Special Functions

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Dedicated

To

My Honorable Teacher

Dr. Muhey-U-Din

&

My Parents

Lecture # 01

Recommended Book:

Rainville, Earl David, Special function 2nd Edition Chelsea publishing Co. 1971

Gamma Function:

The Gamma function is denoted by Γn , where n > 0 and it is defined as

$$\Gamma n = \int_0^\infty e^{-x}.\, x^{n-1} dx \quad , n > 0$$

Example: For n = 1

$$\Gamma 1 = \int_0^\infty e^{-x}.\, x^{1-1} dx = \int_0^\infty e^{-x}.\, x^{0} dx$$

$$\Gamma 1 = \int_0^\infty e^{-x}\, dx$$

$$\Gamma 1 = \left.\frac{e^{-x}}{-1}\right|_0^\infty = -\left(e^{-x}\right)_0^\infty$$

$$\Gamma 1 = -\left(e^{-\infty} - e^{-0}\right)$$

$$\Gamma 1 = -(0-1)$$

$$\Gamma 1 = 1$$

For n = 2
$$\Gamma 2 = \int_0^\infty e^{-x}.\, x^{2-1} dx = \int_0^\infty e^{-x}.\, x\, dx$$

Integrating by parts

$$\Gamma 2 = x\left.\frac{e^{-x}}{-1}\right|_0^\infty - \int_0^\infty \frac{e^{-x}}{-1}(1)\, dx$$

$$\Gamma 2 = -\left(xe^{-x}\right)_0^\infty + \int_0^\infty e^{-x}dx$$

$$\Gamma 2 = -\left(xe^{-x}\right)_0^\infty + 1 \qquad \_\_\_\_\_\_(i) \qquad \because (i)\ by\ first\ proof$$

$$\because \lim_{x\to\infty} \frac{x}{e^x}, \left(\frac{\infty}{\infty}\right)$$

By L-Hospital rule

$$= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

$$eq(i) \Rightarrow \Gamma 2 = -(0-0)+1$$

$$\Rightarrow \Gamma 2 = 1$$

Some properties of Gamma function:

(i) $\Gamma(n+1) = n\Gamma n = n! \quad , \quad n > 0$

(ii) $\Gamma n = z^n \int_0^\infty e^{-zx}.x^{n-1}dx \quad ; n, z > 0$

(iii) $\Gamma n = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$

(iv) $\Gamma n = \int_0^\infty e^{-y^{\frac{1}{n}}} dy$

(v) $\Gamma\frac{1}{2} = \sqrt{\pi}$

Proof: (i) By definition of Gamma function

$$\Gamma n = \int_0^\infty e^{-x}.\, x^{n-1}dx$$

Replace n by n+1

$$\Gamma(n+1) = \int_0^\infty e^{-x}.\, x^{n+1-1}dx = \int_0^\infty e^{-x}.\, x^n dx$$

Integrating by parts

$$\left\lceil n+1 \right. = x^n \left.\frac{e^{-x}}{-1}\right|_0^\infty - \int_0^\infty \frac{e^{-x}}{-1}.n.\, x^{n-1}dx$$

$$\left\lceil n+1 \right. = -\left(x^n e^{-x}\right)_0^\infty + n\int_0^\infty e^{-x}.\, x^{n-1}dx$$

$$\left\lceil n+1 \right. = -\left(x^n e^{-x}\right)_0^\infty + n\left\lceil n \right. \qquad \_\_\_\_(i)$$

Consider $\lim\limits_{x\to\infty} \frac{x^n}{e^x}, \left(\frac{\infty}{\infty}\right) form$

$$= \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} \quad , \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \quad , \left(\frac{\infty}{\infty}\right)$$

Up to so on

$$= \lim_{x\to\infty} \frac{n(n-1)(n-2).....3.2.1.x^0}{e^x}$$

$$= \frac{n!}{e^\infty} = \frac{n!}{\infty} = 0$$

Eq (i) $\Rightarrow$ $\left\lceil n+1 \right. = -(0-0) + n\left\lceil n \right.$

$$\left\lceil n+1 \right. = n\left\lceil n \right.$$

Now $\left\lceil n+1 \right. = n\left\lceil n \right.$

$$\left\lceil n+1 \right. = n(n-1)\left\lceil n-1 \right.$$

$$\left\lceil n+1 \right. = n(n-1)(n-2)\left\lceil n-2 \right.$$

$$\left\lceil n+1 \right. = n(n-1)(n-2)...3.2.1\left\lceil 1 \right.$$

$$\left\lceil n+1 \right. = n!$$

As $\left\lceil n+1 \right. = n\left\lceil n \right.$

$$\left\lceil n+1 \right. = n\left\lceil n-1+1 \right.$$

$$\left\lceil n+1 \right. = n(n-1)\left\lceil n-1 \right.$$

Upto so on

Proof: (ii) By the definition of Gamma function

$$\Gamma n = \int_0^\infty e^{-x}.\, x^{n-1}dx \quad ; \quad n > 0 \qquad \_\_\_\_\_(i)$$

Let $x = zy$

$dx = z \,.\, dy$

$y \to 0$ as $x \to 0$, $y \to \infty$ as $x \to \infty$

Eq (i) $\Rightarrow$

$$\Gamma n = \int_0^\infty e^{-zy}.\,(zy)^{n-1}\, zdy$$

$$\Gamma n = \int_0^\infty e^{-zy}.\, z^{n-1}.\mathrm{y}^{n-1}\, zdy$$

$$\Gamma n = z^n \int_0^\infty e^{-zy}.\, \mathrm{y}^{n-1}\, dy$$

Replace 'y' by 'x'

