum_{n=1}^{\infty} \int_{D_n} f\,d\mu.$$

Theorem

Let $(X, \mathcal{A}, \mu)$ be a measureable space. Let f_1 & f_2 be bounded real value functions defined on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$. If $f_1 = f_2$ a.e on D then show that

$$\int_D f_1\,d\mu = \int_D f_2\,d\mu.$$

Proof: Let Ω_i be the collection of all simple function ϕ_i on $D \in \mathcal{A}$ with $\mu(D) < \infty$ s.t

$$\phi_i \leq f_i \quad \forall \; i = 1, 2, 3, \cdots, n$$

Then

$$\int_D f_1\,d\mu = \sup_{\phi_1 \leq f_1} \left\{ \int_D \phi_1\,d\mu \right\} \text{ where } \phi_1 \in \Omega_1. \quad \because f_1 \text{ is Lebesque Integrable.}$$

&

$$\int_D f_2\,d\mu = \sup_{\phi_2 \leq f_2} \left\{ \int_D \phi_2\,d\mu : \phi_2 \in \Omega_2 \right\}$$

First we show that corrosponding to every simple function $\phi_1 \in \Omega_1$ and $\phi_2 \in \Omega_2$ s.t

$$\int_D \phi_1\,d\mu = \int_D \phi_2\,d\mu$$

Since $f_1 = f_2$ a.e on D Then $\exists$ a null set $D_0 \subseteq D$ s.t $f_1 = f_2$ on $D \setminus D_0$.

Since f_1 & f_2 are bounded on D. Therefore $\exists$ $M > 0$ s.t

$$f_1(x), f_2(x) \in [-M, M] \quad \text{i.e}$$
$$-M \leq f_1(x), f_2(x) \leq M \quad \forall\ x \in D.$$

Define a simple function $\phi_2 : D \to \mathbb{R}$ s.t

$$\phi_2(x) = \begin{cases} \phi_1(x) \ ; & x \in D \setminus D_0 \\ -M \ ; & x \in D_0 \end{cases}$$

Then

$\phi_2 \leq f_2 \qquad \because \phi_1 \leq f_1$ and $f_1 = f_2$ on $D \setminus D_0$

$\Rightarrow \phi_2 \in \Omega_2 \qquad$ so that $\phi_1 \leq f_2$; $-M \leq f_2$

NOW $\displaystyle\int_D \phi_1 d\mu = \int_{D \setminus D_0} \phi_1 d\mu + \int_{D_0} \phi_1 d\mu$

$$= \int_{D \setminus D_0} \phi_1 d\mu \quad \because \mu(D_0) = 0 \Rightarrow \int_{D_0} \phi_1 d\mu = 0$$

$$= \int_{D \setminus D_0} \phi_2 d\mu + \int_{D_0} \phi_2 d\mu$$
$$= \int_D \phi_2 d\mu .$$

$$\Rightarrow \int_D \phi_1 d\mu = \int_D \phi_2 d\mu .$$

$\therefore \displaystyle\sup_{\phi_1 \leq f_1} \int_D \phi_1 d\mu = \sup_{\phi_2 \leq f_2} \int_D \phi_2 d\mu$ where $\phi_1 \in \Omega_1$ & $\phi_2 \in \Omega_2$.

$\Rightarrow \boxed{\displaystyle\int_D f_1 d\mu = \int_D f_2 d\mu}$ is ~~required~~.

Lemma:

let $(X, \mathcal{A}, \mu)$ be measureable space, f and g are bounded real value functions defined on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$.

(1) If $f \geq 0$ a.e on D & $\int_D f d\mu = 0$ Then $f = 0$ a.e on D.

(2) If $f \leq g$ a.e on D & $\int_D f d\mu = \int_D f d\mu$ Then $f = g$ a.e on D.

Proof (1):

Consider first the case that $f \geq 0$ on D

s.t

$D_0 = \{x \in D \mid f(x) = 0\}$ and $D_1 = \{x \in D \mid f(x) > 0\}$.

Then $D_0 \cap D_1 = \phi$ and $D_0 \cup D_1 = D$

we claim that

$$f = 0 \text{ a.e on } D \iff \mu(D_1) = 0 \quad \text{——(A)}$$

Suppose that $f = 0$ a.e on D. Then $\exists$ a null set $E \subseteq D$ such that $f = 0$ on $D \setminus E$.

Since $E \subset D$

$\therefore D \setminus E \subseteq D_0$

$\Rightarrow D \setminus E \subset D \setminus D_1 \quad \because D_0 = D \setminus D_1$

$D_1 \subseteq E \quad \because$ If $A \subseteq B$ then $A^c \supseteq B^c$.

so by monotonicity property

$$\mu(D_1) \leq \mu(E) = 0 \quad \because E \text{ is null set.}$$

$$\Rightarrow \mu(D_1) = 0$$

conversely Suppose that $\mu(D_1) = 0$ we are to show that $f=0$ a.e on D. Since $\mu(D_1)=0$ Then D_1 is a null set in $(X, \mathcal{A}, \mu)$. But

$$f=0 \text{ on } D_0 = D \setminus D_1$$

$\Rightarrow f=0$ a.e on D by "almost every where property".

we note here that If $\underline{\underline{\mu(D)=0}}$ from $D_1 \subseteq D$ we have $\mu(D_1) \leq \mu(D) = 0$

$$\Rightarrow \mu(D_1) = 0.$$

So that $f=0$ a.e on D. when $\underline{\underline{\mu(D)=0}}$.

