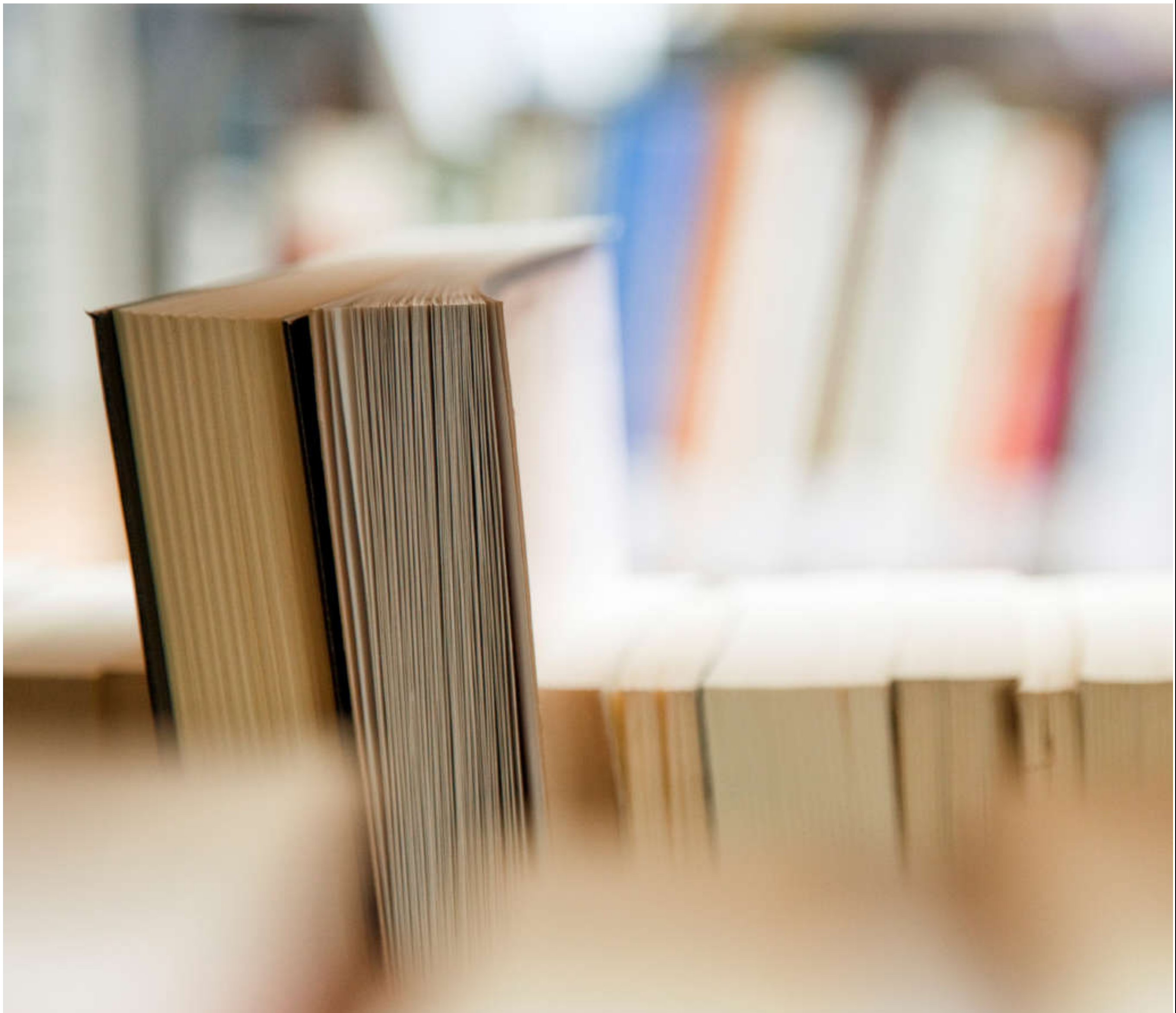

Partial Differential Equations

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Dedicated

To

My Honorable Teacher

&

My Parents

Lecture # 01

Differential Equations:

An equation containing the derivatives of one dependent variable, with respect to one or more independent variables, is said to be a differential equation (D.E).

Types: D.E has two types

(i) Ordinary differential equations (ODEs)

(ii) Partial differential equations (PDEs)

Ordinary differential equations:

If an equation contains only ordinary derivatives of the dependent variables with respect to a single independent variable called ordinary differential equation. For example

$$\frac{dy}{dx}+5y=e^{x} \quad , \frac{d^2y}{dx^2}-\frac{dy}{dx}+6y=0$$

Partial Differential Equations:

An equation involving the derivatives of an unknown function or dependent variable w.r.t two or more independent variables is called a partial differential equation. For example

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad , \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial t}$$

Method of separation of variable or product method or Fourier method:

In this method the unknown function of a partial differential equation is written as a product of functions. Each function in the product depends only on single independent variable which is involved in to equation. Number of these functions is equal number of independent variables.

Some useful techniques for solving problems:

Consider a differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

$$\text{Let } y = e^{mx}$$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

$$\Rightarrow \quad am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$\left(am^2 + bm + c\right)e^{mx} = 0$$

The auxiliary equation

$$am^2 + bm + c = 0$$

Let m_1 and m_2 be its roots. Then

$$y(x) = Ae^{m_1 x} + Be^{m_2 x}$$

$$Let \;\; m_1 = \alpha + \sqrt{\beta} \;\; and \;\; m_2 = \alpha - \sqrt{\beta} \;\; be \; roots$$

$$y(x) = Ae^{\left(\alpha+\sqrt{\beta}\right)x} + Be^{\left(\alpha-\sqrt{\beta}\right)x}$$

$$y(x) = Ae^{\alpha x+\sqrt{\beta}x} + Be^{\alpha x-\sqrt{\beta}x}$$

$$y(x) = Ae^{\alpha x}.e^{\sqrt{\beta}x} + Be^{\alpha x}.e^{-\sqrt{\beta}x}$$

$$y(x) = e^{\alpha x}\left(Ae^{\sqrt{\beta}x} + Be^{-\sqrt{\beta}x}\right)$$

$$\because e^{y} = \cosh y + \sinh y \quad \& \; e^{-y} = \cosh y - \sinh y$$

$$y(x) = e^{\alpha x}\left[A\left(\cosh\sqrt{\beta}x + \sinh\sqrt{\beta}x\right) + B\left(\cosh\sqrt{\beta}x - \sinh\sqrt{\beta}x\right)\right]$$

$$y(x) = e^{\alpha x}\left[(A+B)\cosh\sqrt{\beta}x + (A-B)\sinh\sqrt{\beta}x\right]$$

$$y(x) = e^{\alpha x}\left[C\cosh\sqrt{\beta}x + D\sinh\sqrt{\beta}x\right] \quad \because A+B=C \;\; \& \, i(A-B)=D$$

If $\alpha = 0$
$$y(x) = C\cosh\sqrt{\beta}x + D\sinh\sqrt{\beta}x$$

Roots are complex:

$If\ \ m_1 = \alpha + i\beta\ \ ,\ m_2 = \alpha - i\beta$

$$y(x) = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x}$$

$$y(x) = Ae^{\alpha x+i\beta x} + Be^{\alpha x-i\beta x}$$

$$y(x) = Ae^{\alpha x}.e^{\alpha x} + Be^{\alpha x}.e^{-i\beta x}$$

$$y(x) = e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right]$$

$$y(x) = e^{\alpha x}\left[A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)\right]$$

$$y(x) = e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right]$$

$$y(x) = e^{\alpha x}\left[C\cos\beta x + D\sin\beta x\right] \quad \because\ A+B=C\ \ \&\ \ i(A-B)=D$$

If $\alpha = 0$, $\quad y(x) = C\cosh\beta x + D\sinh\beta x$

Question: Solve the D.E $\quad \dfrac{d^2y}{dx^2} + \alpha^2 y = 0$

Solution: Let $\quad y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

$$\Rightarrow\ \ m^2e^{mx} + \alpha^2e^{mx} = 0$$

$$\Rightarrow\ \ (m^2+\alpha^2)e^{mx} = 0$$

The auxiliary equation

$$m^2 + \alpha^2 = 0$$

$$m^2 = -\alpha^2 = i^2\alpha^2$$

$$m = \pm\alpha \;\Rightarrow\; y(x) = A\cos\alpha x + B\sin\alpha x$$

Eigenvalues and Eigen function:

When a linear operator x on a function convert the function into another function. The obtained function is some scalar multiple of the original function. The function is called Eigen function or characteristic or proper function. The scalar is called on Eigen value or characteristic or proper value or latent value e.g.

$$y = \cos(mx)$$

$$\frac{dy}{dx} = -m\sin(mx)$$

$$\frac{d^2y}{dx^2} = -m^2\cos(mx)$$

$$\frac{d^2y}{dx^2} = -m^2 y$$

$\because \; -m^2$ is called Eigen value.

Solution of Non-Homogeneous equation:

$$ax + by = c$$

$$a_1x + b_1y = c_1$$

has unique solution if $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0$

has no solution if $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$

Solution of Homogenous equation:

$$ax + by = 0$$

$$a_1x + b_1y = 0$$

has trivial and unique solution (0,0) if $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0$

has infinite many solution if $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$

Question: Determine Eigenvalues and Eigen function associated with the following boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \ ; \ \mathrm{y}(0) = 0 \ , \ y(L) = 0$$

Solution: Here arises three cases

(i) $\lambda < 0$, (ii) $\lambda = 0$ (iii) $\lambda > 0$

Case-I: $\lambda < 0$

Let $\lambda = -\beta^2 \ ; \ \beta \neq 0$

$$\left(\frac{d^2}{dx^2} - \beta^2\right) y = 0$$

The auxiliary equation

$$\frac{d^2}{dx^2} - \beta^2 = 0$$

$$\frac{d^2}{dx^2} = \beta^2$$

$$\frac{d}{dx} = \pm\beta$$

$$\Rightarrow \ y(x) = Ae^{\beta x} + Be^{-\beta x} \quad \because \ y(0) = 0$$

$$y(0) = Ae^0 + Be^0 \ \Rightarrow \ A + B = 0 \quad \_\_\_\_\_ (i)$$

$$Ae^{\beta L} + Be^{-\beta L} = y(L)$$

$$\because \ y(L) = 0$$

$$Ae^{\beta L} + Be^{-\beta L} = 0 \qquad \_\_\_\_(ii)$$

Eq (i) and (ii) are two homogeneous equation

$$\begin{vmatrix} 1 & 1 \\ e^{\beta L} & e^{-\beta L} \end{vmatrix} = e^{-\beta L} - e^{\beta L} \neq 0 \text{ has unique solution } (0,0)$$

$$\Rightarrow A = B = 0$$

$$\Rightarrow y(x) = 0$$

Case-II: $\lambda = 0$

$$\frac{d^2y}{dx^2} = 0$$

The auxiliary equation

$$\frac{d^2}{dx^2} = 0$$

$$\frac{d}{dx} = 0,0$$

$$\Rightarrow \; y(x) = (A + Bx)e^{0x} = A + Bx$$

$$\because \; y(0) = 0$$

$$0 = (A + B0) \Rightarrow A = 0$$

$$\because \; y(L) = 0$$

$$0 = A + B(L) \Rightarrow \; B(L) = 0 \;\because\; A = 0$$

$$B = 0 \;, L \neq 0$$

$$\Rightarrow \; y(x) = 0 \quad \text{trivial solution}$$

Case-III: $\lambda > 0$

Let $\lambda = \beta^2$

$$\left(\frac{d^2}{dx^2}+\beta^2\right)y=0$$

The auxiliary equation

$$\frac{d^2}{dx^2}+\beta^2=0$$

$$\frac{d^2}{dx^2}=-\beta^2$$

$$\frac{d}{dx}=\pm i\beta$$

$$y(x)=A\cos\beta x+B\sin\beta x \qquad \because\ y(0)=0$$

$$0=A\cos 0+B\sin 0 \Rightarrow\ A=0$$

$$y(x)=B\sin\beta x$$

$$\because\ y(L)=0$$

$$0=B\sin\beta L$$

For non-trivial solution

$$B\neq 0\ ,\ \sin\beta L=0$$

$$\beta L=n\pi$$

$$\because\ \sin x=0\ \Rightarrow x=\sin^{-1}(0)$$

$$\Rightarrow\ x=n\pi\ \ ,\ n\in Z$$

$$\beta=\frac{n\pi}{L}\quad ;\quad n=0,\pm1,\pm2,\pm3,\pm4,.....$$

$$\lambda=\left(\frac{n\pi}{L}\right)^2\quad ;\quad n=0,\pm1,\pm2,\pm3,\pm4,.....$$

$$\lambda=\left(\frac{n\pi}{L}\right)^2\quad ;\quad n=0,1,2,3,4,.....\qquad \because -\text{ve } \textit{neglect due to square}$$

$$\Rightarrow \quad y(x) = B\sin\beta x$$

$$\Rightarrow \quad y(x) = B\sin\left(\frac{n\pi}{L}\right)x \qquad ; \; n = 0,1,2,3,....$$

For n = 0

$$y(x) = B\sin 0 = 0$$

$$\Rightarrow \quad \lambda = \beta^2 = \left(\frac{n\pi}{L}\right)^2 \qquad ; \; n = 1,2,3,....$$

$$\Rightarrow \quad y(x) = B\sin\left(\frac{n\pi}{L}\right)x \qquad ; \; n = 1,2,3,....$$

Cauchy Euler Equation:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x\frac{dy}{dx} + a_0 y = f(x)$$

This is called Cauchy Euler equation or Equidimensional equation.

Question: Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + n^2 y = 0$

Solution: let $y = x^m$

$$\frac{dy}{dx} = mx^{m-1}$$

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} + xmx^{m-1} + n^2x^m = 0$$

$$\left(m^2 - m\right)x^{m-2+2} + mx^{m-1+1} + n^2x^m = 0$$

$$\left(m^2 - m + m + n^2\right)x^m = 0$$

$$\left(m^2 + n^2\right)x^m = 0$$

$$m^2 + n^2 = 0 \ , \ x^m \neq 0$$

$$m^2 = -n^2$$

$$\Rightarrow m = \pm in$$

$$\Rightarrow \quad y(x) = Ax^{ni} + Bx^{-ni}$$

$$\Rightarrow y(x) = Ae^{\ln x^{ni}} + Be^{\ln x^{-ni}}$$

$$\Rightarrow y(x) = Ae^{i(n \ln x)} + Be^{-i(n \ln x)}$$

$$\Rightarrow y(x) = A\left[\cos(n \ln x) + i \sin(n \ln x)\right] + B\left[\cos(n \ln x) - i \sin(n \ln x)\right]$$

$$\Rightarrow \quad y(x) = (A+B)\cos(n \ln x) + i(A-B)\sin(n \ln x)$$

$$\Rightarrow \quad y(x) = C\cos(n \ln x) + iD\sin(n \ln x) \quad \because A + B = C \ , \ i(A-B) = D$$

$\because a = e^{\ln a}$

Proof:

Let $z = e^{\ln a}$

$\ln z = \ln e^{\ln a}$

$\ln z = \ln a \ln e$

$\ln z = \ln a$

$z = a$

$\Rightarrow a = e^{\ln a}$

Lecture # 02

Question: Solve the PDE $\phi_{xx} = \alpha^2\phi_t$

B.C: $\phi(0,t) = 0 = \phi(L,t)$

I.C: $\phi(x,0) = f(x)$

Solution: The given PDE $\phi_{xx} = \alpha^2\phi_t$ ______(i)

Suppose that the solution of equation (i) is

$$\phi(x,t) = X(x)T(t) \quad \_\_\_\_(ii)$$

Substituting (ii) in (i), we have

$$\frac{\partial^2}{\partial x^2}(XT) = \alpha^2\frac{\partial}{\partial t}(XT)$$

$$\Rightarrow T\frac{d^2X}{dx^2} = \alpha^2 X\frac{dT}{dt}$$

Divide both side by XT

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{\alpha^2}{T}\frac{dT}{dt} \quad \_\_\_(iii)$$

L.H.S of (iii) is a function of X and R.H.S is of T. This only possible if both functions are equation to some constant (say $-\lambda^2$).

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{\alpha^2}{T}\frac{dT}{dt} = -\lambda^2 \quad \_\_\_(iv)$$

From (iv) $\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$$\Rightarrow \left(\frac{d^2}{dx^2} + \lambda^2\right)X = 0$$

The auxiliary equation $\frac{d^2}{dx^2} + \lambda^2 = 0$

$$\frac{d}{dx} = \pm i\lambda$$

$$\Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \_\_\_\_ (v)$$

Transformed B.C X(0) = 0 = X(L)

(v) $\Rightarrow$ 0 = Acos0 + Bsin0 $\Rightarrow$ A = 0

(v) $\Rightarrow$ X(x) = B sin(λ x) _____ (vi)

$$0 = B\sin(\lambda L) \Rightarrow B \neq 0 \ , \ \sin \lambda L = 0$$

$$\Rightarrow \lambda L = n\pi \quad ; \ n = 1,2,3,....$$

$$\Rightarrow \lambda = \frac{n\pi}{L} \quad ; \ n = 1,2,3,....$$

(vi) $\Rightarrow$

$$X(x) = B\sin\left(\frac{n\pi x}{L}\right)$$

Again from (iv)

$$\frac{\alpha^2}{T}\frac{dT}{dt} = -\lambda^2$$

$$\frac{dT}{dt} + \frac{\lambda^2}{\alpha^2}T = 0$$

$$\left(\frac{d}{dt} + \frac{\lambda^2}{\alpha^2}\right)T = 0$$

The auxiliary equation

$$\frac{d}{dt} + \frac{\lambda^2}{\alpha^2} = 0$$

$$\frac{d}{dt} = -\frac{\lambda^2}{\alpha^2}$$

$$\Rightarrow T(t) = Ce^{-\frac{\lambda^2}{\alpha^2}t}$$

Put the value of λ

$$\Rightarrow T(t) = Ce^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}$$

Put the value of X(x) , T(t) in (ii)

$$\phi(x,t) = B\sin\left(\frac{n\pi x}{L}\right).Ce^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}$$

$$\phi(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t} \qquad \because BC = A_n$$

Now by applying Principle of superposition

$$\phi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t} \qquad \_\_\_\_(viii)$$

Put t = 0 in (viii)

$$\phi(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)e^{0}$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ both side and integrate w.r.t x from 0 → L, we get

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{-1}{2}\sum_{n=1}^{\infty} A_n \int_0^L -2.\sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{-1}{2}\sum_{n=1}^{\infty}A_n\left(\int_0^L \cos(n+m)\frac{\pi x}{L} - \cos(n-m)\frac{\pi x}{L}\right)dx$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{-1}{2}\sum_{n=1}^{\infty}A_n\left|\frac{\sin(n+m)\frac{\pi x}{L}}{(n+m)} - \frac{\sin(n-m)\frac{\pi x}{L}}{(n-m)}\right|_0^L$$

Where n , m = 1,2,3,….

Using orthogonality condition.

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = 0 \quad if \quad m \neq n$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}A_n\int_0^L \sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{n\pi x}{L}\right)dx \quad if \quad m = n$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}A_n\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}A_n\int_0^L \left(\frac{1-\cos 2\left(\frac{n\pi x}{L}\right)}{2}\right)dx$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\sum_{n=1}^{\infty}A_n\left|x - \frac{\sin 2\left(\frac{n\pi x}{L}\right)}{\frac{2n\pi}{L}}\right|_0^L$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\sum_{n=1}^{\infty}A_n(L-0)$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\sum_{n=1}^{\infty}A_n$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

Put in (viii)

$$\phi(x,t) = \sum_{n=1}^{\infty}\left[\left\{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}\right]$$

Which is the required solution of the given PDE.

Question: Solve the PDE $\phi_{xx} = \alpha^2\phi_{tt}$

B.C: $\phi(0,t) = 0 = \phi(\mathrm{L},t)$

I.C: $\phi(x,0) = f(x)$ & $\phi_t(x,0) = g(x)$

Solution: The given wave equation $\phi_{xx} = \alpha^2\phi_{tt}$ _____(i)

Suppose that the solution of equation (i) is

$$\phi(x,t) = X(x)T(t) \quad \_\_\_\_(ii)$$

Substituting (ii) in (i), we have

$$\frac{\partial^2}{\partial x^2}(XT) = \alpha^2\frac{\partial}{\partial t}(XT)$$

$$\Rightarrow T\frac{d^2X}{dx^2} = \alpha^2 X\frac{dT}{dt}$$

Divide both side by XT

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{\alpha^2}{T}\frac{dT}{dt}$$

L.H.S of (iii) is a function of X and R.H.S is of T. This only possible if both functions are equation to some constant (say $-\lambda^2$).

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{\alpha^2}{T}\frac{dT}{dt} = -\lambda^2 \quad \_\_\_(iii)$$

From (iii) $\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$$\Rightarrow \left(\frac{d^2}{dx^2} + \lambda^2\right)X = 0$$

The auxiliary equation

$$\frac{d^2}{dx^2} + \lambda^2 = 0$$

$$\frac{d}{dx} = \pm i\lambda$$

$$\Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \_\_\_\_ (iv)$$

Transformed B.C X(0) = 0

(iv) $\Rightarrow$ $0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$

(iv) $\Rightarrow$ $X(x) = B\sin(\lambda x)$ _____ (v)

Put X(L) = 0

$0 = B\sin(\lambda L) \Rightarrow B \neq 0$, $\sin \lambda L = 0$

$$\Rightarrow \lambda L = n\pi \quad ; n = 1,2,3,....$$

$$\Rightarrow \lambda = \frac{n\pi}{L} \quad ; n = 1,2,3,....$$

(v) $\Rightarrow$ $X(x) = B\sin\left(\frac{n\pi x}{L}\right)$

Again from (iii) $\frac{\alpha^2}{T}\frac{d^2T}{dt^2} = -\lambda^2$

$$\frac{d^2T}{dt^2} + \frac{\lambda^2}{\alpha^2}T = 0$$

$$\left(\frac{d^2}{dt^2}+\frac{\lambda^2}{\alpha^2}\right)T=0$$

The auxiliary equation

$$\frac{d^2}{dt^2}+\frac{\lambda^2}{\alpha^2}=0$$

$$\frac{d^2}{dt^2}=-\frac{\lambda^2}{\alpha^2}$$

$$\frac{d}{dt}=\pm i\frac{\lambda}{\alpha}$$

$$\Rightarrow T(t)=C\cos\left(\frac{\lambda}{\alpha}t\right)+D\sin\left(\frac{\lambda}{\alpha}t\right)$$

$$\because \lambda=\frac{n\pi}{L} \quad T(t)=C\cos\left(\frac{n\pi}{\alpha L}\right)t+D\sin\left(\frac{n\pi}{\alpha L}\right)t \qquad ;\ n=1,2,3,....$$

Put the value X(x), T(t) in (ii)

$$\phi(x,t)=B\sin\left(\frac{n\pi x}{L}\right).\left[C\cos\left(\frac{n\pi}{\alpha L}\right)t+D\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

$$\phi(x,t)=\sin\left(\frac{n\pi x}{L}\right)\left[BC\cos\left(\frac{n\pi}{\alpha L}\right)t+BD\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

$$\phi(x,t)=\sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi}{\alpha L}\right)t+B_n\sin\left(\frac{n\pi}{\alpha L}\right)t\right] \quad \because BC=A_n \ \& BD=B_n$$

Now by applying Principle of superposition

$$\phi(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi}{\alpha L}\right)t+B_n\sin\left(\frac{n\pi}{\alpha L}\right)t\right] \qquad \_\_\_\_(vii)$$

Transformed I.C, t = 0 $\quad \phi(x,0)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos(0)+B_n\sin(0)\right]$

$$f(x)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)[\ A_n+0\ ]$$

$$f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ both side and integrate w.r.t x from $0 \to$ L, we get

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \ =\int_0^L\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \ =\sum_{n=1}^{\infty}A_n\int_0^L\sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{m\pi x}{L}\right)dx$$

By orthogonality principle

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx=0\quad if\ \ n\neq m$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx=\frac{L}{2}A_m\quad if\ \ n=m$$

$$A_m=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

Put the value of A_n in (vii)

$$\phi(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{m\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx\ \right\}\cos\left(\frac{m\pi}{\alpha L}\right)t+B_n\sin\left(\frac{m\pi}{\alpha L}\right)t\ \right]\_\_\_(viii)$$

Now differentiate (vii) w.r.t 't'

$$\phi_t(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{m\pi x}{L}\right)\left[-A_n\left(\frac{m\pi}{\alpha L}\right)\sin\left(\frac{m\pi}{\alpha L}\right)t+B_n\left(\frac{m\pi}{\alpha L}\right)\cos\left(\frac{m\pi}{\alpha L}\right)t\ \right]$$

Applying I.C $\ \phi_t(x,0)=g(x)$

$$\phi_t(x,0)=\sum_{n=1}^{\infty}\sin\left(\frac{m\pi x}{L}\right)\left[-A_m\left(\frac{m\pi}{\alpha L}\right)\sin(0)+B_m\left(\frac{m\pi}{\alpha L}\right)\cos(0)\right]$$

$$g(x)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right).B_n\left(\frac{n\pi}{\alpha L}\right)$$

Multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ both side and integrate w.r.t x from 0 $\rightarrow$ L, Also applying the orthogonality principle,

$$\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx=\int_0^L\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{\alpha L}\right)\sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx=0 \quad if \;\; m\neq n$$

$$\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx=\frac{L}{2}B_n\left(\frac{n\pi}{\alpha L}\right) \quad if \;\; m=n$$

$$B_n=\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Put the value of B_n in (viii)

$$\phi(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\cos\left(\frac{n\pi}{\alpha L}\right)t+\left\{\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

is the required solution of the given PDE.

Lecture # 03

Question: Solve the PDE $\phi_{xx} + \phi_{yy} = 0 \quad with$

B.C: $\phi(0,\mathrm{y}) = 0 = \phi(\mathrm{a},\mathrm{y})$

I.C: $\phi(x,0) = 0$, $\phi_y(x,\mathrm{b}) = f(x)$

Solution: The given PDE is $\phi_{xx} + \phi_{yy} = 0$ _____ (i)

Suppose that the solution of equation (i) is

$$\phi(x,y) = X(x)Y(y) \quad \_\_\_\_(ii)$$

Substituting (ii) in (i), we have

$$\frac{\partial^2}{\partial x^2}(XY) + \frac{\partial^2}{\partial y^2}(XY) = 0$$

$$\Rightarrow Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both side by XY

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda^2\ (say) \quad \_\_\_(iii)$$

From (iii)
$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\frac{d^2X}{dx^2} = -\lambda^2 X$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$$\Rightarrow \left(\frac{d^2}{dx^2} + \lambda^2\right)X = 0$$

The auxiliary equation

$$\frac{d^2}{dx^2}+\lambda^2=0$$

$$\frac{d}{dx}=\pm i\lambda$$

$$\Rightarrow X(x)=A\cos(\lambda x)+B\sin(\lambda x) \quad \_\_\_\_(iv)$$

Transformed B.C X(0) = 0

$0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$

(iv) $\Rightarrow$ $X(x) = B\sin(\lambda x)$ _____ (v)

Put X(a) = 0

$0 = B\sin(\lambda a) \Rightarrow B \neq 0$, $\sin \lambda a = 0$

$$\Rightarrow \lambda a = n\pi \quad ; \; n=1,2,3,....$$

$$\Rightarrow \lambda=\frac{n\pi}{a} \quad ; \; n=1,2,3,....$$

(v) $\Rightarrow$ $$X(x)=B\sin\left(\frac{n\pi x}{a}\right) \quad \_\_\_\_\_(vi)$$

Again from (iii) $$-\frac{1}{Y}\frac{d^2Y}{dy^2}=-\lambda^2$$

$$\frac{d^2Y}{dy^2}=\lambda^2 Y$$

$$\Rightarrow \frac{d^2Y}{dy^2}-\lambda^2 Y=0$$

$$\Rightarrow \left(\frac{d^2}{dy^2}-\lambda^2\right)Y=0$$

The auxiliary equation $$\frac{d^2}{dy^2}-\lambda^2=0$$

$$\Rightarrow \frac{d^2}{dy^2} = \lambda^2$$

$$\Rightarrow \frac{d}{dy} = \pm\lambda$$

$$\Rightarrow Y(y) = C\cosh(\lambda y) + D\sinh(\lambda y) \quad \_\_\_\_(vii) \quad \because \textit{in hyperbolic form}$$

Put B.C Y(0) = 0

(vii) $\Rightarrow$ $0 = C\cosh 0 + D\sinh 0 \Rightarrow C = 0$

(vii) $\Rightarrow$ $Y(y) = D\sinh(\lambda y)$ _____ (viii)

Put Y(a) = 0

$$0 = D\sinh(\lambda a) \;\Rightarrow D \neq 0 \;,\; \sinh(\lambda a) = 0$$

$$\Rightarrow \lambda a = n\pi \quad ; \; n = 1,2,3,....$$

$$\Rightarrow \lambda = \frac{n\pi}{a} \quad ; \; n = 1,2,3,....$$

(viii) $\Rightarrow$ $$Y(y) = D\sinh\left(\frac{n\pi y}{a}\right)$$

Put the value of X(x) , Y(y) in (ii)

$$\phi(x,y) = B\sin\left(\frac{n\pi x}{a}\right).D\sinh\left(\frac{n\pi y}{a}\right)$$

$$\phi(x,y) = A_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \quad \because BD = A_n \; ; \; n = 1,2,3,...$$

Now by the principle of superposition

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \quad \_\_\_(ix)$$

$$\phi_y(x,y) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{a}\right).\left(\frac{n\pi}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)$$

$$\phi_y(x,\text{b}) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right).\left(\frac{n\pi}{a}\right)\cosh\left(\frac{n\pi b}{a}\right)$$

$$\Rightarrow \; f(x) = \sum_{n=1}^{\infty} \frac{n\pi}{a}.A_n \cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right) \_\_\_\_ (x)$$

Multiplying by $\sin\left(\frac{m\pi x}{a}\right)$ both side and integrate w.r.t x from $0 \to$ a,

$$\int_0^a f(x)\sin\left(\frac{m\pi x}{a}\right)dx = \sum_{n=1}^{\infty} \frac{n\pi}{a}.A_n \cosh\left(\frac{n\pi b}{a}\right)\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx$$

Using orthogonality condition

$= 0$ if $m \neq n$

$$= \frac{m\pi}{a}\text{A}_n\cosh\left(\frac{n\pi b}{a}\right).\frac{a}{2} \qquad if \;\; m = n$$

$$\text{A}_n = \frac{2}{m\pi \;\; \cosh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{m\pi x}{a}\right)dx$$

Put in (ix)

$$\phi(x,y) = \sum_{n=1}^{\infty}\left[\frac{2}{m\pi \;\; \cosh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{m\pi x}{a}\right)dx\right]\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

Where n = m = 1,2,3,…..

Which is the required solution of given PDE.