$$\Gamma n = z^n \int_0^\infty e^{-zx}.\, x^{n-1}dx$$

Proof: (iii) By the definition of Gamma function

$$\Gamma n = \int_0^\infty e^{-x}.\, x^{n-1}dx \quad ; \quad n > 0$$

$$Let \quad x = \log\frac{1}{y}$$

$$\Rightarrow \quad e^x = \frac{1}{y}$$

$$y = \frac{1}{e^x} = e^{-x}$$

$$dy = -e^{-x}dx$$

$$-dy = e^{-x}dx$$

$$y \to 1 \;\; as \;\; x \to 0$$

$$y \to 0 \;\; as \;\; x \to \infty$$

$$\Gamma n = \int_1^0 \left(\log \frac{1}{y} \right)^{n-1} (-dy)$$

$$\Gamma n = \int_0^1 \left(\log \frac{1}{y} \right)^{n-1} dy$$

Proof: (iv) By the definition of Gamma function

$$\Gamma n = \int_0^\infty e^{-x}.\, x^{n-1} dx \quad ; \;\; n > 0$$

$$Let \quad x = y^{\frac{1}{n}}$$

$$\Rightarrow \quad x^n = y$$

$$nx^{n-1}dx = dy$$

$$x^{n-1}dx = \frac{dy}{n}$$

$$y \to 0 \;\; as \;\; x \to 0$$

$$y \to \infty \;\; as \;\; x \to \infty$$

$$\Gamma n = \int_0^\infty e^{-y^{\frac{1}{n}}} \frac{dy}{n}$$

$$n\Gamma n = \int_0^\infty e^{-y^{\frac{1}{n}}} dy$$

$$n\Gamma(n+1) = \int_0^\infty e^{-y^{\frac{1}{n}}} dy \qquad \because n\Gamma n = \Gamma(n+1)$$

Question: Evaluate $\int_0^\infty e^{-x^2}dx$

Solution: $$\int_0^\infty e^{-x^2}dx \qquad \_\_\_\_(i)$$

By the definition of Gamma function

$$\overline{|n} = \int_0^\infty e^{-x}.\, x^{n-1}dx \quad ; \quad n > 0$$

$$Let \quad y = x^2 \qquad \Rightarrow \quad x = \sqrt{y}$$

$$dy = 2xdx$$

$$dy = 2\sqrt{y}dx$$

$$\frac{dy}{2\sqrt{y}} = dx$$

$$y \to 0 \;\; as \;\; x \to 0$$

$$y \to \infty \;\; as \;\; x \to \infty$$

Put in (i) $$\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y}\frac{dy}{2\sqrt{y}}$$

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-y}.y^{\frac{-1}{2}}dy$$

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-y}.y^{\frac{1}{2}-1}dy$$

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\overline{\left|\frac{1}{2}\right.} \qquad \because \overline{|n} = \int_0^\infty e^{-x}.\, x^{n-1}dx$$

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \qquad \because \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$$

Question: Evaluate $\int_0^\infty e^{-4x} x^{\frac{5}{2}} dx$

Solution: $$\int_0^\infty e^{-4x} x^{\frac{5}{2}} dx \qquad \_\_\_\_(i)$$

By the definition of Gamma function

$$\overline{|n} = \int_0^\infty e^{-x}.\, x^{n-1} dx \quad ; \quad n > 0$$

$$Let \quad y = 4x \qquad \Rightarrow \quad x = \frac{y}{4}$$

$$dy = 4dx$$

$$\frac{dy}{4} = dx$$

$$y \to 0 \ \ as \ \ x \to 0$$

$$y \to \infty \ \ as \ \ x \to \infty$$

Put in (i) $$\int_0^\infty e^{-4x} x^{\frac{5}{2}} dx = \int_0^\infty e^{-y}.\left(\frac{y}{4}\right)^{5/2} \frac{dy}{4}$$

$$\int_0^\infty e^{-4x} x^{\frac{5}{2}} dx = \int_0^\infty e^{-y}.\frac{y^{5/2}}{4^{5/2}} \frac{dy}{4}$$

$$\int_0^\infty e^{-4x} x^{\frac{5}{2}} dx = \frac{1}{4^{7/2}} \int_0^\infty e^{-y}.y^{5/2} dy$$

$$\int_0^\infty e^{-4x} x^{\frac{5}{2}} dx = \frac{1}{128} \int_0^\infty e^{-y}.y^{7/2-1} dy$$

$$\int_0^\infty e^{-4x} x^{\frac{5}{2}} dx = \frac{1}{128} \overline{\left|\frac{7}{2}\right.} = \frac{1}{128} \overline{\left|\frac{5}{2}+1\right.}$$

$$\int_0^\infty e^{-4x}x^{\frac{5}{2}}dx = \frac{1}{128}.\frac{5}{2}\Gamma\left(\frac{5}{2}\right) \qquad \because \Gamma(n+1) = n\Gamma(n)$$

$$\int_0^\infty e^{-4x}x^{\frac{5}{2}}dx = \frac{5}{256}\Gamma\left(\frac{3}{2}+1\right)$$

$$\int_0^\infty e^{-4x}x^{\frac{5}{2}}dx = \frac{5}{256}.\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{15}{512}\Gamma\left(\frac{1}{2}+1\right)$$

$$\int_0^\infty e^{-4x}x^{\frac{5}{2}}dx = \frac{15}{512}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\int_0^\infty e^{-4x}x^{\frac{5}{2}}dx = \frac{15}{1024}\sqrt{\pi}$$

Question: Evaluate $\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx$

Solution:

$$\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx \qquad \_\_\_\_(i)$$

By the definition of Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x}.\, x^{n-1}dx \quad ; \; n > 0$$

$$Let \quad y = 3\sqrt{x} \quad \Rightarrow \quad \sqrt{x} = \frac{y}{3}$$

$$\frac{1}{2}x^{\frac{-1}{2}}dx = \frac{dy}{3}$$

$$dx = \frac{2}{3}\sqrt{x}\frac{dy}{3} = \frac{2}{3}.\frac{y}{3}dy$$

$$y \to 0 \;\; as \;\; x \to 0$$

$$y \to \infty \;\; as \;\; x \to \infty$$

Put in (i) $$\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx = \int_0^\infty \frac{y}{3}.e^{-y}\frac{2y}{9}dy$$

$$\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx = \frac{2}{27}\int_0^\infty e^{-y}y^2dy$$

$$\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx = \frac{2}{27}\int_0^\infty e^{-y}y^{3-1}dy = \frac{2}{27}.\overline{|3}$$

$$\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx = \frac{2}{27}.\overline{|2+1}$$

$$\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx = \frac{2}{27}.2\overline{|2}$$

$$\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx = \frac{4}{27}.1 \qquad \because \overline{|2} = 1$$

$$\int_0^\infty \sqrt{x}.e^{-3\sqrt{x}}dx = \frac{4}{27}$$

Beta function:

The Beta function denoted by $\beta(m,\mathrm{n})$ where m,n > 0 and is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$$

Some properties of Beta function:

(i) $\beta(m,n) = \beta(n,m)$ (symmetric property)

(ii) $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}}dx$

(iii) $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$

Important calculus property $\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof: (i) $L.H.S = \beta(m,n)$

$$= \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}.(1-(1-x))^{n-1}\,dx \qquad \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 (1-x)^{m-1}.(1-1+x)^{n-1}\,dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$\beta(m,n) = \beta(n,m)$$

Proof: (ii) By the definition of Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx \quad \_\_\_(i)$$

$$Let \quad x = \frac{1}{1+y} \quad \Rightarrow dx = \frac{-1}{(1+y)^2}dy$$

$$1+y = \frac{1}{x}$$

$$y \to \infty \;\; as \;\; x \to 0$$

$$y \to 0 \;\; as \;\; x \to 1$$

Put all values in (i)

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}.\left(1-\frac{1}{1+y}\right)^{n-1}.\frac{-1}{(1+y)^2}dy$$

$$\beta(m,n)=\int_0^\infty \frac{1}{(1+y)^{m-1}}\cdot\left(\frac{1+y-1}{1+y}\right)^{n-1}\cdot\frac{1}{(1+y)^2}dy$$

$$\beta(m,n)=\int_0^\infty \frac{1}{(1+y)^{m-1}}\cdot\left(\frac{y}{1+y}\right)^{n-1}\cdot\frac{1}{(1+y)^2}dy$$

$$\beta(m,n)=\int_0^\infty \frac{1}{(1+y)^{m-1}}\cdot\frac{y^{n-1}}{(1+y)^{n-1}}\cdot\frac{1}{(1+y)^2}dy$$

$$\beta(m,n)=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m-1+n-1+2}}dy$$

$$\beta(m,n)=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

Replace y by x

$$\beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$\because \beta(m,n)=\beta(n,m)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}}dx$$

$$\because \beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx=\int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}}dx$$

Proof: (iii) By definition of Beta function

$$\beta(m,n)=\int_0^1 x^{m-1}.(1-x)^{n-1}\,dx \quad \_\_\_(i)$$

$$Let \quad x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta d\theta$$

$$\theta\to 0 \ as \ x\to 0$$

$$\theta\to\frac{\pi}{2} \ as \ x\to 1$$

Put these values in (i)

$$\beta(m,n)=\int_0^{\frac{\pi}{2}}\left(\sin^2\theta\right)^{m-1}.\left(1-\sin^2\theta\right)^{n-1}2\sin\theta\cos\theta d\theta$$

$$\beta(m,n)=\int_0^{\frac{\pi}{2}}\left(\sin\theta\right)^{2m-2}.\left(\cos^2\theta\right)^{n-1}2\sin\theta\cos\theta d\theta$$

$$\beta(m,n)=2\int_0^{\frac{\pi}{2}}\left(\sin\theta\right)^{2m-2+1}.\left(\cos\theta\right)^{2n-2+1}d\theta$$

$$\beta(m,n)=2\int_0^{\frac{\pi}{2}}\left(\sin\theta\right)^{2m-1}.\left(\cos\theta\right)^{2n-1}d\theta$$

Relation between Beta and Gamma Function:

$$\beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma m+n}\quad ; m,n>0$$

As we know that

$$\beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma m+n}\quad ; m,n>0\qquad \_\_\_\_(i)$$

Also $$\beta(m,n)=2\int_0^{\frac{\pi}{2}}\left(\sin\theta\right)^{2m-1}.\left(\cos\theta\right)^{2n-1}d\theta\qquad \_\_\_\_(ii)$$

From (i) and (ii)

$$\frac{\Gamma m\,\Gamma n}{\Gamma m+n}=2\int_0^{\frac{\pi}{2}}\left(\sin\theta\right)^{2m-1}.\left(\cos\theta\right)^{2n-1}d\theta$$

Put in m = 1/2 and n = 1/2 in above, we get

$$\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2\left(\frac{1}{2}\right)-1}.(\cos\theta)^{2\left(\frac{1}{2}\right)-1}\,d\theta$$

$$\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{2}{2}}=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{1-1}.(\cos\theta)^{1-1}\,d\theta=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{0}.(\cos\theta)^{0}\,d\theta$$

$$\frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1}=2\int_0^{\frac{\pi}{2}}1\,d\theta$$

$$\left(\Gamma\frac{1}{2}\right)^2=2\left|\theta\right|_0^{\frac{\pi}{2}}=2\left(\frac{\pi}{2}-0\right)$$

$$\left(\Gamma\frac{1}{2}\right)^2=\pi$$

$$\Gamma\frac{1}{2}=\sqrt{\pi}$$

Theorem: Prove that $\beta(m,n)=\beta(m,n+1)+\beta(m+1,n)$

Proof: R.H.S = $\beta(m,n+1)+\beta(m+1,n)$

$$=\frac{\Gamma m\,\Gamma(n+1)}{\Gamma(m+n+1)}+\frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)}$$

$$=\frac{\Gamma m.n\Gamma n}{(m+n)\Gamma(m+n)}+\frac{m\Gamma m.\Gamma n}{(m+n)\Gamma(m+n)}$$

$$=\frac{\Gamma m.n\Gamma n+m\Gamma m.\Gamma n}{(m+n)\Gamma(m+n)}=\frac{\Gamma m\Gamma n(n+m)}{(m+n)\Gamma(m+n)}$$

$$\beta(m,n+1)+\beta(m+1,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}=\beta(m,n)=L.H.S$$

Theorem: Prove that $\dfrac{\beta(m+1,n)}{m}=\dfrac{\beta(m,n+1)}{n}=\dfrac{\beta(m,n)}{m+n}$