Now we consider the case when $\mu(D) \in (0, \infty)$.

To show that $f=0$ a.e on D. Suppose on contrary that $f=0$ a.e on D is false.

Then by (A) $\mu(D_1) > 0$.

Now $D_1 = \{x \in D \mid f(x) > 0\}$

$$D_1 = \bigcup_{k=1}^{\infty} \{x \in D \mid f(x) \geq \tfrac{1}{k}\}$$

operating 'μ' on both sides we have

$$\mu(D_1) = \mu\left(\bigcup_{k=1}^{\infty} \{x \in D \mid f(x) \geq \tfrac{1}{k}\}\right)$$

$$0 < \mu(D_1) \leq \sum_{k=1}^{\infty} \mu\{x \in D \mid f(x) \geq \tfrac{1}{k}\}$$ by Countably Sub additive property of 'μ'.

$\therefore \exists\ k_0 \in N$ s.t $\mu(\{x \in D \mid f(x) > \frac{1}{k_0}\}) > 0$.

Define a simple function φ on D by setting

$$\varphi(x) = \begin{cases} \frac{1}{k_0} & \text{if } x \in \{x \in D \mid f(x) > \frac{1}{k_0}\} \\ 0 & \text{if } x \notin \{x \in D \mid f(x) > \frac{1}{k_0}\} \end{cases}$$

Then $\phi(x) \leq f(x) \quad \because f(x) \geq \frac{1}{k_0}$

on D. So that

$$\int_D f(x)\,d\mu \geq \int_D \phi\,d\mu = 0 \cdot \mu\left(\{x \in D \mid f(x) \geq \tfrac{1}{k_0}\}^c\right) + \frac{1}{k_0}\mu\left(\{x \in D \mid f(x) \geq \tfrac{1}{k_0}\}\right)$$

$$= \frac{1}{k_0}\mu\left(\{x \in D \mid f(x) \geq \tfrac{1}{k_0}\}\right) > 0$$

$\Rightarrow \int_D f(x)\,d\mu > 0.$

which is controduction to the fact that

$\int_D f\,d\mu = 0$ Hence $f = 0$ a.e on D.

So far we have proved that

If $f \geq 0$ on D and $\int_D f\,d\mu = 0$ Then

$f = 0$ a.e on D ——— (B)

Now we considee the case that

$f \geq 0$ a.e on D and $\int_D f\,d\mu = 0$

Then $\exists$ a null set E in $(X, \mathcal{A}, \mu)$ s.t

$f \geq 0$ on $D \setminus E$ Then

$$0 = \int_D f\,d\mu = \int_{D \setminus E} f\,d\mu + \underbrace{\int_E f\,d\mu}_{0}$$

i.e $\int_{D \setminus E} f\,d\mu = 0$ Now $f \geq 0$ on $D \setminus E$ and

$\int_{D \setminus E} f\,d\mu = 0 \Rightarrow f = $ a.e on $D \setminus E$ by (B)

Then $\exists$ a null set F in $(X, \mathcal{A}, \mu)$ s.t

$F \subset D \setminus E$ and $f = 0$ on $(D \setminus E) \setminus F$

i.e $f = 0$ on $D \setminus E \cup F$. $\because A \setminus B = A \cap B^c$.

$\Rightarrow$ $f = 0$ a.e on D $\because$ $E \cup F$ is null set being the union of two null set.

(2) Proof:

If $f \leq g$ a.e on D Then

$g - f \geq 0$ a.e on D. In addition

$\int_D f d\mu = \int_D g d\mu$

$$\Rightarrow \int_D (g-f) d\mu = 0$$

Then by First of Theorem-(1) we have

$g - f = 0$ a.e on D

i.e $f = g$ a.e on D.

Uniform Convergence:

Let $(X, \mathcal{A}, \mu)$ be a measureable space, A sequence of e.r.v function $\{f_n\}_{n=1}^{\infty}$ converge uniformly on a set D to e.r.v function 'f' If for every $\varepsilon > 0$ $\exists$ $n_0 \in N$ depending upon ε but not on $x \in D$ s.t

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in D \text{ whenever } n \geq n_0 \in N$$

Equivalently $\forall m \in N$ s.t $|f_n(x) - f(x)| < \frac{1}{m}$ $\forall x \in D$ when $n \geq N$.

Almost uniform convergence:

Let $(X, \mathcal{A}, \mu)$ be a measure space. A sequence $\{f_n\}_{n=1}^{\infty}$ of e.r.v defined on set $D \in \mathcal{A}$ is said to be almost uniformly convergent to e.r.v $\mathcal{A}$-measureable function 'f' defined on set $D \in \mathcal{A}$ if $\exists$ $\mathcal{A}$-measureable subset E of A s.t $\mu(E) < \frac{1}{\eta}$ s.t $\{f_n\}_{n=1}^{\infty}$ converge to 'f' uniformly on $D \setminus E$.