Question: Solve the PDE $\phi_{xx} + \phi_{yy} + \phi_{zz} = \alpha^2 \phi_t$

B.C: $\phi(0, y, z, t) = 0 = \phi(a, y, z, t)$

$\phi(x, 0, z, t) = 0 = \phi(x, b, z, t)$

$\phi(x, y, 0, t) = 0 = \phi(x, y, c, t)$

I.C: $\phi(x, y, z, 0) = f(x, y, z)$

Solution: The given PDE is $\phi_{xx} + \phi_{yy} + \phi_{zz} = \alpha^2 \phi_t$ _____ (i)

Suppose that the solution of equation (i) is

$$\phi(x, y, z, t) = X(x)Y(y)Z(z)T(t) \quad \_\_\_\_(ii)$$

Substituting (ii) in (i), we have

$$\frac{\partial^2}{\partial x^2}(XYZT) + \frac{\partial^2}{\partial y^2}(XYZT) + \frac{\partial^2}{\partial z^2}(XYZT) = \alpha^2 \frac{\partial}{\partial t}(XYZT)$$

$$YZT\frac{\partial^2 X}{\partial x^2} + XZT\frac{\partial^2 Y}{\partial y^2} + XYT\frac{\partial^2 Z}{\partial z^2} = \alpha^2 XYZ\frac{\partial T}{\partial t}$$

Divide both side by XYZT

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = \frac{\alpha^2}{T}\frac{dT}{dt}$$

$$\Rightarrow \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} - \frac{\alpha^2}{T}\frac{dT}{dt} = -\frac{1}{X}\frac{d^2 X}{dx^2} = \lambda^2 \, (say) \quad \_\_\_(iii)$$

From (iii) $$\Rightarrow -\frac{1}{X}\frac{d^2 X}{dx^2} = \lambda^2$$

$$\Rightarrow \frac{d^2 X}{dx^2} + \lambda^2 X = 0$$

$$\Rightarrow \left(\frac{d^2}{dx^2} + \lambda^2\right) X = 0$$

The auxiliary equation

$$\frac{d^2}{dx^2}+\lambda^2=0$$

$$\frac{d}{dx}=\pm i\lambda$$

$$\Rightarrow X(x)=A\cos(\lambda x)+B\sin(\lambda x) \qquad \_\_\_\_(iv)$$

Transformed B.C X(0) = 0 = X(a)

$$0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$$

(iv) $\Rightarrow$ $X(x) = B\sin(\lambda x)$

Put X(a) = 0

$$0 = B\sin(\lambda a) \Rightarrow B\neq 0\ ,\ \sin\lambda a = 0$$

$$\Rightarrow \lambda a = l\pi$$

$$\Rightarrow \lambda = \frac{l\pi}{a}$$

(v) $\Rightarrow$ $$X(x)=B\sin\left(\frac{l\pi x}{a}\right) \qquad \_\_\_\_\_(v)$$

Again from (iii) $$\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}-\frac{\alpha^2}{T}\frac{dT}{dt}=\lambda^2$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}-\frac{\alpha^2}{T}\frac{dT}{dt}-\lambda^2=-\frac{1}{Y}\frac{d^2Y}{dy^2}=\mu^2 \qquad \_\_\_(vi)$$

From (vi) $$-\frac{1}{Y}\frac{d^2Y}{dy^2}=\mu^2$$

$$\Rightarrow \frac{d^2Y}{dy^2}+\mu^2Y=0$$

$$\Rightarrow \left(\frac{d^2}{dy^2}+\mu^2\right)Y=0$$

The auxiliary equation

$$\frac{d^2}{dy^2}+\mu^2=0$$

$$\Rightarrow \frac{d}{dy}=\pm i\mu$$

$$\Rightarrow Y(y)=C\cos(\mu y)+D\sin(\mu y) \quad \_\_\_\_(vii)$$

Transform B.C

C = 0 , $\mu=\frac{n\pi}{b}$, $m=1,2,3,.....$

$$Y(y)=D\sin\left(\frac{m\pi y}{b}\right)\_\_\_\_\_(viii)$$

From (vi) $\Rightarrow$ $\frac{1}{Z}\frac{d^2Z}{dz^2}-\frac{\alpha^2}{T}\frac{dT}{dt}-\lambda^2=\mu^2$

$$-\frac{\alpha^2}{T}\frac{dT}{dt}-\lambda^2-\mu^2=-\frac{1}{Z}\frac{d^2Z}{dz^2}=\nu^2 \quad \_\_\_(ix)$$

From (ix) $-\frac{1}{Z}\frac{d^2Z}{dz^2}=\nu^2$

$$\frac{d^2Z}{dz^2}+\nu^2 Z=0$$

$$\left(\frac{d^2}{dz^2}+\nu^2\right)Z=0$$

The auxiliary equation

$$\frac{d^2}{dz^2}+\nu^2=0$$

$$\frac{d}{dz}=\pm i\nu$$

$$Z(z)=E\cos(\nu z)+F\sin(\nu z) \quad \_\_\_\_(x)$$

Transformed B.C $\quad Z(0) = , Z(c) = 0$

$$\Rightarrow E = 0 \;, \nu = \frac{n\pi}{c} \;\; ; \; n = 1,2,3,....$$

$$\Rightarrow Z(z) = F\sin\left(\frac{n\pi z}{c}\right) \quad \_\_\_\_\_ (xi)$$

From (ix)
$$\frac{-\alpha^2}{T}\frac{dT}{dt} - \lambda^2 - \mu^2 = \nu^2$$

$$\frac{dT}{dt}\left(\frac{\lambda^2 + \mu^2 + \nu^2}{\alpha^2}\right)T = 0$$

$$\left[\frac{d}{dt}\left(\frac{\lambda^2 + \mu^2 + \nu^2}{\alpha^2}\right)\right]T = 0$$

The auxiliary equation

$$\frac{d}{dt}\left(\frac{\lambda^2 + \mu^2 + \nu^2}{\alpha^2}\right) = 0$$

$$\Rightarrow \frac{d}{dt} = -\left(\frac{\lambda^2 + \mu^2 + \nu^2}{\alpha^2}\right)$$

$$\Rightarrow T(t) = Ge^{-\left(\frac{\lambda^2 + \mu^2 + \nu^2}{\alpha^2}\right)t}$$

Using the value of $\lambda, \mu \, and \, \nu$

$$\Rightarrow T(t) = Ge^{-\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\frac{\pi^2}{\alpha^2}t} \quad \_\_\_\_ (xii)$$

Put the value of X(x), Y(y), Z(z) & T(t) in (ii)

$$\phi(x,y,z,t) = B\sin\left(\frac{l\pi x}{a}\right)\mathrm{D}\sin\left(\frac{m\pi y}{b}\right)\mathrm{F}\sin\left(\frac{n\pi z}{c}\right)Ge^{-\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\frac{\pi^2}{\alpha^2}t} \quad ; l,m,n = 1,2,3,...$$

Let BDFG $= A_l B_m C_n$

$$\phi(x,y,z,t)=A_lB_mC_n\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)e^{-\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\frac{\pi^2}{\alpha^2}t}\quad ;l,m,n=1,2,3,...$$

By the Principle of Superposition

$$\phi(x,y,z,t)=\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_lB_mC_n\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)e^{-\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\frac{\pi^2}{\alpha^2}t}$$

Let $A_lB_mC_n = P_{lmn}$

$$\phi(x,y,z,t)=\sum_{l,m,n=1}^{\infty}P_{lmn}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)e^{-\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\frac{\pi^2}{\alpha^2}t}\quad \_\_\_\_(xiii)$$

Put I.C ; t = 0 $\quad \phi(x,y,z,0)=\sum_{l,m,n=1}^{\infty}P_{lmn}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)e^{0}$

$$f(x,y,z)=\sum_{l,m,n=1}^{\infty}P_{lmn}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\_\_\_\_(xiv)$$

Multiplying (iv) by $\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)$ both side and integrate

w.r.t x,y,z form 0 $\rightarrow$ a , 0$\rightarrow$b ,0$\rightarrow$c and using condition of orthogonality.

$$\int_0^a\int_0^b\int_0^c f(x,y,z)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)dzdydx=P_{lmn}\int_0^a\sin^2\left(\frac{l\pi x}{a}\right)dx\int_0^b\sin^2\left(\frac{m\pi y}{b}\right)dx\int_0^c\sin^2\left(\frac{n\pi z}{c}\right)dz$$

$$\int_0^a\int_0^b\int_0^c f(x,y,z)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)dzdydx=P_{lmn}\frac{a}{2}.\frac{b}{2}.\frac{c}{2}$$

$$P_{lmn}=\frac{8}{abc}\int_0^a\int_0^b\int_0^c f(x,y,z)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)dzdydx$$

Put in (xiii)

$$\phi(x,y,z,t)=\sum_{l,m,n=1}^{\infty}\left[\left\{\frac{8}{abc}\int_0^a\int_0^b\int_0^c f(x,y,z)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)dzdydx\right\}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right)e^{-\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\frac{\pi^2}{\alpha^2}t}\right]$$

Which is the required solution of given PDE.

Question: Solve the PDE $\phi_{xxxx} + \phi_{yyyy} = \alpha^2 \phi_{tt}$

Solution: The given PDE is $\phi_{xxxx} + \phi_{yyyy} = \alpha^2 \phi_{tt}$ _____(i)

Suppose that the solution of equation (i) is

$$\phi(x,y,t) = X(x)Y(y)T(t) \quad \_\_\_\_(ii)$$

Substituting (ii) in (i), we have

$$\frac{\partial^4}{\partial x^4}(XYT) + \frac{\partial^4}{\partial y^4}(XYT) = \alpha^2 \frac{\partial^2}{\partial t^2}(XYT)$$

$$YT\frac{\partial^4 X}{\partial x^4} + XT\frac{\partial^4 Y}{\partial y^4} = \alpha^2 XZ\frac{\partial^2 T}{\partial t^2}$$

Divide both side by XYT

$$\Rightarrow \frac{1}{X}\frac{\partial^4 X}{\partial x^4} + \frac{1}{Y}\frac{\partial^4 Y}{\partial y^4} = \frac{\alpha^2}{T}\frac{\partial^2 T}{\partial t^2} = \lambda^4 \quad \_\_\_(iii)$$

From (iii) $\Rightarrow \frac{\alpha^2}{T}\frac{\partial^2 T}{\partial t^2} = \lambda^4$

$$\Rightarrow \frac{\partial^2 T}{\partial t^2} - \frac{T}{\alpha^2}\lambda^4 = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} - \frac{\lambda^4}{\alpha^2}\right)T = 0$$

The auxiliary equation

$$\frac{\partial^2}{\partial t^2} - \frac{\lambda^4}{\alpha^2} = 0$$

$$\frac{d}{dt} = \pm\frac{\lambda^2}{\alpha}$$

$$\Rightarrow T(t) = A_1 e^{\frac{\lambda^2}{\alpha}t} + A_2 e^{-\frac{\lambda^2}{\alpha}t} \quad \_\_\_\_(iv)$$

From (iii) $$\frac{1}{X}\frac{\partial^4 X}{\partial x^4}+\frac{1}{Y}\frac{\partial^4 Y}{\partial y^4}=\lambda^4$$

$$\frac{1}{Y}\frac{\partial^4 Y}{\partial y^4}-\lambda^4=-\frac{1}{X}\frac{\partial^4 X}{\partial x^4}=-\mu^4 \quad \_\_\_(v)$$

From (v) $$-\frac{1}{X}\frac{\partial^4 X}{\partial x^4}=-\mu^4$$

$$\Rightarrow \frac{d^4 X}{dx^4}-\mu^4 X=0$$

$$\Rightarrow \left(\frac{d^4}{dx^4}-\mu^4\right)X=0$$

The auxiliary equation

$$\frac{d^4}{dx^4}-\mu^4=0$$

$$\Rightarrow \frac{d}{dx}=\pm\mu\ ,\pm i\mu$$

$$\Rightarrow X(x)=B_1e^{\mu x}+B_2e^{-\mu x}+B_3\cos(\mu x)+B_4\sin(\mu x) \quad \_\_\_(vi)$$

From (v)

$$\frac{1}{Y}\frac{\partial^4 Y}{\partial y^4}-\lambda^4=-\mu^4$$

$$\Rightarrow \frac{d^4 Y}{dy^4}+\left(\mu^4-\lambda^4\right)Y=0 \quad \_\_\_\_(vii)$$

Here arise three cases.

Case-I:

$$\mu^4-\lambda^4<0$$

Let $\mu^4-\lambda^4=-\beta^4$

Equation (vii) becomes

$$\frac{d^4Y}{dy^4}-\beta^4Y=0$$

$$\left(\frac{d^4}{dy^4}-\beta^4\right)Y=0$$

The auxiliary equation

$$\frac{d^4}{dy^4}-\beta^4=0$$

$$\Rightarrow\frac{d}{dy}=\pm\beta\ ,\pm i\beta$$

$$\Rightarrow Y(y)=C_1e^{\beta y}+C_2e^{-\beta y}+C_3\cos(\beta y)+C_4\sin(\beta y)\quad \_\_\_(vii)$$

Case-II:

$$\mu^4-\lambda^4=0$$

Equation (vii) becomes

$$\frac{d^4Y}{dy^4}=0$$

The auxiliary equation

$$\frac{d^4}{dy^4}=0$$

$$\Rightarrow\frac{d}{dy}=0,0,0,0$$

$$Y(y)=\left(D_1+D_2y+D_3y^2+D_4y^3\right)e^0$$

$$Y(y)=D_1+D_2y+D_3y^2+D_4y^3\qquad \_\_\_\_\_\_(ix)$$

Case-I:

$$\mu^4-\lambda^4<0$$

Let $\mu^4-\lambda^4=\nu^4$

Equation (vii) becomes

$$\frac{d^4Y}{dy^4}+\nu^4Y=0$$

$$\left(\frac{d^4}{dy^4}+\nu^4\right)Y=0$$

The Auxiliary equation

$$\frac{d^4}{dy^4}+\nu^4=0$$

$$\Rightarrow \frac{d}{dy}=\frac{\nu}{\sqrt{2}}\pm i\frac{\nu}{\sqrt{2}},\frac{-\nu}{\sqrt{2}}\pm i\frac{\nu}{\sqrt{2}}$$

$$\Rightarrow Y(y)=e^{\frac{\nu}{\sqrt{2}}y}\left[E_1\cos\left(\frac{\nu y}{\sqrt{2}}\right)+E_2\sin\left(\frac{\nu y}{\sqrt{2}}\right)\right]+e^{\frac{-\nu}{\sqrt{2}}y}\left[E_3\cos\left(\frac{\nu y}{\sqrt{2}}\right)+E_4\sin\left(\frac{\nu y}{\sqrt{2}}\right)\right]\_\_\_(x)$$

Equation (ii) becomes

$$\phi(x,y,t)=\left[A_1\,e^{\frac{\lambda^2}{\alpha}t}+A_2\,e^{-\frac{\lambda^2}{\alpha}t}\right].\left[B_1e^{\mu x}+B_2e^{-\mu x}+B_3\cos(\mu x)+B_4\sin(\mu x)\right].$$

$$\left[C_1e^{\beta y}+C_2e^{-\beta y}+C_3\cos(\beta y)+C_4\sin(\beta y)\right]\_\_\_\_(xi)$$

$$\phi(x,y,t)=\left[A_1\,e^{\frac{\lambda^2}{\alpha}t}+A_2\,e^{-\frac{\lambda^2}{\alpha}t}\right].\left[B_1e^{\mu x}+B_2e^{-\mu x}+B_3\cos(\mu x)+B_4\sin(\mu x)\right].$$

$$\left[D_1+D_2y+D_3y^2+D_4y^3\right]\_\_\_\_\_(xii)$$

$$\phi(x,y,t)=\left[A_1\,e^{\frac{\lambda^2}{\alpha}t}+A_2\,e^{-\frac{\lambda^2}{\alpha}t}\right].\left[B_1e^{\mu x}+B_2e^{-\mu x}+B_3\cos(\mu x)+B_4\sin(\mu x)\right].$$

$$\left[e^{\frac{\nu}{\sqrt{2}}y}\left\{E_1\cos\left(\frac{\nu y}{\sqrt{2}}\right)+E_2\sin\left(\frac{\nu y}{\sqrt{2}}\right)\right\}+e^{\frac{-\nu}{\sqrt{2}}y}\left\{E_3\cos\left(\frac{\nu y}{\sqrt{2}}\right)+E_4\sin\left(\frac{\nu y}{\sqrt{2}}\right)\right\}\right]\_\_(xiii)$$

Equation (xi) , (xii) , (xiii) are the required solution of given PDE.

Lecture # 04

Remark: $a\phi_{xx}+b\phi_{xy}+c\phi_{yy}+d\phi_x+e\phi_y+f\phi=g(x,y)$

It is homogeneous if g(x,y) = 0 and Non-Homogeneous if g(x,y) $\neq$ 0

Question: Solve the PDE $\phi_{xx}=\alpha^2\phi_t$

B.C $\phi(0,t)=0$, $\phi(\mathrm{L},t)=P$

I.C $\phi(x,0)=f(x)$

Solution: Given PDE $\phi_{xx}=\alpha^2\phi_t$ ____ (i)

We convert the given non-homogeneous problem into homogeneous by considering the following transformation.

$$\phi(x,\mathrm{t})=\psi(x,t)+\frac{x}{L}P \quad \_\_\_(ii)$$

Put x = 0 on (i)

$$\phi(0,\mathrm{t})=\psi(0,t)+0$$

$$\Rightarrow \quad 0=\psi(0,t) \qquad \because \phi(0,\mathrm{t})$$

Put x = L in (ii)

$$\phi(\mathrm{L},\mathrm{t})=\psi(\mathrm{L},t)+\frac{L}{L}P$$

$$P=\psi(\mathrm{L},t)+P \quad \Rightarrow \psi(\mathrm{L},t)=0$$

Put t = 0 in (ii)

$$\phi(x,0)=\psi(x,0)+\frac{x}{L}P$$

$$f(x)=\psi(x,0)+\frac{x}{L}P$$

$$\Rightarrow \psi(x,0)= f(x)-\frac{x}{L}P= f_1(x)\,(say)$$

$$\Rightarrow \psi(x,0) = f_1(x)\ (say)$$

Substitute (ii) in (i)

$$\frac{\partial^2}{\partial x^2}\left[\psi(x,t) + \frac{x}{L}P\right] = \alpha^2 \frac{\partial}{\partial t}\left[\psi(x,t) + \frac{x}{L}P\right]$$

$$\frac{\partial^2 \psi}{\partial x^2} = \alpha^2 \frac{\partial \psi}{\partial t} \quad \_\_\_(iii)$$

With B.C $\psi(0,t) = 0 = \psi(L,t)$

I.C $\psi(x,0) = f_1(x)$

Suppose that solution of equation (iii) is

$$\psi(x,t) = X(x)T(t) \quad \_\_\_\_\_(iv)$$

Substitute (iv) in (iii) we have

$$\frac{\partial^2}{\partial x^2}(XT) = \alpha^2 \frac{\partial}{\partial t}(XT)$$

$$\Rightarrow T\frac{d^2X}{dx^2} = \alpha^2 X \frac{dT}{dt}$$

Divide both side by XT

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{\alpha^2}{T}\frac{dT}{dt}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{\alpha^2}{T}\frac{dT}{dt} = -\lambda^2 \quad \_\_\_(v)$$

From (v) $$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$$\Rightarrow \left(\frac{d^2}{dx^2} + \lambda^2\right)X = 0$$

The auxiliary equation

$$\frac{d^2}{dx^2}+\lambda^2 = 0$$

$$\frac{d}{dx}=\pm i\lambda$$

$$\Rightarrow X(x)=A\cos(\lambda x)+B\sin(\lambda x)$$

Transformed B.C X(0) = 0 = X(L)

(v) $\Rightarrow$ 0 = Acos0 + Bsin0 $\Rightarrow$ A = 0

(v) $\Rightarrow$ X(x) = B sin(λ x) _____ (vi)

$$0 = B\sin(\lambda L) \Rightarrow B\neq 0 \ , \ \sin\lambda L = 0$$

$$\Rightarrow \lambda L = n\pi \quad ; \ n = 1,2,3,....$$

$$\Rightarrow \lambda = \frac{n\pi}{L} \quad ; \ n = 1,2,3,....$$

$$\Rightarrow X(x)=B\sin\left(\frac{n\pi x}{L}\right)$$

Again from (v)

$$\frac{\alpha^2}{T}\frac{dT}{dt}=-\lambda^2$$

$$\frac{dT}{dt}+\frac{\lambda^2}{\alpha^2}T=0$$

$$\left(\frac{d}{dt}+\frac{\lambda^2}{\alpha^2}\right)T=0$$

The auxiliary equation

$$\frac{d}{dt}+\frac{\lambda^2}{\alpha^2}=0$$

$$\frac{d}{dt}=-\frac{\lambda^2}{\alpha^2}$$

$$\Rightarrow T(t) = Ce^{-\frac{\lambda^2}{\alpha^2}t}$$

Put the value of λ

$$\Rightarrow T(t) = Ce^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}$$

Put the value of X(x) , T(t) in (iv)

$$\psi(x,t) = B\sin\left(\frac{n\pi x}{L}\right).Ce^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}$$

$$\psi(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t} \_\_\_(vi) \qquad \because BC = A_n$$

Now by applying Principle of superposition

$$\psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t} \qquad \_\_\_\_(vii)$$

Put t = 0 in (viii)

$$\psi(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)e^{0}$$

$$f_1(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ both side and integrate w.r.t x from 0 → L, and also apply orthogonality condition.

$$\int_0^L f_1(x)\sin\left(\frac{m\pi x}{L}\right)dx \quad = \int_0^L \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= 0 \qquad if \quad m \neq n$$

$$= \frac{L}{2}A_n \qquad if \;\; m = n$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

Replace 'm' by 'n'

$$\Rightarrow A_n = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Put in (vii)

$$\psi(x,t) = \sum_{n=1}^{\infty}\left[\left\{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}\right]$$

Put the value of $f_1(x) = f(x) - \frac{x}{L}P$

$$\psi(x,t) = \sum_{n=1}^{\infty}\left[\left\{\frac{2}{L}\int_0^L \left(f(x) - \frac{x}{L}P\right)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}\right]$$

Put the value of ψ (x,t) in (ii)

$$\phi(x,\mathrm{t}) = \sum_{n=1}^{\infty}\left[\left\{\frac{2}{L}\int_0^L \left(f(x) - \frac{x}{L}P\right)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}\right] + \frac{x}{L}P$$

Question: Solve the PDE $\phi_{xx} = \alpha^2\phi_t$

B.C $\phi(0,t) = P$, $\phi(\mathrm{L},t) = q$

I.C $\phi(x,0) = f(x)$

Solution: Given PDE $\phi_{xx} = \alpha^2\phi_t$ ___(i)

We convert the given non-homogeneous problem into homogeneous by considering the following transformation.

$$\phi(x,\mathrm{t}) = \psi(x,t) + \left(\frac{L-x}{L}\right)P + \frac{x}{L}q \quad \_\_\_(ii)$$

Put x = 0 on (i)

$$\phi(0,\mathrm{t})=\psi(0,t)+\left(\frac{L-0}{L}\right)P+\frac{0}{L}q$$

$$P=\psi(0,t)+P \quad \Rightarrow \psi(0,t)=0$$

Put x = L in (ii)

$$\phi(\mathrm{L},\mathrm{t})=\psi(\mathrm{L},t)+\left(\frac{L-L}{L}\right)P+\frac{L}{L}q$$

$$q=\psi(\mathrm{L},t)+0+q \quad \Rightarrow \psi(\mathrm{L},t)=0$$

Put t = 0 in (ii)

$$\phi(x,0)=\psi(x,0)+\left(\frac{L-x}{L}\right)P+\frac{x}{L}q$$

$$f(x)=\psi(x,0)+\left(\frac{L-x}{L}\right)P+\frac{x}{L}q$$

$$\psi(x,0)=f(x)-\left(\frac{L-x}{L}\right)P-\frac{x}{L}q=f_1(x)\ (say)$$

$$\Rightarrow \psi(x,0)=f_1(x)\ (say)$$

Substitute (ii) in (i)

$$\frac{\partial^2}{\partial x^2}\left[\psi(x,t)+\left(\frac{L-x}{L}\right)P+\frac{x}{L}q\right]=\alpha^2\frac{\partial}{\partial t}\left[\psi(x,t)+\left(\frac{L-x}{L}\right)P+\frac{x}{L}q\right]$$

$$\frac{\partial^2\psi}{\partial x^2}=\alpha^2\frac{\partial\psi}{\partial t} \quad \_\_\_(iii)$$

Suppose that solution of equation (iii) is

$$\psi(x,t)=X(x)T(t) \quad \_\_\_\_\_(iv)$$

Substitute (iv) in (iii) we have

$$\frac{\partial^2}{\partial x^2}(XT)=\alpha^2\frac{\partial}{\partial t}(XT)$$

$$\Rightarrow T\frac{d^2X}{dx^2}=\alpha^2X\frac{dT}{dt}$$

Divide both side by XT

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=\frac{\alpha^2}{T}\frac{dT}{dt}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=\frac{\alpha^2}{T}\frac{dT}{dt}=-\lambda^2 \qquad \_\_\_(v)$$

From (v) $\quad \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=-\lambda^2$

$$\Rightarrow \frac{d^2X}{dx^2}+\lambda^2X=0$$

$$\Rightarrow \left(\frac{d^2}{dx^2}+\lambda^2\right)X=0$$

The auxiliary equation

$$\frac{d^2}{dx^2}+\lambda^2=0$$

$$\frac{d}{dx}=\pm i\lambda$$

$$\Rightarrow X(x)=A\cos(\lambda x)+B\sin(\lambda x)$$

Transformed B.C X(0) = 0 = X(L)

(v) $\Rightarrow$ $\quad 0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$

(v) $\Rightarrow$ $\quad X(x) = B\sin(\lambda x) \quad \_\_\_\_\_$ (vi)

$$0=B\sin(\lambda L) \Rightarrow B\neq 0\ ,\ \sin\lambda L=0$$

$$\Rightarrow \lambda L=n\pi \quad ;\ n=1,2,3,....$$

$$\Rightarrow \lambda=\frac{n\pi}{L} \quad ;\ n=1,2,3,....$$

$$\Rightarrow \quad X(x) = B\sin\left(\frac{n\pi x}{L}\right)$$

Again from (v)

$$\frac{\alpha^2}{T}\frac{dT}{dt} = -\lambda^2$$

$$\frac{dT}{dt} + \frac{\lambda^2}{\alpha^2}T = 0$$

$$\left(\frac{d}{dt} + \frac{\lambda^2}{\alpha^2}\right)T = 0$$

The auxiliary equation

$$\frac{d}{dt} + \frac{\lambda^2}{\alpha^2} = 0$$

$$\frac{d}{dt} = -\frac{\lambda^2}{\alpha^2}$$

$$\Rightarrow T(t) = Ce^{-\frac{\lambda^2}{\alpha^2}t}$$

Put the value of λ

$$\Rightarrow T(t) = Ce^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}$$

Put the value of X(x) , T(t) in (iv)

$$\psi(x,t) = B\sin\left(\frac{n\pi x}{L}\right).Ce^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t}$$

$$\psi(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2 L^2}t} \_\_\_(vi) \qquad \because BC = A_n$$

Now by applying Principle of superposition

$$\psi(x,t)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2L^2}t} \qquad \_\_\_\_(vii)$$

Put t = 0 in (viii)

$$\psi(x,0)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)e^{0}$$

$$f_1(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ both side and integrate w.r.t x from 0 → L, and also apply orthogonality condition.

$$\int_0^L f_1(x)\sin\left(\frac{m\pi x}{L}\right)dx \;=\int_0^L\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{m\pi x}{L}\right)dx$$

$$=0 \qquad if \;\; m\neq n$$

$$=\frac{L}{2}A_n \qquad if \;\; m=n$$

$$\Rightarrow A_n=\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

Replace 'm' by 'n'

$$\Rightarrow A_n=\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Put in (vii)

$$\psi(x,t)=\sum_{n=1}^{\infty}\left[\left\{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2L^2}t}\right]$$

Put the value of $f_1(x)=f(x)-\left(\frac{L-x}{L}\right)P-\frac{x}{L}q$

$$\psi(x,t)=\sum_{n=1}^{\infty}\left[\left\{\frac{2}{L}\int_0^L\left(f(x)-\left(\frac{L-x}{L}\right)P-\frac{x}{L}q\right)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2L^2}t}\right]$$

Put the value of ψ (x,t) in (ii)

$$\phi(x,\text{t})=\sum_{n=1}^{\infty}\left[\left\{\frac{2}{L}\int_0^L\left(f(x)-\left(\frac{L-x}{L}\right)P-\frac{x}{L}q\right)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\lambda^2}{\alpha^2L^2}t}\right]+\left(\frac{L-x}{L}\right)P+\frac{x}{L}q$$

is the required solution of given PDE.

Question: Solve the PDE $\phi_{xx}=\alpha^2\phi_{tt}$

B.C $\phi(0,t)=0$, $\phi(\text{L},t)=P$

I.C $\phi(x,0)=f(x)$, $\phi_t(x,0)=g(x)$

Solution: Given PDE $\phi_{xx}=\alpha^2\phi_{tt}$ ____ (i)

We convert the given non-homogeneous problem into homogeneous by considering the following transformation.

$$\phi(x,\text{t})=\psi(x,t)+\frac{x}{L}P \quad \_\_\_(ii)$$

Put x = 0 on (i)

$$\phi(0,\text{t})=\psi(0,t)+0$$

$$\Rightarrow \quad 0=\psi(0,t) \qquad \because \phi(0,\text{t})$$

Put x = L in (ii)

$$\phi(\text{L},\text{t})=\psi(\text{L},t)+\frac{L}{L}P$$

$$P=\psi(\text{L},t)+P \quad \Rightarrow \psi(\text{L},t)=0$$

Put t = 0 in (ii)

$$\phi(x,0)=\psi(x,0)+\frac{x}{L}P$$

$$f(x)=\psi(x,0)+\frac{x}{L}P$$

$$\Rightarrow \psi(x,0)= f(x)-\frac{x}{L}P= f_1(x)\,(say)$$

$$\Rightarrow \psi(x,0)=f_1(x)\,(say)$$

Diff. (ii) w.r.t 't'

$$\Rightarrow \phi_t(x,\mathrm{t})=\psi_t(x,\mathrm{t})$$

Put t = 0

$$\phi_t(x,0)=\psi_t(x,0)$$

$$\Rightarrow \psi_t(x,0)=g(x)$$

Substitute (ii) in (i)

$$\frac{\partial^2}{\partial x^2}\left[\psi(x,t)+\frac{x}{L}P\right]=\alpha^2\frac{\partial^2}{\partial t^2}\left[\psi(x,t)+\frac{x}{L}P\right]$$

$$\frac{\partial^2\psi}{\partial x^2}=\alpha^2\frac{\partial^2\psi}{\partial t^2}\quad \_\_\_(iii)$$

With B.C $\psi(0,t) = 0 = \psi(L,t)$

I.C $\psi(x,0) = f_1(x)$, $\psi_t(x,0) = g(x)$

Suppose that solution of equation (iii) is

$$\psi(x,t)=X(x)T(t)\quad \_\_\_\_\_(iv)$$

Substitute (iv) in (iii) we have

$$\frac{\partial^2}{\partial x^2}(XT)=\alpha^2\frac{\partial}{\partial t}(XT)$$

$$\Rightarrow T\frac{d^2X}{dx^2}=\alpha^2X\frac{dT}{dt}$$

Divide both side by XT

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=\frac{\alpha^2}{T}\frac{dT}{dt}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=\frac{\alpha^2}{T}\frac{dT}{dt}=-\lambda^2 \qquad \_\_\_(v)$$

From (v) $\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=-\lambda^2$

$$\Rightarrow \frac{d^2X}{dx^2}+\lambda^2X=0$$

$$\Rightarrow \left(\frac{d^2}{dx^2}+\lambda^2\right)X=0$$

The auxiliary equation

$$\frac{d^2}{dx^2}+\lambda^2=0$$

$$\frac{d}{dx}=\pm i\lambda$$

$$\Rightarrow X(x)=A\cos(\lambda x)+B\sin(\lambda x)$$

Transformed B.C X(0) = 0

$0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$

$X(x) = B\sin(\lambda x)$

Put X(L) = 0

$0 = B\sin(\lambda L) \Rightarrow B \neq 0 \ , \ \sin\lambda L = 0$

$$\Rightarrow \lambda L = n\pi \quad ; \ n=1,2,3,....$$

$$\Rightarrow \lambda=\frac{n\pi}{L} \quad ; \ n=1,2,3,....$$

$$\Rightarrow \qquad X(x)=B\sin\left(\frac{n\pi x}{L}\right)$$

Again from (v) $$\frac{\alpha^2}{T}\frac{d^2T}{dt^2}=-\lambda^2$$

$$\frac{d^2T}{dt^2}+\frac{\lambda^2}{\alpha^2}T=0$$

$$\left(\frac{d^2}{dt^2}+\frac{\lambda^2}{\alpha^2}\right)T=0$$

The auxiliary equation

$$\frac{d^2}{dt^2}+\frac{\lambda^2}{\alpha^2}=0$$

$$\frac{d^2}{dt^2}=-\frac{\lambda^2}{\alpha^2}$$

$$\frac{d}{dt}=\pm i\frac{\lambda}{\alpha}$$

$$\Rightarrow T(t)=C\cos\left(\frac{\lambda}{\alpha}t\right)+D\sin\left(\frac{\lambda}{\alpha}t\right) \qquad \because \lambda=\frac{n\pi}{L}$$

$$T(t)=C\cos\left(\frac{n\pi}{\alpha L}\right)t+D\sin\left(\frac{n\pi}{\alpha L}\right)t \qquad ; n=1,2,3,\ldots.$$

Put the value X(x), T(t) in (iv)

$$\psi(x,t)=B\sin\left(\frac{n\pi x}{L}\right).\left[C\cos\left(\frac{n\pi}{\alpha L}\right)t+D\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

$$\psi(x,t)=\sin\left(\frac{n\pi x}{L}\right)\left[BC\cos\left(\frac{n\pi}{\alpha L}\right)t+BD\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

$$\psi(x,t)=\sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi}{\alpha L}\right)t+B_n\sin\left(\frac{n\pi}{\alpha L}\right)t\right] \quad \because BC=A_n \ \& BD=B_n$$

Now by applying Principle of superposition

$$\psi(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\ A_n\cos\left(\frac{n\pi}{\alpha L}\right)t+B_n\sin\left(\frac{n\pi}{\alpha L}\right)t\ \right] \qquad \_\_\_\_(vi)$$

t = 0 in (vi) $\phi(x,0)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\ A_n\cos(0)+B_n\sin(0)\ \right]$

$$f_1(x)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\ A_n+0\ \right]$$

$$f_1(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{n\pi x}{L}\right)$ both side and integrate w.r.t x from 0 $\rightarrow$ L, we get

$$\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx=\int_0^L\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{L}{2}A_n$$

$$\Rightarrow A_n=\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Put in (vi)

$$\psi(x,\mathrm{t})=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx\ \right\}\cos\left(\frac{n\pi}{\alpha L}\right)t+B_n\sin\left(\frac{n\pi}{\alpha L}\right)t\ \right]\_\_\_(vii)$$

Now differentiate (vii) w.r.t ‘t’

$$\psi_t(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[-A_n\left(\frac{n\pi}{\alpha L}\right)\sin\left(\frac{n\pi}{\alpha L}\right)t+B_n\left(\frac{n\pi}{\alpha L}\right)\cos\left(\frac{n\pi}{\alpha L}\right)t\ \right]$$

Put t = 0

$$\psi_t(x,0)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[-A_n\left(\frac{n\pi}{\alpha L}\right)\sin(0)+B_n\left(\frac{n\pi}{\alpha L}\right)\cos(0)\ \right]$$

$$g(x)=\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{\alpha L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{n\pi x}{L}\right)$ both side and integrate w.r.t x from $0\to$ L, Also applying the orthogonality principle,

$$\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx=\int_0^L\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{\alpha L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\left(\frac{n\pi}{\alpha L}\right).\frac{L}{2}.B_n$$

$$B_n=\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Put the value of B_n in (viii)

$$\psi(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\cos\left(\frac{n\pi}{\alpha L}\right)t+\left\{\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

Put the value of $f_1(x) = f(x) - xP/L$

$$\psi(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L\left(x-\frac{x}{L}P\right)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\cos\left(\frac{n\pi}{\alpha L}\right)t\ +\right.$$

$$\left.\left\{\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

Put the value of $\psi(x,t)$ in (ii)

$$\phi(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L\left(x-\frac{x}{L}P\right)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\cos\left(\frac{n\pi}{\alpha L}\right)t\ +\right.$$

$\left.\left\{\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi}{\alpha L}\right)t\right]+\frac{x}{L}P$ is the required solution given PDE.