Proof: Consider

$$\frac{\beta(m+1,n)}{m}=\frac{1}{m}\cdot\frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)}=\frac{1}{m}\cdot\frac{m\,\Gamma m\,\Gamma n}{(m+n)\,\Gamma(m+n)}$$

$$\frac{\beta(m+1,n)}{m}=\frac{\Gamma m\,\Gamma n}{(m+n)\,\Gamma(m+n)}=\frac{\beta(m,n)}{m+n}\qquad \_\_\_\_(i)$$

Again consider

$$\frac{\beta(m,n+1)}{n}=\frac{1}{n}\cdot\frac{\Gamma m\,\Gamma(n+1)}{\Gamma(m+n+1)}=\frac{1}{n}\cdot\frac{\Gamma m\; n\,\Gamma n}{(m+n)\,\Gamma(m+n)}$$

$$\frac{\beta(m,n+1)}{n}=\frac{\Gamma m\,\Gamma n}{(m+n)\,\Gamma(m+n)}=\frac{\beta(m,n)}{m+n}\qquad \_\_\_\_(ii)$$

From (i) and (ii)

$$\frac{\beta(m+1,n)}{m}=\frac{\beta(m,n+1)}{n}=\frac{\beta(m,n)}{m+n}$$

Lecture # 02

Relation between Beta & Gamma function:

Theorem: Prove that $\beta(m,n)=\dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$; $m,n>0$

Proof: By the definition of Gamma function

$$\Gamma n=\int_0^\infty e^{-x}.\,x^{n-1}dx \quad ; \quad n>0$$

Also
$$\Gamma n=z^n\int_0^\infty e^{-zx}.x^{n-1}dx \quad ; n,z>0 \qquad \_\_\_(i)$$

$$\frac{\Gamma n}{z^n}=\int_0^\infty e^{-zx}.x^{n-1}dx \quad ; n,z>0 \qquad \_\_\_(ii)$$

Multiplying (i) by e^{-z} and z^{m-1} then integrate with respect to z from $0\to\infty$

$$\Gamma n\int_0^\infty e^{-z}.\,z^{m-1}dz=\int_0^\infty e^{-z}.\,z^{m-1}\left\{z^n\int_0^\infty e^{-zx}.x^{n-1}dx\right\}dz$$

$$\Gamma n.\Gamma m=\int_0^\infty\int_0^\infty e^{-z(1+x)}.\,z^{m+n-1}.x^{n-1}dxdz$$

By changing the order of the integration

$$\Gamma n.\Gamma m=\int_0^\infty x^{n-1}\left\{\int_0^\infty e^{-z(1+x)}.\,z^{m+n-1}\;dz\right\}dx$$

$$\Gamma n.\Gamma m=\int_0^\infty x^{n-1}\frac{\Gamma(m+n)}{(1+x)^{m+n}}dx \quad \because \textit{from } (ii)$$

$$\Gamma n.\Gamma m=\Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$\frac{\Gamma n \,\Gamma m}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Since by the property of Beta function $\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \beta(m,n)$ *proved*

Duplication formula:

Prove that $\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}.\Gamma 2m \quad ; m > 0$

Proof: By the relation between Beta and Gamma function

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \quad \text{\_\_\_\_\_}(i)$$

Also, by the property of Beta function.

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}.(\cos\theta)^{2n-1}\,d\theta \quad \text{\_\_\_\_}(ii)$$

Compare (i) and (ii)

$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}.(\cos\theta)^{2n-1}\,d\theta = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \quad \text{\_\_\_\_\_}(iii)$$

Put n = 1/2 $\Rightarrow$ 2n–1= 0 in (iii)

$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}.(\cos\theta)^{0}\,d\theta = \frac{\Gamma m\,\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}\,d\theta = \frac{\Gamma m\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \quad \text{\_\_\_\_\_}(iv) \quad \because \Gamma\frac{1}{2} = \sqrt{\pi}$$

Put n = m in (iii) $\Rightarrow \quad 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}.(\cos\theta)^{2m-1}\, d\theta = \dfrac{\Gamma m\,\Gamma m}{\Gamma m+m}$

$$2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1}\, d\theta = \frac{\left(\Gamma m\right)^2}{\Gamma 2m}$$

$$2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta = \frac{\left(\Gamma m\right)^2}{\Gamma 2m}$$

$$\frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1}\, d\theta = \frac{\left(\Gamma m\right)^2}{\Gamma 2m}$$

Put 2θ = t $\Rightarrow$ 2dθ = dt

$$t \to 0 \ as \ \theta \to 0$$

$$t \to \pi \ as \ \theta \to \frac{\pi}{2}$$

Now $\quad \dfrac{2}{2^{2m-1}}\displaystyle\int_0^{\pi} (\sin t)^{2m-1} \frac{dt}{2} = \frac{\left(\Gamma m\right)^2}{\Gamma 2m}$

$$\frac{1}{2^{2m-1}}\int_0^{\pi} (\sin\theta)^{2m-1}\, d\theta = \frac{\left(\Gamma m\right)^2}{\Gamma 2m} \qquad \because \int_a^b f(t)\,dt = \int_a^b f(\theta)\,d\theta$$

$$\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad if \ \ f(2a-x) = f(x)$$

$$\frac{1}{2^{2m-1}}.2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}\, d\theta = \frac{\left(\Gamma m\right)^2}{\Gamma 2m}$$

$$\frac{1}{2^{2m-1}}.\frac{\Gamma m\,\sqrt{\pi}}{\Gamma m+\frac{1}{2}} = \frac{\left(\Gamma m\right)^2}{\Gamma 2m} \qquad \because \ by \ (iv)$$

$$\frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}} = \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)$$

Hypergeometric function:

Learning objectives:

(i) To solve Hypergeometric equation and obtain its solution.
(ii) Differentiation of Hypergeometric function.
(iii) Integral representation of Hypergeometric function.
(iv) Some applications of Hypergeometric function.