Theorem (Egoroff's Theorem) (without Proof)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}_{n=1}^{\infty}$ be sequence of $\mathcal{A}$-measureable functions defined on set $D \in \mathcal{A}$ with $\mu(D) < \infty$ and let f be e.r.v $\mathcal{A}$-measureable function on D. If $\{f_n\}_{n=1}^{\infty}$ converges to 'f' a.e on D, then $\{f_n\}_{n=1}^{\infty}$ converges to f almost uniformly on D.

Theorem (Bounded Convergence Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence of r.v $\mathcal{A}$-measureable functions defined on set $D \in \mathcal{A}$ with $\mu(D) < \infty$. Let f be a bounded r.v $\mathcal{A}$-measureable function on D. If $\{f_n\}_{n=1}^{\infty}$ converges to 'f' a.e on D then

$$\lim_{n \to \infty} \int_D |f_n - f| \, d\mu = 0$$

and in particular $\lim_{n \to \infty} \int_D f_n \, d\mu = \int_D \lim_{n \to \infty} f_n \, d\mu = \int_D f \, d\mu.$

Proof: Since $\{f_n\}_{n=1}^{\infty}$ is bounded on D, Therefore $\exists\ M>0$ s.t $|f_n(x)| < M \quad \forall\ x\in D\ \&\ \forall\ n\in N$.

Since f is also bounded, we assume that $M>0$ be so choosen that

$$|f(x)| \leq M \quad \forall\ x\in D.$$

Now since $\{f_n\}_1^{\infty}$ converges to 'f' a.e on D with $\mu(D)<\infty$. Therefore by "Egoroff's Theorem" $\{f_n\}_1^{\infty}$ converges to f almost uniformly on D then $\forall\ \eta>0\ \exists$ a subset E of D with $\mu(E)<\eta$ s.t $\{f_n\}_{n=1}^{\infty}$ converge to f uniformly on $D\setminus E$.

Therefore by definition of "uniform convergence" then $\forall\ \varepsilon>0$, $\exists\ n_0\in N$ which depends on ε but not on x s.t

$$|f_n(x)-f(x)|<\varepsilon \quad \forall\ x\in D\setminus E \text{ and } n\geq n_0\in N.$$

Now for $n\geq n_0\in N$ we have

$$\int_D |f_n-f|\,d\mu = \int_{D\setminus E} |f_n-f|\,d\mu + \int_E |f_n-f|\,d\mu$$

$$\leq \int_{D\setminus E} \varepsilon\,d\mu + \int_E 2M\,d\mu \quad \because |f_n-f| \leq |f_n|+|f| \leq M+M=2M.$$

$$= \varepsilon\int_{D\setminus E} 1\,d\mu + 2M\int_E d\mu$$

$$= \varepsilon\mu(D\setminus E) + 2m\mu(E)$$

$$\leq \varepsilon\mu(D)+2m\eta$$

Since this's hold $\forall\ n \geq n_0 \in N$ we have

$$\lim_{n\to\infty} \int_D |f_n - f|\, d\mu \leq \epsilon\mu(D) + 2\eta M.$$

Since this is true for every $\epsilon > 0$ and $\eta > 0$ therefore

$$\lim_{n\to\infty} \int_D |f_n - f|\, d\mu = 0 \quad \text{--- (1)}$$

Now consider

$$\left|\int_D f_n\, d\mu - \int_D f\, d\mu\right| = \left|\int_D (f_n - f)\, d\mu\right|$$

$$\leq \int_D |f_n - f|\, d\mu$$

$\therefore \lim_{n\to\infty} \left|\int_D f_n\, d\mu - \int_D f\, d\mu\right| \leq \lim_{n\to\infty} \int_D |f_n - f|\, d\mu = 0$ by (1)

$$\Rightarrow \lim_{n\to\infty} \left|\int_D f_n\, d\mu - \int_D f\, d\mu\right| = 0$$

$$\Rightarrow \lim_{n\to\infty} \int_D f_n\, d\mu - \lim_{n\to\infty} \int_D f\, d\mu = 0$$

$\Rightarrow \lim_{n\to\infty} \int_D f_n\, d\mu - \int_D f\, d\mu = 0 \quad \because \lim_{n\to\infty} k = k.$

$$\Rightarrow \lim_{n\to\infty} \int f_n\, d\mu = \int_D f\, d\mu.$$

Non-negative function: Let $(X, \mathcal{A}, \mu)$ be measure space. A real value $f : D \to \mathbb{R}$, $D \in \mathcal{A}$ said to be non-negative if
$$f(x) \geq 0 \quad \forall\, x \in D. \quad \text{with } \mu(D) < \infty.$$

Lebesgue Integral of Non-negative function:

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let 'f' be non-negative e.r.v $\mathcal{A}$-measureble function on $D \in \mathcal{A}$ with $\mu(D) < \infty$ we defined Lebesgue integral of f on D w.r.t 'μ' by

$$\int_D f\,d\mu = \sup_{0 \leq \phi \leq f} \int_D \phi\,d\mu.$$

where supremum is taken over all non-negative simple function ϕ on D s.t $\phi \leq f$.