Question: Solve the PDE $\phi_{xx} = \alpha^2 \phi_{tt}$

B.C $\phi(0,t) = 0$, $\phi(\mathrm{L},t) = q$

I.C $\phi(x,0) = f(x)$, $\phi_t(x,0) = g(x)$

Solution: $\phi_{xx} = \alpha^2 \phi_{tt}$ ___ (i)

We convert the given non-homogeneous problem into homogeneous by considering the following transformation.

$$\phi(x,\mathrm{t}) = \psi(x,t) + \left(\frac{L-x}{L}\right)P + \frac{x}{L}q \quad \_\_\_(ii)$$

Put x = 0 on (i)

$$\phi(0,\mathrm{t}) = \psi(0,t) + \left(\frac{L-0}{L}\right)P + \frac{0}{L}q$$

$$P = \psi(0,t) + P \quad \Rightarrow \psi(0,t) = 0$$

Put x = L in (ii)

$$\phi(\mathrm{L},\mathrm{t}) = \psi(\mathrm{L},t) + \left(\frac{L-L}{L}\right)P + \frac{L}{L}q$$

$$q = \psi(\mathrm{L},t) + 0 + q \quad \Rightarrow \psi(\mathrm{L},t) = 0$$

Put t = 0 in (ii)

$$\phi(x,0) = \psi(x,0) + \left(\frac{L-x}{L}\right)P + \frac{x}{L}q$$

$$f(x) = \psi(x,0) + \left(\frac{L-x}{L}\right)P + \frac{x}{L}q$$

$$\psi(x,0) = f(x) - \left(\frac{L-x}{L}\right)P - \frac{x}{L}q = f_1(x)\ (say)$$

$$\Rightarrow \psi(x,0) = f_1(x)\ (say)$$

Differentiate (ii) w.r.t 't'

$$\phi_t(x,t)=\psi_t(x,t)$$

Put t = 0

$$\Rightarrow \psi_t(x,0)=g(x)$$

Substitute (ii) in (i)

$$\frac{\partial^2}{\partial x^2}\left[\psi(x,t)+\left(\frac{L-x}{L}\right)P+\frac{x}{L}q\right]=\alpha^2\frac{\partial^2}{\partial t^2}\left[\psi(x,t)+\left(\frac{L-x}{L}\right)P+\frac{x}{L}q\right]$$

$$\frac{\partial^2\psi}{\partial x^2}=\alpha^2\frac{\partial^2\psi}{\partial t^2} \quad \_\_\_\_\_(iii)$$

With B.C $\quad \psi(0,t)=0=\psi(L,t)$

I.C $\quad \psi(x,0)=f_1(x)\ ,\ \psi_t(x,0)=g(x)$

Suppose that the general solution of (iii) is

$$\psi(x,t)=X(x)T(t) \quad \_\_\_\_\_(iv)$$

Substitute (iv) in (iii) we have

$$\frac{\partial^2}{\partial x^2}(XT)=\alpha^2\frac{\partial}{\partial t}(XT)$$

$$\Rightarrow T\frac{d^2X}{dx^2}=\alpha^2X\frac{dT}{dt}$$

Divide both side by XT

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=\frac{\alpha^2}{T}\frac{dT}{dt}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=\frac{\alpha^2}{T}\frac{dT}{dt}=-\lambda^2 \quad \_\_\_(v)$$

From (v) $\quad \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=-\lambda^2$

$$\Rightarrow \frac{d^2X}{dx^2}+\lambda^2X=0$$

$$\Rightarrow \left(\frac{d^2}{dx^2}+\lambda^2\right)X=0$$

The auxiliary equation

$$\frac{d^2}{dx^2}+\lambda^2=0$$

$$\frac{d}{dx}=\pm i\lambda$$

$$\Rightarrow X(x)=A\cos(\lambda x)+B\sin(\lambda x)$$

Transformed B.C X(0) = 0

$$0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$$

$$X(x) = B\sin(\lambda x)$$

Put X(L) = 0

$$0=B\sin(\lambda L) \Rightarrow B\neq 0\ ,\ \sin\lambda L=0$$

$$\Rightarrow \lambda L=n\pi \quad ;\ n=1,2,3,....$$

$$\Rightarrow \lambda=\frac{n\pi}{L} \quad ;\ n=1,2,3,....$$

$$\Rightarrow \quad X(x)=B\sin\left(\frac{n\pi x}{L}\right)$$

Again from (v) $\quad \dfrac{\alpha^2}{T}\dfrac{d^2T}{dt^2}=-\lambda^2$

$$\frac{d^2T}{dt^2}+\frac{\lambda^2}{\alpha^2}T=0$$

$$\left(\frac{d^2}{dt^2}+\frac{\lambda^2}{\alpha^2}\right)T=0$$

The auxiliary equation $\quad \dfrac{d^2}{dt^2}+\dfrac{\lambda^2}{\alpha^2}=0$

$$\frac{d^2}{dt^2} = -\frac{\lambda^2}{\alpha^2}$$

$$\frac{d}{dt} = \pm i\frac{\lambda}{\alpha}$$

$$\Rightarrow T(t) = C\cos\left(\frac{\lambda}{\alpha}t\right) + D\sin\left(\frac{\lambda}{\alpha}t\right) \qquad \because \lambda = \frac{n\pi}{L}$$

$$T(t) = C\cos\left(\frac{n\pi}{\alpha L}\right)t + D\sin\left(\frac{n\pi}{\alpha L}\right)t \qquad ; n = 1, 2, 3, ...$$

Put the value X(x), T(t) in (iv)

$$\psi(x,t) = B\sin\left(\frac{n\pi x}{L}\right).\left[C\cos\left(\frac{n\pi}{\alpha L}\right)t + D\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

$$\psi(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[BC\cos\left(\frac{n\pi}{\alpha L}\right)t + BD\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

$$\psi(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi}{\alpha L}\right)t + B_n\sin\left(\frac{n\pi}{\alpha L}\right)t\right] \quad \because BC = A_n \;\&\; BD = B_n$$

Now by applying Principle of superposition

$$\psi(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi}{\alpha L}\right)t + B_n\sin\left(\frac{n\pi}{\alpha L}\right)t\right] \qquad \_\_\_\_(vi)$$

t = 0 in (vi) $\phi(x,0) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos(0) + B_n\sin(0)\right]$

$$f_1(x) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[A_n + 0\right]$$

$$f_1(x) = \sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{n\pi x}{L}\right)$ both side and integrate w.r.t x from 0 $\rightarrow$ L, we get

$$\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx \;=\int_0^L\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right).\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{L}{2}A_n$$

$$\Rightarrow A_n=\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Put in (vi)

$$\psi(x,\mathrm{t})=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\cos\left(\frac{n\pi}{\alpha L}\right)t+B_n\sin\left(\frac{n\pi}{\alpha L}\right)t\right]\_\_\_(vii)$$

Now differentiate (vi) w.r.t 't'

$$\psi_t(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[-A_n\left(\frac{n\pi}{\alpha L}\right)\sin\left(\frac{n\pi}{\alpha L}\right)t+B_n\left(\frac{n\pi}{\alpha L}\right)\cos\left(\frac{n\pi}{\alpha L}\right)t\right]$$

Put t = 0

$$\psi_t(x,0)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[-A_n\left(\frac{n\pi}{\alpha L}\right)\sin(0)+B_n\left(\frac{n\pi}{\alpha L}\right)\cos(0)\right]$$

$$g(x)=\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{\alpha L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Multiplying by $\sin\left(\frac{n\pi x}{L}\right)$ both side and integrate w.r.t x from 0 → L, Also applying the orthogonality principle,

$$\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx=\int_0^L\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{\alpha L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\left(\frac{n\pi}{\alpha L}\right).\frac{L}{2}.B_n$$

$$B_n = \frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Put the value of B_n in (viii)

$$\psi(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\cos\left(\frac{n\pi}{\alpha L}\right)t + \left\{\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

Put the value of $f_1(x) = f(x) - \left(\frac{L-x}{L}\right)P - \frac{x}{L}q$

$$\psi(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L\left(f(x) - \left(\frac{L-x}{L}\right)P - \frac{x}{L}q\right)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\cos\left(\frac{n\pi}{\alpha L}\right)t + \right.$$

$$\left.\left\{\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi}{\alpha L}\right)t\right]$$

Put the value of $\psi(x,t)$ in (ii)

$$\phi(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[\left\{\frac{2}{L}\int_0^L\left(f(x) - \left(\frac{L-x}{L}\right)P - \frac{x}{L}q\right)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\cos\left(\frac{n\pi}{\alpha L}\right)t + \right.$$

$$\left.\left\{\frac{2\alpha}{n\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\}\sin\left(\frac{n\pi}{\alpha L}\right)t\right] + \frac{x}{L}P$$ is the required solution given PDE.

Laplace equation:

$\phi_{xx} + \phi_{yy} = 0$ is called the Laplace equation in two dimension.

Polar form of Laplace equation:

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = 0$$

And its solution is $\phi(r,\theta) = R(r)\theta(\theta)$

Question: Solve the polar form of Laplace equation $\phi_{rr}+\frac{1}{r}\phi_r+\frac{1}{r^2}\phi_{\theta\theta}=0$

Solution: Given $\phi_{rr}+\frac{1}{r}\phi_r+\frac{1}{r^2}\phi_{\theta\theta}=0$ ___(i)

Consider the general solution of (i) is

$$\phi(r,\theta)=R(r)\theta(\theta) \quad \text{\_\_\_\_(ii)}$$

Substitute (ii) in (i) we have

$$\frac{\partial^2}{\partial r^2}(R\theta)+\frac{1}{r}\frac{\partial}{\partial r}(R\theta)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(R\theta)=0$$

$$\theta\frac{d^2R}{dr^2}+\frac{\theta}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\theta}{d\theta^2}=0$$

Multiplying by $\frac{r^2}{R\theta}$

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+\frac{1}{\theta}\frac{d^2\theta}{d\theta^2}=0$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}=-\frac{1}{\theta}\frac{d^2\theta}{d\theta^2}=\lambda^2 \quad \text{\_\_\_\_}(iii)$$

From (iii)

$$-\frac{1}{\theta}\frac{d^2\theta}{d\theta^2}=\lambda^2$$

$$\Rightarrow\frac{d^2\theta}{d\theta^2}+\lambda^2\theta=0$$

$$\Rightarrow\left(\frac{d^2}{d\theta^2}+\lambda^2\right)\theta=0$$

$$\frac{d^2}{d\theta^2}+\lambda^2=0 \quad \because \textit{The Auxiliary equation}$$

Here arise two cases

Case-I: When $\lambda = 0$

$$\Rightarrow \frac{d^2}{d\theta^2} = 0$$

$$\Rightarrow \frac{d}{d\theta} = 0, 0$$

$$\theta(\theta) = (A + B\theta)e^0 = A + B\theta$$

Case-II: When $\lambda > 0$

$$\Rightarrow \frac{d^2}{d\theta^2} = -\lambda^2$$

$$\Rightarrow \frac{d}{d\theta} = \pm i\lambda$$

$$\Rightarrow \theta(\theta) = C\cos(\lambda\theta) + D(\sin\lambda\theta)$$

Again from (iii)

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} = \lambda^2 \quad \text{\_\_\_\_}(iv)$$

When $\lambda = 0$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} = 0$$

$$\frac{r}{R}\left(r\frac{d^2R}{dr^2} + \frac{dR}{dr}\right) = 0$$

$$\frac{r}{R} \neq 0 \quad \& \quad \left(r\frac{d^2R}{dr^2} + \frac{dR}{dr}\right) = 0$$

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

$$r\frac{dR}{dr} = D \qquad \because \ on \text{ integration}$$

$$\frac{dR}{dr} = \frac{D}{r}$$

$$\Rightarrow R(r) = C_1 + D_1 \ln r \quad \because \ on \quad \text{integration}$$

When $\lambda > 0$ from (iv)

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} = \lambda^2$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} = \lambda^2 R$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 R = 0$$

Which is the Cauchy Euler equation.

$$Let \quad R(r) = r^k$$

$$\frac{dR}{dr} = kr^{k-1}$$

$$\frac{d^2R}{dr^2} = k(k-1)r^{k-2}$$

$$\Rightarrow \quad r^2k(k-1)r^{k-2} + rkr^{k-1} - \lambda^2 r^k = 0$$

$$\Rightarrow \ (k^2 - k)r^k + kr^{k-1} - \lambda^2 r^k = 0$$

$$\Rightarrow \ (k^2 - k + k - \lambda^2)r^k = 0$$

$$\Rightarrow \ (k^2 - \lambda^2)r^k = 0$$

$$r^k \neq 0 \quad , \quad (k^2 - \lambda^2) = 0$$

$$k^2 = \lambda^2$$

$$k = \pm\lambda$$

$$R(r) = C_2 r^{\lambda} + C_3 r^{-\lambda}$$

Hence $\qquad \phi(r,\theta) = (A + B\theta)(C_1 + D_1 \ln r) \; when \;\; \lambda = 0$

$$\phi(r,\theta) = \left(C\cos(\theta\lambda) + D\sin(\theta\lambda)\right)\left(C_2 r^{\lambda} + C_3 r^{-\lambda}\right) \; when \; \lambda > 0$$

$$\phi(r,\theta) = (A + B\theta)(C_1 + D_1 \ln r) + \left(C\cos(\theta\lambda) + D\sin(\theta\lambda)\right)\left(C_2 r^{\lambda} + C_3 r^{-\lambda}\right) \; when \; \lambda \geq 0$$

Question: Solve the polar form of Laplace equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = \frac{1}{c^2}\phi_{tt}$$

Solution: Given $\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = \frac{1}{c^2}\phi_{tt}$ ___(i)

Let its solution is of the form

$$\phi(r,\theta,z,t) = R(r)\theta(\theta)Z(z)T(t) \quad \_\_\_(ii)$$

Substitute (ii) in (i) we have

$$\frac{\partial^2}{\partial r^2}(R\theta ZT) + \frac{1}{r}\frac{\partial}{\partial r}(R\theta ZT) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R\theta ZT) + \frac{\partial^2}{\partial z^2}(R\theta ZT) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(R\theta ZT)$$

$$\theta ZT\frac{d^2R}{dr^2} + \frac{\theta ZT}{r}\frac{dR}{dr} + \frac{RZT}{r^2}\frac{d^2\theta}{d\theta^2} + \frac{R\theta T}{r^2}\frac{d^2Z}{dz^2} = \frac{R\theta Z}{c^2}\frac{d^2T}{dt^2}$$

Divide by RθZT

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\theta}\frac{d^2\theta}{d\theta^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -\omega^2 (say) \quad \_\_\_(iii)$$

From (iii)

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = -\omega^2$$

$$\Rightarrow \quad \frac{d^2T}{dt^2} + c^2\omega^2 T = 0$$

$$\Rightarrow \left(\frac{d^2}{dt^2}+c^2\omega^2\right)T=0$$

The Auxiliary equation

$$\frac{d^2}{dt^2}+c^2\omega^2=0$$

$$\frac{d}{dt}=\pm ic\omega$$

$$\Rightarrow T(t)=A_1\cos(\omega cx)+A_2\sin(\omega cx) \qquad \_\_\_(iv)$$

Again from (iii)

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{rR}\frac{dR}{dr}+\frac{1}{r^2\theta}\frac{d^2\theta}{d\theta^2}+\omega^2=-\frac{1}{Z}\frac{d^2Z}{dz^2}=m^2 \qquad \_\_\_(v)$$

From (v)

$$-\frac{1}{Z}\frac{d^2Z}{dz^2}=m^2$$

$$\Rightarrow \frac{d^2Z}{dz^2}+m^2\ Z=0$$

$$\left(\frac{d^2}{dz^2}+m^2\right)Z=0$$

$$\frac{d^2}{dz^2}+m^2=0 \quad \because \textit{The Auxiliary equation}$$

$$\frac{d}{dz}=\pm im$$

$$Z(z)=A_3\cos(mz)+A_4\sin(mz) \qquad \_\_\_(vi)$$

Again from (v)

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{rR}\frac{dR}{dr}+\frac{1}{r^2\theta}\frac{d^2\theta}{d\theta^2}+\omega^2=m^2$$

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{rR}\frac{dR}{dr}+\omega^2-m^2=-\frac{1}{r^2\theta}\frac{d^2\theta}{d\theta^2}$$

Multiplying by r^2

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+\left(\omega^2-m^2\right)r^2=-\frac{1}{\theta}\frac{d^2\theta}{d\theta^2}=n^2 \quad \_\_\_(vii)$$

$$From\ (vii)\ \Rightarrow\ -\frac{1}{\theta}\frac{d^2\theta}{d\theta^2}=n^2$$

$$\frac{d^2\theta}{d\theta^2}+n^2\theta=0$$

$$\Rightarrow\left(\frac{d^2}{d\theta^2}+n^2\right)\theta=0$$

The auxiliary equation

$$\Rightarrow\frac{d^2}{d\theta^2}+n^2=0$$

$$\Rightarrow\frac{d}{d\theta}=\pm in$$

$$\Rightarrow\theta(\theta)=A_5\cos(n\theta)+A_6\sin(n\theta) \quad \_\_\_\_\_(viii)$$

Again from (vii)

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+\left(\omega^2-m^2\right)r^2=n^2$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+\left(\omega^2-m^2\right)r^2-n^2=0$$

Multiplying by R

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+R\left[\left(\omega^2-m^2\right)r^2-n^2\right]=0$$

Which is parametric Bessel equation. Its solution is

$$R(r)=A_7J_\nu\left(\sqrt{\omega^2-m^2}\,r\right)+A_8r^2Y_\nu\left(\sqrt{\omega^2-m^2}\,r\right)\quad \_\_\_\_(ix)$$

Put the value of R(r), θ(θ), Z(z) and T(t) in (ii)

$$\phi(r,\theta,z,t)=\left[A_7J_\nu\left(\sqrt{\omega^2-m^2}\,r\right)+A_8r^2Y_\nu\left(\sqrt{\omega^2-m^2}\,r\right)\right].\left[A_5\cos(n\theta)+A_6\sin(n\theta)\right]$$

$$.\left[A_3\cos(mz)+A_4\sin(mz)\right].\left[A_1\cos(\omega cx)+A_2\sin(\omega cx)\ \right]$$

is the required solution of given equation.

Some information:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\nu^2\right)y=0$$

if ν is not an integer then its general solution is $y=A_1J_\nu(x)+A_2J_\nu(x)$

if ν is an integer then its general solution is $y=A_1J_\nu(x)+A_4Y_\nu(x)$

Where $J_\nu(x)$ is a Bessel function of first kind and $Y_\nu(x)$ is a Bessel function of 2nd kind.

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\nu^2\right)y=0$ which is parametric Bessel equation. Its general solution is $y=A_1J_\nu(\lambda x)+A_2Y_\nu(\lambda x)$.

Lecture # 05

Consider a second order PDE of the form

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi = g(x,y) \quad \_\_\_(i)$$

Where a $\rightarrow$ f are coefficient functions of x and y or they may be constant.
Where $a\phi_{xx} + b\phi_{xy} + c\phi_{yy}$ is called Principle part of equation.

Define the discriminant Δ at (x_0, y_0) is

$$\Delta = b^2(x_0,y_0) - 4(x_0,y_0)c(x_0,y_0)$$

The equation (i) is called Hyperbolic if $\Delta > 0$ is called Parabolic if $\Delta = 0$ and is called Elliptic if $\Delta < 0$.

Canonical form / Normal form / Standard form:

In this form the second order derivatives are reduced to two or one second order derivatives with another set of independent variables.

General transform:

Let $\xi \; and \; \eta$ be twice differentiable function i.e.

$$\xi = \xi(x,y) \;,\; \eta = \eta(x,y)$$

Using chain rule

$$\phi_x = \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\xi}.\frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial\eta}.\frac{\partial\eta}{\partial x}$$

$$\phi_x = \phi_\xi \xi_x + \phi_\eta \eta_x \quad \_\_\_\_(ii)$$

Similarly, $\phi_y = \phi_\xi \xi_y + \phi_\eta \eta_y$

Now $\phi_{xy} = \frac{\partial}{\partial y}(\phi_x) = \frac{\partial}{\partial y}\left[\phi_\xi \xi_x + \phi_\eta \eta_x\right]$

$$\phi_{xy} = \frac{\partial}{\partial y}\left[\phi_\xi \xi_x\right] + \frac{\partial}{\partial y}\left[\phi_\eta \eta_x\right]$$

$$\phi_{xy} = \xi_x \frac{\partial}{\partial y}\left[\phi_\xi\right] + \phi_\xi \frac{\partial}{\partial y}\left[\xi_x\right] + \eta_x \frac{\partial}{\partial y}\left[\phi_\eta\right] + \phi_\eta \frac{\partial}{\partial y}\left[\eta_x\right]$$

$$\phi_{xy} = \left[\frac{\partial}{\partial \xi}(\phi_\xi)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(\phi_\xi)\frac{\partial \eta}{\partial y}\right]\xi_x + \phi_\xi.\xi_{xy} + \left[\frac{\partial}{\partial \xi}(\phi_\eta)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(\phi_\eta)\frac{\partial \eta}{\partial y}\right]\eta_x + \phi_\eta.\eta_{xy}$$

$$\phi_{xy} = \phi_{\xi\xi}\xi_x\xi_y + \phi_{\xi\eta}\xi_x\eta_y + \phi_\xi\xi_{xy} + \phi_{\eta\xi}\xi_y\eta_x + \phi_{\eta\eta}\eta_x\eta_y + \phi_\eta\eta_{xy} \quad \_\_\_\_\_(iii)$$

Using $\phi_{\eta\xi} = \phi_{\xi\eta}$

$$\phi_{xy} = \phi_{\xi\xi}\xi_x\xi_y + \phi_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + \phi_{\eta\eta}\eta_x\eta_y + \phi_\xi\xi_{xy} + \phi_\eta\eta_{xy} \quad \_\_\_\_\_(iv)$$

Replace ‘y’ by ‘x’

$$\phi_{xx} = \phi_{\xi\xi}\xi_x^2 + 2\phi_{\xi\eta}\xi_x\eta_x + \phi_{\eta\eta}\eta_x^2 + \phi_\xi\xi_{xx} + \phi_\eta\eta_{xx} \quad \_\_\_\_\_(v)$$

Similarly, replace ‘x’ by ‘y’ in (v)

$$\phi_{yy} = \phi_{\xi\xi}\xi_y^2 + 2\phi_{\xi\eta}\xi_y\eta_y + \phi_{\eta\eta}\eta_y^2 + \phi_\xi\xi_{yy} + \phi_\eta\eta_{yy} \quad \_\_\_\_\_(vi)$$

Put all these values in (i)

$$a\left[\phi_{\xi\xi}\xi_x^2 + 2\phi_{\xi\eta}\xi_x\eta_x + \phi_{\eta\eta}\eta_x^2 + \phi_\xi\xi_{xx} + \phi_\eta\eta_{xx}\right]$$

$$+b\left[\phi_{\xi\xi}\xi_x\xi_y + \phi_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + \phi_{\eta\eta}\eta_x\eta_y + \phi_\xi\xi_{xy} + \phi_\eta\eta_{xy}\right]$$

$$+c\left[\phi_{\xi\xi}\xi_y^2 + 2\phi_{\xi\eta}\xi_y\eta_y + \phi_{\eta\eta}\eta_y^2 + \phi_\xi\xi_{yy} + \phi_\eta\eta_{yy}\right]$$

$$+d\left[\phi_\xi\xi_x + \phi_\eta\eta_x\right] + e\left[\phi_\xi\xi_y + \phi_\eta\eta_y\right] + f\phi = g(x,y)$$

$$\left(a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2\right)\phi_{\xi\xi} + \left[2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y\right]\phi_{\xi\eta}$$

$$+\left(a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2\right)\phi_{\eta\eta} + \left[a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y\right]\phi_\xi$$

$$+\left(a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y\right)\phi_\eta + f\phi = g(x,y)$$

$$\Rightarrow \quad A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} + D\phi_\xi + E\phi_\eta + E\phi = G \quad \_\_\_\_\_(vii)$$

Where $A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$

$$B = 2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y$$

$$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

$$D = a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y$$

$$E = a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y$$

$$F = f,\ G = g$$

The form of equation (i) and (vii) is same except independent variable and coefficient function. Under this transformation the kind or nature of PDE does not change.

Jacobian transformation:

$$|J| = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

From this transformation it is assumed that $|J| \neq 0$. This condition is imposed to ensure that the transformation of a PDE can be converted to original form

$$\text{Let } \Delta = b^2 - 4ac \;\rightarrow\; for\;\; equation\,(i)$$

$$\Delta^* = B^2 - 4AC \;\rightarrow\; for\;\; equation\,(vii)$$

$$\text{Here } \Delta^* = B^2 - 4AC$$

$$= \left[2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y\right]^2 - 4\left[a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2\right]\left[a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2\right]$$

$$= 4a^2\xi_x^2\eta_x^2 + b^2\left(\xi_x\eta_y + \xi_y\eta_x\right)^2 + 4c^2\xi_y^2\eta_y^2 + 4ab\left[\xi_x^2\eta_x\eta_y + \xi_x\xi_y\eta_x^2\right]$$

$$+4bc\left[\xi_x\xi_y\eta_y^2 + \xi_x^2\eta_x\eta_y\right] + 8ac\xi_x\xi_y\eta_x\eta_y - 4a^2\xi_x^2\eta_x^2 - 4ab\xi_x^2\eta_x\eta_y - 4ac\xi_x^2\eta_y^2$$

$$-4ab\xi_x^2\eta_x\eta_y - 4ab\xi_x^2\eta_y^2 - 4b^2\xi_x\xi_y\eta_x\eta_y - 4bc\xi_x\xi_y\eta_y^2 - 4ac\xi_y^2\eta_x^2 - 4bc\xi_y^2\eta_x\eta_y - 4c^2\xi_y^2\eta_y^2$$

$$= b^2\left[\left(\xi_x\eta_y + \xi_y\eta_x\right)^2 - 4\xi_x\xi_y\eta_x\eta_y\right] - 4ac\xi_x^2\eta_y^2 - 4ac\xi_y^2\eta_x^2 + 8ac\xi_x\xi_y\eta_x\eta_y$$

$$= b^2\left[\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2 + 2\xi_x\xi_y\eta_x\eta_y - 4\xi_x\xi_y\eta_x\eta_y\right] - 4ac\left[\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2 - 2\xi_x\xi_y\eta_x\eta_y\right]$$

$$= b^2\left[\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2 - 2\xi_x\xi_y\eta_x\eta_y\right] - 4ac\left(\xi_x\eta_y - \xi_y\eta_x\right)^2$$

$$= b^2\left(\xi_x\eta_y - \xi_y\eta_x\right)^2 - 4ac\left(\xi_x\eta_y - \xi_y\eta_x\right)^2$$

$$\Delta^* = \left(\xi_x\eta_y - \xi_y\eta_x\right)^2\left(b^2 - 4ac\right)$$

$$\Rightarrow \quad \Delta^* = |J|^2\,\Delta$$

Now $|J| \neq 0$ i.e. if $\Delta \gtrless 0$ *then* $\Delta^* \gtrless 0$ under this transformation the nature of PDE does not change.

e.g.
$$\phi_{xx} - \phi_{yy} = 0$$

a = 0 , b = 0 , c = −1

$\Delta = b^2 - 4ac = (0)^2 - 4(1)(-1) = 4 > 0$ which is Hyperbolic.

$$\phi_{xx} - \phi_{yy} = 0$$

a = 1 , b = 0 , c = 0

$\Delta = b^2 - 4ac = (0)^2 - 4(1)(0) = 0$ which is Parabolic.

$$\phi_{xx} - \phi_{yy} = 0$$

a = 1 , b = 0 , c = 1

$\Delta = b^2 - 4ac = (0)^2 - 4(1)(1) = -4 < 0$ which is Elliptic.

Classify the equation in the Right Half plane;

$$\phi_{xx} + \phi_{xy} - x\phi_{yy} = 0 \qquad (x \leq 0)$$

a = 1 , b = 1 , c = −x

$$\Delta = b^2 - 4ac = (1)^2 - 4(1)(-x) = 1 + 4x$$

When x = 0 $\quad \Delta = 1 + 4(0) = 1 > 0$

The equation is Hyperbolic.

Now the equation is Hyperbolic if $\Delta > 0$

$$\Rightarrow 1 + 4x > 0 \qquad \Rightarrow\ 4x > -1$$

$$\Rightarrow x > \frac{-1}{4}$$

The equation is parabolic if $\Delta = 0$

$\Rightarrow 1 + 4x = 0$

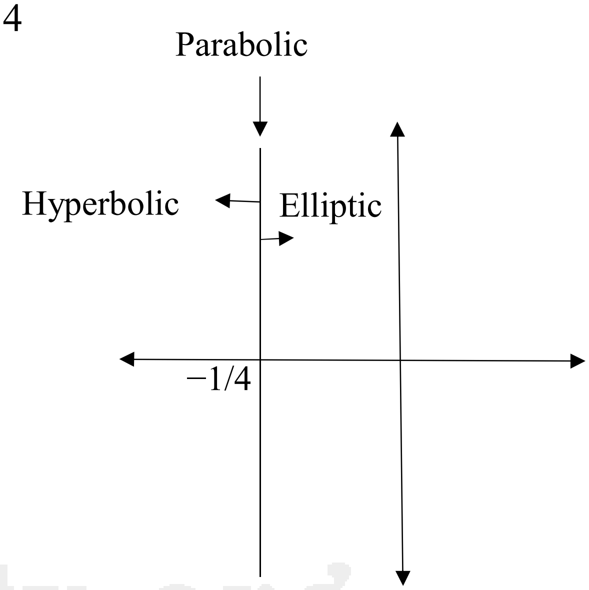
$\Rightarrow x = \frac{-1}{4}$

The equation is elliptic if $\Delta < 0$

$\Rightarrow 1 + 4x < 0$

$\Rightarrow x < \frac{-1}{4}$

i.e. $x \in]-\infty, \frac{-1}{4}[$



Classify the equation in the Right Half plane;

$$\phi_{xx} + \phi_{xy} - x\phi_{yy} = 0 \qquad (x > 0)$$

$$a = 1\ ,\ b = 1\ ,\ c = -y$$

$$\Delta = b^2 - 4ac = (1)^2 - 4(1)(-y) = 1 + 4y$$

Equation is Hyperbolic if $\Delta > 0$

$\Rightarrow 1 + 4y > 0 \qquad \Rightarrow \ y > -\frac{1}{4}$

The equation is parabolic if $\Delta = 0$

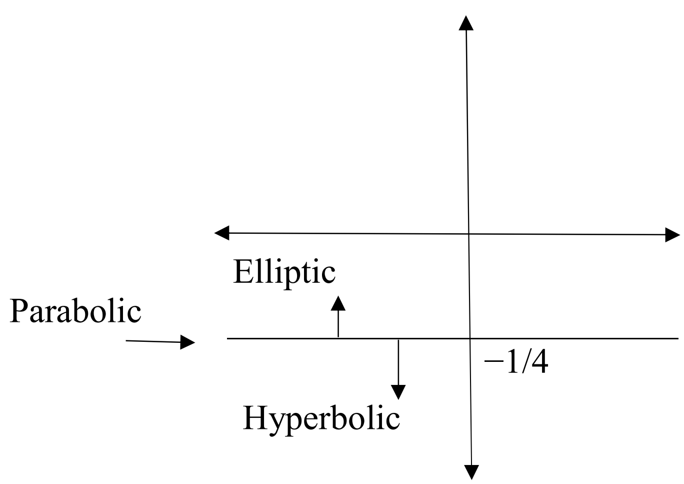
$\Rightarrow 1 + 4y = 0$

$\Rightarrow y = \frac{-1}{4}$

The equation is elliptic if $\Delta < 0$

$\Rightarrow 1 + 4y < 0$

$\Rightarrow y < \frac{-1}{4}$



Canonical form of Hyperbolic equation:

In the general transformation structures of A and C are similar i.e.

$$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$$

$$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

Let z denoted either ξ or η

i.e. $az_x^2 + bz_xz_y + cz_y^2$

We select A and C such that they are annihilated i.e. A=C=0

Now z = constant $\Rightarrow$ z (x,y) = constant $\Rightarrow$ dz = 0

$$\Rightarrow \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$\Rightarrow z_x dx + z_y dy = 0$$

$$\Rightarrow \frac{z_x}{z_y} = -\frac{dy}{dx} \quad \text{\_\_\_\_\_}(i)$$

Also, we have $az_x^2 + bz_xz_y + cz_y^2 = 0$

$$\Rightarrow a\frac{z_x^2}{z_y^2} + b\frac{z_xz_y}{z_x^2} + c\frac{z_y^2}{z_x^2} = 0 \qquad divide\ by\ z_y^2$$

$$\Rightarrow a\left(\frac{z_x}{z_y}\right)^2 + b\left(\frac{z_x}{z_y}\right) + c = 0$$

$$\Rightarrow a\left(\frac{-dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

Which is called Characteristic equation.

Lecture # 06

Question: Show that the equation $y^2\phi_{xx} - x^2\phi_{yy} = 0$, $x > 0$, $y > 0$

Is the hyperbolic and determine its canonical form.

Solution: The given equation is $y^2\phi_{xx} - x^2\phi_{yy} = 0$

Compare with $$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi = g$$

$$\Rightarrow a = y^2, b = 0, c = -x^2, d = e = f = g = 0$$

Now $\Delta = b^2 - 4ac = (0)^2 - 4(y^2)(-x^2)$

$$\Delta = 4y^2x^2 > 0$$

So, the equation is hyperbolic

Now the characteristic D.E is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{2a} = \frac{0 \pm \sqrt{4x^2y^2}}{2y^2}$$

$$\frac{dy}{dx} = \pm\frac{2xy}{2y^2} = \pm\frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y} \ , \ \frac{dy}{dx} = -\frac{x}{y}$$

$$ydy = xdx \ , \ ydy = -xdx$$

$$ydy - xdx = 0 \ , \ ydy + xdx = 0$$

On integration

$$y^2 - x^2 = c_1 \ , \ y^2 + x^2 = c_2$$

There are two characteristics curves

Suppose $$\xi = y^2 + x^2 \ , \eta = y^2 - x^2$$

$$\xi_x = 2x \ , \eta_x = -2x$$

$$\xi_{xx} = 2 \ , \eta_{xx} = -2$$

$$\xi_{xy} = 0 \ , \eta_{xy} = 0$$

$$\xi_{y} = 2y \ , \eta_{y} = 2y$$

$$\xi_{yy} = 2 \ , \eta_{yy} = 2$$

Now $$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$$

$$A = y^2(2x)^2 + (0)(2x)(2y) + (-x^2)(2y)^2$$

$$A = 4x^2y^2 - 4x^2y^2 = 0$$

$$B = 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y$$

$$B = 2(y^2)(2x)(-2x) + 0 + 2(-x^2)(2y)(2y)$$

$$B = -8x^2y^2 - 8x^2y^2 = -16x^2y^2$$

$$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

$$C = y^2(-2x)^2 + 0 + (-x^2)(2y)^2$$

$$C = 4x^2y^2 - 4x^2y^2 = 0$$

$$D = a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y$$

$$D = y^2(2) + 0 + (-x^2)(2) + 0 + 0$$

$$D = 2(y^2 - x^2)$$

$$E = a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y$$

$$E = y^2(-2) + 0 + (-x^2)(2) + 0 + 0$$

$$E = -2(y^2 + x^2)$$

F = f =0 , G = g = 0

Now $$A\phi_{\xi\xi}+B\phi_{\xi\eta}+C\phi_{\eta\eta}+D\phi_{\xi}+E\phi_{\eta}+F\phi=G$$

$$0.\phi_{\xi\xi}+\left(-16x^2y^2\right)\phi_{\xi\eta}+0.\phi_{\eta\eta}+2\left(y^2-x^2\right)\phi_{\xi}+\left(-2\left(y^2+x^2\right)\right)\phi_{\eta}+0=0$$

$$-16x^2y^2\phi_{\xi\eta}+2\left(y^2-x^2\right)\phi_{\xi}-2\left(y^2+x^2\right)\phi_{\eta}=0 \quad \_\_\_*$$

As $$\xi=y^2+x^2\ ,\ \eta=y^2-x^2$$

By adding		By subtracting
$\xi=y^2+x^2$	,	$\xi=y^2+x^2$
$\eta=y^2-x^2$	,	$\pm\eta=\pm y^2\mp x^2$
$2y^2=\xi+\eta$	,	$2x^2=\xi-\eta$
$y^2=\dfrac{\xi+\eta}{2}$	,	$x^2=\dfrac{\xi-\eta}{2}$

$$x^2y^2=\frac{\xi-\eta}{2}\cdot\frac{\xi+\eta}{2}=\frac{\xi^2-\eta^2}{4}$$

Put in * $\Rightarrow$ $$-16\left(\frac{\xi^2-\eta^2}{4}\right)\phi_{\xi\eta}+2\eta\phi_{\xi}-2\xi\phi_{\eta}=0$$

$$4\left(\xi^2-\eta^2\right)\phi_{\xi\eta}+2\eta\phi_{\xi}-2\xi\phi_{\eta}=0$$

$\phi_{\xi\eta}-\dfrac{\eta}{2\left(\xi^2-\eta^2\right)}\phi_{\xi}+\dfrac{\xi}{2\left(\xi^2-\eta^2\right)}\phi_{\eta}=0$ is the required canonical form of given equation.