Historical Background of Hypergeometric Function:

The term Hypergeometric function was introduced by John Wallis in his book 1655, which also appear in the work of Euler, Gauss, Riemann and Kummer. Mellin-Barnes studied their integral representations and some other special properties, was discuss by Schwarz and Govrsat.

However, the first in depth treatment was executed by Gauss in 1813, where he presented most of the properties of Hypergeometric functions that we se today. Hypergeometric function has motivated the development of several domains such as complex functions, Riemann surfaces, differential equations and so forth. Here we concentrated on the Hypergeometric function of single variable.

To understand the Hypergeometric function, we must know about the Pochhammer symbol.

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2).....(\alpha+n-1)$$

Where $\alpha > 0$ and 'n' is a non-negative integer.

Note: Pochhammer symbol is actually the generalization of factorial.

We can also write the Pochhammer symbol in the form of gamma function

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad ; \quad \alpha > 0\ ,\ n \geq 0$$

Proof: By the definition of Pochhammer symbol

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2).....(\alpha+n-1)$$

$$(\alpha)_n = \frac{\alpha(\alpha+1)(\alpha+2).....(\alpha+n-1)}{(\alpha-1)(\alpha-2).....3.2.1}.(\alpha-1)(\alpha-2).....3.2.1$$

$$(\alpha)_n = \frac{(\alpha+n-1)(\alpha+n-2)....\alpha(\alpha-1)(\alpha-2).....3.2.1}{(\alpha-1)!}$$

$$(\alpha)_n = \frac{(\alpha+n-1)!}{(\alpha-1)!}$$

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \qquad \because \Gamma(n+1) = n!$$

Note: $(\alpha)_0 = 1 \ \because (\alpha)_0 = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$

Hypergeometric function of one variable:

Hypergeometric function of one variable can be define as

$${}_2F_1(\alpha,\beta,\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$= 1 + \frac{\alpha.\beta}{\gamma}x + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{x^3}{3!} + ...$$

Remark: If α = 1 and β = γ the above expression become the geometric series

$$F(1,\gamma,\gamma;x) = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

The Hypergeometric equation is given by

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0 \qquad \_\_\_(i)$$

This equation was studied in detail by Gauss in connection with this theory of the Hypergeometric series but it was Euler who had worked with this equation and its solution at an earlier date. Using Frobenius method we solve this Hypergeometric differential equation.

Lecture # 03

Hypergeometric equation:

The Hypergeometric equation is given by

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{dy}{dx}-\alpha\beta y=0 \qquad \_\_\_(i)$$

Using Frobenius method, we now solve the Hypergeometric equation.

Eq(i) can be written as

$$x(1-x)y''+\{\gamma-(\alpha+\beta+1)x\}y'-\alpha\beta y=0 \qquad \_\_\_(ii)$$

$$Also \quad y''+p(x)y'-Q(x)y=0 \qquad \_\_\_(iii)$$

Where $P(x)=\dfrac{\gamma-(\alpha+\beta+1)x}{x(1-x)}, Q(x)=\dfrac{-\alpha\beta}{x(1-x)}$

We observe that xP(x) and x^2Q(x) are both analytic x = 0, implying that x = 0 is the regular singular point of (iii).

Therefore, we can apply Frobenius method

Let $$y=\sum_{m=0}^{\infty}C_m x^{k+m}, C_0\neq 0 \qquad \_\_\_(iv)$$

$$y'=\sum_{m=0}^{\infty}C_m(k+m)x^{k+m-1}$$

$$y''=\sum_{m=0}^{\infty}C_m(k+m-1)(k+m)x^{k+m-2}$$

Put all of these value in (ii)

(ii) $\Rightarrow$

$$(x-x^2)\sum_{m=0}^{\infty}C_m(k+m-1)(k+m)x^{k+m-2}+\{\gamma-(\alpha+\beta+1)x\}\sum_{m=0}^{\infty}C_m(k+m)x^{k+m-1}$$

$$-\alpha\beta\sum_{m=0}^{\infty}C_m x^{k+m}=0$$

$$\sum_{m=0}^{\infty} C_m (k+m-1)(k+m) x^{k+m-1} + \sum_{m=0}^{\infty} C_m (k+m-1)(k+m) x^{k+m} + \gamma \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1}$$

$$-(\alpha+\beta+1) \sum_{m=0}^{\infty} C_m (k+m) x^{k+m} \quad - \alpha\beta \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

$$\sum_{m=0}^{\infty} C_m (k+m-1+\gamma)(k+m) x^{k+m-1} - \sum_{m=0}^{\infty} C_m \{(k+m-1)(k+m) + (\alpha+\beta+1)(k+m) + \alpha\beta\} x^{k+m} = 0$$

____(v)

$$Now (k+m)(k+m-1) + (\alpha+\beta+1)(k+m) + \alpha\beta = (k+m)^2 - (k+m) + (\alpha+\beta+1)(k+m) + \alpha\beta$$

$$= (k+m)^2 + (k+m)(\alpha+\beta+1-1) + \alpha\beta$$

$$= (k+m)^2 + (k+m)(\alpha+\beta) + \alpha\beta$$

$$= (k+m)^2 + \alpha(k+m) + \beta(k+m) + \alpha\beta$$

$$= (k+m)(k+m+\alpha) + \beta(k+m+\alpha)$$

$$= (k+m+\alpha)(k+m+\beta)$$

Put in (v)

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1+\gamma) x^{k+m-1} - \sum_{m=0}^{\infty} C_m \{(k+m+\alpha)(k+m+\beta)\} x^{k+m} = 0 \quad \_\_(vi)$$

Now we equate to zero the coefficient of the smallest power of x, namely x^{k-1} to get the indicial equation as

$$C_0 \,.\, k(k+\gamma-1) = 0 \quad \because when\, we\, put\, m = 0\, in (vi)$$

$$C_0 \neq 0\,, \quad k(k+\gamma-1) = 0$$

$$k = 0 \;\; \& \; (k+\gamma-1) = 0$$

$$k = 1-\gamma$$

We equate to zero the coefficients of x^{k+m-1} (for the recurrence relation)

$$C_m(k+m)(k+m-1+\gamma)-C_{m-1}\{(k+m-1+\alpha)(k+m-1+\beta)\}=0$$

$$C_m(k+m)(k+m-1+\gamma)=C_{m-1}\{(k+m-1+\alpha)(k+m-1+\beta)\}$$

$$C_m=\frac{(k+m-1+\alpha)(k+m-1+\beta)}{(k+m)(k+m-1+\gamma)}C_{m-1} \quad \_\_\_\_(vii)$$