Remark: A non-negative e.r.v function need not be bounded and therefore there may not be simple function 'ψ' s.t $f \leq \psi$ Then the $\int_D f\,d\mu = \inf_{f \leq \psi} \int_D \psi\,d\mu$ (for bounded function) may not exist for non-negative e.r.v $\mathcal{A}$-measureable function f. This fact has the consequences that while the integral of a non-negative e.r.v can be approximated by integral of simple functions from below. It can't be approximated by integral of simple functions from above.

<u>Lemma</u> (Without Proof)

Let $(X, \mathcal{A}, \mu)$ be measure space, let f, f_1, f_2 be non-negative e.r.v functions defined on a set $D \in \mathcal{A}$ Then

(1) If $\int_D f d\mu = 0$ Then $f = 0$ a.e on D

(2) If D_0 is $\mathcal{A}$-measureable subset of D Then $\int_{D_0} f d\mu \leq \int_D f d\mu$.

(3) If $f \geq 0$ a.e on D & $\int_D f \cdot d\mu = 0$ Then $\mu(D) = 0$

(4) If $f_1 \leq f_2$ on D Then $\int_D f_1 d\mu \leq \int_D f_2 d\mu$.

(5) If $f_1 = f_2$ a.e on D Then $\int_D f_1 d\mu = \int_D f_2 d\mu$.

where f, f_1, f_2 are integrable on a set D.

<u>Note:</u> Lebesgue integral of non-negative function is defined

$$\int_D f d\mu = \sup_{0 \leq \phi \leq f} \int_D \phi \, d\mu.$$

Proposition:

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let ψ be non-negative simple function on X. Then show that a set function

$\nu : \mathcal{A} \longrightarrow [0, \infty]$ defined as

$$\nu(A) = \int_A \psi \, d\mu \quad \forall \; A \in \mathcal{A} \quad \text{is}$$

measure on $\mathcal{A}$.

Proof

To show that $\nu : \mathcal{A} \to [0, \infty]$ is measure we are to show that

(i) $\nu(\phi) = 0$, $\phi \in \mathcal{A}$

(ii) for disjoint sequence $\{E_j\}_{j=1}^{\infty}$ in $\mathcal{A}$ we have $\nu\left(\bigcup_{j=1}^{\infty} E_i\right) = \sum_{j=1}^{\infty} \nu(E_j)$.

(i) Since $\phi \in \mathcal{A}$ therefore by definition of set function 'ν' we get

$$\nu(\phi) = \int_{\phi} \psi \, d\mu$$

$$= \mu(\phi)$$

$$\nu(\phi) = 0 \quad \because \quad \mu \text{ is measure on } \mathcal{A}.$$

(ii) Since ψ is simple & let $\{D_i\}_{i=1}^{m}$ be disjoint sequence in $(X, \mathcal{A}, \mu)$ s.t $X = \bigcup_{i=1}^{m} D_i$

and $a_1, a_2, \ldots, a_m$ are distinct real number

s.t

$$\psi(x) = \sum_{i=1}^{m} a_i 1_{D_i}(x)$$ is canonical representation of ψ. on X. Then the restriction of ψ on $A \in \mathcal{A}$ is given by

$$\psi(x) = \sum_{i=1}^{m} a_i 1_{D_i \cap A}(x).$$

Then $\nu(A) = \int_A \psi(x) d\mu = \sum_{i=1}^{m} a_i \mu(D_i \cap A)$ by def. of Lebesgue integral of Simple function.

let $\{E_j\}_{j=1}^{\infty}$ be disjoint sequence in $(X, \mathcal{A}, \mu)$ Then by defintion of set function 'ν' we have

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{i=1}^{m} a_i \mu\left(D \cap \left(\bigcup_{i=1}^{\infty} E_j\right)\right) \quad \because \nu(A) = \sum_{i=1}^{m} a_i \mu(D_i \cap A)$$

$$\Rightarrow \nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{i=1}^{m} a_i \mu\left(\bigcup_{i=1}^{\infty} (D \cap E_j)\right) \text{ by Dist. Property.}$$

$$= \sum_{i=1}^{m} a_i \cdot \sum_{i=1}^{\infty} \mu(D \cap E_j) \quad \because \mu \text{ is measure.}$$

$$= \sum_{j=1}^{\infty} \left[\sum_{i=1}^{m} a_i \mu(D \cap E_j) \right.$$

$$= \sum_{j=1}^{\infty} \left[\nu(E_j)\right] \text{ by } \nu(A) = \sum_{i=1}^{m} a_i \mu(D_i \cap A).$$

$$\Rightarrow \nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j).$$ Hence ν is measure on $\mathcal{A}$. //