Question: Show that the equation $y^2\phi_{xx}-x^2\phi_{yy}=0$

Is the hyperbolic and determine its canonical form.

Solution: The given equation is $y^2\phi_{xx}-x^2\phi_{yy}=0$

Compare with $$a\phi_{xx}+b\phi_{xy}+c\phi_{yy}+d\phi_{x}+e\phi_{y}+f\phi=g$$

$$\Rightarrow a=x^2,b=0,c=-y^2,d=e=f=g=0$$

$$\text{Now } \Delta = b^2 - 4ac = (0)^2 - 4(x^2)(-y^2)$$

$$\Delta = 4y^2x^2 > 0$$

So, the equation is hyperbolic

Now the characteristic D.E is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{2a} = \frac{0 \pm \sqrt{4x^2y^2}}{2x^2}$$

$$\frac{dy}{dx} = \pm\frac{2xy}{2x^2} = \pm\frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x} \ , \ \frac{dy}{y} = -\frac{dx}{x}$$

$$\frac{dy}{y} - \frac{dx}{x} = 0 \ , \ \frac{dy}{y} + \frac{dx}{x} = 0$$

On integration

$$\ln y - \text{lnx} = \ln c_1 \ , \ \ln y + \text{lnx} = \ln c_2$$

$$\ln\left(\frac{y}{x}\right) = \ln c_1 \ , \ln xy = \ln c_2$$

$$\frac{y}{x} = c_1 \ , xy = c_2$$

There are two characteristics curves

Suppose $$\xi = xy \ , \ \eta = \frac{y}{x}$$

$$\xi_x = y \ , \ \eta_x = \frac{-y}{x^2}$$

$$\xi_{xx} = 0 \ , \ \eta_{xx} = \frac{2y}{x^3}$$

$$\xi_{xy} = 1 \ , \eta_{xy} = \frac{-1}{x^2}$$

$$\xi_y = x \ , \eta_y = \frac{1}{x}$$

$$\xi_{yy} = 0 \ , \eta_{yy} = 0$$

Now $$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$$

$$A = \left(x^2\right)\left(y^2\right) + 0 + \left(-y^2\right)\left(x\right)^2$$

$$A = x^2y^2 - x^2y^2 = 0$$

$$\text{B} = 2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y$$

$$\text{B} = 2\left(x^2\right)\left(y\right)\left(\frac{-y}{x^2}\right) + 0 + 2\left(-y^2\right)\left(x\right)\left(\frac{1}{x}\right)$$

$$\text{B} = -2y^2 - 2y^2 = -4y^2$$

$$\text{C} = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

$$\text{C} = x^2\left(\frac{-y}{x^2}\right)^2 + 0 + \left(-y^2\right)\left(\frac{1}{x}\right)^2$$

$$\text{C} = \frac{y^2}{x^2} - \frac{y^2}{x^2} = 0$$

$$\text{D} = a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y$$

$$\text{D} = x^2\left(0\right) + 0 + \left(-y^2\right)\left(0\right) + 0 + 0$$

$$\text{D} = 0$$

$$\text{E} = a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y$$

$$\text{E} = x^2\left(\frac{2y}{x^3}\right) + 0 + \left(-y^2\right)\left(0\right) + 0 + 0$$

$$E = \frac{2y}{x}$$

$$F = f = 0 \text{ , } G = g = 0$$

Now $$A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} + D\phi_{\xi} + E\phi_{\eta} + F\phi = G$$

$$0.\phi_{\xi\xi} + \left(-4y^2\right)\phi_{\xi\eta} + 0.\phi_{\eta\eta} + 0.\phi_{\xi} + \frac{2y}{x}\phi_{\eta} + 0 = 0$$

$$-4y^2\phi_{\xi\eta} + \frac{2y}{x}\phi_{\eta} = 0$$

$$-4(xy)\left(\frac{y}{x}\right)\phi_{\xi\eta} + 2\left(\frac{y}{x}\right)\phi_{\eta} = 0 \quad \because y^2 = (xy)\left(\frac{y}{x}\right)$$

$$-4\xi\eta\phi_{\xi\eta} + 2\eta\phi_{\eta} = 0$$

$$4\xi\eta\phi_{\xi\eta} = 2\eta\phi_{\eta}$$

$\phi_{\xi\eta} = \dfrac{\phi_{\eta}}{2\xi}$ is the required canonical form of given equation.

Question: Show that the equation $e^y\phi_{xx} - e^x\phi_{yy} = 0$

Is the hyperbolic and determine its canonical form.

Solution: The given equation is $e^y\phi_{xx} - e^x\phi_{yy} = 0$

Compare with $$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_{x} + e\phi_{y} + f\phi = g$$

$$\Rightarrow a = e^y, b = 0, c = -e^x, d = e = f = g = 0$$

$$\text{Now } \Delta = b^2 - 4ac = (0)^2 - 4\left(e^y\right)\left(-e^x\right)$$

$$\Delta = 4e^{x+y} > 0$$

So, the equation is hyperbolic

Now the characteristic D.E is given by

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{2a} = \frac{0 \pm \sqrt{4e^{x+y}}}{2e^y}$$

$$\frac{dy}{dx}=\pm\frac{2e^{\frac{x+y}{2}}}{2e^{y}}=\pm\frac{e^{\frac{x}{2}}.e^{\frac{y}{2}}}{e^{y}}=\pm\frac{e^{\frac{x}{2}}}{e^{\frac{y}{2}}}$$

$$\frac{dy}{dx}=-\frac{e^{\frac{x}{2}}}{e^{\frac{y}{2}}}\ ,\ \frac{dy}{dx}=\frac{e^{\frac{x}{2}}}{e^{\frac{y}{2}}}$$

$$e^{\frac{y}{2}}dy=-e^{\frac{x}{2}}dx\ ,\ e^{\frac{y}{2}}dy=e^{\frac{x}{2}}dx$$

$$e^{\frac{y}{2}}dy+e^{\frac{x}{2}}dx=0\ ,\ e^{\frac{y}{2}}dy-e^{\frac{x}{2}}dx=0$$

On integration

$$\frac{e^{\frac{y}{2}}}{\frac{1}{2}}+\frac{e^{\frac{x}{2}}}{\frac{1}{2}}=c_1\ ,\ \frac{e^{\frac{y}{2}}}{\frac{1}{2}}-\frac{e^{\frac{x}{2}}}{\frac{1}{2}}=c_2$$

$$e^{\frac{y}{2}}+e^{\frac{x}{2}}=\frac{1}{2}c_1\ ,\ e^{\frac{y}{2}}-e^{\frac{x}{2}}=\frac{1}{2}c_2$$

There are two characteristics curves

Suppose $$\xi=e^{\frac{y}{2}}+e^{\frac{x}{2}}\ ,\ \eta=e^{\frac{y}{2}}-e^{\frac{x}{2}}$$

$$\xi_x=\frac{1}{2}e^{\frac{x}{2}}\ ,\ \eta_x=-\frac{1}{2}e^{\frac{x}{2}}$$

$$\xi_{xx}=\frac{1}{4}e^{\frac{x}{2}}\ ,\ \eta_{xx}=-\frac{1}{4}e^{\frac{x}{2}}$$

$$\xi_{xy}=0\ ,\ \eta_{xy}=0$$

$$\xi_y=\frac{1}{2}e^{\frac{y}{2}}\ ,\ \eta_y=\frac{1}{2}e^{\frac{y}{2}}$$

$$\xi_{yy}=\frac{1}{4}e^{\frac{y}{2}}\ ,\ \eta_{yy}=\frac{1}{4}e^{\frac{y}{2}}$$

Now $$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$$

$$A = e^y\left(\frac{1}{2}e^{\frac{x}{2}}\right)^2 + (0) + \left(-e^x\right)\left(\frac{1}{2}e^{\frac{y}{2}}\right)^2$$

$$A = \frac{1}{4}e^x e^y - \frac{1}{4}e^x e^y = 0$$

$$\mathrm{B} = 2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y$$

$$\mathrm{B} = 2e^y\left(\frac{1}{2}e^{\frac{x}{2}}\right)\left(-\frac{1}{2}e^{\frac{x}{2}}\right) + 0 + 2\left(-e^x\right)\left(\frac{1}{2}e^{\frac{y}{2}}\right)\left(\frac{1}{2}e^{\frac{y}{2}}\right)$$

$$\mathrm{B} = -\frac{1}{2}e^y.e^x - \frac{1}{2}e^y.e^x = -e^{y+x}$$

$$\mathrm{C} = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

$$\mathrm{C} = e^y\left(-\frac{1}{2}e^{\frac{x}{2}}\right)^2 + 0 + \left(-e^x\right)\left(\frac{1}{2}e^{\frac{y}{2}}\right)^2$$

$$\mathrm{C} = \frac{1}{4}e^y e^x - \frac{1}{4}e^x e^y = 0$$

$$\mathrm{D} = a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y$$

$$\mathrm{D} = e^y\left(\frac{1}{4}e^{\frac{x}{2}}\right) + 0 + \left(-e^x\right)\left(\frac{1}{4}e^{\frac{y}{2}}\right) + 0 + 0$$

$$\mathrm{D} = \frac{1}{4}e^{\frac{x}{2}}.e^{\frac{y}{2}}\left(e^{\frac{y}{2}} - e^{\frac{x}{2}}\right) = \frac{1}{4}e^{\frac{x+y}{2}}\left(e^{\frac{y}{2}} - e^{\frac{x}{2}}\right)$$

$$\mathrm{E} = a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y$$

$$\mathrm{E} = e^y\left(-\frac{1}{4}e^{\frac{x}{2}}\right) + 0 + \left(-e^x\right)\left(\frac{1}{4}e^{\frac{y}{2}}\right) + 0 + 0$$

$$\mathrm{E} = -\frac{1}{4}e^{\frac{x}{2}}.e^{\frac{y}{2}}\left(e^{\frac{y}{2}} + e^{\frac{x}{2}}\right) = -\frac{1}{4}e^{\frac{x+y}{2}}\left(e^{\frac{y}{2}} - e^{\frac{x}{2}}\right)$$

F = f =0 , G = g = 0

Now $$A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} + D\phi_{\xi} + E\phi_{\eta} + F\phi = G$$

$$0.\phi_{\xi\xi} + \left(-e^{x+y}\right)\phi_{\xi\eta} + 0.\phi_{\eta\eta} + \frac{1}{4}e^{\frac{x+y}{2}}\left(e^{\frac{y}{2}} - e^{\frac{x}{2}}\right)\phi_{\xi} + \left(-\frac{1}{4}e^{\frac{x+y}{2}}\left(e^{\frac{y}{2}} + e^{\frac{x}{2}}\right)\right)\phi_{\eta} + 0 = 0$$

$$-e^{x+y}\phi_{\xi\eta} + \frac{1}{4}e^{\frac{x+y}{2}}\left(e^{\frac{y}{2}} - e^{\frac{x}{2}}\right)\phi_{\xi} - \frac{1}{4}e^{\frac{x+y}{2}}\left(e^{\frac{y}{2}} + e^{\frac{x}{2}}\right)\phi_{\eta} = 0$$

Divide by $e^{\frac{x+y}{2}}$

$$-e^{\frac{x+y}{2}}\phi_{\xi\eta} + \frac{1}{4}\left(e^{\frac{y}{2}} - e^{\frac{x}{2}}\right)\phi_{\xi} - \frac{1}{4}\left(e^{\frac{y}{2}} + e^{\frac{x}{2}}\right)\phi_{\eta} = 0 \quad \underline{\qquad}*$$

As $\xi = e^{\frac{y}{2}} + e^{\frac{x}{2}}$, $\eta = e^{\frac{y}{2}} - e^{\frac{x}{2}}$

By adding		By subtracting
$\xi = e^{\frac{y}{2}} + e^{\frac{x}{2}}$	,	$\xi = e^{\frac{y}{2}} + e^{\frac{x}{2}}$
$\eta = e^{\frac{y}{2}} - e^{\frac{x}{2}}$	,	$\eta = e^{\frac{y}{2}} - e^{\frac{x}{2}}$
$2e^{\frac{y}{2}} = \xi + \eta$	,	$2e^{\frac{x}{2}} = \xi - \eta$
$e^{\frac{y}{2}} = \frac{\xi + \eta}{2}$	,	$e^{\frac{x}{2}} = \frac{\xi - \eta}{2}$

$$e^{\frac{x}{2}}e^{\frac{y}{2}} = \frac{\xi - \eta}{2}.\frac{\xi + \eta}{2}$$

$$e^{\frac{x+y}{2}} = \frac{\xi^2 - \eta^2}{4}$$

Put in * $\Rightarrow$ $$-\left(\frac{\xi^2-\eta^2}{4}\right)\phi_{\xi\eta}+\frac{1}{4}\eta\phi_\xi-\frac{1}{4}\xi\phi_\eta=0$$

$\Rightarrow\ \phi_{\xi\eta}=\dfrac{\eta}{\xi^2-\eta^2}\phi_\xi-\dfrac{\xi}{\xi^2-\eta^2}\phi_\eta$ is the required canonical form of given equation.

Question: Show that the equation $e^x\phi_{xx}-e^y\phi_{yy}=0$

Is the hyperbolic and determine its canonical form.

Solution: The given equation is $e^x\phi_{xx}-e^y\phi_{yy}=0$

Compare with $$a\phi_{xx}+b\phi_{xy}+c\phi_{yy}+d\phi_x+e\phi_y+f\phi=g$$

$$\Rightarrow a=e^x, b=0, c=-e^y, d=e=f=g=0$$

$$\text{Now } \Delta=b^2-4ac=(0)^2-4\left(e^x\right)\left(-e^y\right)$$

$$\Delta=4e^{x+y}>0$$

So, the equation is hyperbolic

Now the characteristic D.E is given by

$$\frac{dy}{dx}=\frac{b\pm\sqrt{\Delta}}{2a}=\frac{0\pm\sqrt{4e^{x+y}}}{2e^x}$$

$$\frac{dy}{dx}=\pm\frac{e^{\frac{x}{2}}.e^{\frac{y}{2}}}{e^x}=\pm\frac{e^{\frac{y}{2}}}{e^{\frac{x}{2}}}$$

$$\frac{dy}{dx}=-\frac{e^{\frac{y}{2}}}{e^{\frac{x}{2}}}\ ,\ \frac{dy}{dx}=\frac{e^{\frac{y}{2}}}{e^{\frac{x}{2}}}$$

$$e^{\frac{-y}{2}}dy=-e^{\frac{-x}{2}}dx\ ,\ e^{\frac{-y}{2}}dy=e^{\frac{-x}{2}}dx$$

$$e^{\frac{-y}{2}}dy+e^{\frac{-x}{2}}dx=0\ ,\ e^{\frac{-y}{2}}dy-e^{\frac{-x}{2}}dx=0$$

On integration

$$\frac{e^{-\frac{y}{2}}}{-\frac{1}{2}}+\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}=c_1\ ,\ \frac{e^{-\frac{y}{2}}}{-\frac{1}{2}}-\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}=c_2$$

$$e^{\frac{-y}{2}}+e^{\frac{-x}{2}}=-\frac{1}{2}c_1\ ,\ e^{\frac{-y}{2}}-e^{\frac{-x}{2}}=-\frac{1}{2}c_2$$

There are two characteristics curves

Suppose $$\xi=e^{\frac{-y}{2}}+e^{\frac{-x}{2}}\ ,\ \eta=e^{\frac{-y}{2}}-e^{\frac{-x}{2}}$$

$$\xi_x=-\frac{1}{2}e^{\frac{-x}{2}}\quad ,\ \eta_x=\frac{1}{2}e^{\frac{-x}{2}}$$

$$\xi_{xx}=\frac{1}{4}e^{\frac{-x}{2}}\quad ,\ \eta_{xx}=-\frac{1}{4}e^{\frac{-x}{2}}$$

$$\xi_{xy}=0\ ,\ \eta_{xy}=0$$

$$\xi_y=\frac{-1}{2}e^{\frac{-y}{2}}\ ,\ \eta_y=\frac{-1}{2}e^{\frac{-y}{2}}$$

$$\xi_{yy}=\frac{1}{4}e^{\frac{-y}{2}}\ ,\ \eta_{yy}=\frac{1}{4}e^{\frac{-y}{2}}$$

Now $$A=a\xi_x^2+b\xi_x\xi_y+c\xi_y^2$$

$$A=e^x\left(\frac{-1}{2}e^{\frac{-x}{2}}\right)^2+(0)+\left(-e^y\right)\left(\frac{-1}{2}e^{\frac{-y}{2}}\right)^2$$

$$A=\frac{1}{4}e^xe^{-x}-\frac{1}{4}e^ye^{-y}=0$$

$$\mathrm{B}=2a\xi_x\eta_x+b\left(\xi_x\eta_y+\xi_y\eta_x\right)+2c\xi_y\eta_y$$

$$\mathrm{B}=2e^x\left(\frac{-1}{2}e^{\frac{-x}{2}}\right)\left(\frac{1}{2}e^{\frac{-x}{2}}\right)+0+2\left(-e^y\right)\left(\frac{-1}{2}e^{\frac{-y}{2}}\right)\left(\frac{-1}{2}e^{\frac{-y}{2}}\right)$$

$$B = -\frac{1}{2}e^{x}.e^{-x} - \frac{1}{2}e^{y}.e^{-y} = -\frac{1}{2} - \frac{1}{2} = -1$$

$$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

$$C = e^{x}\left(\frac{1}{2}e^{\frac{-x}{2}}\right)^2 + 0 + \left(-e^{y}\right)\left(\frac{-1}{2}e^{\frac{-y}{2}}\right)^2$$

$$C = \frac{1}{4}e^{x}e^{-x} - \frac{1}{4}e^{y}e^{-y} = \frac{1}{4} - \frac{1}{4} = 0$$

$$D = a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y$$

$$D = e^{x}\left(\frac{1}{4}e^{\frac{-x}{2}}\right) + 0 + \left(-e^{y}\right)\left(\frac{1}{4}e^{\frac{-y}{2}}\right) + 0 + 0$$

$$D = \frac{1}{4}e^{x-\frac{x}{2}} - \frac{1}{4}e^{y-\frac{y}{2}} = \frac{1}{4}\left(e^{\frac{x}{2}} - e^{\frac{y}{2}}\right)$$

$$E = a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y$$

$$E = e^{x}\left(-\frac{1}{4}e^{\frac{-x}{2}}\right) + 0 + \left(-e^{y}\right)\left(\frac{1}{4}e^{\frac{-y}{2}}\right) + 0 + 0$$

$$E = -\frac{1}{4}e^{x-\frac{x}{2}} - \frac{1}{4}e^{y-\frac{y}{2}} = -\frac{1}{4}\left(e^{\frac{x}{2}} + e^{\frac{y}{2}}\right)$$

$$F = f = 0\ ,\ G = g = 0$$

Now $$A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} + D\phi_{\xi} + E\phi_{\eta} + F\phi = G$$

$$0.\phi_{\xi\xi} + (-1)\phi_{\xi\eta} + 0.\phi_{\eta\eta} + \frac{1}{4}\left(e^{\frac{x}{2}} - e^{\frac{y}{2}}\right)\phi_{\xi} - \frac{1}{4}\left[e^{\frac{y}{2}} + e^{\frac{x}{2}}\right]\phi_{\eta} + 0 = 0$$

$$-e^{x+y}\phi_{\xi\eta} + \frac{1}{4}e^{\frac{x+y}{2}}\left(e^{\frac{y}{2}} - e^{\frac{x}{2}}\right)\phi_{\xi} - \frac{1}{4}e^{\frac{x+y}{2}}\left(e^{\frac{y}{2}} + e^{\frac{x}{2}}\right)\phi_{\eta} = 0$$

$$-\phi_{\xi\eta}+\frac{1}{4}\left(e^{\frac{x}{2}}-e^{\frac{y}{2}}\right)\phi_{\xi}-\frac{1}{4}\left(e^{\frac{x}{2}}+e^{\frac{y}{2}}\right)\phi_{\eta}=0 \quad \_\_\_*$$

As $\xi=e^{\frac{-y}{2}}+e^{\frac{-x}{2}}$, $\eta=e^{\frac{-y}{2}}-e^{\frac{-x}{2}}$

By adding | By subtracting

$$\xi=e^{\frac{-y}{2}}+e^{\frac{-x}{2}} \quad , \quad \xi=e^{\frac{-y}{2}}+e^{\frac{-x}{2}}$$

$$\eta=e^{\frac{-y}{2}}-e^{\frac{-x}{2}} \quad , \quad \eta=e^{\frac{-y}{2}}-e^{\frac{-x}{2}}$$

$$2e^{\frac{-y}{2}}=\xi+\eta \quad , \quad 2e^{\frac{-x}{2}}=\xi-\eta$$

$$e^{\frac{-y}{2}}=\frac{\xi+\eta}{2} \quad , \quad e^{\frac{-x}{2}}=\frac{\xi-\eta}{2}$$

$$e^{\frac{y}{2}}=\frac{2}{\xi+\eta} \quad , \quad e^{\frac{x}{2}}=\frac{2}{\xi-\eta}$$

$$e^{\frac{x}{2}}-e^{\frac{y}{2}}=\frac{2}{\xi-\eta}-\frac{2}{\xi+\eta}=\frac{2(\xi+\eta-\xi+\eta)}{\xi^2-\eta^2}=\frac{4\eta}{\xi^2-\eta^2}$$

$$e^{\frac{x}{2}}+e^{\frac{y}{2}}=\frac{2}{\xi-\eta}+\frac{2}{\xi+\eta}=\frac{2(\xi+\eta+\xi-\eta)}{\xi^2-\eta^2}=\frac{4\xi}{\xi^2-\eta^2}$$

Put in * $\Rightarrow$ $$-\phi_{\xi\eta}+\frac{1}{4}\left(\frac{4\eta}{\xi^2-\eta^2}\right)\phi_{\xi}-\frac{1}{4}\left(\frac{4\xi}{\xi^2-\eta^2}\right)\phi_{\eta}=0$$

$\phi_{\xi\eta}=\left(\frac{\eta}{\xi^2-\eta^2}\right)\phi_{\xi}-\left(\frac{\xi}{\xi^2-\eta^2}\right)\phi_{\eta}$ is the required canonical form of given equation.

Question: Show that the equation $\phi_{xx} - \sec^4 x\phi_{yy} = 0$

Is the hyperbolic and determine its canonical form.

Solution: The given equation is $\phi_{xx} - \sec^4 x\phi_{yy} = 0$

Compare with $$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi = g$$

$$\Rightarrow a = 1, b = 0, c = -\sec^4 \text{x}, d = e = f = g = 0$$

Now $$\Delta = b^2 - 4ac = (0)^2 - 4(1)(-\sec^4 x) = 4\sec^4 x > 0$$

So, the equation is hyperbolic

Now the characteristic D.E is given by

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{2a} = \frac{0 \pm \sqrt{\sec^4 x}}{2(1)}$$

$$\frac{dy}{dx} = \pm\sec^2 x$$

$$\frac{dy}{dx} = -\sec^2 x \ , \ \frac{dy}{dx} = \sec^2 x$$

$$dy = -\sec^2 xdx \ , \ dy = \sec^2 xdx$$

$$dy + \sec^2 xdx = 0 \ , \ dy - \sec^2 xdx = 0$$

On integration

$$y + \tan x = c_1 \ , \ y - \tan x = c_2$$

Suppose $$\xi = y + \tan x \ , \eta = y - \tan x$$

$$\xi_x = \sec^2 x \ , \eta_x = -\sec^2 x$$

$$\xi_{xx} = 2\sec x.\sec x \tan x \ , \eta_{xx} = -\sec^2 x \tan\text{x}$$

$$\xi_{xy} = 0 \ , \eta_{xy} = 0$$

$$\xi_y = 1 , \eta_y = 1$$

$$\xi_{yy}=0\ ,\ \eta_{yy}=0$$

Now $$A=a\xi_x^2+b\xi_x\xi_y+c\xi_y^2$$

$$A=(1)\left(\sec^2 x\right)^2+(0)+\left(-\sec^4 x\right)(1)^2$$

$$A=\sec^4 x-\sec^4 x=0$$

$$\text{B}=2a\xi_x\eta_x+b\left(\xi_x\eta_y+\xi_y\eta_x\right)+2c\xi_y\eta_y$$

$$\text{B}=2(1)\left(\sec^2 x\right)\left(-\sec^2 x\right)+0+2\left(-\sec^4 x\right)(1)(1)$$

$$\text{B}=-2\sec^4 x-2\sec^4 x=-4\sec^4 x$$

$$\text{C}=a\eta_x^2+b\eta_x\eta_y+c\eta_y^2$$

$$\text{C}=\left(e^x\right)\left(-\sec^2 x\right)^2+0+\left(-\sec^4 x\right)(1)^2$$

$$\text{C}=\sec^4 x-\sec^4 x=0$$

$$\text{D}=a\xi_{xx}+b\xi_{xy}+c\xi_{yy}+d\xi_x+e\xi_y$$

$$\text{D}=(1)\left(2\sec^2 x\tan x\right)+0+\left(-\sec^4 x\right)(0)+0+0$$

$$\text{D}=2\sec^2 x\tan x$$

$$\text{E}=a\eta_{xx}+b\eta_{xy}+c\eta_{yy}+d\eta_x+e\eta_y$$

$$\text{E}=-2\sec^2 x\tan x$$

$$\text{E}=-\frac{1}{4}e^{x-\frac{x}{2}}-\frac{1}{4}e^{y-\frac{y}{2}}=-\frac{1}{4}\left(e^{\frac{x}{2}}+e^{\frac{y}{2}}\right)$$

F = f =0 , G = g = 0

Now $$A\phi_{\xi\xi}+B\phi_{\xi\eta}+C\phi_{\eta\eta}+D\phi_\xi+E\phi_\eta+F\phi=G$$

$$0.\phi_{\xi\xi}+\left(-4\sec^4 x\right)\phi_{\xi\eta}+0.\phi_{\eta\eta}+\left(2\sec^2 x\tan x\right)\phi_\xi+\left(-2\sec^2 x\tan x\right)\phi_\eta+0=0$$

$$\left(4\sec^4 x\right)\phi_{\xi\eta}=\left(2\sec^2 x\tan x\right)\phi_\xi-\left(2\sec^2 x\tan x\right)\phi_\eta$$

$$\left(4\sec^4 x\right)\phi_{\xi\eta} = 2\sec^2 x\tan x\left(\phi_\xi - \phi_\eta\right)$$

$$\phi_{\xi\eta} = \frac{\tan x}{2\sec^2 x}\left(\phi_\xi - \phi_\eta\right) \quad \_\_\_\_*$$

As $\xi = y + \text{tanx}$, $\eta = y - \text{tanx}$

By subtracting

$$\xi = y + \text{tanx}$$

$$\pm\eta = \pm y \mp \text{tanx}$$

$$\xi - \eta = 2\tan x$$

$$\tan x = \frac{\xi - \eta}{2}$$

Put in * $\Rightarrow$

$$\phi_{\xi\eta} = \frac{\frac{\xi - \eta}{2}}{2\left(1 + \left(\frac{\xi - \eta}{2}\right)^2\right)}\left(\phi_\xi - \phi_\eta\right)$$

$$\phi_{\xi\eta} = \frac{\xi - \eta}{4\left(1 + \frac{(\xi - \eta)^2}{4}\right)}\left(\phi_\xi - \phi_\eta\right)$$

$$\phi_{\xi\eta} = \frac{\xi - \eta}{4\left(\frac{4 + (\xi - \eta)^2}{4}\right)}\left(\phi_\xi - \phi_\eta\right)$$

$\phi_{\xi\eta} = \frac{\xi - \eta}{4 + (\xi - \eta)^2}\left(\phi_\xi - \phi_\eta\right)$ is the required canonical form of given equation.

Canonical form or Normal form of parabolic PDE's

Since for parabolic PDE's

$$\Delta = 0 \quad \& \ \Delta^* = 0$$

$$b^2 - 4ac = 0 \quad , \ B^2 - 4AC = 0$$

$$\Rightarrow \ b = \pm 2\sqrt{ac} \ \_\_\_\_ (i) \quad , \ B = \pm 2\sqrt{AC}$$

This shows that we cannot arrange both (a&c) or (A&C) to be zero. Now the characteristic equation

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{2a}$$

$$\Rightarrow \frac{dy}{dx} = \frac{b}{2a} \qquad \because \Delta = 0$$

So, the characteristic D.E has only one solution. This means that parabolic equation has only one characteristic curve. Suppose that we choose $\phi(x,y) = \xi$ as the solution of equation (i)

Let we select A = 0 for parabolic

$$a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2 = 0$$

From (i)

$$a\xi_x^2 + 2\sqrt{ac}\,\xi_x\xi_y + c\xi_y^2 = 0$$

$$\left(\sqrt{a}\xi_x\right)^2 + 2\sqrt{a}\xi_x\sqrt{c}\xi_y + \left(\sqrt{c}\xi_y\right)^2 = 0$$

$$\left(\sqrt{a}\xi_x + \sqrt{c}\xi_y\right)^2 = 0$$

$$\Rightarrow \sqrt{a}\xi_x + \sqrt{c}\xi_y = 0 \quad \_\_\_\_ (ii)$$

$$B = 2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y$$

$$B = 2a\xi_x\eta_x + 2\sqrt{ac}\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y$$

$$B = 2\left[a\xi_x\eta_x + \sqrt{a}\sqrt{c}\xi_x\eta_y + \sqrt{a}\sqrt{c}\xi_y\eta_x + c\xi_y\eta_y \right]$$

$$B = 2\left[\sqrt{a}\xi_x\left\{\sqrt{a}\eta_x + \sqrt{c}\eta_y\right\} + \sqrt{c}\xi_y\left\{\sqrt{a}\eta_x + \sqrt{c}\eta_y\right\}\right]$$

$$B = 2\left[\left(\sqrt{a}\xi_x + \sqrt{c}\xi_y\right)\left(\sqrt{a}\eta_x + \sqrt{c}\eta_y\right)\right]$$

$$B = 2\left[(0)\left(\sqrt{a}\eta_x + \sqrt{c}\eta_y\right)\right]$$

$$B = 0$$

So, the canonical form of parabolic PDE is written as

$$A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} + D\phi_\xi + E\phi_\eta + F\phi = G$$

$$\Rightarrow \; C\phi_{\eta\eta} + D\phi_\xi + E\phi_\eta + F\phi = G$$

Where the characteristic curve $\eta = \eta(x, y)$ is taken as arbitrary such that the Jacobian $|J| \neq 0$

$$\Delta^* = |J|^2 \Delta$$

$$\Rightarrow \; \Delta = \frac{1}{|J|^2}\Delta^*$$

Lecture # 07

Canonical form of Elliptic equation:

For elliptic equation $\Delta = b^2 - 4ac < 0$ then the characteristic D.E for elliptic PDE's

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{2a} \_\_\_\_\_ (i)$$

So, there are no real characteristic for elliptic equation. For this we show that

$$A^* = C^* \ , \ B^* = 0$$

Let ξ and η be the solution of equation (i) and ξ = α+iβ , η = α−iβ

Where α and β are the real and imaginary parts of ξ and η is the complex conjugate of ξ i.e. $\eta = \bar{\xi}$

Here we take A = C = 0

$$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$$

Put $\xi = \alpha + i\beta$

$$\xi_x = \alpha_x + i\beta_x \ , \ \xi_y = \alpha_y + i\beta_y$$

$$a\left(\alpha_x + i\beta_x\right)^2 + b\left(\alpha_x + i\beta_x\right)\left(\alpha_y + i\beta_y\right) + c\left(\alpha_y + i\beta_y\right)^2 = 0$$

$$a\left(\alpha_x^2 - \beta_x^2 + 2i\alpha_x\beta_x\right) + b\left(\alpha_x\alpha_y + i\alpha_x\beta_y + i\beta_x\alpha_y - \beta_x\beta_y\right) + c\left(\alpha_y^2 - \beta_y^2 + 2i\alpha_y\beta_y\right) = 0$$

$$a\alpha_x^2 - a\beta_x^2 + 2ia\alpha_x\beta_x + b\alpha_x\alpha_y + ib\alpha_x\beta_y + ib\beta_x\alpha_y - b\beta_x\beta_y + c\alpha_y^2 - c\beta_y^2 + 2ic\alpha_y\beta_y = 0$$

$$\left(a\alpha_x^2 + b\alpha_x\alpha_y + c\alpha_y^2\right) - \left(a\beta_x^2 + b\beta_x\beta_y + c\beta_y^2\right) + \left(2a\alpha_x\beta_x + b\left(\alpha_x\beta_y + \beta_x\alpha_y\right) + 2c\alpha_y\beta_y\right) = 0$$

$$\Rightarrow A^* - C^* + iB^* = 0$$

$$\Rightarrow A^* - C^* = 0 \, , \, B^* = 0$$

$$\Rightarrow A^* = C^*, B^* = 0$$

So, the canonical form of elliptic equation is written as

$$A^*\phi_{\alpha\alpha}+B^*\phi_{\alpha\beta}+C^*\phi_{\beta\beta}+D^*\phi_\alpha+E^*\phi_\beta+F^*\phi=G^*$$

$$A^*\phi_{\alpha\alpha}+0.\phi_{\alpha\beta}+A^*\phi_{\beta\beta}+D^*\phi_\alpha+E^*\phi_\beta+F^*\phi=G^*$$

$$A^*\left(\phi_{\alpha\alpha}+\phi_{\beta\beta}\right)+D^*\phi_\alpha+E^*\phi_\beta+F^*\phi=G^*$$

$$\phi_{\alpha\alpha}+\phi_{\beta\beta}=\frac{G^*-D^*\phi_\alpha-E^*\phi_\beta-F^*\phi}{A^*}$$

$$\Rightarrow\ \phi_{\alpha\alpha}+\phi_{\beta\beta}=\frac{H^*\left(\alpha,\beta,\phi,\phi_\alpha,\phi_\beta\right)}{A^*}$$

Question: Solve the PDE $y^2\phi_{xx}+x^2\phi_{yy}=0\ ,\ x>0\ ,\ y>0$

Also write its canonical form.