Case-I: For k = 0 and substituting m = 1,2,3,4,…

m = 1 $\quad C_1=\frac{\alpha.\beta}{\gamma}C_0$

m = 2 $\quad C_2=\frac{(\alpha+1).(\beta+1)}{2(\gamma+1)}.C_1$

$$C_2=\frac{(\alpha+1).(\beta+1)}{2(\gamma+1)}.\frac{\alpha.\beta}{\gamma}C_0$$

$$C_2=\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{C_0}{2!}$$

m = 3 $\quad C_3=\frac{(\alpha+2).(\beta+2)}{3(\gamma+2)}.C_2$

$$C_3=\frac{(\alpha+2).(\beta+2)}{3(\gamma+2)}.\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{C_0}{2!}$$

$$C_3=\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{C_0}{3!}$$

. .

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Put the values of these coefficient in (iv)

$$\Rightarrow \quad y=\sum_{m=0}^{\infty}C_m x^{k+m}=\sum_{m=0}^{\infty}C_m x^m \qquad \because k=0$$

$$y = C_0 x^0 + C_1 x^1 + C_2 x^2 + \ldots\ldots.$$

$$y = C_0 + \frac{\alpha\beta}{\gamma}C_0 x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{C_0}{2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{C_0}{3!}x^3 + \ldots$$

$$y = C_0\left[1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{x^3}{3!} + \ldots\right]$$

If we put $C_0 = 1$, the series obtained is called Hypergeometric series and is denoted by $F(\alpha, \beta, \gamma; x)$

Case-II: For k = 1−γ

Eq (vii) ⇒ $$C_m = \frac{(1-\gamma+m-1+\alpha)(1-\gamma+m-1+\beta)}{(1-\gamma+m)(1-\gamma+m-1+\gamma)}C_{m-1}$$

$$Let \quad \alpha' = \alpha+1-\gamma \ , \beta' = \beta+1-\gamma$$

$$C_m = \frac{(\alpha'+m-1)(\beta'+m-1)}{(1+1-\gamma+m-1)m}C_{m-1}$$

$$Let \quad \gamma' = 2-\gamma$$

$$C_m = \frac{(\alpha'+m-1)(\beta'+m-1)}{(\gamma'+m-1)m}C_{m-1}$$

$$m = 1 \quad , \quad C_1 = \frac{\alpha'\beta'}{\gamma'}C_0$$

$$m = 2 \quad , \quad C_2 = \frac{(\alpha'+1)(\beta'+1)}{2(\gamma'+1)}C_1$$

$$C_2 = \frac{(\alpha'+1)(\beta'+1)}{2(\gamma'+1)}.\frac{\alpha'\beta'}{\gamma'}C_0$$

$$C_2 = \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{\gamma'(\gamma'+1)}.\frac{C_0}{2!}$$

$$m = 3 \quad , \quad C_3 = \frac{(\alpha'+2)\beta'(\beta'+2)}{3(\gamma'+2)}.C_2$$

$$C_3 = \frac{(\alpha'+2)\beta'(\beta'+2)}{3(\gamma'+2)}.\frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{\gamma'(\gamma'+1)}.\frac{C_0}{2!}$$

$$C_3 = \frac{\alpha'(\alpha'+1)(\alpha'+2).\beta'(\beta'+1)(\beta'+2)}{\gamma'(\gamma'+1)(\gamma'+2)}.\frac{C_0}{3!}$$

. .

. .

. .

Put the values of these coefficients in (iv)

(iv). $\Rightarrow y = \sum_{m=0}^{\infty} C_m x^{k+m} = \sum_{m=0}^{\infty} C_m x^{1-\gamma+m} \qquad \because k = 1-\gamma$

$$y = C_0 x^{1-\gamma} + C_1 x^{1-\gamma+1} + C_2 x^{1-\gamma+2} + ...$$

$$y = x^{1-\gamma}\left[C_0 + C_1 x + C_2 x^2 + ...\right]$$

$$= x^{1-\gamma}\left[C_0 + \frac{\alpha'\beta'}{\gamma'}C_0\, x + \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{\gamma'(\gamma'+1)}.\frac{C_0}{2!}x^2 + \frac{\alpha'(\alpha'+1)(\alpha'+2).\beta'(\beta'+1)(\beta'+2)}{\gamma'(\gamma'+1)(\gamma'+2)}.\frac{C_0}{3!}x^3 + ...\right]$$

$$= C_0 x^{1-\gamma}\left[1 + \frac{\alpha'\beta'}{\gamma'}x + \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{\gamma'(\gamma'+1)}.\frac{x^2}{2!} + \frac{\alpha'(\alpha'+1)(\alpha'+2).\beta'(\beta'+1)(\beta'+2)}{\gamma'(\gamma'+1)(\gamma'+2)}.\frac{x^3}{3!} + ...\right]$$

If we put $C_0 = 1$ the series obtained is called Hypergeometric series and is denoted by $F(\alpha', \beta', \gamma'; x)$. Hence the Hypergeometric equation is

$y = AF(\alpha, \beta, \gamma; x) + BF(\alpha', \beta', \gamma'; x)$

Lecture # 04

Symmetric Property of Hypergeometric Function:

Theorem: Prove that $F(\alpha,\beta,\gamma;x)=F(\beta,\alpha,\gamma;x)$

Proof: As we know that

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\beta)_n(\alpha)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=F(\beta,\alpha,\gamma;x)$$

Differentiation of Hypergeometric function:

Theorem: Prove that

(i) $\frac{d}{dx}F(\alpha,\beta,\gamma;x)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;x)$

(ii) $\frac{d^2}{dx^2}F(\alpha,\beta,\gamma;x)=\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}F(\alpha+2,\beta+2,\gamma+2;x)$