Theorem: (Monoton Convergence Theorem)

let $(X, \mathcal{A}, \mu)$ be a measure space & $\{f_n\}_{n=1}^{\infty}$ be an increasing sequence of non-negative e.r.v $\mathcal{A}$ - measureable functions on a set $D \in \mathcal{A}$ and $\lim_{n \to \infty} f_n = f$ on D Then

$$\lim_{n \to \infty} \int_D f_n \, d\mu = \int_D f \, d\mu.$$

Proof: Since $\{f_n\}_1^{\infty}$ is $\uparrow$ (increasing) sequence of non-negative e.r.v functions on D. Therefore

$f_n \leq f_{n+1} \quad \forall \ n \in N$

$\Rightarrow \int_D f_n \, d\mu \leq \int_D f_{n+1} \, d\mu \quad \forall \ n \in N.$

So $\left\{ \int_D f_n \, d\mu \right\}_{n=1}^{\infty}$ is an increasing sequence of extended real numbers bounded above by $\int_D f d\mu$. Also $\lim_{n \to \infty} f_n(x)$ exist in $[0, \infty]$ $\forall \ x \in \mathbb{R}$. So that $\lim_{n \to \infty} f_n = f$ is non-negative e.r.v function on D which is $\mathcal{A}$-measureable on D because $\{f_n\}$ is $\mathcal{A}$-measureable. Since $\int_D f_n \, d\mu \leq \int_D f d\mu$.

Hence

$$\lim_{n \to \infty} \int_D f_n \, d\mu \leq \int_D f \, d\mu \quad \text{——} (1)$$

To prove the reverse inequality of (1) let ϕ be an arbitrary non-negative simple function on D s.t $0 \le \phi \le f$ with $\alpha \in (0,1)$ arbitrary fixed

$$0 \le \alpha\phi \le \phi \le f \quad \text{on } D \quad \because \alpha \in (0,1)$$

Define a sequence $\{E_n\}_{n=1}^{\infty}$ of subsets of D s.t

$$E_n = \{x \in D \mid f_n(x) \ge \alpha\phi(x)\} \quad \text{——} (2) \quad \forall n \in N.$$

Since f_n and $\alpha\phi$ are $\mathcal{A}$-measureable. Therefore $E_n \in \mathcal{A} \quad \forall n \in N$.

Now for $f_n \le f_{n+1} \quad \forall n \in N$

$$\Rightarrow E_n \subseteq E_{n+1} \quad \forall n \in N.$$

So that $\{E_n\}_{n=1}^{\infty}$ is increasing sequence in $\mathcal{A}$.

Since $E_n \subset D \quad \forall n \in N$

Therefore $\bigcup_{n=1}^{\infty} E_n \subset D$ —— (3)

Conversely let $x \in D$.

If $f(x) = 0$

then $\phi(x) = 0 \quad \because 0 < \phi < f$

also since $0 \le f_n < f$

$\Rightarrow f_n(x) = 0 \quad \because f(x) = 0 \ \& \ 0 < \phi < f_n < f.$

$\Rightarrow f_n(x) = 0 = \alpha \cdot \phi(x)$ and $x \in E_n$

$$\Rightarrow x \in \bigcup_{n=1}^{\infty} E_n$$

$$\Rightarrow D \subseteq \bigcup_{n=1}^{\infty} E_n \quad \text{——} (4)$$

from (3) & (4) we get

$$D = \bigcup_{n=1}^{\infty} E_n.$$

If $f(x) > 0$ then since $0 \leq \phi \leq f$ and $\alpha \in (0,1)$ we have $f(x) > \alpha \phi(x)$.

Since $\{f_n\}$ is increasing sequence, $\exists\ n \in N$ s.t $f_n(x) > \alpha \phi(x)$

and so $x \in E_n \Rightarrow x \in \bigcup_{n=1}^{\infty} E_n$

$\Rightarrow D \subseteq \bigcup_{n=1}^{\infty} E_n$ — (5)

from (3) & (5)

$$D = \bigcup_{n=1}^{\infty} E_n .$$

Define a set function $\nu : \mathcal{A} \to [0, \infty]$ s.t

$$\nu(A) = \int_A \phi \, d\mu \text{ then } \nu \text{ is measure.}$$

NOW $\int_D f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} \alpha \phi \, d\mu = \alpha \int_{E_n} \phi \, d\mu = \alpha \nu(E_n)$

$$\Rightarrow \lim_{n\to\infty} \int_D f_n \, d\mu \geq \alpha \lim_{n\to\infty} \nu(E_n)$$

$$= \alpha \nu\left(\lim_{n\to\infty} E_n\right)$$

$$= \alpha \nu\left(\bigcup_{n=1}^{\infty} E_n\right) \quad \because \{E_n\}_1^{\infty} \uparrow \quad \therefore \lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$$

$$= \alpha \nu(D)$$

$$= \alpha \int_D \phi \, d\mu$$

i.e

$$\lim_{n\to\infty} \int_D f_n \, d\mu \geq \alpha \int_D \phi \, d\mu$$

Since this holds for arbitrary non-negative simple function ϕ on D s.t
$0 \le \phi \le f$ we have

$$\lim_{n\to\infty} \int_D f_n \, d\mu \ge \alpha \int_D f \, d\mu$$

Let $\alpha \to 1$ we obtain

$$\lim_{n\to\infty} \int_D f_n \, d\mu \ge \int_D f \, d\mu \quad \text{--- (6)}$$

from (1) & (6) we get

$$\lim_{n\to\infty} \int_D f_n \, d\mu = \int_D f \, d\mu .$$

Remark: Prove that Monotone convergence Theorem is not valid for decreasing sequence.