Solution: The given equation is $y^2\phi_{xx}+x^2\phi_{yy}=0$

$$Here\ \ a=y^2, b=0, c=x^2, d=e=f=g=0$$

$$\text{Now } \Delta=b^2-4ac=(0)^2-4\left(y^2\right)\left(x^2\right)$$

$\Delta=-4y^2x^2<0$ So, the equation is elliptic.

Now the characteristic D.E is

$$\frac{dy}{dx}=\frac{b\pm\sqrt{\Delta}}{2a}=\frac{0\pm\sqrt{-4x^2y^2}}{2y^2}$$

$$\frac{dy}{dx}=\pm\frac{i2xy}{2y^2}=\pm\frac{ix}{y}$$

$$\frac{dy}{dx}=-\frac{ix}{y}\ ,\ \frac{dy}{dx}=\frac{ix}{y}$$

$$2ydy+2ixdx=0\ ,\ 2ydy-2ixdx=0$$

On integration

$$y^2+ix^2=c_1\ ,\ y^2-ix^2=c_2$$

There are two characteristics curves

Suppose $$\xi = y^2 + ix^2 \ , \eta = y^2 - ix^2$$

$$Let\ \alpha = y^2 \ , \beta = x^2$$

$$\alpha_x = 0 \ , \beta_x = 2x$$

$$\alpha_{xx} = 0 \ , \beta_{xx} = 2$$

$$\alpha_{xy} = 0 \ , \beta_{xy} = 0$$

$$\alpha_y = 2y \ , \beta_y = 0$$

$$\alpha_{yy} = 2 \ , \beta_{yy} = 0$$

Now $$A = a\alpha_x^2 + b\alpha_x\alpha_y + c\alpha_y^2$$

$$A = y^2(0) + 0 + (x^2)(2y)^2$$

$$A = 4x^2y^2$$

$$\text{B} = 2a\alpha_x\beta_x + b(\alpha_x\beta_y + \alpha_y\beta_x) + 2c\alpha_y\beta_y$$

$$\text{B} = 2(y^2)(0)(2x) + 0 + 2(x^2)(2y)(0)$$

$$\text{B} = 0$$

$$\text{C} = \text{A} = 4x^2y^2$$

$$\text{D} = a\alpha_{xx} + b\alpha_{xy} + c\alpha_{yy} + d\alpha_x + e\alpha_y$$

$$\text{D} = y^2(0) + 0 + x^2(2) + 0 + 0$$

$$\text{D} = 2x^2$$

$$\text{E} = a\beta_{xx} + b\beta_{xy} + c\beta_{yy} + d\beta_x + e\beta_y$$

$$\text{E} = y^2(2) + 0 + c(0) + 0 + 0$$

$$\text{E} = 2y^2$$

$$F = f = 0 \;,\; G = g = 0$$

Now $$A\left(\phi_{\alpha\alpha} + \phi_{\beta\beta}\right) + D\phi_\alpha + E\phi_\beta + F\phi = G$$

$$4x^2y^2\left(\phi_{\alpha\alpha} + \phi_{\beta\beta}\right) + 2x^2\phi_\alpha + 2y^2\phi_\beta + 0 = 0$$

Divide by $4x^2y^2$

$$\left(\phi_{\alpha\alpha} + \phi_{\beta\beta}\right) + \frac{1}{2y^2}\phi_\alpha + \frac{1}{2x^2}\phi_\beta = 0$$

$\left(\phi_{\alpha\alpha} + \phi_{\beta\beta}\right) + \dfrac{1}{2\alpha}\phi_\alpha + \dfrac{1}{2\beta}\phi_\beta = 0$ is the required canonical form of given PDE.

Question: : Solve the PDE $e^y\phi_{xx} + e^x\phi_{yy} = 0$ Also write its canonical form.

Solution: The given equation is $e^y\phi_{xx} + e^x\phi_{yy} = 0$

$$\Rightarrow a = e^y, b = 0, c = e^x, d = e = f = g = 0$$

Now $\Delta = b^2 - 4ac = (0)^2 - 4\left(e^y\right)\left(e^x\right)$

$\Delta = -4e^{x+y} < 0$ So, the equation is elliptic.

Now the characteristic D.E is given by

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{2a} = \frac{0 \pm \sqrt{-4e^{x+y}}}{2e^y}$$

$$\frac{dy}{dx} = \pm\frac{i2e^{\frac{x+y}{2}}}{2e^y} = \pm\frac{ie^{\frac{x}{2}}.e^{\frac{y}{2}}}{e^y} = \pm\frac{ie^{\frac{x}{2}}}{e^{\frac{y}{2}}}$$

$$\frac{dy}{dx} = -\frac{ie^{\frac{x}{2}}}{e^{\frac{y}{2}}} \;,\; \frac{dy}{dx} = \frac{ie^{\frac{x}{2}}}{e^{\frac{y}{2}}}$$

$$e^{\frac{y}{2}}dy + ie^{\frac{x}{2}}dx = 0 \;,\; e^{\frac{y}{2}}dy - ie^{\frac{x}{2}}dx = 0$$

On integration , On integration

$$\frac{e^{\frac{y}{2}}}{\frac{1}{2}}+\frac{ie^{\frac{x}{2}}}{\frac{1}{2}}=c_1 \ , \ \frac{e^{\frac{y}{2}}}{\frac{1}{2}}-\frac{ie^{\frac{x}{2}}}{\frac{1}{2}}=c_2$$

$$e^{\frac{y}{2}}+ie^{\frac{x}{2}}=\frac{1}{2}c_1 \ , \ e^{\frac{y}{2}}-ie^{\frac{x}{2}}=\frac{1}{2}c_2$$

There are two characteristics curves

Suppose $\xi=e^{\frac{y}{2}}+ie^{\frac{x}{2}} \ , \eta=e^{\frac{y}{2}}-ie^{\frac{x}{2}}$

Let $\alpha=e^{\frac{y}{2}} \ , \ \beta=e^{\frac{x}{2}}$

$$\alpha_x=0 \ , \beta_x=\frac{1}{2}e^{\frac{x}{2}}$$

$$\alpha_{xx}=0 \ , \beta_{xx}=\frac{1}{4}e^{\frac{x}{2}}$$

$$\alpha_{xy}=0 \ , \beta_{xy}=0$$

$$\alpha_y=\frac{1}{2}e^{\frac{y}{2}} \ , \beta_y=0$$

$$\alpha_{yy}=\frac{1}{4}e^{\frac{y}{2}} \ , \beta_{yy}=0$$

Now $$A=a\alpha_x^2+b\alpha_x\alpha_y+c\alpha_y^2$$

$$A=e^y(0)^2+(0)+(e^x)\left(\frac{1}{2}e^{\frac{y}{2}}\right)^2 \Rightarrow \ A=\frac{1}{4}e^x e^y$$

$$\text{B}=2a\alpha_x\beta_x+b\left(\alpha_x\beta_y+\alpha_y\beta_x\right)+2c\alpha_y\beta_y$$

$$\text{B}=2e^y(0)\left(\frac{1}{2}e^{\frac{x}{2}}\right)+0+2\left(e^x\right)\left(\frac{1}{2}e^{\frac{y}{2}}\right)(0)$$

$$B = 0$$

$$C = A = \frac{1}{4}e^x e^y$$

$$D = a\alpha_{xx} + b\alpha_{xy} + c\alpha_{yy} + d\alpha_x + e\alpha_y$$

$$D = e^y(0) + 0 + (e^x)\left(\frac{1}{4}e^{\frac{y}{2}}\right) + 0 + 0$$

$$D = = \frac{1}{4}e^x.e^{\frac{y}{2}}$$

$$E = a\beta_{xx} + b\beta_{xy} + c\beta_{yy} + d\beta_x + e\beta_y$$

$$E = e^y\left(\frac{1}{4}e^{\frac{x}{2}}\right) + 0 + (e^x)(0) + 0 + 0$$

$$E = \frac{1}{4}e^{\frac{x}{2}}.e^y$$

$$F = f = 0, \; G = g = 0$$

Now $A(\phi_{\alpha\alpha} + \phi_{\beta\beta}) + D\phi_\alpha + E\phi_\beta + F\phi = G$

$$\frac{1}{4}e^x e^y(\phi_{\alpha\alpha} + \phi_{\beta\beta}) + \frac{1}{4}e^x e^{\frac{y}{2}}\phi_\alpha + \frac{1}{4}e^{\frac{x}{2}}e^y\phi_\beta + 0 = 0$$

$$\frac{1}{4}e^x e^y(\phi_{\alpha\alpha} + \phi_{\beta\beta}) + \frac{1}{4}e^x e^{\frac{y}{2}}\phi_\alpha + \frac{1}{4}e^{\frac{x}{2}}e^y\phi_\beta = 0$$

Divide by $\frac{1}{4}e^x e^y$

$$\phi_{\alpha\alpha} + \phi_{\beta\beta} + \frac{\phi_\alpha}{e^{\frac{y}{2}}} + \frac{\phi_\beta}{e^{\frac{x}{2}}} = 0$$

$\phi_{\alpha\alpha} + \phi_{\beta\beta} + \frac{\phi_\alpha}{\alpha} + \frac{\phi_\beta}{\beta} = 0$ is the required canonical form of given PDE.

P.D.E's with constant coefficient:

Let $a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi = g$

$$And\ \ \Delta = b^2 - 4ac \text{ is also constant.}$$

For Hyperbolic P.D.E's:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{2a}$$

$$\frac{dy}{dx} = \frac{b + \sqrt{\Delta}}{2a}\ ,\ \ \frac{dy}{dx} = \frac{b - \sqrt{\Delta}}{2a}$$

On integration

$$y = \left(\frac{b + \sqrt{\Delta}}{2a}\right)x + c_1\ ,\ \ y = \left(\frac{b - \sqrt{\Delta}}{2a}\right)x + c_2$$

$$y - \left(\frac{b + \sqrt{\Delta}}{2a}\right)x = c_1\ ,\ \ y - \left(\frac{b - \sqrt{\Delta}}{2a}\right)x = c_2$$

$$Let\ \xi = y - \left(\frac{b + \sqrt{\Delta}}{2a}\right)x\ ,\ \ Let\ \eta = y - \left(\frac{b - \sqrt{\Delta}}{2a}\right)x$$

If a = 0 $\quad \dfrac{dy}{dx} = \dfrac{b \pm \sqrt{\Delta}}{2a}$ is not applicable.

Remove this difficulty

$$az_x^2 + bz_xz_y + cz_y^2 = 0$$

$$bz_xz_y + cz_y^2 = 0 \qquad \because\ a = 0$$

Divide by z_x^2

$$b\frac{z_xz_y}{z_x^2} + c\frac{z_y^2}{z_x^2} = 0$$

$$b\left(\frac{z_y}{z_x}\right)+c\left(\frac{z_y}{z_x}\right)^2=0$$

Let $z_x dx + z_y dy = 0$

$$\Rightarrow \frac{z_y}{z_x}=-\frac{dx}{dy}$$

Put in above

$$b\left(-\frac{dx}{dy}\right)+c\left(-\frac{dx}{dy}\right)^2=0$$

$$-\frac{dx}{dy}\left[b-c\frac{dx}{dy}\right]=0$$

$$-\frac{dx}{dy}=0 \;,\; b-c\frac{dx}{dy}=0$$

On integration

$$\Rightarrow x=c_1 \;,\; by-cx=c_2$$

Let $\xi = x \;,\; \eta = by - cx$

For Parabolic P.D.E's

$$\frac{dy}{dx}=\frac{b\pm\sqrt{\Delta}}{2a}=\frac{b\pm\sqrt{0}}{2a}=\frac{b}{2a}$$

On integration

$$y=\frac{b}{2a}x+c_1$$

$$\Rightarrow y-\frac{b}{2a}x=c_1$$

Let $\xi = y-\dfrac{b}{2a}x$ and $\eta =$arbitrary

For Elliptic P.D.E's

$$\frac{dy}{dx}=\frac{b\pm\sqrt{\Delta}}{2a}=\frac{b\pm i\sqrt{-\Delta}}{2a}$$

On integration

$$y=\left(\frac{b\pm i\sqrt{-\Delta}}{2a}\right)x+c$$

$$\Rightarrow\ y=\left(\frac{b+i\sqrt{-\Delta}}{2a}\right)x+c_1\ ,\ \ y=\left(\frac{b-i\sqrt{-\Delta}}{2a}\right)x+c_2$$

$$\Rightarrow\ y=\frac{b}{2a}x+\frac{i\sqrt{-\Delta}}{2a}x+c_1\ ,\ \ y=\frac{b}{2a}x-\frac{i\sqrt{-\Delta}}{2a}x+c_2$$

$$\Rightarrow\ y-\frac{b}{2a}x-\frac{i\sqrt{-\Delta}}{2a}x=c_1\ ,\ \ y-\frac{b}{2a}x+\frac{i\sqrt{-\Delta}}{2a}x=c_2$$

Let $\xi=y-\frac{b}{2a}x-\frac{i\sqrt{-\Delta}}{2a}x\ ,\ \ \eta=y-\frac{b}{2a}x+\frac{i\sqrt{-\Delta}}{2a}x$

Question: Solve the P.D.E $4\phi_{xx}+5\phi_{xy}+\phi_{yy}+\phi_x+\phi_y=2$ also find its canonical form.

Solution: Given $4\phi_{xx}+5\phi_{xy}+\phi_{yy}+\phi_x+\phi_y=2$

Here $a=4,b=5,c=1,d=e=1,f=0,g=2$

$$\Delta=b^2-4ac=25-16=9>0$$

So, the equation is Hyperbolic.

Now the characteristic equation

$$\frac{dy}{dx}=\frac{b\pm\sqrt{\Delta}}{2a}=\frac{5\pm\sqrt{9}}{2(4)}=\frac{5\pm3}{8}=1,\frac{1}{4}$$

$$\frac{dy}{dx}=1\ ,\ \ \frac{dy}{dx}=\frac{1}{4}$$

On integration

$$y = x + c_1 \ , \ \ y = \frac{1}{4}x + c_2$$

$$y - x = c_1 \ , \ \ y - \frac{1}{4}x = c_2$$

Let $\xi = y - x \ , \ \ \eta = y - \frac{1}{4}x$

$$\xi_x = -1 \quad , \ \eta_x = -\frac{1}{4}$$

$$\xi_{xx} = 0 \quad , \ \eta_{xx} = 0$$

$$\xi_{xy} = 0 \quad , \ \eta_{xy} = 0$$

$$\xi_y = 1 \quad , \ \eta_y = 1$$

$$\xi_{yy} = 0 \quad , \ \eta_{yy} = 0$$

$$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$$

$$A = 4(-1)^2 + 5(-1)(1) + (1)(1)^2 = 4 - 5 + 1 = 0$$

$$A = \sec^4 x - \sec^4 x = 0$$

B = $2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y$

B = $2(4)(-1)\left(\frac{-1}{4}\right) + 5\left((-1)(1) + (1)\left(\frac{-1}{4}\right)\right) + 2(1)(1)(1)$

B = $2 + 5\left(-1 - \frac{1}{4}\right) + 2 = 4 + 5\left(-\frac{5}{4}\right) = \frac{16 - 25}{4} = \frac{-9}{4}$

C = $a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$

C = $4\left(\frac{-1}{4}\right)^2 + 5\left(\frac{-1}{4}\right)(1) + 1(1)^2 = 4\left(\frac{1}{16}\right) - \frac{5}{4} + 1$

$$C = \frac{1}{4} - \frac{5}{4} + 1 = \frac{1-5+4}{4} = 0$$

$$D = a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y$$

$$D = 4(0) + 5(0) + 1(0) + 1(-1) + 1(1) = 0$$

$$E = a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y$$

$$E = 4(0) + 5(0) + 1(0) + 1\left(\frac{-1}{4}\right) + 1(1) = -\frac{1}{4} + 1 = \frac{3}{4}$$

$$F = f = 0\ ,\ G = g = 2$$

Now $$A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} + D\phi_{\xi} + E\phi_{\eta} + F\phi = G$$

$$0.\phi_{\xi\xi} + \left(\frac{-9}{4}\right)\phi_{\xi\eta} + 0.\phi_{\eta\eta} + 0.\phi_{\xi} + \frac{3}{4}\phi_{\eta} + 0\phi = 2$$

$$\frac{-9}{4}\phi_{\xi\eta} + \frac{3}{4}\phi_{\eta} = 2 \quad \text{Divide by } \frac{-9}{4}$$

$$\phi_{\xi\eta} - \frac{1}{3}\phi_{\eta} = \frac{-8}{9}$$ is the required canonical form of given P.D.E

Let $\phi_{\eta} = \psi$ $$\Rightarrow \phi_{\xi\eta} - \frac{1}{3}\phi_{\eta} = \frac{-8}{9}$$

$$\phi_{\eta\xi} - \frac{1}{3}\phi_{\eta} = \frac{-8}{9}$$

$$\frac{\partial}{\partial \xi}(\phi_{\eta}) - \frac{1}{3}\phi_{\eta} = \frac{-8}{9}$$

$$\frac{\partial}{\partial \xi}(\psi) - \frac{1}{3}(\psi) = \frac{-8}{9} \quad \text{\_\_\_\_\_(i)}$$

$$I.F = e^{\int \frac{-1}{3}d\xi} = e^{\frac{-1}{3}\xi}$$

Multiplying (i) with I.F

$$e^{\frac{-1}{3}\xi}\frac{\partial}{\partial\xi}(\psi)-e^{\frac{-1}{3}\xi}\frac{1}{3}(\psi)=\frac{-8}{9}e^{\frac{-1}{3}\xi}$$

$$\Rightarrow \frac{\partial}{\partial\xi}\left(\psi e^{\frac{-1}{3}\xi}\right)=\frac{-8}{9}e^{\frac{-1}{3}\xi}$$

On integration

$$e^{\frac{-1}{3}\xi}\psi=\frac{-8}{9}\int e^{\frac{-1}{3}\xi}d\xi+h_1(\eta)$$

$$e^{\frac{-1}{3}\xi}\psi=\frac{-8}{9}\frac{e^{\frac{-1}{3}\xi}}{\frac{-1}{3}}+h_1(\eta)$$

$$e^{\frac{-1}{3}\xi}\psi=\frac{8}{3}e^{\frac{-1}{3}\xi}+h_1(\eta)$$

$$\Rightarrow \psi=\frac{8}{3}+h_1(\eta)e^{\frac{1}{3}\xi}$$

$$\Rightarrow \phi_\eta=\frac{8}{3}+h_1(\eta)e^{\frac{1}{3}\xi} \qquad \because \psi=\phi_\eta$$

$$\Rightarrow \frac{\partial\phi}{\partial\eta}=\frac{8}{3}+h_1(\eta)e^{\frac{1}{3}\xi}$$

On integration

$$\Rightarrow \phi=\frac{8}{3}\int d\eta+e^{\frac{1}{3}\xi}\int h_1(\eta)d\eta+h_2(\xi)$$

$$\Rightarrow \phi=\frac{8}{3}\eta+e^{\frac{1}{3}\xi}h_3(\eta)+h_2(\xi)$$

$$\Rightarrow \phi(x,y)=\frac{8}{3}\left(y-\frac{x}{4}\right)+e^{\frac{y-x}{3}}h_3\left(y-\frac{x}{4}\right)+h_2(y-x)$$

Lecture # 08

Question: $\int_0^\infty e^{-x^2}dx$

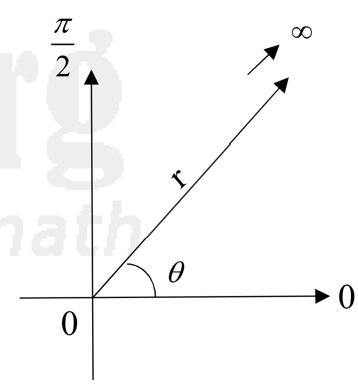
Solution: Let $I=\int_0^\infty e^{-x^2}dx=\int_0^\infty e^{-y^2}dy \quad \because \int_a^b f(x)dx=\int_a^b f(t)dt$

$$I^2=\left(\int_0^\infty e^{-x^2}dx\right)^2$$

$$I^2=\int_0^\infty e^{-x^2}dx.\int_0^\infty e^{-x^2}dx$$

$$I^2=\int_0^\infty e^{-x^2}dx.\int_0^\infty e^{-y^2}dy$$

$$I^2=\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dydx$$



Let x = rcosθ , y = rsinθ

$$r^2=x^2+y^2,\ \theta=\tan^{-1}\left(\frac{y}{x}\right)$$

In first quadrant

$$0\le\theta\le\frac{\pi}{2}$$

$$0\le r\le\infty$$

dxdy = rdrdθ

$$\Rightarrow I^2=\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta$$

$$\Rightarrow I^2 = \int_0^{\frac{\pi}{2}} d\theta . \left(\frac{-1}{2}\right) \int_0^{\infty} e^{-r^2}(-2r)\,dr$$

$$\Rightarrow I^2 = \theta\Big|_0^{\frac{\pi}{2}} . \left(\frac{-1}{2}\right) e^{-r^2}\Big|_0^{\infty}$$

$$\Rightarrow I^2 = \left(\frac{\pi}{2} - 0\right) . \left(\frac{-1}{2}\right)\left[\underset{r\to\infty}{Lim}\, e^{-r^2} - e^0\right] \quad \because \underset{x\to\infty}{Lim}\, e^{mx} = \begin{cases} 0 \text{ if } m < 0 \\ \infty \text{ if } m \geq 0 \end{cases}$$

$$\Rightarrow I^2 = \frac{\pi}{2} . \left(\frac{-1}{2}\right)[0-1] = \frac{\pi}{4}$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{2}$$

$$I = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Some Basics:

f(x,y) = 0

if f(−x,−y) = f(x,y) symmetric w.r.t origin

if f(−x,y) = f(x,y) symmetric w.r.t y-axis

if f(x,−y) = f(x,y) symmetric w.r.t x-axis

Let y = f(x)

f(−x) = f(x) even function

f(−x) = −f(x) odd function

[0,1] unit interval

[−a,a] symmetric interval

Question: $\int_{-a}^{a} f(x)dx$

Solution: $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

Put x = −y ⇒ dx = −dy in the first integral

$$\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-y)(-dy)$$

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-y)(dy) = \int_{0}^{a} f(-x)(dx)$$

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$

If f(x) is odd function i.e. f(−x) = −f(x)

$$\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$

If f(x) is even function i.e. f(−x) = f(x)

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Question: $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

Solution: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,dx$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \lim_{a\to\infty} 2\int_{0}^{a} e^{-x^2}\,dx \qquad \because \text{as } e^{-x^2} \text{ is even function}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_{0}^{a} e^{-x^2}\,dx$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Gamma Function:

$$\Gamma(\alpha) = \int_{0}^{\infty} e^{-x}.x^{\alpha-1}\,dx \;,\; \alpha > 0$$

$$\Gamma(\alpha+1) = \int_{0}^{\infty} e^{-x}.x^{\alpha}\,dx$$

$$\Gamma(\alpha+1) = \frac{e^{-x}}{-1}.x^{\alpha}\Big|_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-x}}{-1}.\alpha x^{\alpha-1}\,dx$$

$$\Gamma(\alpha+1) = -\left[\lim_{x\to\infty} e^{-x}.x^{\alpha} - e^{0}.0^{\alpha}\right] - \alpha\int_{0}^{\infty} e^{-x}\,.\,x^{\alpha-1}\,dx$$

$$\Gamma(\alpha+1) = -[0-0] - \alpha\Gamma(\alpha)$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$\Gamma(1) = \int_{0}^{\infty} e^{-x}\,.x^{1-1}\,dx = \int_{0}^{\infty} e^{-x}\,dx = \frac{e^{-x}}{-1}\Big|_{0}^{\infty} = -\left[\lim_{x\to\infty} e^{-x} - e^{0}\right]$$

$$\Gamma(1) = -[0-1] = 1$$

If n is a positive integer then $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n+1) = n(n-1)\Gamma(n-1)$$

$$\Gamma(n+1) = n(n-1)(n-2)...3.2.1\Gamma(1)$$

$$\Gamma(n+1) = n!$$

$$\Gamma(n) = (n-1)!$$

If α =1/2

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} \, .x^{\frac{1}{2}-1} \, dx = \int_0^\infty e^{-x} .\text{x}^{\frac{-1}{2}} \, dx$$

Let $x = z^2 \quad \Rightarrow \quad dx = 2zdz$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-z^2} \, .\left(z^2\right)^{\frac{-1}{2}} .2zdz = 2\int_0^\infty e^{-z^2} \, .z^{-1} \, zdz$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-z^2} \, dz$$

$$\Gamma\left(\frac{1}{2}\right) = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3.1}{2.2}\sqrt{\pi} = \frac{3.1}{2^2}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5.3.1}{2.2.2}\sqrt{\pi} = \frac{5.3.1}{2^3}\sqrt{\pi}$$

$$\Gamma\left(\frac{9}{2}\right) = \frac{7.5.3.1}{2^4}\sqrt{\pi}$$

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$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)(2n-3)...7.5.3.1}{2^n}\sqrt{\pi}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)(2n-1)(2n-2)(2n-3)...7.6.5.4.3.2.1}{2^{n}.2n(2n-2)(2n-4)...6.4.2}\sqrt{\pi}$$

$$\Gamma\left(\frac{2n+1}{2}\right)=\frac{(2n)!}{2^{n}.2^{n}\,n(n-1)(n-2)...3.2.1}\sqrt{\pi}$$

$$\Gamma\left(\frac{2n+1}{2}\right)=\frac{(2n)!}{2^{2n}.\text{n}!}\sqrt{\pi}$$

Piecewise Continuous function:

A function f(x) is said to be piecewise continuous throughout an interval except at a finite number of points.

Laplace Transform:

Let f(t) be a piecewise continuous function then its Laplace transform is defined as $\mathcal{L}\left[f(t)\right]=\int_0^\infty e^{-st}f(t)\,dt$, s > 0 .And it is denoted by F(s)

$$\mathcal{L}\left[f(t)\right]=F(s)$$

Question: f(t) = c

Solution: $\mathcal{L}\left[f(t)\right]=\mathcal{L}[c]=\int_0^\infty e^{-st}c\,dt$

$$\mathcal{L}[c]=c\int_0^\infty e^{-st}\,dt\;=c\left|\frac{e^{-st}}{-s}\right|_0^\infty$$

$$\mathcal{L}[c]=\frac{-c}{s}\left[\lim_{t\to\infty}e^{-st}-e^{0}\right]=\frac{-c}{s}[0-1]$$

$$\mathcal{L}[c]=\frac{c}{s}$$

$$\mathcal{L}[1]=\frac{1}{s}\qquad\because c=1$$

Question: f(t) $f\left(t\right)=t^{\alpha}$

Solution: $$\mathcal{L}\left[f\left(t\right)\right]=\mathcal{L}\left[t^{\alpha}\right]=\int_0^{\infty}e^{-st}t^{\alpha}dt$$

Let z = st ⇒ t = z/s

$$\frac{dz}{s}=dt$$

$$\mathcal{L}\left[t^{\alpha}\right]=\int_0^{\infty}e^{-z}\left(\frac{z}{s}\right)^{\alpha}\frac{dz}{s}$$

$$\mathcal{L}\left[t^{\alpha}\right]=\frac{1}{s^{\alpha+1}}\int_0^{\infty}e^{-z}z^{\alpha}\,dz$$

$$\mathcal{L}\left[t^{\alpha}\right]=\frac{1}{s^{\alpha+1}}\int_0^{\infty}e^{-z}z^{\alpha+1-1}\,dz=\frac{1}{s^{\alpha+1}}.\left\lceil\alpha+1\right.$$

If α is a positive integer

$$\mathcal{L}\left[t^{n}\right]=\frac{\left\lceil n+1\right.}{s^{n+1}}=\frac{n!}{s^{n+1}}$$

If $\alpha=n+\dfrac{1}{2}$

$$\mathcal{L}\left[t^{n+\frac{1}{2}}\right]==\frac{\left(n+\frac{1}{2}\right)\left\lceil n+\frac{1}{2}\right.}{s^{\frac{2n+1+2}{2}}}$$

$$\mathcal{L}\left[t^{n+\frac{1}{2}}\right]==\frac{\left(\frac{2n+1}{2}\right)}{s^{\frac{2n+3}{2}}}.\frac{\left(2n\right)!\sqrt{\pi}}{2^{2n}.n!}=\frac{\left(2n+1\right)!\sqrt{\pi}}{2^{2n+1}.n!.s^{\frac{2n+3}{2}}}$$

Question: $f(t)=e^{at}$

Solution: $$\mathcal{L}[e^{at}]=\int_0^\infty e^{-st}e^{at}\,dt$$

$$=\int_0^\infty e^{-st+at}\,dt$$

$$=\int_0^\infty e^{-(s-a)t}\,dt$$

$$=\left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty$$

$$=0-\left[\frac{-1}{-(s-a)}\right]$$

$$\mathcal{L}[e^{at}]=\frac{1}{s-a}\qquad,\qquad s>a$$

Similarly, $\mathcal{L}[e^{-at}]=\dfrac{1}{s+a}$

Question: $f(t)=\cos at$

Solution: $$\mathcal{L}[\cos at]=\int_0^\infty e^{-st}\cos at\,dt$$

$$\because\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\cos bx+b\sin bx\right]$$

$$=\frac{e^{-st}}{(-s)^2+a^2}\left[-s\cos at+a\sin at\right]_0^\infty$$

$$=0-\frac{1}{s^2+a^2}[-s+0]\ \Rightarrow \mathcal{L}[\cos at]=\frac{s}{s^2+a^2}$$

Question: $f(t)=\sin at$

Solution: $$\mathcal{L}[\sin at]=\int_0^\infty e^{-st}\sin at\,dt$$

$$\because \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\sin bx-b\cos bx\right]$$

$$=\frac{e^{-st}}{(-s)^2+a^2}\left[-s\,\text{sinat}-a\,\text{cosat}\right]_0^\infty$$

$$=0-\frac{1}{s^2+a^2}[0-a]$$

$$\mathcal{L}[\sin at]=\frac{a}{s^2+a^2}$$

Linearity Property:

Let f(t) and g(t) be two piecewise continuous function c_1 and c_2 are two scalars then

$$\mathcal{L}\left[c_1 f(t)+c_2 g(t)\right]=c_1\mathcal{L}\left[f(t)\right]+c_2\mathcal{L}\left[g(t)\right]$$

Proof: $$\mathcal{L}\left[c_1 f(t)+c_2 g(t)\right]$$

$$=\int_0^\infty e^{-st}\left(c_1 f(t)+c_2 g(t)\right)dt$$

$$=\int_0^\infty\left(c_1 e^{-st} f(t)+c_2 e^{-st} g(t)\right)dt$$

$$=c_1\int_0^\infty e^{-st} f(t)\,dt+c_2\int_0^\infty e^{-st} g(t)\,dt$$

$$\mathcal{L}\left[c_1 f(t)+c_2 g(t)\right]=c_1\mathcal{L}\left[f(t)\right]+c_2\mathcal{L}\left[g(t)\right]$$

First Shifting property:

Suppose that $\mathcal{L}\left[f(t)\right]=F(s)$ then prove that $\mathcal{L}\left[e^{at}f(t)\right]=F(s-a)$

Proof: As $\mathcal{L}\left[f(t)\right]=\int_0^\infty e^{-st}f(t)dt=F(s)$

$$\mathcal{L}\left[e^{at}f(t)\right]=\int_0^\infty e^{-st}.e^{at}f(t)dt$$

$$\mathcal{L}\left[e^{at}f(t)\right]=\int_0^\infty e^{-(s-a)t}f(t)dt=F(s-a)$$

Unit function / The Heaviside unit function:

Let a ≥ 0 then the unit function denoted by $u_a(t)$ is defined as

$$u_a(t)=u(t-a)=\begin{cases}0 & if\ t<a\\ 1 & if\ t>a\end{cases}$$

$$u_0(t)=u(t)=\begin{cases}0 & if\ t<0\\ 1 & if\ t>0\end{cases}$$

$$\mathcal{L}\left[u_a(t)\right]=\int_0^\infty e^{-st}.u_a(t)dt$$

$$\mathcal{L}\left[u_a(t)\right]=\int_0^a e^{-st}.u_a(t)dt+\int_a^\infty e^{-st}.u_a(t)dt$$

$$\mathcal{L}\left[u_a(t)\right]=\int_0^a e^{-st}.(0)dt+\int_a^\infty e^{-st}.(1)dt$$

$$\mathcal{L}\left[u_a(t)\right]=0+\left.\frac{e^{-st}}{-s}\right|_0^\infty=\frac{-1}{s}\left[\lim_{t\to\infty}e^{-st}-e^{-sa}\right]=\frac{-1}{s}\left[0-e^{-sa}\right]=\frac{e^{-sa}}{s}$$

If a = 0 $\mathcal{L}\left[u_0(t)\right]=\frac{e^0}{s}=\frac{1}{s}$

Lecture # 11

Second Shifting property:

Let f(t) be a piecewise continuous function. Then

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=\int_0^\infty e^{-st}u_a(t)f(t-a)\,dt$$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=\int_0^a e^{-st}u_a(t)f(t-a)\,dt+\int_a^\infty e^{-st}u_a(t)f(t-a)\,dt$$

$t < a \qquad + \qquad t \geq a$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=0+\int_a^\infty e^{-st}u_a(t)f(t-a)\,dt$$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=\int_a^\infty e^{-st}u_a(t)f(t-a)\,dt$$

Let $z = t - a \Rightarrow t = z + a$

$dz = dt$

$z \to 0$ as $t \to a$

$z \to \infty$ as $t \to \infty$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=\int_a^\infty e^{-s(z+a)}f(z)\,dz$$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=\int_a^\infty e^{-s(z+a)}f(z)\,dz=\int_a^\infty e^{-sz-as}.f(z)\,dz$$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=\int_a^\infty e^{-sz}.e^{-as}.f(z)\,dz$$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=e^{-as}\int_a^\infty e^{-sz}.f(z)\,dz$$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=e^{-as}\int_a^\infty e^{-st}.f(t)dt \quad \because \text{ z is dummy variable}$$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=e^{-as}F(s)$$

Differential formula for Laplace transform:

Let $f'(t)$ be the derivative of f(t). Then

$$\mathcal{L}\left[f'(t)\right]=\int_0^\infty e^{-st}f'(t)dt$$

$$\mathcal{L}\left[f'(t)\right]=e^{-st}.f(t)\Big|_0^\infty-\int_0^\infty e^{-st}(-s)f(t)dt$$

$$\mathcal{L}\left[f'(t)\right]=e^{-st}.f(t)\Big|_0^\infty\left[\lim_{t\to\infty}e^{-st}.\lim_{t\to\infty}f(t)-e^0f(0)\right]+s\int_0^\infty e^{-st}f(t)dt$$

$$\mathcal{L}\left[f'(t)\right]=0-f(0)+s\mathcal{L}\left[f(t)\right]$$

$$\mathcal{L}\left[f'(t)\right]=s\mathcal{L}\left[f(t)\right]-f(0) \qquad \_\_\_\_(i)$$

$$\mathcal{L}\left[f''(t)\right]=s\mathcal{L}\left[f'(t)\right]-f'(0)$$

$$\mathcal{L}\left[f''(t)\right]=s\left[s\mathcal{L}\left[f(t)\right]-f(0)\right]-f'(0) \qquad \because by\ (i)$$

$$\mathcal{L}\left[f''(t)\right]=s^2\mathcal{L}\left[f(t)\right]-sf(0)-f'(0) \qquad \_\_\_\_(ii)$$

$$\mathcal{L}\left[f'''(t)\right]=s\mathcal{L}\left[f''(t)\right]-f''(0)$$

$$\mathcal{L}\left[f'''(t)\right]=s\left[s^2\mathcal{L}\left[f(t)\right]-sf(0)-f'(0)\right]-f''(0)$$

$$\mathcal{L}\left[f'''(t)\right]=s^3\mathcal{L}\left[f(t)\right]-s^2f(0)-sf'(0)-f''(0)$$

In general, $\mathcal{L}\left[f^n(t)\right]=s^n\mathcal{L}\left[f(t)\right]-s^{n-1}f(0)-sf^{n-2}(0)-f^{n-1}(0)$

Inverse Laplace Transform:

If F(s) is the Laplace transform of f(t) then f(t) is called inverse Laplace transform of F(s). i.e.

$$\mathcal{L}\left[f(t)\right]=F(s) \quad \Leftrightarrow \quad f(t)=\mathcal{L}^{-1}\left[F(s)\right]$$

Some Functions:

$$\mathcal{L}\left[t^n\right]=\frac{n!}{s^{n+1}} \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$$

$$\mathcal{L}\left[e^{\pm at}\right]=\frac{1}{s\mp a} \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{1}{s\mp a}\right]=e^{\pm at}$$

$$\mathcal{L}\left[\cos at\right]=\frac{s}{s^2+a^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$$

$$\mathcal{L}\left[\sin at\right]=\frac{a}{s^2+a^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$$

$$\mathcal{L}\left[\cosh at\right]=\frac{s}{s^2-a^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$$

$$\mathcal{L}\left[\sinh at\right]=\frac{a}{s^2-a^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right]=\sinh at$$

$$\mathcal{L}\left[u_a(t)\right]=\frac{e^{-as}}{s} \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{e^{-as}}{s}\right]=u_a(t)$$

$$\mathcal{L}\left[u_a(t)f(t-a)\right]=e^{-as}F(s) \quad \Rightarrow \quad \mathcal{L}^{-1}\left[e^{-as}F(s)\right]=u_a(t)f(t-a)$$

Question: Solve the following D.E

$$\frac{dy}{dt}+y=\sin t \quad ; \; I.C \quad y(0)=1$$

Solution: Taking Laplace Transform on both side

$$\mathcal{L}\left[\frac{dy}{dt}+y\right]=\mathcal{L}[\sin t]$$

$$\mathcal{L}\left[\frac{dy}{dt}\right]+\mathcal{L}[y]=\frac{1}{s^2+1}$$

$$s\mathcal{L}[y]-y(0)+\mathcal{L}[y]=\frac{1}{s^2+1}$$

$$(s+1)\mathcal{L}[y]-1=\frac{1}{s^2+1}$$

$$(s+1)\mathcal{L}[y]=1+\frac{1}{s^2+1}$$

$$(s+1)\mathcal{L}[y]=\frac{s^2+1+1}{s^2+1}=\frac{s^2+2}{s^2+1}$$

$$\mathcal{L}[y]=\frac{s^2+2}{(s+1)(s^2+1)} \qquad \_\_\_(i)$$

$$Let \; \frac{s^2+2}{(s+1)(s^2+1)}=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+1)}$$

$$s^2+2=A(s^2+1)+Bs(s+1)+C(s+1)$$

$$Put \;\; s+1=0 \;\Rightarrow s=-1$$

$$(-1)^2+2=A\left((-1)^2+1\right)+Bs(-1+1)+C(-1+1)$$

$$A=\frac{3}{2}$$

Comparing s^2

$$1 = A + B \quad \Rightarrow B = 1 - A = 1 - \frac{3}{2}$$

$$B = -\frac{1}{2}$$

Comparing s

$$0 = B + C \quad \Rightarrow C = -B \quad \Rightarrow C = \frac{1}{2}$$

$$\frac{s^2+2}{(s+1)(s^2+1)} = \frac{\frac{3}{2}}{(s+1)} + \frac{\frac{-1}{2}s + \frac{1}{2}}{(s^2+1)}$$

Put in (i) $\Rightarrow \quad \mathcal{L}[y] = \dfrac{3}{2(s+1)} - \dfrac{s}{2(s^2+1)} + \dfrac{1}{2(s^2+1)}$

Taking Laplace inverse

$$\mathcal{L}\mathcal{L}^{-1}[y] = \mathcal{L}^{-1}\left[\frac{3}{2(s+1)} - \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}\right]$$

$$y = \frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{(s+1)}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)}\right]$$

$$y = \frac{3}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

Question: Solve the following D.E

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = u(t-1) \quad ; \quad I.C \quad y(0) = \alpha \, , \, y'(0) = \beta$$

Solution: Taking Laplace Transform on both side

$$\mathcal{L}\left[\frac{d^2y}{dt^2} + \frac{dy}{dt}\right] = \mathcal{L}[u(t-1)]$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right]+\mathcal{L}\left[\frac{dy}{dt}\right]=\frac{e^{-s}}{s}$$

$$s^2\mathcal{L}[y]-sy(0)-y'(0)+s\mathcal{L}[y]-y(0)=\frac{e^{-s}}{s}$$

$$(s^2+s)\mathcal{L}[y]-s(\alpha)-\beta-\alpha=\frac{e^{-s}}{s}$$

$$s(s+1)\mathcal{L}[y]=\alpha+\beta+s(\alpha)+\frac{e^{-s}}{s}$$

$$\mathcal{L}[y]=\frac{\alpha+\beta+s\alpha}{s(s+1)}+\frac{e^{-s}}{s^2(s+1)} \qquad \_\_\_\_\_(i)$$

$$Let\ \frac{\alpha+\beta+s\alpha}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}$$

$$\alpha+\beta+s\alpha=A(s+1)+B(s)$$

Put s = 0 $\Rightarrow$ $\alpha+\beta+0=A(0+1)+B(0) \quad \Rightarrow A=\alpha+\beta$

Put s+1 = 0 $\Rightarrow$ s = − 1

$$\alpha+\beta-\alpha=A(-1+1)+B(-1) \quad \Rightarrow B=-\beta$$

$$\frac{\alpha+\beta+s\alpha}{s(s+1)}=\frac{\alpha+\beta}{s}-\frac{\beta}{s+1}$$

$$Now\ \frac{1}{s^2(s+1)}=\frac{C}{s}+\frac{D}{s^2}+\frac{E}{s+1}$$

$$1=Cs(s+1)+D(s+1)+Es^2$$

Put s = 0 $\Rightarrow$ D = 1

Put s = − 1 $\Rightarrow$ E = 1

Compare s^2 $\Rightarrow$ 0 = C + E $\Rightarrow$ C = −1

$$\frac{1}{s^2(s+1)}=\frac{-1}{s}+\frac{1}{s^2}+\frac{1}{s+1}$$

Put in (i) $\Rightarrow$ $\mathcal{L}[y]=\frac{\alpha+\beta}{s}-\frac{\beta}{s+1}-\frac{e^{-s}}{s}+\frac{e^{-s}}{s^2}+\frac{e^{-s}}{s+1}$

Taking inverse Laplace

$$\mathcal{L}^{-1}\mathcal{L}[y]=\mathcal{L}^{-1}\left[\frac{\alpha+\beta}{s}-\frac{\beta}{s+1}-\frac{e^{-s}}{s}+\frac{e^{-s}}{s^2}+\frac{e^{-s}}{s+1}\right]$$

$$y=(\alpha+\beta)\mathcal{L}^{-1}\left[\frac{1}{s}\right]-\beta\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]-\mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right]+\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right]+\mathcal{L}^{-1}\left[\frac{e^{-s}}{s+1}\right] \quad \_\_\_(ii)$$

$$\mathcal{L}^{-1}\left[e^{-s}.\frac{1}{s^2}\right]=u(t-1)f(t-1) \quad where\ f(t)=\mathcal{L}^{-1}[F(s)]\ \&\ f(t)=\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]=t$$

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right]=u(t-1)f(t-1) \quad where\ f(t-1)=t-1$$

$$\mathcal{L}^{-1}\left[e^{-s}.\frac{1}{s+1}\right]=u(t-1)f(t-1) \quad where\ f(t)=\mathcal{L}^{-1}[F(s)]\ \&\ f(t)=\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]=e^{-t}$$

$$\mathcal{L}^{-1}\left[e^{-s}.\frac{1}{s+1}\right]=u(t-1).e^{-(t-1)} \quad where\ f(t-1)=e^{-(t-1)}$$

Put in (ii) $y=(\alpha+\beta)(1)-\beta e^{-t}-u(t-1)+u(t-1)(t-1)+u(t-1)e^{-(t-1)}$

$$y=\alpha+\beta-\beta e^{-t}-u(t-1)\left[1-t+1-e^{-(t-1)}\right]$$

$$y=\alpha+\beta-\beta e^{-t}-u(t-1)\left[2-t+-e^{-(t-1)}\right]$$

Convolution for Laplace transformation:

Let f(t) and g(t) be two piecewise continuous function. Their convolution is denoted by f*g where

$$f*g=\int_0^t f(\tau)g(t-\tau)d\tau$$

$$g*f=\int_0^t g(\tau)f(t-\tau)d\tau$$

Then $f*g=g*f$

Proof: $$g*f=\int_0^t g(\tau)f(t-\tau)d\tau$$

Let $t-\tau=z \quad \Rightarrow \tau=t-z$

$$d\tau=-dz$$

$$z\to t \ \ as \ \tau\to 0$$

$$z\to 0 \ \ as \ \tau\to t$$

$$g*f=\int_t^0 g(t-z)f(z)(-dz)$$

$$g*f=\int_0^t g(t-z)f(z)dz$$

$$g*f=\int_0^t g(t-\tau)f(\tau)d\tau \qquad \because \text{ z is dummy variable}$$

$$g*f=\int_0^t f(\tau)g(t-\tau)d\tau$$

$$g*f=f*g$$

Convolution Theorem:

Let $\mathcal{L}\left[f(t)\right]=F(s) \ \ and \ \ \mathcal{L}\left[g(t)\right]=G(s)$ then prove that

$$f*g=\mathcal{L}^{-1}\left[F(s).G(s)\right]$$

Proof: By definition $\quad f*g=\int_0^t f(\tau)g(t-\tau)d\tau$

$$f*g=\int_0^t 1.f(\tau)g(t-\tau)d\tau+\int_0^\infty 0.f(\tau)g(t-\tau)d\tau$$

t > τ t < τ

$$f*g=\int_0^t 1.f(\tau)g(t-\tau)d\tau+\int_t^\infty 0.f(\tau)g(t-\tau)d\tau$$

$$f*g=\int_0^t u(t-\tau)f(\tau)g(t-\tau)d\tau+\int_t^\infty u(t-\tau)f(\tau)g(t-\tau)d\tau$$

$$f*g=\int_0^\infty u(t-\tau)f(\tau)g(t-\tau)d\tau$$

Taking Laplace transform

$$\mathcal{L}[f*g]=\mathcal{L}\left[\int_0^\infty u(t-\tau)f(\tau)g(t-\tau)d\tau\right]$$

$$\mathcal{L}[f*g]=\int_0^\infty e^{-st}\left[\int_0^\infty u(t-\tau)f(\tau)g(t-\tau)d\tau\right]dt$$

Interchange the order of integration

$$\mathcal{L}[f*g]=\int_0^\infty f(\tau)\left[\int_0^\infty e^{-st}u(t-\tau)g(t-\tau)dt\right]d\tau$$

$$\mathcal{L}[f*g]=\int_0^\infty f(\tau)\left[\int_0^\tau e^{-st}u(t-\tau)g(t-\tau)dt+\int_\tau^\infty e^{-st}u(t-\tau)g(t-\tau)dt\right]d\tau$$

t < τ t ≥ τ

$$\mathcal{L}[f*g]=\int_0^\infty f(\tau)\left[0+\int_\tau^\infty e^{-st}u(t-\tau)g(t-\tau)dt\right]d\tau$$

$$\mathcal{L}[f*g]=\int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st}g(t-\tau)dt\right]d\tau$$

Let $t-\tau=z \Rightarrow t=\tau+z$

dt = dz

$$z\to 0 \text{ as } t\to\tau \;\&\; z\to\infty \text{ as } t\to\infty$$

$$\mathcal{L}[f*g]=\int_0^\infty f(\tau)\left[\int_0^\infty e^{-s(\tau+z)}g(z)dz\right]d\tau$$

$$\mathcal{L}[f*g]=\int_0^\infty f(\tau)\left[\int_0^\infty e^{-s\tau}.e^{-sz}g(z)dz\right]d\tau$$

$$\mathcal{L}[f*g]=\int_0^\infty f(\tau)\left[e^{-s\tau}\int_0^\infty e^{-sz}g(z)dz\right]d\tau$$

$$\mathcal{L}[f*g]=\left(\int_0^\infty f(\tau)e^{-s\tau}d\tau\right)\left(\int_0^\infty e^{-sz}g(z)dz\right)$$

$$\mathcal{L}[f*g]=\left(\int_0^\infty f(t)e^{-st}dt\right)\left(\int_0^\infty e^{-st}g(t)dt\right)$$

$\because z$ & τ dummy variable

$$\mathcal{L}[f*g]=\mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

$$\mathcal{L}[f*g]=F(s).G(s)$$

$$\mathcal{L}[f*g]=\mathcal{L}^{-1}[F(s).G(s)]$$

Question: Solve $\mathcal{L}^{-1}\left[\dfrac{s}{(s+1)(s^2+1)}\right]$

Solution: $\mathcal{L}^{-1}\left[\dfrac{s}{(s+1)(s^2+1)}\right]=\mathcal{L}^{-1}\left[\dfrac{1}{(s+1)}.\dfrac{s}{(s^2+1)}\right]$

Let $F(s)=\dfrac{s}{(s+1)}$, $G(s)=\dfrac{s}{(s^2+1)}$

$$f(t)=\mathcal{L}^{-1}[F(s)],\ g(t)=\mathcal{L}^{-1}[G(s)]$$

$$f(t)=\mathcal{L}^{-1}\left[\frac{s}{s+1}\right],\ g(t)=\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]$$

$$f(t)=e^{-t}\ ,\ g(t)=\cos t$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s+1)(s^2+1)}\right]=\mathcal{L}^{-1}\left[F(s).G(s)\right]$$

$$f*g=\int_0^t f(\tau)g(t-\tau)d\tau$$

$$f*g=\int_0^t e^{-\tau}\cos(t-\tau)d\tau=\int_0^t e^{-\tau}\cos(-\tau+t)d\tau$$

$$f*g=\frac{e^{-\tau}}{(-1)^2+(-1)^2}\left[(-1)\cos(-\tau+t)+(-1)\sin(-\tau+t)\right]_0^t$$

$$f*g=\frac{-e^{-\tau}}{2}\left[\cos(t-\tau)+\sin(t-\tau)\right]_0^t$$

$$f*g=\frac{-e^{-t}}{2}\left[\cos(t-t)+\sin(t-t)\right]+\frac{e^{0}}{2}\left[\cos(t-0)+\sin(t-0)\right]$$

$$f*g=\frac{-e^{-t}}{2}\left[\cos(0)+\sin(0)\right]+\frac{1}{2}\left[\cos t+\sin t\right]$$

$$f*g=\frac{-e^{-t}}{2}+\frac{1}{2}\cos t+\frac{1}{2}\sin t$$

Question: Solve $y(t)=1+\int\limits_0^t y(\tau)(t-\tau)d\tau$

Solution: $$y(t)=1+\int\limits_0^t y(\tau)(t-\tau)d\tau$$

Compare with $$y(t)=1+\int\limits_0^t f(\tau)g(t-\tau)d\tau$$

$$y(t)=1+f*g$$

Where $f(\tau)=y(\tau)\quad\Rightarrow f(t)=y(t)$

$$g(t-\tau)=(t-\tau) \quad \Rightarrow g(t)=t$$

$$\Rightarrow \; y(t)=1+y*t$$

Taking Laplace on both sides

$$\Rightarrow \; \mathcal{L}[y(t)]=\mathcal{L}[1+y*t]$$

$$\mathcal{L}[y(t)]=\mathcal{L}[1]+\mathcal{L}[y*t]$$

$$\mathcal{L}[y(t)]=\frac{1}{s}+\mathcal{L}[y]\mathcal{L}[t]$$

$$\mathcal{L}[y(t)]=\frac{1}{s}+\mathcal{L}[y].\frac{1}{s^2} \quad \because \; \mathcal{L}[t]=\frac{1}{s^2}$$

$$\mathcal{L}[y(t)]-\mathcal{L}[y].\frac{1}{s^2}=\frac{1}{s}$$

$$\left(1-\frac{1}{s^2}\right)\mathcal{L}[y]=\frac{1}{s}$$

$$\left(\frac{s^2-1}{s^2}\right)\mathcal{L}[y]=\frac{1}{s}$$

$$\mathcal{L}[y]=\frac{1}{s}.\frac{s^2}{s^2-1}=\frac{s}{s^2-1}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\,\mathcal{L}[y]=\mathcal{L}^{-1}\left[\frac{s}{s^2-1}\right]$$

$$y=\cosh t$$

Question: Solve $y(t)=t+\frac{1}{6}\int_0^t y(\tau)(t-\tau)^3\,d\tau$

Solution:
$$y(t)=t+\frac{1}{6}\int_0^t y(\tau)(t-\tau)^3\,d\tau$$

Compare with
$$y(t)=t+\frac{1}{6}\int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$y(t)=t+\frac{1}{6}(f*g)$$

Where
$$f(\tau)=y(\tau)\quad\Rightarrow f(t)=y(t)$$

$$g(t-\tau)=(t-\tau)^3\quad\Rightarrow g(t)=t^3$$

$$\Rightarrow\ y(t)=t+\frac{1}{6}\left(y*t^3\right)$$

Taking Laplace on both sides

$$\Rightarrow\ \mathcal{L}[y(t)]=\mathcal{L}\left[t+\frac{1}{6}\left(y*t^3\right)\right]$$

$$\mathcal{L}[y(t)]=\mathcal{L}[t]+\frac{1}{6}\mathcal{L}\left[\left(y*t^3\right)\right]$$

$$\mathcal{L}[y(t)]=\frac{1}{s^2}+\frac{1}{6}\mathcal{L}[y]\mathcal{L}\left[t^3\right]$$

$$\mathcal{L}[y(t)]=\frac{1}{s^2}+\frac{1}{6}\mathcal{L}[y]\frac{6}{s^4}$$

$$\mathcal{L}[y(t)]-\frac{1}{s^4}\mathcal{L}[y]=\frac{1}{s^2}$$

$$\left(1-\frac{1}{s^4}\right)\mathcal{L}[y]=\frac{1}{s^2}$$

$$\left(\frac{s^4-1}{s^4}\right)\mathcal{L}[y]=\frac{1}{s^2}$$

$$\mathcal{L}[y]=\frac{1}{s^2}.\frac{s^4}{s^4-1}=\frac{s^2}{s^4-1} \qquad \_\_\_\_(i)$$

Let $\dfrac{s^2}{s^4-1}=\dfrac{r}{r^2-1}=\dfrac{r}{(r-1)(r+1)} \qquad \because\ r=s^2$

$$\frac{r}{(r-1)(r+1)}=\frac{A}{(r-1)}+\frac{B}{(r+1)}$$

$$r=A(r+1)+B(r-1)$$

Put r = 1 $\Rightarrow$ $\quad 1=A(1+1)+B(1-1) \ \Rightarrow\ A=\dfrac{1}{2}$

Put r = −1 $\Rightarrow$ $\quad -1=A(-1+1)+B(-1-1) \ \Rightarrow\ B=\dfrac{1}{2}$

$$\frac{r}{(r-1)(r+1)}=\frac{1}{2(r-1)}+\frac{1}{2(r+1)}$$

$$\Rightarrow \frac{s^2}{(s^2-1)(s^2+1)}=\frac{1}{2(s^2-1)}+\frac{1}{2(s^2+1)}$$

Put in (i) $\quad\Rightarrow\quad \mathcal{L}[y]=\dfrac{1}{2(s^2-1)}+\dfrac{1}{2(s^2+1)}$

Taking Laplace inverse $\Rightarrow \mathcal{L}^{-1}\mathcal{L}[y]=\mathcal{L}^{-1}\left[\dfrac{1}{2(s^2-1)}+\dfrac{1}{2(s^2+1)}\right]$

$$y=\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right]+\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$y=\frac{1}{2}\sinh t+\frac{1}{2}\sin t$$

Question: Solve $\dfrac{dy}{dt}=1+\int_0^t y(\tau)(t-\tau)\,d\tau \;\; ; y(0)=1$

Solution: $$\frac{dy}{dt}=1+\int_0^t y(\tau)(t-\tau)\,d\tau$$

Compare with $$\frac{dy}{dt}=1+\int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\frac{dy}{dt}=1+f*g$$

Where $$f(\tau)=y(\tau) \quad \Rightarrow f(t)=y(t)$$

$$g(t-\tau)=(t-\tau) \quad \Rightarrow g(t)=t$$

$$\Rightarrow \frac{dy}{dt}=1+y*t$$

Taking Laplace on both sides

$$\Rightarrow \mathcal{L}\left[\frac{dy}{dt}\right]=\mathcal{L}[1+y*t]$$

$$s\mathcal{L}[y]-y(0)=\mathcal{L}[1]+\mathcal{L}[y*t]$$

$$s\mathcal{L}[y]-1=\frac{1}{s}+\mathcal{L}[y]\mathcal{L}[t]$$

$$s\mathcal{L}[y]-1=\frac{1}{s}+\mathcal{L}[y].\frac{1}{s^2}$$

$$s\mathcal{L}[y]-\mathcal{L}[y].\frac{1}{s^2}=\frac{1}{s}+1$$

$$\left(s-\frac{1}{s^2}\right)\mathcal{L}[y]=\frac{s+1}{s}$$

$$\mathcal{L}[y]=\frac{s+1}{s}.\left(\frac{s^2}{s^3-1}\right)=\frac{s(s+1)}{s^3-1}$$

$$\mathcal{L}[y]=\frac{s^2+s}{(s-1)(s^2+s+1)} \qquad \_\_\_\_\_(i)$$

Let $\dfrac{s^2+s}{(s-1)(s^2+s+1)}=\dfrac{A}{s-1}+\dfrac{Bs+C}{s^2+s+1}$

$$s^2+s=A(s^2+s+1)+Bs(s-1)+C(s-1)$$

Put s = 1 $\Rightarrow (1)^2+1=A\left((1)^2+1+1\right)+B(1)(1-1)+C(1-1)$

$$A=\frac{2}{3}$$

Comparing s^2 $\Rightarrow$ $1=A+B \Rightarrow B=1-A=1-\dfrac{2}{3} \Rightarrow B=\dfrac{1}{3}$

Comparing s $\Rightarrow$ 1 = A−B+C $\Rightarrow$ C = 1−A+B

$$C=1-\frac{2}{3}+\frac{1}{3}=\frac{3-2+1}{3} \Rightarrow C=\frac{2}{3}$$

$$\frac{s^2+s}{(s-1)(s^2+s+1)}=\frac{2}{3(s-1)}+\frac{\frac{1}{3}s+\frac{2}{3}}{s^2+s+1}$$

Put in (i) $\Rightarrow$ $\mathcal{L}[y]=\dfrac{2}{3(s-1)}+\dfrac{\frac{1}{3}s+\frac{2}{3}}{s^2+s+1}$

Let $s^2+s+1=s^2+s+\left(\dfrac{1}{2}\right)^2-\left(\dfrac{1}{2}\right)^2+1$

$$s^2+s+1=\left(s+\frac{1}{2}\right)^2-\left(\frac{1}{4}\right)+1=\left(s+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$s^2+s+1=\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

$$\mathcal{L}[y]=\frac{2}{3(s-1)}+\frac{1}{3}\frac{\left(s+\frac{1}{2}\right)+\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\mathcal{L}[y]=\frac{2}{3(s-1)}+\frac{1}{3}\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{3}.\frac{3}{2}\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\mathcal{L}[y]=\frac{2}{3(s-1)}+\frac{1}{3}\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{2}\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\mathcal{L}[y]=\mathcal{L}^{-1}\left[\frac{2}{3(s-1)}+\frac{1}{3}\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{2}\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$y=\frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]+\frac{1}{3}\mathcal{L}^{-1}\left[\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]+\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$y=\frac{2}{3}e^t+\frac{1}{3}\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Lecture # 10

Question: Find Laplace transform of

$$\frac{\partial \phi}{\partial x}+x\frac{\partial \phi}{\partial t}=0$$

Boundary Condition $\qquad \phi(0,t)=t$

Initial Condition $\qquad \phi(x,0)=0$

Solution: Apply Laplace transform

$$\mathcal{L}\left[\frac{\partial \phi}{\partial x}\right]+\mathcal{L}\left[\frac{\partial \phi}{\partial t}\right]=\mathcal{L}[0]$$

$$\frac{\partial}{\partial x}\mathcal{L}\left[\phi(x,t)\right]+x\left[s\mathcal{L}\left[\phi(x,t)\right]-\phi(x,0)\right]=0$$

By putting initial condition

$$\frac{\partial}{\partial x}\mathcal{L}\left[\phi(x,t)\right]+x\left[s\mathcal{L}\left[\phi(x,t)\right]-0\right]=0$$

$$\frac{\partial}{\partial x}\mathcal{L}\left[\phi(x,t)\right]+xs\mathcal{L}\left[\phi(x,t)\right]=0$$

$$\text{let} \quad \mathcal{L}\left[\phi(x,\text{t})\right]=\bar{\phi}(x,s) \quad \_\_\_\_(i)$$

$$\frac{\partial \bar{\phi}(x,s)}{\partial x}+xs\bar{\phi}(x,s)=0 \qquad \_\_\_\_(ii)$$

Which is linear equation

I.F $= e^{\int sx\,dx}=e^{\frac{x^2 s}{2}}$

Multiplying (ii) by I.F

$$e^{\frac{x^2 s}{2}}\frac{\partial \bar{\phi}(x,s)}{\partial x}+xse^{\frac{x^2 s}{2}}\bar{\phi}(x,s)=0$$

$$\frac{\partial}{\partial x}\left[e^{\frac{x^2 s}{2}}\bar{\phi}(x,s)\right]=0$$

On integration

$$\bar{\phi}(x,s)=e^{\frac{-x^2 s}{2}}A(s) \quad \_\_\_\_(iii)$$

Now by B.C $\quad \phi(0,t)=t \quad \Rightarrow \mathcal{L}[\phi(0,t)]=\mathcal{L}[t]$

$$\bar{\phi}(0,t)=\frac{1}{s^2} \quad \_\_\_\_\_(iv)$$

Put x = 0 in (iii) and compare with (iv)

$$\bar{\phi}(0,s)=e^0 A(s)=A(s)$$

$$\Rightarrow A(s)=\frac{1}{s^2}$$

Put in (iii) $\quad \bar{\phi}(x,s)=e^{\frac{-x^2 s}{2}}.\frac{1}{s^2}$

$$\mathcal{L}[\phi(x,\text{t})]=e^{\frac{-x^2 s}{2}}.\frac{1}{s^2} \quad \because by\ (i)$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\mathcal{L}[\phi(x,\text{t})]=\mathcal{L}^{-1}\left[e^{\frac{-x^2 s}{2}}.\frac{1}{s^2}\right]$$

$$\phi(x,\text{t})=u\left(t-\frac{x^2}{2}\right)f\left(t-\frac{x^2}{2}\right)$$

Where $f(t)=\mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]=t \Rightarrow f\left(t-\frac{x^2}{2}\right)=t-\frac{x^2}{2}$

$$\phi(x,\text{t})=u\left(t-\frac{x^2}{2}\right).\left(t-\frac{x^2}{2}\right)$$

Question: $\dfrac{\partial^2\phi}{\partial x^2}=c^2\dfrac{\partial^2\phi}{\partial t^2}$

Initial Condition $\phi\left(x,0\right)=0$, $\dfrac{\partial\phi\left(x,0\right)}{\partial t}=0$

Boundary Condition $\phi\left(0,\mathrm{t}\right)=\sin t$, $\lim\limits_{x\to\infty}\phi\left(x,\mathrm{t}\right)=0$

Solution: Apply Laplace transform on both sides

$$\mathcal{L}\left[\frac{\partial^2\phi}{\partial x^2}\right]=c^2\mathcal{L}\left[\frac{\partial^2\phi}{\partial t^2}\right]$$

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right]=c^2\left[s^2\mathcal{L}\left[\phi(x,t)\right]-s\phi(x,0)-\frac{\partial\phi\left(x,0\right)}{\partial t}\right]$$

By Initial condition

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right]=c^2\left[s^2\mathcal{L}\left[\phi(x,t)\right]-s\left(0\right)-0\right]$$

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right]=c^2\,s^2\mathcal{L}\left[\phi(x,t)\right]$$

Say $\mathcal{L}\left[\phi(x,t)\right]=\overline{\phi}\left(x,s\right)$ ______(i)

$$\frac{\partial^2}{\partial x^2}\overline{\phi}\left(x,s\right)=c^2\,s^2\overline{\phi}\left(x,s\right)$$

$$\frac{\partial^2}{\partial x^2}\overline{\phi}\left(x,s\right)-c^2\,s^2\overline{\phi}\left(x,s\right)=0$$

$$\left(\frac{\partial^2}{\partial x^2}-c^2\,s^2\right)\overline{\phi}\left(x,s\right)=0$$

$$\frac{\partial^2}{\partial x^2}-c^2\,s^2=0$$

$$\frac{\partial^2}{\partial x^2} = c^2 s^2$$

$$\frac{\partial}{\partial x} = \pm c s$$

$$\bar{\phi}(x,s) = A(s)e^{csx} + B(s)e^{-csx} \quad \_\_\_\_(ii)$$

Now by boundary condition

$$\phi(0,t) = \sin t$$

Applying Laplace transformation

$$\mathcal{L}\left[\phi(0,t)\right] = \mathcal{L}\left[\sin t\right]$$

$$\bar{\phi}(0,s) = \frac{1}{s^2+1} \quad \_\_\_\_(\text{iii})$$

$$\lim_{x\to\infty} \phi(x,t) = 0$$

$$\mathcal{L}\left[\lim_{x\to\infty} \phi(x,t)\right] = \mathcal{L}[0]$$

$$\lim_{x\to\infty} \bar{\phi}(x,s) = 0 \quad \_\_\_\_(iv)$$

Put x = 0 in (ii) and compare with (iii)

$$\bar{\phi}(x,0) = A(s)e^{0} + B(s)e^{-0} = A(s) + B(s)$$

$$A(s) + B(s) = \frac{1}{s^2+1}$$

Take limit x $\to\infty$ of (ii)

$$\lim_{x\to\infty} \bar{\phi}(x,s) = \lim_{x\to\infty} A(s)e^{csx} + \lim_{x\to\infty} B(s)e^{-csx}$$

$$0 = A(s) \lim_{x\to\infty} e^{csx} + 0$$

$$A(s) \lim_{x\to\infty} e^{csx} = 0$$

Only possible if A(s) = 0

Put in (iv) $\Rightarrow$ $$0+B(s)=\frac{1}{s^2+1} \Rightarrow B(s)=\frac{1}{s^2+1}$$

Put in (ii) $\Rightarrow$ $$\bar{\phi}(x,s)=0+\frac{1}{s^2+1}e^{-csx}$$

$$\bar{\phi}(x,s)=e^{-csx}\ \frac{1}{s^2+1} \quad \because by\ (i)$$

$$\mathcal{L}^{-1}\mathcal{L}\left[\phi(x,\mathrm{t})\right]=\mathcal{L}^{-1}\left[e^{-csx}\ \frac{1}{s^2+1}\right]$$

$$\phi(x,\mathrm{t})=u(t-cx)\,f(t-cx) \qquad Where\ f(t)=\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]=\sin t$$