(iii) $\frac{d^3}{dx^3}F(\alpha,\beta,\gamma;x)=\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}F(\alpha+3,\beta+3,\gamma+3;x)$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d^n}{dx^n}F(\alpha,\beta,\gamma;x)=\frac{\alpha(\alpha+1)(\alpha+2)...(\alpha+n-1)\beta(\beta+1)(\beta+2)...(\beta+n-1)}{\gamma(\gamma+1)(\gamma+2)....(\gamma+n-1)}F(\alpha+n,\beta+n,\gamma+n;x)$$

$$=\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+n,\gamma+n;x)$$

Proof:(i) As we know that $F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$

$$=1+\frac{\alpha.\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{x^3}{3!}+\dots$$

Diff. w.r.t 'x'

$$\frac{d}{dx}F(\alpha,\beta,\gamma,x)=0+\frac{\alpha\beta}{\gamma}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{2x}{2}+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{3x^2}{3.2.1}+\dots$$

$$\frac{d}{dx}F(\alpha,\beta,\gamma,x)=\frac{\alpha\beta}{\gamma}\left[1+\frac{(\alpha+1)(\beta+1)}{(\gamma+1)}.x+\frac{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}{(\gamma+1)(\gamma+2)}.\frac{x^2}{2.1}+\dots\right]\text{---}(i)$$

$$\frac{d}{dx}F(\alpha,\beta,\gamma;x)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;x)$$

(ii) Again Diff. eq (i) w.r.t 'x'

$$\frac{d^2}{dx^2}F(\alpha,\beta,\gamma;x)=\frac{\alpha\beta}{\gamma}\left[\frac{(\alpha+1)(\beta+1)}{(\gamma+1)}+\frac{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}{(\gamma+1)(\gamma+2)}.\frac{2x}{2.1}+\dots\right]$$

$$\frac{d^2}{dx^2}F(\alpha,\beta,\gamma;x)=\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\left[1+\frac{(\alpha+2)(\beta+2)}{(\gamma+2)}.x+\dots\right]$$

$$\frac{d^2}{dx^2}F(\alpha,\beta,\gamma;x)=\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}F(\alpha+2,\beta+2,\gamma+2;x)$$

Similarly, in the same way we get the following results

$$\frac{d^3}{dx^3}F(\alpha,\beta,\gamma;x)=\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}F(\alpha+3,\beta+3,\gamma+3;x)$$

. .

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$$\frac{d^n}{dx^n}F(\alpha,\beta,\gamma;x)=\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+n,\gamma+n;x)$$

Integral representation of Hypergeometric function:

Theorem: Prove that $F(\alpha,\beta,\gamma;x)=\dfrac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\displaystyle\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$

Proof: By the definition of Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\cdot\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}(\alpha)_n\frac{\Gamma\beta+n}{\Gamma\beta}\frac{\Gamma\gamma}{\Gamma\gamma+n}\cdot\frac{x^n}{n!}\qquad\because(\alpha)_n=\frac{\Gamma\alpha+n}{\Gamma\alpha}$$

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\beta}\sum_{n=0}^{\infty}(\alpha)_n\frac{\Gamma\beta+n}{\Gamma\gamma+n}\cdot\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\sum_{n=0}^{\infty}(\alpha)_n\frac{\Gamma\beta+n\;\Gamma\gamma-\beta}{\Gamma\gamma+n}\cdot\frac{x^n}{n!}\qquad\because \textit{Multiplying and divide by } \Gamma\gamma-\beta$$

$$\because\ \beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\sum_{n=0}^{\infty}(\alpha)_n B(\beta+n,\gamma-\beta)\cdot\frac{x^n}{n!}\qquad\because B \text{ for } \beta$$

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\sum_{n=0}^{\infty}(\alpha)_n\left[\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}\,dt\right]\cdot\frac{x^n}{n!}$$

$$\because\ \beta(m,\mathrm{n})=\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left\{\sum_{n=0}^{\infty}(\alpha)_n\cdot\frac{(xt)^n}{n!}\right\}dt$$

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma-\beta}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt\qquad\because\sum_{n=0}^{\infty}(\alpha)_n\cdot\frac{(xt)^n}{n!}=(1-xt)^{-\alpha}$$

Gauss Theorem:

Prove that $$F(\alpha,\beta,\gamma;1)=\frac{\Gamma\gamma\ .\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\ \Gamma(\gamma-\beta)}\ ;\gamma>\alpha+\beta>0$$

Proof: By the integral representation of Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$$

Put x = 1 $$F(\alpha,\beta,\gamma;1)=\frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t)^{-\alpha}\,dt$$

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\alpha-\beta-1}\,dt$$

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)}\beta(\beta,\gamma-\alpha-\beta)\quad\because\beta(m,\text{n})=\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma\gamma}{\Gamma\beta\ \Gamma(\gamma-\beta)}.\frac{\Gamma\beta\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)}\quad\because\beta(m,\text{n})=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$$

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma\gamma\ .\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\ \Gamma(\gamma-\beta)}$$

Vandermonde's Theorem:

Prove that $$F(-\text{n},\beta,\gamma;1)=\frac{(\gamma-\beta)_n}{(\gamma)_n}$$

Proof: By the Gauss theorem

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma\gamma\ .\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\ \Gamma(\gamma-\beta)}\qquad\_\_\_\_(i)$$