Proof:
Consider the Lebesgue measure space (R, m_L, μ_L).
Let $\{f_n\}_{n=1}^{\infty}$ be decreasing sequence of non-negative e.r.v functions on R defined $f_n = 1_{[n,\infty)}$ $\forall n \in N$
we have

$$\int_D f_n \, d\mu_L = \int 1_{[n,\infty)}(x) \, d\mu$$

$$= \mu([n,\infty)) = \infty$$

$$\Rightarrow \lim_{n\to\infty} \int_D f_n \, d\mu = \infty$$

NOW
$\lim_{n\to\infty} f_n = 0$ $\downarrow \{f_n\}$ But $\lim_{n\to\infty} f_n = f$
so that $\int_D f d\mu = \int 0 d\mu$.

How $\lim_{n\to\infty} \int_D f_n \, d\mu \neq \int_D f \, d\mu$ for $\{f_n\}_1^{\infty} \downarrow$

Lemma (Without Proof)

Let $(X, \mathcal{A}, \mu)$ be measure space and $f : X \to \bar{R}$ be a non-negative e.r.v $\mathcal{A}$-measureable function on X Then $\exists$ an increasing sequence of non-negative simple functions $\{\phi_n\}_{n=1}^{\infty}$ on X such that

(i) $\phi_n \longrightarrow f$ on X mean that ϕ_n approch to f.

(ii) $\phi_n \longrightarrow f$ uniformly on an arbitrary subset E of X on which f is bounded.

(iii) $\lim_{n \to \infty} \int_D \phi_n \, d\mu = \int_D f \, d\mu$.

Proposition :

Let $(X, \mathcal{A}, \mu)$ be a measure space and $D \in \mathcal{A}$

(a) If $f_1, f_2, \cdots, f_n$ are non-negative e.r.v $\mathcal{A}$ measureable function defined on D Then Show that
$$\int_D \left(\sum_{i=1}^{n} f_i \right) d\mu = \sum_{i=1}^{n} \int_D f_i \, d\mu .$$

Proof : Let f_1 & f_2 be non-negative e.r.v $\mathcal{A}$-measureable functions defined on $D \in \mathcal{A}$ Then by "above Lemma" $\exists$ two increasing Sequences of non-negative simple functions i.e $\{\phi_{n,1}\}_1^{\infty}$ and $\{\phi_{n,2}\}_1^{\infty}$ on X s.t $\phi_{n,1} \longrightarrow f$ and $\phi_{n,2} \longrightarrow f$ Then clearly $\{\phi_{n,1} + \phi_{n,2}\}$ is non-negative increasing sequence of simple functions on X s.t

$$\phi_{n,1} + \phi_{n,2} \longrightarrow f_1 + f_2 \quad \text{as } n \to \infty$$

Then by "Monotone convergence theorem" we have

$$\lim_{n\to\infty} \int_D (\phi_{n,1} + \phi_{n,2})\, d\mu = \int_D (f_1+f_2)\, d\mu \quad \text{--- (1)}$$

Now consider

$$\lim_{n\to\infty} \int_D (\phi_{n,1} + \phi_{n,2})\, d\mu = \lim_{n\to\infty} \left[\int_D \phi_{n,1}\, d\mu + \int_D \phi_{n,2}\, d\mu \right]$$

$$= \lim_{n\to\infty} \int_D \phi_{n,1}\, d\mu + \lim_{n\to\infty} \int_D \phi_{n,2}\, d\mu$$

$$= \int_D f_1\, d\mu + \int_D f_2\, d\mu$$ by Monotone Convergent theorem.

i.e $$\lim_{n\to\infty} \int_D (\phi_{n,1} + \phi_{n,2})\, d\mu = \int_D f_1\, d\mu + \int f_2\, d\mu \quad \text{--- (2)}$$

from (1) & (2)

$$\int_D (f_1+f_2)\, d\mu = \int_D f_1\, d\mu + \int_D f_2\, d\mu. \quad \text{--- (3)}$$

By repeated application of (3) to the sequence $f_1, f_2, \cdots, f_n$ we obtain

$$\int_D \left(\sum_{i=1}^{n} f_i \right) d\mu = \sum_{i=1}^{n} \int_D f_i\, d\mu.$$

Proposition

(b) If $\{f_n\}_{n=1}^{\infty}$ is sequence of non-negative e.r.v $\mathcal{A}$-measureable functions defined on $D \in \mathcal{A}$ Then

$$\int_D \left(\sum_{i=1}^{\infty} f_i\right) d\mu = \sum_{i=1}^{\infty} \int_D f_i \, d\mu.$$