$\Rightarrow$ f(t−cx) = sin(t−cx)

$$\phi(x,\mathrm{t})=u(t-cx).\sin(t-cx)$$

Question: Find Laplace transform of

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial\phi}{\partial t}$$

Boundary Condition $\phi(0,t)=1=\phi(1,t)$

Initial Condition $\phi(x,0)=1+\sin(\pi x)$

Solution: Apply Laplace transform

$$\mathcal{L}\left[\frac{\partial^2\phi}{\partial x^2}\right]=\mathcal{L}\left[\frac{\partial\phi}{\partial t}\right]$$

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right]=s\mathcal{L}\left[\phi(x,t)\right]-\phi(x,0)$$

By initial condition

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi\left(x,t\right)\right]=s\mathcal{L}\left[\phi\left(x,t\right)\right]-1-\sin\left(\pi x\right)$$

let $\mathcal{L}\left[\phi(x,\text{t})\right]=\bar{\phi}(x,s)$

$$\frac{\partial^2}{\partial x^2}\bar{\phi}(x,s)=s\bar{\phi}(x,s)-1-\sin\left(\pi x\right)$$

$$\left(\frac{\partial^2}{\partial x^2}-s\right)\bar{\phi}(x,s)=-1-\sin\left(\pi x\right) \qquad \_\_\_\_(ii)$$

For complementary solution we have

$$\left(\frac{\partial^2}{\partial x^2}-s\right)\bar{\phi}(x,s)=0$$

$$\frac{\partial^2}{\partial x^2}-s=0$$

$$\frac{\partial^2}{\partial x^2}=s$$

$$\frac{\partial}{\partial x}=\pm\sqrt{s}$$

$$\overline{\phi_c}\left(x,s\right)=Ae^{\sqrt{s}\,x}+Be^{-\sqrt{s}\,x}$$

For particular solution

$$\left(\frac{\partial^2}{\partial x^2}-s\right)\overline{\phi_p}(x,s)=-1-\sin\left(\pi x\right)$$

$$\left(D^2-s\right)\bar{\phi}_p(x,s)=-1-\sin\pi x$$

$$\bar{\phi}_p(x,s)=\frac{1}{\left(D^2-s\right)}.\left(-1-\sin\pi x\right)$$

$$\bar{\phi}_p(x,s)=\frac{1}{\left(D^2-s\right)}\left(-1\right)+\frac{1}{\left(D^2-s\right)}\left(-\sin\pi x\right)$$

$$\bar{\phi}_p(x,s) = \frac{-1}{-s\left(1-\frac{D^2}{s}\right)} - \frac{1}{\left(D^2-s\right)}\left(\sin \pi x\right)$$

$$\bar{\phi}_p(x,s) = \frac{\left(1-\frac{D^2}{s}\right)^{-1}}{s} - \frac{1}{\left(\pi^2-s\right)}\left(\sin \pi x\right)$$

$$\bar{\phi}_p(x,s) = \frac{1}{s}\left(1+\frac{D^2}{s}+..\right)(1) + \frac{1}{\pi^2+s}\left(\sin \pi x\right)$$

$$\bar{\phi}_p(x,s) = \frac{1}{s}\left(1+0\right) + \frac{1}{\pi^2+s}\left(\sin \pi x\right)$$

$$\bar{\phi}_p(x,s) = \frac{1}{s} + \frac{\sin \pi x}{\pi^2+s}$$

The general solution is $\bar{\phi}(x,s) = \bar{\phi}_c(x,s) + \bar{\phi}_p(x,s)$

$$\bar{\phi}(x,s) = A(s)e^{\sqrt{s}\,x} + B(s)e^{-\sqrt{s}\,x} + \frac{1}{s} + \frac{\sin(\pi x)}{\pi^2+s} \quad \_\_\_\_\_(iii)$$

Now by boundary condition

$$\phi(0,t) = 1$$

$$\mathcal{L}\left[\phi(0,t)\right] = \mathcal{L}[1]$$

$$\bar{\phi}(0,\mathrm{s}) = \frac{1}{s} \qquad \_\_\_\_\_(\mathrm{iv})$$

Now $\phi(1,t) = 1 \qquad \Rightarrow \qquad \mathcal{L}\left[\phi(1,t)\right] = \mathcal{L}[1]$

$$\bar{\phi}(1,\mathrm{s}) = \frac{1}{s} \qquad \_\_\_\_(\mathrm{v})$$

Put x = 0 in (iii) and compare with (iv)

$$\bar{\phi}(0,\mathrm{s}) = A(s) + B(s) + \frac{1}{s}$$

$$\Rightarrow \frac{1}{s} = A(s) + B(s) + \frac{1}{s}$$

$$\Rightarrow A(s) + B(s) = 0 \qquad \_\_\_(vi)$$

Put x = 1 in eq (iii) and compare with (v)

$$\bar{\phi}(1,\mathrm{s}) = A(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} + \frac{1}{s}$$

$$\frac{1}{s} = A(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} + \frac{1}{s}$$

$$A(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} = 0$$

Put A = – B

$$-B(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} = 0$$

$$\left(-e^{\sqrt{s}} + e^{-\sqrt{s}}\right)B(s) = 0$$

$$\Rightarrow \quad \mathrm{B(s)} = 0 \Rightarrow \quad \mathrm{A(s)} = 0$$

Put in (iii) $\Rightarrow$ $\bar{\phi}(x,s) = 0 + 0\frac{1}{s} + \frac{\sin(\pi x)}{\pi^2 + s}$

$$\mathcal{L}[\phi(\mathrm{x,t})] = \frac{1}{s} + \frac{\sin(\pi x)}{\pi^2 + s} \qquad \because \bar{\phi}(x,s) = \mathcal{L}[\phi(x,t)]$$

$$\mathcal{L}^{-1}\mathcal{L}[\phi(x,t)] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{\sin(\pi x)}{\pi^2 + s}\right]$$

$$\mathcal{L}^{-1}\mathcal{L}[\phi(x,t)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{\sin(\pi x)}{\pi^2 + s}\right]$$

$$\phi(x,t) = 1 + \sin(\pi x).e^{-\pi^2 t}$$

Question: Solve by Laplace transform

$$\phi_{tt} = a^2\phi_{xx} - g$$

Initial Condition $\qquad \phi(x,0)=0 \;,\; \dfrac{\partial\phi(x,0)}{\partial t}=0$

Boundary Condition $\qquad \phi(0,t)=0 \;,\; \lim\limits_{x\to\infty}\dfrac{\partial}{\partial x}\phi(x,\mathrm{t})=0$

Solution: Apply Laplace transform

$$\mathcal{L}[\phi_{tt}] = \mathcal{L}\left[a^2\phi_{xx}\right] - \mathcal{L}[g]$$

$$\mathcal{L}\left[\frac{\partial^2\phi(x,t)}{\partial t^2}\right] = a^2\mathcal{L}\left[\frac{\partial^2\phi(x,t)}{\partial x^2}\right] - g\mathcal{L}[1]$$

$$s^2\mathcal{L}\left[\phi(x,t)\right] - s\phi(x,0) - \frac{\partial\phi(x,0)}{\partial t} = a^2\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right] - g.\frac{1}{s}$$

By initial condition

$$s^2\mathcal{L}\left[\phi(x,t)\right] - s(0) - 0 = a^2\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right] - \frac{g}{s}$$

$$s^2\mathcal{L}\left[\phi(x,t)\right] = a^2\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right] - \frac{g}{s}$$

$$a^2\frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi(x,t)\right] - s^2\mathcal{L}\left[\phi(x,t)\right] = \frac{g}{s}$$

let $\quad \mathcal{L}[\phi(x,t)] = \bar{\phi}(x,s) \qquad$ ______(i)

$$a^2\frac{\partial^2}{\partial x^2}\bar{\phi}(x,s) - s^2\bar{\phi}(x,s) = \frac{g}{s}$$

$$a^2\frac{\partial^2}{\partial x^2}\bar{\phi}(x,s) - s^2\bar{\phi}(x,s) - \frac{g}{s} = 0$$

$$\frac{\partial^2}{\partial x^2}\bar{\phi}(x,s) - \frac{g}{a^2 s} - \frac{s^2}{a^2}\bar{\phi}(x,s) = 0$$

$$\frac{\partial^2}{\partial x^2}\bar{\phi}(x,s)-\frac{s^2}{a^2}\left[\bar{\phi}(x,s)+\frac{g}{s^3}\right]=0 \quad \_\_\_\_\_(ii)$$

Let $\psi(x,s)=\bar{\phi}(x,s)+\dfrac{g}{s^3} \quad \_\_\_(iii)$

$$\frac{\partial}{\partial x}\psi(x,s)=\frac{\partial}{\partial x}\bar{\phi}(x,s)$$

$$\frac{\partial^2}{\partial x^2}\psi(x,s)=\frac{\partial^2}{\partial x^2}\bar{\phi}(x,s)$$

Eq (ii) $\Rightarrow$

$$\frac{\partial^2}{\partial x^2}\psi(x,s)-\frac{s^2}{a^2}\psi(x,s)=0$$

$$\left(\frac{\partial^2}{\partial x^2}-\frac{s^2}{a^2}\right)\psi(x,s)=0$$

$$\frac{\partial^2}{\partial x^2}-\frac{s^2}{a^2}=0$$

$$\frac{\partial^2}{\partial x^2}=\frac{s^2}{a^2}$$

$$\frac{\partial}{\partial x}=\pm\frac{s}{a}$$

$$\psi(x,s)=Ae^{\frac{s}{a}x}+Be^{-\frac{s}{a}x}$$

$$\bar{\phi}(x,s)+\frac{g}{s^3}=Ae^{\frac{s}{a}x}+Be^{-\frac{s}{a}x}$$

$$\bar{\phi}(x,s)=Ae^{\frac{s}{a}x}+Be^{-\frac{s}{a}x}-\frac{g}{s^3} \quad \_\_\_\_\_(iv)$$

Now by B.C $\quad \phi(0,t)=0$

$$\mathcal{L}\left[\phi(0,t)\right]=\mathcal{L}[0]$$

$$\bar{\phi}(0,\mathrm{s})=0 \qquad \_\_\_\_\_(\mathrm{v})$$

$$\lim_{x\to\infty}\frac{\partial}{\partial x}\phi(x,\mathrm{t})=0$$

$$\mathcal{L}\left[\lim_{x\to\infty}\frac{\partial}{\partial x}\phi(x,\mathrm{t})\right]=\mathcal{L}[0]$$

$$\lim_{x\to\infty}\frac{\partial}{\partial x}\mathcal{L}\left[\phi(x,\mathrm{t})\right]=0$$

$$\lim_{x\to\infty}\frac{\partial}{\partial x}\bar{\phi}(x,s)=0 \qquad (vi)$$

Put x = 0 in (iv) and compare with (v)

$$\bar{\phi}(0,\mathrm{s})=A(s)e^{0}+B(s)e^{0}-\frac{g}{s^{3}}$$

$$\Rightarrow \quad A(s)+B(s)=\frac{g}{s^{3}}=0 \qquad \_\_\_\_\_(vii)$$

Diff. eq (iv) w.r.t x

$$\overline{\phi_x}(x,\mathrm{s})=\frac{s}{a}Ae^{\frac{s}{a}x}-\frac{s}{a}Be^{-\frac{s}{a}x}$$

Taking $\lim_{x\to\infty}$ on both side and compare with (vi)

$$\lim_{x\to\infty}\overline{\phi_x}(x,\mathrm{s})=A(s)\frac{s}{a}\lim_{x\to\infty}e^{\frac{s}{a}x}-\frac{s}{a}B(s)\lim_{x\to\infty}e^{-\frac{s}{a}x}$$

$$0=A(s)\frac{s}{a}\lim_{x\to\infty}e^{\frac{s}{a}x}-0$$

$$A(s)\frac{s}{a}\lim_{x\to\infty}e^{\frac{s}{a}x}=0$$

Which is only possible if A(s) = 0

Put A = 0 in (vii)

$$B(s)-\frac{g}{s^3}=0 \quad \Rightarrow \quad B(s)=\frac{g}{s^3}$$

Put the value of A and B in Eq (iv)

$$\bar{\phi}(x,\text{s})=\frac{g}{s^3}e^{-\frac{s}{a}x}-\frac{g}{s^3}$$

$$\mathcal{L}\left[\phi(x,\text{t})\right]=\frac{g}{s^3}e^{-\frac{s}{a}x}-\frac{g}{s^3}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\mathcal{L}\left[\phi(x,\text{t})\right]=\mathcal{L}^{-1}\left[\frac{g}{s^3}e^{-\frac{s}{a}x}\right]-\mathcal{L}^{-1}\left[\frac{g}{s^3}\right]$$

$$\phi(x,\text{t})=g\mathcal{L}^{-1}\left[\frac{1}{s^3}e^{\frac{-x}{a}s}\right]-g\mathcal{L}^{-1}\left[\frac{1}{s^3}\right]$$

$$\phi(x,\text{t})=gu\left(t-\frac{x}{a}\right)f\left(t-\frac{x}{a}\right)-g.\frac{t^2}{2}$$

Where f(t) = $\mathcal{L}^{-1}\left[\frac{1}{s^3}\right]=\frac{t^2}{2} \Rightarrow f\left(t-\frac{x}{a}\right)=\frac{\left(t-\frac{x}{a}\right)^2}{2}$

$$\phi(x,\text{t})=gu\left(t-\frac{x}{a}\right)\frac{\left(t-\frac{x}{a}\right)^2}{2}-g.\frac{t^2}{2}$$

$$\phi(x,\text{t})=\frac{g}{2}\left[u\left(t-\frac{x}{a}\right)\left(t-\frac{x}{a}\right)^2-t^2\right]$$

Periodic Function:

Let f(t) be a periodic function with period l , then

$$f(t \pm l) = f(t) \quad ; \; l > 0$$

Period of f(t):

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}[f(t)] = \int_0^l e^{-st} f(t)\,dt + \int_l^\infty e^{-st} f(t)\,dt \qquad \_\_\_\_\_(i)$$

Put t = z + l in 2^{nd} integral

dt = dz

$$z \to 0 \; as \; t \to l \qquad \& \quad z \to \infty \; as \; t \to \infty$$

$$\int_l^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(z+l)} f(z+l)\,dz$$

$$\int_l^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-sz-sl} f(z+l)\,dz = \int_0^\infty e^{-sz}.e^{-sl} f(z+l)\,dz$$

$$\int_l^\infty e^{-st} f(t)\,dt = e^{-sl}\int_0^\infty e^{-sz}.f(z+l)\,dz$$

Since f is periodic function with period 'l' i.e $f(t \pm l) = f(t)$

$$\int_l^\infty e^{-st} f(t)\,dt = e^{-sl}\int_0^\infty e^{-sz}.f(t)\,dt \qquad \because \text{ z is dummy variable}$$

$$\int_l^\infty e^{-st} f(t)\,dt = e^{-sl}\mathcal{L}[f(t)] \qquad Put\; in\; (i)$$

$$\mathcal{L}[f(t)] = \int_0^l e^{-st} f(t)\,dt + e^{-sl}\mathcal{L}[f(t)]$$

$$\mathcal{L}\left[f(t)\right]-e^{-sl}\mathcal{L}\left[f(t)\right] \;=\int_0^l e^{-st}f(t)dt$$

$$\left(1-e^{-sl}\ \right)\ \mathcal{L}\left[f(t)\right]=\int_0^l e^{-st}f(t)dt$$

$$\mathcal{L}\left[f(t)\right]=\frac{1}{1-e^{-sl}}\int_0^l e^{-st}f(t)dt$$

Question: Find the period of sin(at)

Solution: $f(t)=\sin(at)=\sin(at+2\pi)=\sin a\left(t+\dfrac{2\pi}{a}\right)$

Period $l=\dfrac{2\pi}{a}$

As we know that $\mathcal{L}\left[f(t)\right]=\dfrac{1}{1-e^{-sl}}\displaystyle\int_0^l e^{-st}f(t)dt$

$$\mathcal{L}\left[\sin(at)\right]=\frac{1}{1-e^{-s\left(\frac{2\pi}{a}\right)}}\int_0^{\frac{2\pi}{a}} e^{-st}\sin(at)dt$$

$$\mathcal{L}\left[\sin(at)\right]=\frac{1}{1-e^{\frac{-2\pi s}{a}}}\left[\frac{e^{-st}}{(-s)^2+a^2}\left\{-s\ \sin(at)-a\cos(at)\right\}\right]_0^{\frac{2\pi}{a}}$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{a}}}\left[\frac{e^{-s.\frac{2\pi}{a}}}{s^2+a^2}\left\{-s\ \sin a\left(\frac{2\pi}{a}\right)-a\cos a\left(\frac{2\pi}{a}\right)\right\}-\frac{e^0}{s^2+a^2}\left\{-s\ \sin(0)-a\cos(0)\right\}\right]$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{a}}}\left[\frac{e^{-s.\frac{2\pi}{a}}}{s^2+a^2}\left\{-s\ \sin 2\pi-a\cos 2\pi\right\}-\frac{1}{s^2+a^2}\left\{-0-a\right\}\right]$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{a}}}\left[\frac{e^{\frac{-2\pi s}{a}}}{s^2+a^2}\left\{-s\,(0)-a(1)\right\}+\frac{a}{s^2+a^2}\right]$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{a}}}\left[\frac{-ae^{\frac{-2\pi s}{a}}}{s^2+a^2}+\frac{a}{s^2+a^2}\right]$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{a}}}\left[\frac{a}{s^2+a^2}\left(1-e^{\frac{-2\pi s}{a}}\right)\right]$$

$$\mathcal{L}\left[\sin(at)\right]=\frac{a}{s^2+a^2}$$

Lecture # 11

Fourier Transform:

Let f(x) be a real valued function s.t f(x) → 0 as x →±∞. Then its Fourier transform is defined as

$$\Im[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx = F(k)$$

And is denoted as F(k) i.e. $\Im$[f(x)] = F(k)

Then its inverse Fourier transform is defined as

$$\Im^{-1}[F(k)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} F(k)dk = F(x)$$

Fourier transform of Gaussian function:

$$f(x) = Ne^{-\alpha x^2}, \alpha > 0$$

Solution:

We know $\Im[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx = F(k)$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} Ne^{-\alpha x^2}dx$$

$$=\frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha x^2+ikx}dx$$

$$=\frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(\alpha x^2-ikx)}dx \qquad \_\_\_\_\_\_\_(i)$$

$$Let\ \alpha x^2 - ikx = \left(\sqrt{\alpha}x\right)^2 - 2x\sqrt{\alpha}\left(\frac{ik}{2\sqrt{\alpha}}\right) + \left(\frac{ik}{2\sqrt{\alpha}}\right)^2 - \left(\frac{ik}{2\sqrt{\alpha}}\right)^2$$

$$=\left(\sqrt{\alpha}x - \frac{ik}{2\sqrt{\alpha}}\right)^2 - \left(\frac{ik}{2\sqrt{\alpha}}\right)^2 \quad put\ in\ \text{(i)}$$

$$=\frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(\sqrt{\alpha}x-\frac{ik}{2\sqrt{\alpha}}\right)^2-\left(\frac{ik}{2\sqrt{\alpha}}\right)^2\right]}dx$$

$$=\frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha}x-\frac{ik}{2\sqrt{\alpha}}\right)^2} e^{\left(\frac{ik}{2\sqrt{\alpha}}\right)^2} dx$$

$$=\frac{N}{\sqrt{2\pi}}e^{\left(\frac{ik}{2\sqrt{\alpha}}\right)^2}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha}x-\frac{ik}{2\sqrt{\alpha}}\right)^2} dx$$

$$Put \quad z=\sqrt{\alpha}x-\frac{ik}{2\sqrt{\alpha}}$$

$$dz=\sqrt{\alpha}dx \quad \Rightarrow \quad dx=\frac{1}{\sqrt{\alpha}}dz$$

$$z\to\pm\infty \quad as \quad x\to\pm\infty$$

$$=\frac{N}{\sqrt{2\pi}}e^{\frac{i^2k^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-z^2}.\frac{1}{\sqrt{\alpha}}dz$$

$$=\frac{N}{\sqrt{2\pi}}e^{\frac{-k^2}{4\alpha}}.\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-z^2}dz$$

$$=\frac{N}{\sqrt{2\pi}}e^{\frac{-k^2}{4\alpha}}.\frac{1}{\sqrt{\alpha}}.\sqrt{\pi} \qquad \because \int_{-\infty}^{\infty} e^{-z^2}dz=\sqrt{\pi}$$

$$\Im[f(x)]=\frac{N}{\sqrt{2\alpha}}e^{\frac{-k^2}{4\alpha}}$$

Contour Integration: $\int_{-\infty}^{\infty} g(x)dx = 2\pi i \times$ Residue of g(x) at x = α

$\int_{-\infty}^{\infty} g(x)dx = 2\pi i \,.\lim_{x\to\alpha}\left[(x-\alpha)g(x)\right]$ where α is the simple pole of g(x) under the contour.

Question: Find the Fourier transform of $\frac{a}{a^2+x^2}$.

Solution: Applying Fourier transform on both side

$$\Im\left[f(x)\right]=\Im\left[\frac{a}{a^2+x^2}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}.\frac{a}{a^2+x^2}dx$$

$$=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ikx}}{a^2+x^2}dx$$

Here arise three cases

(i) k > 0 (ii) k < 0 (iii) k = 0

Case-I: K > 0

Consider the integral $\int_{-\infty}^{\infty}\frac{e^{ikx}}{a^2+x^2}dx=\int_{-\infty}^{\infty}g(x)dx \quad Where \quad g(x)=\frac{e^{ikx}}{a^2+x^2}$

Put $a^2+x^2=0 \quad \Rightarrow x=\pm ia$

Here $x=ia$ is the simple pole and lies under the contour then Residue of g(x) at $x=ia$ is

$$\text{Res}(g(x),x=ia)=\lim_{x\to ia}\left[(x-ia)g(x)\right]$$

$$\text{Res}(g(x),x=ia)=\lim_{x\to ia}\left[(x-ia).\frac{e^{ikx}}{(x-ia)(x+ia)}\right]$$

$$\text{Res}(g(x),x=ia)=\frac{e^{ik(ia)}}{ai+ia}$$

$$\text{Res}(g(x),x=ia)=\frac{e^{-ka}}{2ia}$$

Hence, $$\int_{-\infty}^{\infty}\frac{e^{ikx}}{a^2+x^2}dx=2\pi i\times\text{Res}(g(x),x=ia)$$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2+x^2}dx = 2\pi i \times \frac{e^{-ka}}{2ia} = \frac{\pi e^{-ka}}{a}$$

Hence (i) becomes as

$$\Im\left[f(x)\right] = \frac{a}{\sqrt{2\pi}}.\frac{\pi e^{-ka}}{a} = \sqrt{\frac{\pi}{2}}.e^{-ka}$$

Case-I: K < 0

Consider the integral $\int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2+x^2}dx = \int_{-\infty}^{\infty} g(x)dx \quad Where \;\; g(x) = \frac{e^{ikx}}{a^2+x^2}$

Put $a^2+x^2=0 \quad \Rightarrow x = \pm ia$

Here $x=-ia$ is the simple pole and lies under the contour then Residue of g(x) at $x=-ia$ is

$$\operatorname{Re}s\left(g(x), x=-ia\right) = \lim_{x\to ia}\left[(x+ia)g(x)\right]$$

$$\operatorname{Re}s\left(g(x), x=-ia\right) = \lim_{x\to ia}\left[(x+ia).\frac{e^{ikx}}{(x-ia)(x+ia)}\right]$$

$$\operatorname{Re}s\left(g(x), x=-ia\right) = \frac{e^{ik(-ia)}}{-ai-ia}$$

$$\operatorname{Re}s\left(g(x), x=-ia\right) = \frac{e^{ka}}{-2ia}$$

Hence, $$\int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2+x^2}dx = 2\pi i \times \operatorname{Re}s\left(g(x), x=-ia\right)$$

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2+x^2}dx = 2\pi i \times \frac{e^{ka}}{-2ia} = \frac{-\pi e^{ka}}{a}$$

Hence (i) becomes as

$$\Im\left[f(x)\right] = \frac{a}{\sqrt{2\pi}}.\frac{-\pi e^{ka}}{a} = -\sqrt{\frac{\pi}{2}}.e^{ka}$$

Case-III: k = 0

From (i)

$$\Im\left[f(x)\right]=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{0}}{a^2+x^2}dx=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{a^2+x^2}dx$$

$$=\frac{a}{\sqrt{2\pi}}.\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Bigg|_{-\infty}^{\infty}$$

$$=\frac{1}{\sqrt{2\pi}}\left[\tan^{-1}(\infty)-\tan^{-1}(-\infty)\right]$$

$$=\left[\frac{\pi}{2}-\left(\frac{-\pi}{2}\right)\right]=\frac{1}{\sqrt{2\pi}}(\pi)=\sqrt{\frac{\pi}{2}}$$

Linearity Property:

Let f(x) and g(x) be two real valued function and α, β be two scalars. Then

$$\Im\left[\alpha f(x)+\beta g(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\left(\alpha f(x)+\beta g(x)\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\alpha\int_{-\infty}^{\infty}e^{ikx}f(x)dx+\beta\int_{-\infty}^{\infty}e^{ikx}g(x)dx\right]$$

$$=\alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx+\beta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}g(x)dx$$

$$=\alpha\Im\left[f(x)\right]+\beta\Im\left[f(x)\right]$$

Attenuation Property:

If Fourier transform $\Im\left[f(x)\right]=F(k)$ then

$\Im\left[f(x-a)\right]=e^{ika}F(k)$ *and* $\Im\left[f(x+a)\right]=e^{-ika}F(k)$

Proof: (i) By the definition

$$\Im[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx$$

Replace f(x) by f(x−a)

$$[f(x-a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x-a)dx$$

In the integral x by x+a

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ik(x+a)}f(x+a-a)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx+ika}f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}.e^{ika}f(x)dx$$

$$=e^{ika}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx$$

$$\Im\left[f(x-a)\right]=e^{ika}F(k)$$

Proof: (ii) By the definition

$$\Im[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx$$

Replace f(x) by f(x+a)

$$\Im[f(x+a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x+a)dx$$

In the integral x by x−a

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ik(x-a)}f(x-a+a)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx-ika} f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}.e^{-ika} f(x)dx$$

$$=e^{-ika}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(x)dx$$

$$\Im\left[f\left(x+a\right)\right]=e^{-ika}F\left(k\right)$$

Question: Find the Fourier transform of $e^{-ax^2}\sin\left(bx\right)$ *where* $a>0$

Solution: By the definition of Fourier transform

$$\Im[\mathrm{f}(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{f}(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}\sin\left(bx\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}\left(\frac{e^{ibx}-e^{-ibx}}{2i}\right)dx$$

$$=\frac{1}{2i\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{ikx}e^{-ax^2}e^{ibx}-e^{ikx}e^{-ax^2}e^{-ibx}\right)dx$$

$$=\frac{1}{2i\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}e^{ibx}dx-\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}e^{-ibx}dx\right]$$

$$Put\quad g(x)=e^{-ax^2}$$

$$=\frac{1}{2i\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{g}(x)e^{ibx}dx-\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{g}(x)e^{-ibx}dx\right]$$

$$=\frac{1}{2i}\left[\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{ikx}e^{ibx}\,\mathrm{g}(x)dx-\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{ikx}e^{-ibx}\,\mathrm{g}(x)dx\right]$$

$$=\frac{1}{2i}\left[\Im\left(e^{ibx}\,\mathrm{g}(x)\right)-\Im\left(e^{-ibx}\,\mathrm{g}(x)\right)\right]$$

$$=\frac{1}{2i}\left[G\left(k-i(ib)\right)-G\left(k-i(-ib)\right)\right] \quad \because\ \Im\left[e^{ax}f(x)\right]=F(k-ia)$$

$$=\frac{1}{2i}\left[G\left(k+b\right)-G\left(k-ib\right)\right] \text{\_\_\_\_\_\_\_\_(i)}$$

Now $\quad \mathrm{G(k)} = \Im[\mathrm{g}(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{g}(x)dx$

$$\mathrm{G}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}dx$$

$$\mathrm{G}(k)=\frac{1}{\sqrt{2\alpha}}e^{\frac{-k^2}{4a}} \qquad \because\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-kx}e^{-ax^2}dx=\frac{1}{\sqrt{2\alpha}}e^{\frac{-k^2}{4a}}$$

$$\mathrm{G}(k+b)=\frac{1}{\sqrt{2a}}e^{\frac{-(k+b)^2}{4a}}$$

$$=\frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2+2kb)}{4a}} = \frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}.e^{\frac{-kb}{2a}}$$

$$\Rightarrow\quad G(k-b)=\frac{1}{\sqrt{2a}}e^{\frac{-(k-b)^2}{4a}}$$

$$=\frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2-2kb)}{4a}} = \frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}.e^{\frac{kb}{2a}}$$

Put these values in (i)

$$=\frac{1}{2i}\left[\frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}.e^{\frac{-kb}{2a}} - \frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4\alpha}}.e^{\frac{kb}{2a}}\right]$$

$$=\frac{1}{i\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}\left[\frac{e^{\frac{-kb}{2a}}-e^{\frac{kb}{2a}}}{2}\right]$$

$$=\frac{1}{i\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}(-1)\left[\frac{e^{\frac{kb}{2a}}-e^{\frac{-kb}{2a}}}{2}\right]$$

$$=\frac{1}{i}.\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+b^2)}{4a}}\left(\mathrm{i}^2\right)\sinh\left(\frac{kb}{2a}\right)$$

$$\Im[\mathrm{f}(x)]=\frac{i}{\sqrt{2\alpha}}e^{\frac{-(k^2+b^2)}{4a}}\sinh\left(\frac{kb}{2a}\right)$$

Theorem: If Fourier transform [f(x)] = F(k) then $\Im[e^{ax}\mathrm{f(x)}]$ = F(k−ia)

Proof: By the definition of Fourier transform

$$\Im\left[f(x)\right]\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}f(x)dx$$

$$\Im\left[e^{ax}f(x)\right]\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}e^{ax}f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{(ikx+ax)}f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{(ikx-i^2ax)}f(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(k-ia)x}f(x)dx$$

= F(k−ia)

Question: Find the Fourier transform of $e^{-ax^2}\cos(bx)\quad where\ a>0$

Solution: By the definition of Fourier transform

$$\Im[\mathrm{f}(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\,\mathrm{f}(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}\cos(bx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}\left(\frac{e^{ibx}+e^{-ibx}}{2}\right)dx$$

$$=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{ikx}e^{-ax^2}e^{ibx}+e^{ikx}e^{-ax^2}e^{-ibx}\right)dx$$

$$=\frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}e^{ibx}dx+\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}e^{-ibx}dx\right]$$

$Put\ \ g(x)=e^{-ax^2}$

$$=\frac{1}{2i\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{ikx}\,g(x)e^{ibx}dx+\int_{-\infty}^{\infty} e^{ikx}\,g(x)e^{-ibx}dx\right]$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{ikx}e^{ibx}\,g(x)dx+\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{ikx}e^{-ibx}\,g(x)dx\right]$$

$$=\frac{1}{2}\left[\Im\left(e^{ibx}\,g(x)\right)+\Im\left(e^{-ibx}\,g(x)\right)\right]$$

$$=\frac{1}{2}\left[G\left(k-i(ib)\right)+G\left(k-i(-ib)\right)\right]\quad\because\ \Im\left[e^{ax}f(x)\right]=F(k-ia)$$

$$=\frac{1}{2}\left[G\left(k+b\right)+G\left(k-ib\right)\right]\_\_\_\_\_\_\_\_(\text{i})$$

Now $\quad G(k)=\ \Im[g(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,g(x)dx$

$$G(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}dx$$

$$G(k)=\frac{1}{\sqrt{2\alpha}}e^{\frac{-k^2}{4a}}\qquad\because\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-kx}e^{-ax^2}dx=\frac{1}{\sqrt{2\alpha}}e^{\frac{-k^2}{4a}}$$

$$G(k+b)=\frac{1}{\sqrt{2a}}e^{\frac{-(k+b)^2}{4a}}$$

$$=\frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2+2kb)}{4a}} = \frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}.e^{\frac{-kb}{2a}}$$

$$\Rightarrow \quad G(k-b)=\frac{1}{\sqrt{2a}}e^{\frac{-(k-b)^2}{4a}}$$

$$=\frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2-2kb)}{4a}} = \frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}.e^{\frac{kb}{2a}}$$

Put these values in (i)

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}.e^{\frac{-kb}{2a}}+\frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4\alpha}}.e^{\frac{kb}{2a}}\right]$$

$$=\frac{1}{\sqrt{2a}}e^{\frac{-(k^2+b^2)}{4a}}\left[\frac{e^{\frac{-kb}{2a}}+e^{\frac{kb}{2a}}}{2}\right]$$

$$=\frac{1}{\sqrt{2\alpha}}e^{\frac{-(k^2+b^2)}{4a}}\cosh\left(\frac{kb}{2a}\right)$$

Fourier Transformation Derivatives of a function:

Let f(x) be a function $\Im[\mathrm{f}(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\,\mathrm{f}(x)dx$

$$\Im[\mathrm{f}'(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\,\mathrm{f}'(x)dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[e^{ikx}f(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}ik\,e^{ikx}\,\mathrm{f}(x)dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[0-ik\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{f}(x)dx\right]$$

$$=(-ik)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{f}(x)dx$$

$$\Im[f^{'}(x)]=(-ik)\,\Im[f(x)]$$

Similarly, $$\Im[f^{''}(x)]=(-ik)^2\,\Im[f(x)]$$

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In general, $$\Im[f^{n}(x)]=(-ik)^n\,\Im[f(x)]$$

Question: If $f\left(x\right)=\sqrt{2}\,e^{\frac{-x^2}{4}}$. Find Fourier transform of $f^{(iv)}\left(x\right)$