Proof:-

If $\{f_n\}_1^{\infty}$ is sequence of non-negative e.r.v $\mathcal{A}$-measureable functions defined on D Then for $\{f_1, f_2, \cdots, f_n\}$, $n \in N$ we have

$$\int_D \left(\sum_{i=1}^{n} f_i\right) d\mu = \sum_{i=1}^{n} \int_D f_i \, d\mu.$$

Now the sum of the series $\sum_{i=1}^{\infty} f_i$ is the limit of the sequence of partial sums $\{S_n = \sum_{i=1}^{n} f_i \mid n \in N\}$.
Since $\{f_n\}_{n=1}^{\infty}$ is sequence of non-negative terms Therefore $\{S_n = \sum_{i=1}^{n} f_i \mid n \in N\}$ is sequence of non-negative terms and $\{S_n\}$ is increasing sequence. Then By Monoton convergence Theorem we have

$$\lim_{n \to \infty} \int_D S_n \, d\mu = \int_D \left(\sum_{i=1}^{\infty} f_i\right) d\mu \qquad \because \lim_{n \to \infty} S_n = \sum_{i=1}^{\infty} f_i$$

$$\lim_{n \to \infty} \int_D \left(\sum_{i=1}^{n} f_i\right) d\mu = \int_D \left(\sum_{i=1}^{\infty} f_i\right) d\mu \qquad \because S_n = \sum_{i=1}^{n} f_i, \ n \in N$$

$$\Rightarrow \lim_{n \to \infty} \sum_{i=1}^{n} \left(\int_D f_i \, d\mu\right) = \int_D \left(\sum_{i=1}^{\infty} f_i\right) d\mu \quad \text{by (a) Part of Proposition.}$$

i.e $$\sum_{i=1}^{\infty}\left(\int_D f_i\,d\mu\right) = \int_D\left(\sum_{i=1}^{\infty} f_i\right)d\mu$$

which is required result.

Proposition:

Let $(X, \mathcal{A}, \mu)$ be measure space and $f: D \to \bar{\mathbb{R}}$ be a non-negative e.r.v $\mathcal{A}$-measureable function defined on a set $D \in \mathcal{A}$

(a) If $\{D_1, D_2, \cdots, D_n\}$ is disjoint collection in $\mathcal{A}$ s.t $\bigcup_{i=1}^{n} D_i = D$ Then

$$\int_D f\,d\mu = \sum_{i=1}^{n}\left(\int_{D_i} f\,d\mu\right)$$

Proof First we prove that let $g: D \to \bar{\mathbb{R}}$ be a non-negative e.r.v $\mathcal{A}$-measureable function on $D \in \mathcal{A}$.

Suppose that $A, B \in \mathcal{A}$ s.t $A \cup B = D$ and $A \cap B = \phi$.

If $g = 0$ on B then

$$\int_D g\,d\mu = \int_A g\,d\mu \quad \text{——— (1)}$$

Since g is non-negative e.r.v $\mathcal{A}$-measureable function defined on D there by lemma "Pag #160" $\exists$ an increasing sequence $\{\phi_n\}_{n=1}^{\infty}$ of non-negative simple function s.t

$$\lim_{n\to\infty} \phi_n = g$$

Since $0 \leq \phi_n < g$ and $g = 0$ on B

Therefore $\phi_n = 0$ on B $\forall$ $n \in N$

Also

$$\int_B \phi_n \, d\mu = 0 \quad \because \phi_n = 0 \text{ on } B.$$

Then

$$\int_D \phi_n \, d\mu = \int_A \phi_n \, d\mu + \int_B \phi_n \, d\mu$$

$$\int_D \phi_n \, d\mu = \int_A \phi_n \, d\mu \quad \because \int_B \phi_n \, d\mu = 0$$

Now $\phi_n \to g$ on D so that $\phi_n \to g$ on A.

Then by "Monotone Convergence Theorem"

$$\int_D g \, d\mu = \lim_{n\to\infty} \int_D \phi_n \, d\mu$$

$$= \lim_{n\to\infty} \int_A \phi_n \, d\mu$$

$$\boxed{\int_D g \, d\mu = \int_A g \, d\mu}$$

Let f be a non-negative e.r.v $\mathcal{A}$-measureable function on D and $\{D_1, D_2, \ldots, D_n\}$ be disjoint collection in $\mathcal{A}$. s.t $\bigcup_{i=1}^{n} D_i = D$.

lets defined a function $f_{D_n} : D \to \bar{R}$ s.t

$$f_{D_n}(x) = \begin{cases} f(x) \ ; & x \in D_n \\ 0 \ ; & x \in D \setminus D_n . \end{cases}$$

Then $f_{D_1}, f_{D_2}, \dots, f_{D_n}$ are non-negative e.r.v $\mathcal{A}$-measureable functions on D and

$$\sum_{i=1}^{n} f_{D_i} = f$$

Then

$$\int_D f d\mu = \int_D \left(\sum_{i=1}^{n} f_{D_i} \right) d\mu$$

$$= \sum_{i=1}^{n} \left(\int_D f_{D_i} \, d\mu \right)$$

$$= \sum_{i=1}^{n} \left(\int_{D_i} f_{D_i} d\mu \right) \quad \because \int_D g d\mu = \int_A g d\mu \text{ if } g = 0 \text{ on } B.$$

$$= \sum_{i=1}^{n} \left(\int_{D_i} f d\mu \right) \quad \because f = f_{D_n} \text{ on } D_n.$$

Hence

$$\int_D f d\mu = \sum_{i=1}^{n} \left(\int_{D_i} f d\mu \right)$$

where $D = \bigcup_{i=1}^{n} D_i$

with $D_i \cap D_j = \phi$

$\forall \, i, j = 1, 2, 3, \dots, n$

~~Proposition~~:

(b) let $\{E_n\}_1^\infty$ be an increasing sequence in $\mathcal{A}$ s.t $\lim_{n\to\infty} E_n = D$ Then

$$\int_D f\,d\mu = \lim_{n\to\infty} \int_{E_n} f\,d\mu.$$

Proof Let $\{E_n\}_1^\infty$ be an increasing sequence in $\mathcal{A}$ with $\lim_{n\to\infty} E_n = D$. For each $n \in N$ define a non-negative e.r.v $\mathcal{A}$-measureable function defined by

$$f_{E_n}(x) = \begin{cases} f(x), & x \in E_n \\ 0, & x \in D \setminus E_n. \end{cases}$$

Then $\{f_{E_n}\}_{n=1}^\infty$ is an increasing sequence with $\lim_{n\to\infty} f_n = f$ on D, so By "Monotone Convergence Theorem" we have

$$\lim_{n\to\infty} \int_D f_{E_n} d\mu = \int_D f\,d\mu$$

$$\Rightarrow \lim_{n\to\infty} \int_{E_n} f_{E_n} d\mu = \int_D f\,d\mu$$

Proposition (c)

If $\{D_n\}_1^\infty$ is a disjoint collection in $\mathcal{A}$ s.t $\bigcup_{n=1}^\infty D_n = D$ Then

$$\int_D f\,d\mu = \sum_{i=1}^\infty \int_{D_i} f$$

Proof:- let $\{D_n\}_{n=1}^\infty$ be sequence of disjoint members of $\mathcal{A}$ s.t $D = \bigcup_{n=1}^\infty D_n$.

Define $E_n = \bigcup_{i=1}^{n} D_i \quad \forall\, n \in N.$

So that $\{E_n\}_{n=1}^{\infty}$ is increasing sequence in $\mathcal{A}$. Then

$$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n = D$$

Then by (b) Part of the Proposition we have

$$\lim_{n\to\infty} \int_{E_n} f\,d\mu = \int_D f\,d\mu$$

$$\lim_{n\to\infty} \int_{\bigcup_{i=1}^{n} D_i} f\,d\mu = \int_D f\,d\mu$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \int_{D_i} f\,d\mu = \int_D f\,d\mu. \quad \text{by (a) Part.}$$

$$\Rightarrow \sum_{i=1}^{\infty} \int_{D_i} f\,d\mu = \int_D f\,d\mu$$

which is the required result.

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State & Prove Fatou's Lemma:

Statement:

Let $(X, \mathcal{A}, \mu)$ be a measure space, then for every sequence $\{f_n\}_{n=1}^{\infty}$ of non-negative e.r.v $\mathcal{A}$-measureable function on set $D \in \mathcal{A}$ Then

$$\int_D \liminf f_n \, d\mu \le \lim_{n \to \infty} \int_D f_n \, d\mu .$$

Proof: we know

$$\lim_{n\to\infty} \inf f_n = \lim_{n\to\infty} \left(\inf_{k \ge n} f_k \right)$$

where $\{ \inf_{k\ge n} f_k \}_1^{\infty}$ is increasing sequence of non-negative e.r.v $\mathcal{A}$-measureable functions on D. Therefore By "Monotone Convergence Theorem" we have

$$\int_D \lim_{n\to\infty} \inf f_n \, d\mu = \lim_{n\to\infty} \int_D \left(\inf_{k\ge n} f_k \right) d\mu \quad \text{—①}$$

Since

$$\left\{ \int_D \left(\inf_{k \ge n} f_k \right) d\mu \right\}_{n=1}^{\infty} \text{ is an increasing}$$

sequence in $\bar{R}$, Therefore its limit exists in $\bar{R}$ and equal to $\lim_{n\to\infty} \inf f_n$. So that from ① we obtain

$$\int_D \lim_{n\to\infty} \inf f_n \, d\mu = \liminf \int_D \left(\inf_{k\ge n} f_k \right) d\mu$$

$$\int_D \liminf f_n \, d\mu = \liminf \int_D \left(\inf_{k \geq n} f_k \right) d\mu$$

$$\leq \liminf \int_D f_n \, d\mu$$

$\because \inf_{k \geq n} f_k \leq f_n \ \forall n \in \mathbb{N}.$

i.e

$$\int_D \liminf f_n \, d\mu \leq \liminf \int_D f_n$$