Solution: We know that $\Im[f^{(iv)}(x)]=(-ik)^4\,\Im\,[f(x)]$

$$\Im[f^{(iv)}(x)]=k^4\left[\sqrt{2}\,e^{\frac{-x^2}{4}}\right] \underline{\qquad}(i)$$

$$\Im\left[\sqrt{2}\,e^{-ax^2}\right]=\frac{N}{\sqrt{2a}}e^{\frac{-k^2}{4a}}$$

$$\Im\left[\sqrt{2}\,e^{\frac{-x^2}{4}}\right]=\frac{\sqrt{2}}{\sqrt{2a}}e^{\frac{-k^2}{4.\frac{1}{4}}}=\frac{\sqrt{2}}{\frac{1}{\sqrt{2}}}e^{-k^2}$$

$$\Im\left[\sqrt{2}\,e^{\frac{-x^2}{4}}\right]=2e^{-k^2}$$

Put in (i) $$\Im[f^{(iv)}(x)]=2k^4\,e^{-k^2}$$

Question: If $f\left(x\right)=e^{-\left(k-1\right)^2}$. Find Fourier transform of $f^{\left(vi\right)}\left(x\right)$

Solution: We know that $\Im[f^{(vi)}(x)]=(-ik)^6\,\Im[f(x)]$

$$\Im[f^{(vi)}(x)]=k^6\,.e^{-(k-1)^2}$$

Lecture # 12

Question: Solve by Fourier Transform

$$\frac{d^2y}{dx^2}+2\alpha\frac{dy}{dx}+w^2y=f(x)$$

Solution: Applying Fourier transform on both side

$$\Im\left[\frac{d^2y}{dx^2}+2\alpha\frac{dy}{dx}+w^2y\right]=\Im\left[f(x)\right]$$

$$\Im\left[\frac{d^2y}{dx^2}\right]+2\alpha\,\Im\left[\frac{dy}{dx}\right]+w^2\Im[y]=F(k)$$

$$(-ik)^2\,\Im[y]+2\alpha(-ik)\Im[y]+w^2\Im[y]=F(k)$$

$$-k^2\Im[y]-2\alpha ik\,\Im[y]+w^2\Im[y]=F(k)$$

$$\left(w^2-2\alpha ik-k^2\right)\Im[y]=F(k)$$

$$\Im[y]=\frac{F(k)}{\left(w^2-2\alpha ik-k^2\right)}$$

Taking Fourier Inverse on both side

$$y(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\frac{F(k)}{\left(w^2-2\alpha ik-k^2\right)}dk$$

Question:

$$\phi_{xx}=\phi_{tt}$$

$$\phi(x,0)=f(x)$$

$$\phi_t(x,0)=0$$

Solution:

$$\Im[\phi_{xx}]=\Im[\phi_{tt}]$$

$$\Im\left[\frac{\partial^2\phi}{\partial x^2}(x,t)\right]=\Im\left[\frac{\partial^2\phi}{\partial t^2}(x,t)\right]$$

$$(-ik)^2 \Im[\phi(x,t)] = \frac{\partial^2}{\partial t^2}\Im[\phi(x,t)]$$

Suppose $\Im[\phi(x,t)] = \overline{\phi}(k,t)$

$$(-ik)^2 \overline{\phi}(k,t) = \frac{\partial^2}{\partial t^2}\overline{\phi}(k,t)$$

$$-k^2 \overline{\phi}(k,t) = \frac{\partial^2}{\partial t^2}\overline{\phi}(k,t)$$

$$\frac{\partial^2}{\partial t^2}\overline{\phi}(k,t) + k^2 \overline{\phi}(k,t) = 0$$

$$\left(\frac{\partial^2}{\partial t^2} + k^2\right)\overline{\phi}(k,t) = 0$$

$$\frac{\partial^2}{\partial t^2} + k^2 = 0 \quad \& \quad \overline{\phi}(k,t) \neq 0$$

$$\frac{\partial^2}{\partial t^2} = -k^2 = i^2 k^2$$

$$\frac{\partial}{\partial t} = \pm ik$$

$$\overline{\phi}(k,t) = A(k)cos(kt) + B(k)\sin(kt) \qquad \_\_\_\_(i)$$

Now $\phi(x,0) = f(x)$

$$\Im[\phi(x,0)] = \Im[f(x)]$$

$$\overline{\phi}(k,0) = F(k) \qquad \_\_\_\_(ii)$$

Put t = 0 in (i) and compare with (ii)

$$F(k) = A(k)cos(0) + B(k)\sin(0)$$

$$F(k) = A(k)$$

Now diff. (ii) partially w.r.t 't'

$$\frac{\partial}{\partial t}\bar{\phi}(x,\mathrm{t}) = -kA(k)\sin(kt) + kB(k)\cos(kt) \qquad \_\_\_\_(iv)$$

Also $\phi_t(x,0)=0$

$$[\phi_t(x,0)]=0$$

$$\Im\left[\frac{\partial}{\partial t}\phi(x,0)\right]=0$$

$$\frac{\partial}{\partial t}\Im[\phi(x,0)]=0$$

$$\frac{\partial}{\partial t}\bar{\phi}(x,0)=0 \qquad \_\_\_\_\_(v)$$

Now put t = 0 in (iv) and compare with (v)

$$\frac{\partial}{\partial t}\bar{\phi}(x,0) = -kA(k)\sin(0) + kB(k)\cos(0)$$

$$0 = -0 + kB(k)(1)$$

$$B(k) = 0$$

Put the value of A(k) , B(k) in (i)

$$\bar{\phi}(x,\mathrm{t}) = F(k)\cos(kt)$$

Taking inverse Fourier on both sides

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}F(k)\cos(kt)\,dk$$

Question: Solve by Fourier Transform

$$\phi_{xxxx} = a^2 \phi_{tt}$$

$$\phi(x,0) = f(x)$$

$$\phi_t(x,0) = ag''(x)$$

Solution: Applying Fourier Transform on both side

$$\Im[\phi_{xxxx}] = \Im[\mathrm{a}^2 \phi_{tt}]$$

$$\Im\left[\frac{\partial^4\phi}{\partial x^4}(x,t)\right] = a^2\Im\left[\frac{\partial^2\phi}{\partial t^2}(x,t)\right]$$

$$-(ik)^4 \Im\big[\phi(x,t)\big] = a^2 \frac{\partial^2}{\partial t^2}\Im\big[\phi(x,t)\big]$$

$$k^4\Im\big[\phi(x,t)\big] = a^2 \frac{\partial^2}{\partial t^2}\Im\big[\phi(x,t)\big]$$

Suppose $\Im\big[\phi(x,t)\big] = \bar{\phi}(k,t)$

$$k^4 \bar{\phi}(k,t) = a^2 \frac{\partial^2}{\partial t^2}\bar{\phi}(k,t)$$

$$a^2 \frac{\partial^2}{\partial t^2}\bar{\phi}(k,t) - k^4 \bar{\phi}(k,t) = 0$$

$$\left(a^2 \frac{\partial^2}{\partial t^2} - k^4\right)\bar{\phi}(k,t) = 0$$

$$a^2 \frac{\partial^2}{\partial t^2} - k^4 = 0 \quad \& \ \bar{\phi}(k,t) \neq 0$$

$$\frac{\partial^2}{\partial t^2} = \frac{k^4}{a^2}$$

$$\frac{\partial}{\partial t} = \pm\frac{k^2}{a}$$

$$\bar{\phi}(k,t)=A(k)\cosh\left(\frac{k^2t}{a}\right)+B(k)\sinh\left(\frac{k^2t}{a}\right) \quad \_\_\_\_\_(i)$$

Given condition $\quad \phi(x,0)=f(x)$

$$\Im[\phi(x,0)]=\Im[f(x)]$$

$$\bar{\phi}(k,0)=F(k) \quad \_\_\_\_\_\_(ii)$$

Put t = 0 in (i) and compare with (ii)

$$\bar{\phi}(k,0)=A(k)\cosh(0)+B(k)\sinh(0)$$

$$F(k)=A(k)$$

Diff. (i) w.r.t. 't'

$$\frac{\partial\,\bar{\phi}(k,t)}{\partial t}=\frac{k^2}{a}A(k)\sinh\left(\frac{k^2t}{a}\right)+\frac{k^2}{a}B(k)\cosh\left(\frac{k^2t}{a}\right) \quad \_\_\_\_(iii)$$

$$\phi_t(x,0)=ag''(x)$$

$$\Im[\phi_t(x,0)]=a\,\Im[g''(x)]$$

$$\frac{\partial}{\partial t}\Im[\phi(x,0)]=a\,\Im[g''(x)]$$

$$\frac{\partial}{\partial t}\bar{\phi}(x,0)=a(-ik)^2\,\Im[g(x)]$$

$$\frac{\partial}{\partial t}\bar{\phi}(k,0)=-ak^2\,\mathrm{G}(k) \quad \_\_\_\_\_(iv)$$

Put t = 0 in (iii) compare with (iv)

$$\frac{\partial}{\partial t}\bar{\phi}(\mathrm{k},0)=\frac{k^2}{a}A(k)\sinh(0)+\frac{k^2}{a}B(k)\cosh(0)$$

$$-ak^2\,\mathrm{G}(k)=\frac{k^2}{a}B(k)$$

$$B(k)=-a^2\,\mathrm{G}(k)$$

Put the value of A(k) , B(k) in (i)

$$\bar{\phi}(k,t)=F(k)\cosh\left(\frac{k^2t}{a}\right)-a^2\,G(k)\sinh\left(\frac{k^2t}{a}\right)$$

Taking inverse Fourier on both sides

$$\phi(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\left[F(k)\cosh\left(\frac{k^2t}{a}\right)-a^2\,G(k)\sinh\left(\frac{k^2t}{a}\right)\right]dk$$

Question: Solve by Fourier Transfomr

$$\phi_{xx}=\alpha^2\phi_t$$

$$\phi(x,0)=e^{-ax^2}\quad,\ a>0$$

Solution: Applying Fourier Transform on both side

$$\Im[\phi_{xx}]=\Im[\alpha^2\phi_t]$$

$$\Im\left[\frac{\partial^2\phi}{\partial x^2}(x,t)\right]=\Im\left[\alpha^2\frac{\partial\phi}{\partial t}(x,t)\right]$$

$$-(ik)^2\,\Im[\phi(x,t)]=\alpha^2\frac{\partial}{\partial t}\Im[\phi(x,t)]$$

$$-k^2\Im[\phi(x,t)]=\alpha^2\frac{\partial}{\partial t}\Im[\phi(x,t)]$$

Suppose $[\phi(x,t)]=\bar{\phi}(k,t)$

$$-k^2\bar{\phi}(k,t)=\alpha^2\frac{\partial}{\partial t}\bar{\phi}(k,t)$$

$$\alpha^2\frac{\partial}{\partial t}\bar{\phi}(k,t)+k^2\bar{\phi}(k,t)=0$$

$$\left(\alpha^2\frac{\partial}{\partial t}+k^2\right)\bar{\phi}(k,t)=0$$

$$\alpha^2 \frac{\partial}{\partial t} + k^2 = 0 \quad \& \ \overline{\phi}(k,t) \neq 0$$

$$\frac{\partial}{\partial t} = -\frac{k^2}{\alpha^2}$$

$$\overline{\phi}(k,t) = A(k) e^{\frac{-k^2 t}{\alpha^2}} \qquad \_\_\_\_\_\_(i)$$

Given condition $\qquad \phi(x,0) = e^{-ax^2}$

$$\Im[\phi(x,0)] = \Im\left[e^{-ax^2}\right]$$

$$\overline{\phi}(k,0) = \frac{e^{\frac{-K^2}{4a}}}{\sqrt{2a}} \qquad \_\_\_\_\_(ii)$$

Put t = 0 in (i) and compare with (ii)

$$\overline{\phi}(k,0) = A(k) e^0$$

$$A(k) = \frac{e^{\frac{-K^2}{4a}}}{\sqrt{2a}}$$

Put in (i)

$$\overline{\phi}(k,t) = \frac{e^{\frac{-K^2}{4a}}}{\sqrt{2a}} e^{\frac{-k^2 t}{\alpha^2}}$$

$$\overline{\phi}(k,t) = \frac{1}{\sqrt{2a}} e^{\frac{-K^2}{4a} - \frac{k^2 t}{\alpha^2}} = \frac{1}{\sqrt{2a}} e^{\frac{-K^2}{4\alpha^2}\left(\frac{\alpha^2 + 4at}{a}\right)}$$

Taking inverse Fourier on both sides

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{\sqrt{2a}} e^{\frac{-K^2}{4\alpha^2}\left(\frac{\alpha^2 + 4at}{a}\right)} dk$$

$$\phi(x,t) = \frac{1}{2\sqrt{a}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-ikx - \frac{K^2}{4\alpha^2}\left(\frac{\alpha^2 + 4at}{a}\right)} dk \qquad \_\_\_\_\_(iii)$$

Consider

$$-ikx-\frac{k^2}{4\alpha^2}\left(\frac{\alpha^2+4at}{a}\right)=-\left[\frac{k^2}{4\alpha^2}\left(\frac{\alpha^2+4at}{a}\right)+ikx\right]$$

$$=-\left[\left(\frac{k}{2\alpha}\left(\sqrt{\frac{\alpha^2+4at}{a}}\right)\right)^2+2\frac{k}{2\alpha}\sqrt{\frac{\alpha^2+4at}{a}}.\alpha\sqrt{\frac{a}{\alpha^2+4at}}(ix)\right.$$

$$\left.+\left(\alpha\sqrt{\frac{a}{\alpha^2+4at}}(ix)\right)^2-\left(\alpha\sqrt{\frac{a}{\alpha^2+4at}}(ix)\right)^2\right]$$

$$=-\left[\left(\frac{k}{2\alpha}\sqrt{\frac{\alpha^2+4at}{a}}+i\alpha x\sqrt{\frac{a}{\alpha^2+4at}}\right)^2+\frac{\alpha^2x^2a}{\alpha^2+4at}\right]$$

$$=-\left(\frac{k}{2\alpha}\sqrt{\frac{\alpha^2+4at}{a}}+i\alpha x\sqrt{\frac{a}{\alpha^2+4at}}\right)^2-\frac{\alpha^2x^2a}{\alpha^2+4at}$$

Put in (iii)

$$\phi(x,t)=\frac{1}{2\sqrt{a}\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\left(\frac{k}{2\alpha}\sqrt{\frac{\alpha^2+4at}{a}}+i\alpha x\sqrt{\frac{a}{\alpha^2+4at}}\right)^2-\frac{\alpha^2x^2a}{\alpha^2+4at}}dk$$

$$\phi(x,t)=\frac{1}{2\sqrt{a}\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\left(\frac{k}{2\alpha}\sqrt{\frac{\alpha^2+4at}{a}}+i\alpha x\sqrt{\frac{a}{\alpha^2+4at}}\right)^2}e^{-\frac{\alpha^2x^2a}{\alpha^2+4at}}dk$$

$$\phi(x,t)=\frac{1}{2\sqrt{a}\sqrt{\pi}}e^{-\frac{\alpha^2x^2a}{\alpha^2+4at}}\int_{-\infty}^{\infty}e^{-\left(\frac{k}{2\alpha}\sqrt{\frac{\alpha^2+4at}{a}}+i\alpha x\sqrt{\frac{a}{\alpha^2+4at}}\right)^2}dk$$

Let $\quad z=\dfrac{k}{2\alpha}\sqrt{\dfrac{\alpha^2+4at}{a}}+i\alpha x\sqrt{\dfrac{a}{\alpha^2+4at}}$

$$dz=\frac{1}{2\alpha}\sqrt{\frac{\alpha^2+4at}{a}}dk \quad \Rightarrow dk=\frac{2\alpha\sqrt{a}}{\sqrt{\alpha^2+4at}}dz$$

$$z\to\pm\infty \quad as \quad k\to\pm\infty$$

$$\phi(x,t)=\frac{1}{2\sqrt{a}\sqrt{\pi}}e^{-\frac{\alpha^2x^2a}{\alpha^2+4at}}\int_{-\infty}^{\infty}e^{-z^2}\frac{2\alpha\sqrt{a}}{\sqrt{\alpha^2+4at}}dz$$

$$\phi(x,t)=\frac{\alpha}{\sqrt{\pi}\sqrt{\alpha^2+4at}}e^{-\frac{\alpha^2x^2a}{\alpha^2+4at}}\int_{-\infty}^{\infty}e^{-z^2}\,dz$$

$$\phi(x,t)=\frac{\alpha}{\sqrt{\pi}\sqrt{\alpha^2+4at}}e^{-\frac{\alpha^2x^2a}{\alpha^2+4at}}\sqrt{\pi}$$

$$\phi(x,t)=\frac{\alpha}{\sqrt{\alpha^2+4at}}e^{-\frac{\alpha^2x^2a}{\alpha^2+4at}}$$

Question: Find Fourier transform of $\frac{1}{\sqrt{2}}e^{\frac{-x^2}{4}}$

Solution: By applying Fourier Transform

$$\Im\left[\frac{1}{\sqrt{2}}e^{\frac{-x^2}{4}}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\frac{1}{\sqrt{2}}e^{\frac{-x^2}{4}}dx$$

$$=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{ikx-\frac{x^2}{4}}dx \qquad \_\_\_\_\_(i)$$

Consider $ikx-\frac{x^2}{4}=-\left[\frac{x^2}{4}-ikx\right]$

$$=-\left[\left(\frac{x}{2}\right)^2-2ik\frac{x}{2}+(ik)^2-(ik)^2\right]$$

$$= -\left[\left(\frac{x}{2} - ik\right)^2 + k^2\right] = -\left(\frac{x}{2} - ik\right)^2 - k^2$$

Put in (i) $\Rightarrow$ $$\Im\left[\frac{1}{\sqrt{2}} e^{\frac{-x^2}{4}}\right] = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{2} - ik\right)^2 - k^2} dx$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{2} - ik\right)^2} e^{-k^2} dx$$

$$= \frac{e^{-k^2}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{2} - ik\right)^2} dx \quad \_\_\_\_\_(ii)$$

Let $z = \frac{x}{2} - ik \quad \Rightarrow dz = \frac{dx}{2} \quad \Rightarrow dx = 2dz$

$$\Im\left[\frac{1}{\sqrt{2}} e^{\frac{-x^2}{4}}\right] = \frac{e^{-k^2}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} 2dz = \frac{e^{-k^2}}{2\sqrt{\pi}} \sqrt{\pi}$$

$$\Im\left[\frac{1}{\sqrt{2}} e^{\frac{-x^2}{4}}\right] = e^{-k^2}$$

Lecture # 13

Fourier Sine and Cosine Transform:

Let f(x) be a real valued function such that f(x) $\to$ 0 as x $\to \infty$ then Fourier sine transform is defined as

$$\Im_s\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(kx)dx$$

And is denoted by $F_s(k)$ i.e. $\Im_s\left[f\left(x\right)\right]=F_s\left(k\right)$

Also, Fourier cosine transform is

$$\Im_c\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(kx)dx$$

And is denoted by F_c (k) i.e. $\Im_c\left[f\left(x\right)\right]=F_c\left(k\right)$

Question: Find Fourier sine transform of $f\left(x\right)=e^{-x}\cos x$

Solution: As we know that

$$\Im_s\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(kx)dx$$

$$\Im_s\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos x\sin(kx)dx \quad \because \cos\alpha\sin\beta=\frac{1}{2}\left[\sin\left(\alpha+\beta\right)-\sin\left(\alpha-\beta\right)\right]$$

$$\Im_s\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left[\frac{1}{2}\left\{\sin\left(x+kx\right)-\sin\left(x-kx\right)\right\}\right]dx$$

$$\Im_s\left[f\left(x\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left\{\sin\left(1+k\right)x-\sin\left(1-k\right)x\right\}dx$$

$$\Im_s\left[f\left(x\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\left(1+k\right)xdx-\int_0^\infty e^{-x}\sin\left(1-k\right)xdx$$

$$\because \int e^{ax}\sin(bx)\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\sin(bx)-b\cos(bx)\right]$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{(-1)^2+(1+k)^2}\left\{-\sin(1+k)x-(1+k)\cos(1+k)x\right\}\Bigg|_0^\infty\right.$$

$$\left.-\frac{e^{-x}}{(-1)^2+(1-k)^2}\left\{-\sin(1-k)x-(1-k)\cos(1-k)x\right\}\Bigg|_0^\infty\right]$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+(1+k)^2}\left\{\lim_{x\to\infty}e^{-x}\lim_{x\to\infty}\left(-\sin(1+k)x-(1+k)\cos(1+k)x\right)-e^{0}\left(-\sin(0)-(1+k)\cos(0)\right)\right\}\right.$$

$$\left.-\frac{1}{1+(1-k)^2}\left\{\lim_{x\to\infty}e^{-x}\lim_{x\to\infty}\left(-\sin(1-k)x-(1-k)\cos(1-k)x\right)-e^{0}\left(-\sin(0)-(1-k)\cos(0)\right)\right\}\right]$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+1+k^2+2k}\left\{0+(1+k)\right\}-\frac{1}{1+1+k^2-2k}\left\{0+(1-k)\right\}\right]$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1+k}{k^2+2k+2}-\frac{1-k}{k^2-2k+2}\right]$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{(1+k)(k^2-2k+2)-(1-k)(k^2+2k+2)}{(k^2+2k+2)(k^2-2k+2)}\right]$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{k^2-2k+2+k^3-2k^2+2k-k^2-2k-2+k^3+2k^2+2k}{(k^2+2)^2-(2k)^2}\right]$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{2k^3}{k^4+4k^2+4-4k^2}\right]$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{k^3}{k^4+4}\right]$$

Question: Find Fourier cosine transform of $f\left(x\right)=e^{-x}\cos x$

Solution: As we know that

$$\Im_c\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(kx)dx$$

$$\Im_c\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos x\cos\left(kx\right)dx \quad \because \cos\alpha\cos\beta=\frac{1}{2}\left[\cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)\right]$$

$$\Im_c\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left[\frac{1}{2}\left\{\cos\left(x+kx\right)+\cos\left(x-kx\right)\right\}\right]dx$$

$$\Im_c\left[f\left(x\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left\{\cos\left(1+k\right)x+\cos\left(1-k\right)x\right\}dx$$

$$\Im_c\left[f\left(x\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos\left(1+k\right)xdx+\int_0^\infty e^{-x}\cos\left(1-k\right)xdx$$

$$\because \int e^{ax}\cos\left(bx\right)dx=\frac{e^{ax}}{a^2+b^2}\left[a\cos\left(bx\right)+b\sin\left(bx\right)\right]$$

$$\Im_c\left[f\left(x\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{\left(-1\right)^2+\left(1+k\right)^2}\left\{-\cos\left(1+k\right)x-\left(1+k\right)\sin\left(1+k\right)x\right\}\Bigg|_0^\infty\right.$$

$$\left.-\frac{e^{-x}}{\left(-1\right)^2+\left(1-k\right)^2}\left\{-\cos\left(1-k\right)x-\left(1-k\right)\sin\left(1-k\right)x\right\}\Bigg|_0^\infty\right]$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+\left(1+k\right)^2}\left\{\lim_{x\to\infty}e^{-x}\lim_{x\to\infty}\left(-\cos\left(1+k\right)x+\left(1+k\right)\sin\left(1+k\right)x\right)-e^0\left(-\cos\left(0\right)+\left(1+k\right)\sin\left(0\right)\right)\right\}\right.$$

$$\left.\frac{1}{1+\left(1-k\right)^2}\left\{\lim_{x\to\infty}e^{-x}\lim_{x\to\infty}\left(-\cos\left(1-k\right)x+\left(1-k\right)\sin\left(1-k\right)x\right)-e^0\left(-\cos\left(0\right)+\left(1-k\right)\sin\left(0\right)\right)\right\}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+1+k^2+2k}\{0+1\}+\frac{1}{1+1+k^2-2k}\{0+1\}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{k^2+2k+2}+\frac{1}{k^2-2k+2}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{k^2-2k+2+k^2+2k+2}{\left(k^2+2k+2\right)\left(k^2-2k+2\right)}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{2k^2+4}{\left(k^2+2\right)^2-\left(2k\right)^2}\right]$$

$$\Im_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\left[\frac{k^2+2}{\left(k^2+2\right)^2-\left(2k\right)^2}\right]$$

$$\Im_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\left[\frac{k^2+2}{k^4+4}\right]$$

Question: Find Fourier sine transform of $f(x)=e^{-x}\sin x$

Solution: As we know that $\Im_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int\limits_0^\infty f(x)\sin(kx)dx$

$$\Im_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int\limits_0^\infty e^{-x}\sin x\sin(kx)dx \qquad \because -2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$$

$$\Im_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int\limits_0^\infty e^{-x}\left[\frac{1}{2}\left\{\cos(x-kx)-\cos(x+kx)\right\}\right]dx$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int\limits_0^\infty e^{-x}\left\{\cos(1-k)x-\cos(1+k)x\right\}dx$$

$$\Im_s\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int\limits_0^\infty e^{-x}\cos(1-k)xdx-\int\limits_0^\infty e^{-x}\cos(1+k)xdx$$

$$\because \int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx)+b\sin(bx)\right]$$

$$\Im_s\left[f(x)\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{(-1)^2+(1-k)^2}\left\{-\cos(1-k)x+(1-k)\sin(1-k)x\right\}\Big|_0^\infty\right.$$

$$\left.-\frac{e^{-x}}{(-1)^2+(1+k)^2}\left\{-\cos(1+k)x-(1+k)\sin(1+k)x\right\}\Big|_0^\infty\right]$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+(1-k)^2}\left\{\lim_{x\to\infty}e^{-x}\lim_{x\to\infty}\left(-\cos(1-k)x+(1-k)\sin(1-k)x\right)-e^{0}\left(-\cos(0)-(1-k)\sin(0)\right)\right\}\right.$$

$$\left.-\frac{1}{1+(1+k)^2}\left\{\lim_{x\to\infty}e^{-x}\lim_{x\to\infty}\left(-\cos(1+k)x+(1+k)\sin(1+k)x\right)-e^{0}\left(-\cos(0)+(1+k)\sin(0)\right)\right\}\right]$$

$$\Im_s\left[f(x)\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+1+k^2-2k}\{0+1\}-\frac{1}{1+1+k^2+2k}\{0+1\}\right]$$

$$\Im_s\left[f(x)\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{k^2-2k+2}-\frac{1}{k^2+2k+2}\right]$$

$$\Im_s\left[f(x)\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{k^2+2k+2-k^2+2k-2}{\left(k^2-2k+2\right)\left(k^2+2k+2\right)}\right]$$

$$\Im_s\left[f(x)\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{4k}{\left(k^2+2\right)^2-(2k)^2}\right]$$

$$\Im_s\left[f(x)\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{2k}{k^4+4k^2+4-4k^2}\right]$$

$$\Im_s\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2k}{k^4+4}\right]$$

Question: Find Fourier cosine transform of $f\left(x\right)=e^{-x}\sin x$

Solution: As we know that

$$\Im_c\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(kx)dx$$

$$\Im_c\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin x\cos\left(kx\right)dx \quad \because 2\sin\alpha\cos\beta=\left[\sin\left(\alpha+\beta\right)+\sin\left(\alpha-\beta\right)\right]$$

$$\Im_c\left[f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left[\frac{1}{2}\left\{\sin\left(x+kx\right)+\sin\left(x-kx\right)\right\}\right]dx$$

$$\Im_c\left[f\left(x\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left\{\sin\left(1+k\right)x+\sin\left(1-k\right)x\right\}dx$$

$$\Im_c\left[f\left(x\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\left(1+k\right)xdx+\int_0^\infty e^{-x}\sin\left(1-k\right)xdx$$

$$\because\int e^{ax}\sin\left(bx\right)dx=\frac{e^{ax}}{a^2+b^2}\left[a\sin\left(bx\right)-b\cos\left(bx\right)\right]$$

$$\Im_c\left[f\left(x\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{\left(-1\right)^2+\left(1+k\right)^2}\left\{-\sin\left(1+k\right)x-\left(1+k\right)\cos\left(1+k\right)x\right\}\Bigg|_0^\infty\right.$$

$$\left.-\frac{e^{-x}}{\left(-1\right)^2+\left(1-k\right)^2}\left\{-\sin\left(1-k\right)x-\left(1-k\right)\cos\left(1-k\right)x\right\}\Bigg|_0^\infty\right]$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+\left(1+k\right)^2}\left\{\lim_{x\to\infty}e^{-x}\lim_{x\to\infty}\left(-\sin\left(1+k\right)x-\left(1+k\right)\cos\left(1+k\right)x\right)-e^0\left(-\sin\left(0\right)+\left(1-k\right)\cos\left(0\right)\right)\right\}\right.$$

$$\left.-\frac{1}{1+\left(1-k\right)^2}\left\{\lim_{x\to\infty}e^{-x}\lim_{x\to\infty}\left(-\sin\left(1-k\right)x-\left(1-k\right)\cos\left(1-k\right)x\right)-e^0\left(-\sin\left(0\right)-\left(1-k\right)\cos\left(0\right)\right)\right\}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+1+k^2+2k}\left\{0+(1+k)\right\}+\frac{1}{1+1+k^2-2k}\left\{1-k\right\}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1+k}{k^2+2k+2}+\frac{1-k}{k^2-2k+2}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{(1+k)(k^2-2k+2)+(1-k)(k^2+2k+2)}{(k^2+2k+2)(k^2-2k+2)}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{k^2+2-2k+k^3+2k-2k^2+k^2+2k+2-k^3-2k-2k^2}{(k^2+2)^2-(2k)^2}\right]$$

$$\Im_c\left[f(x)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{-2k^2+4}{k^4+4k^2+4-4k^2}\right]$$

$$\Im_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\left[\frac{2-k^2}{k^4+4}\right]$$

Fourier Sine and Cosine transform of derivatives:

Let $f'(x)$ be the derivative of real valued function f(x) then

$$\Im_s\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[f'(x)\right]\sin(kx)dx$$

$$\Im_s\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\left[f(x)\sin(kx)\Big|_0^\infty-\int_0^\infty f(x)\cos(kx)\,\mathrm{k}\,dx\right]$$

$$\Im_s\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\left[\underset{x\to\infty}{Lim}f(x)\underset{x\to\infty}{Lim}\sin(kx)-f(0)\sin(0)-k\int_0^\infty f(x)\cos(kx)dx\right]$$

$$\Im_s\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\left[0-f(0)0-k\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(kx)dx\right]$$

$$\Im_s\left[f'(x)\right]=-k\Im_c\left[f(x)\right]$$

And $\Im_c\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[f'(x)\right]\cos(kx)dx$

$$\Im_c\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\left[f(x)\cos(kx)\Big|_0^\infty-\int_0^\infty f(x)\big(-\sin(kx)\mathrm{k}\big)dx\right]$$

$$\Im_c\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\left[\underset{x\to\infty}{Lim}\,f(x)\underset{x\to\infty}{Lim}\cos(kx)-f(0)\cos(0)+k\int_0^\infty f(x)\sin(kx)dx\right]$$

$$\Im_c\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\left[0-f(0)+k\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(kx)dx\right]$$

$$\Im_c\left[f'(x)\right]=-\sqrt{\frac{2}{\pi}}f(0)+k\Im_s\left[f(x)\right]$$

Now $\Im_s\left[f''(x)\right]=-k\Im_c\left[f'(x)\right]$

$$\Im_s\left[f''(x)\right]=-k\left[-\sqrt{\frac{2}{\pi}}f(0)+k\Im_s\left[f(x)\right]\right]$$

$$\Im_s\left[f''(x)\right]=k\sqrt{\frac{2}{\pi}}f(0)-k^2\Im_s\left[f(x)\right]$$

And $\Im_c\left[f''(x)\right]=-\sqrt{\frac{2}{\pi}}f'(0)+k\Im_s\left[f'(x)\right]$

$$\Im_c\left[f''(x)\right]=-\sqrt{\frac{2}{\pi}}f'(0)+k\left[-k\Im_c\left[f(x)\right]\right]$$

$$\Im_c\left[f''(x)\right]=-\sqrt{\frac{2}{\pi}}f'(0)-k^2\Im_c\left[f(x)\right]$$

Question: If $\Im_s\left[f(x)\right]=F(k)$ and f(0) = 1 then find $\Im_s\left[f''(x)\right]$

Solution: As $\Im_s\left[f''(x)\right]=k\sqrt{\frac{2}{\pi}}f(0)-k^2\Im_s\left[f(x)\right]$

$$\Im_s\left[f''(x)\right]=k\sqrt{\frac{2}{\pi}}(1)-k^2F(k)$$

$$\Im_s\left[f''(x)\right]=k\sqrt{\frac{2}{\pi}}-k^2F(k)$$

Convolution for Fourier Transform:

Let f and g be two piecewise continuous function. Their convolution is denoted by $f*g$ where

$$f*g=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\xi)g(x-\xi)\,d\xi$$

$$g*f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(\xi)f(x-\xi)\,d\xi$$

Then $f*g=g*f$

Proof: Let h(x) = $f*g$

Consider $\Im(h(x))=F(k)G(k)$

Applying Fourier inverse transform

$$\Im^{-1}\Im(h(x))=\Im^{-1}\left[F(k)G(k)\right]$$

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}F(k)G(k)\,dk$$

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}F(k)\left(\int_{-\infty}^{\infty}e^{ik\xi}g(\xi)d\xi\right)dk$$

Interchanging the order of integration

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(\xi)\left(\int_{-\infty}^{\infty}e^{-ikx}e^{ik\xi}F(k)dk\right)d\xi$$

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(\xi)\left(\int_{-\infty}^{\infty}e^{-ik(x-\xi)}F(k)dk\right)d\xi$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(\xi)f(x-\xi)d\xi$$

$$h(x)=g*f$$

$$f*g=g*f$$

Question: Find Fourier transform of $f(x)=e^{-(x-a)^2}$

Solution: Applying Fourier Transform on both side

$$\Im\left[f(x)\right]=\Im\left[e^{-(x-a)^2}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}e^{-(x-a)^2}dx$$

Let z = x−a $\Rightarrow$ x = z +a

dx = dz

$$z\to\pm\infty \; as \; x\to\pm\infty$$

$$\Im\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ik(z+a)}e^{-z^2}dz$$

$$\Im\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikz+ika}e^{-z^2}dz$$

$$\Im\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikz}e^{ika}e^{-z^2}dz$$

$$\Im\left[f(x)\right]=e^{ika}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikz}e^{-z^2}dz$$

$$\Im\left[f(x)\right]=e^{ika}\Im\left[e^{-z^2}\right]$$

$$\Im\left[f(x)\right]=e^{ika}\frac{1}{\sqrt{2(1)}}.e^{\frac{-k^2}{4(1)}} \qquad \because \Im\left[Ne^{-ax^2}\right]=\frac{Ne^{\frac{-k^2}{4a}}}{\sqrt{2a}}$$

$$\Im\left[f(x)\right]=\frac{1}{\sqrt{2}}e^{ika}e^{\frac{-k^2}{4}}$$