Put $\alpha=-\text{n}$ in (i) $$F(-\text{n},\beta,\gamma;1)=\frac{\Gamma\gamma\ .\Gamma(\gamma-\beta+n)}{\Gamma(\gamma+n)\ \Gamma(\gamma-\beta)}$$

$$F(-\text{n},\beta,\gamma;1)=\frac{\Gamma\gamma\ .\Gamma\left((\gamma-\beta+n-1)+1\right)}{\Gamma\left((\gamma+n-1)+1\right)\ \Gamma(\gamma-\beta)}$$

$$F(-\text{n},\beta,\gamma;1)=\frac{\Gamma\gamma\ .(\gamma-\beta+n-1)\Gamma(\gamma-\beta+n-1)}{(\gamma+n-1)\Gamma(\gamma+n-1)\ \Gamma(\gamma-\beta)}\qquad \because \Gamma(n+1)=n\Gamma n$$

$$=\frac{\Gamma\gamma\ .(\gamma-\beta+n-1)(\gamma-\beta+n-2)(\gamma-\beta+n-3)....(\gamma-\beta+2)(\gamma-\beta+1)(\gamma-\beta)\Gamma(\gamma-\beta)}{(\gamma+n-1)(\gamma+n-2)(\gamma+n-3)....(\gamma+2)(\gamma+1)\gamma\Gamma\gamma\Gamma(\gamma-\beta)}$$

$$=\frac{(\alpha-\beta)(\gamma-\beta+1)...(\gamma-\beta+n-1)}{\gamma(\gamma+1)(\gamma+2)....(\gamma+n-1)}$$

$$F(-\text{n},\beta,\gamma;1)=\frac{(\gamma-\beta)_n}{(\gamma)_n}$$

Question: Show that $F(1,1,1;x)=F(1,b,b;x)=F(a,1,a;x)=\dfrac{1}{1-x}$

Solution: By the definition of Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=1+\frac{\alpha.\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!}+...\quad \_\_\_\_\_\_(i)$$

Put $\alpha=\beta=\gamma=1$ in (i)

$$F(1,1,1;x)=1+\frac{1.1}{1}\frac{x}{1!}+\frac{1(1+1).1(1+1)}{1(1+1)}.\frac{x^2}{2!}+...$$

$$F(1,1,1;x)=1+x+x^2+x^3+...\quad \_\_\_\_\_(ii)$$

Put $\alpha=1$, $\beta=b$, $\gamma=b$ in (i)

$$F(1,\text{b},\text{b};x)=1+\frac{1.\text{b}}{b}\frac{x}{1!}+\frac{1(1+1).\text{b}(b+1)}{b(b+1)}.\frac{x^2}{2!}+...$$

$$F(1,\text{b},\text{b};x)=1+x+2.\frac{x^2}{2!}+...$$

$$F(1,\text{b},\text{b};x)=1+x+x^2+x^3...\ \_\_\_(iii)$$

Put α = a , β = 1 , γ = a in (i)

$$F(a,1,a;x)=1+\frac{a.1}{a}\frac{x}{1!}+\frac{a(a+1).1(1+1)}{a(a+1)}.\frac{x^2}{2!}+...$$

$$F(a,1,a;x)=1+x+2.\frac{x^2}{2!}+...$$

$$F(a,1,a;x)=1+x+x^2+x^3...\ \_\_\_(iv)$$

$$\text{Also } \frac{1}{1-x}=1+x+x^2+x^3...\ \_\_\_(v)$$

From (ii),(iii),(iv) and (v)

$$F(1,1,1;x)=F(1,b,b;x)=F(a,1,a;x)=\frac{1}{1-x}$$

Question: Show that $F(-\text{n},1,1;-x)=(1+x)^n$

Solution: By the definition of Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=1+\frac{\alpha.\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!}+...$$

Put α = –n , β = 1 , γ = 1 , x = –x in (i)

$$F(-\text{n},1,1;-x)=1+\frac{(-n).1}{1}\frac{(-x)}{1!}+\frac{(-n)((-n)+1).1(1+1)}{1(1+1)}.\frac{(-x)^2}{2!}+...$$

$$F(-\text{n},1,1;-x)=1+nx+\frac{n(n-1)}{2!}x^2+......\ \_\_\_\_(i)$$

$$Also \quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... \quad \_\_\_\_(ii)$$

From (i) and (ii)

$$F(-\text{n},1,1;-x) = (1+x)^n$$

Question: Show that $F(1,1,2;-x) = \dfrac{\ln(1+x)}{x}$

Solution: By the definition of Hypergeometric function

$$F(\alpha,\beta,\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x) = 1 + \frac{\alpha.\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)} \cdot \frac{x^2}{2!} + ...$$

Put α = 1 , β = 1 , γ = 2 , x = –x

$$F(1,1,2;-x) = 1 + \frac{1.1}{2}(-x) + \frac{1(1+1).1(1+1)}{2(2+1)} \cdot \frac{(-x)^2}{2!} + ...$$

$$F(1,1,2;-x) = 1 - \frac{x}{2} + \frac{2}{3}\frac{x^2}{2!} + ...$$

$$F(1,1,2;-x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} ...... \quad \_\_\_(i)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + ... \quad \_\_\_\_(ii)$$

From (i) and (ii)

$$F(1,1,2;-x) = \frac{\ln(1+x)}{x}$$

Question: Show that $F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right)=\frac{\sin^{-1}x}{x}$

Solution: By the definition of Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=1+\frac{\alpha.\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!}+\dots$$

Put $\alpha=\beta=1/2$, $\gamma=3/2$, $x=x^2$

$$F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right)=1+\frac{\frac{1}{2}.\frac{1}{2}}{\frac{3}{2}}.x^2+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right).\frac{1}{2}\left(\frac{1}{2}+1\right)}{\frac{3}{2}\left(\frac{3}{2}+1\right)}.\frac{x^4}{2!}+\dots$$

$$F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right)=1+\frac{1}{2.3}.x^2+\frac{\frac{1}{2}.\frac{3}{2}.\frac{1}{2}.\frac{3}{2}}{\frac{3}{2}.\frac{5}{2}}.\frac{x^4}{2!}+\dots$$

$$F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right)=1+\frac{x^2}{2.3}+\frac{1.3}{2.4.5}.x^4+\dots \quad \_\_\_\_(i)$$

Now
$$\sin^{-1}x=x+\frac{1}{2}.\frac{x^3}{3}+\frac{1}{2}.\frac{3}{4}.\frac{x^5}{5}+\dots$$

$$\frac{\sin^{-1}x}{x}=1+\frac{x^2}{2.3}+\frac{1.3}{2.4.5}.x^4+\dots \quad \_\_\_\_(ii)$$

From (i) and (ii)

$$F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right)=\frac{\sin^{-1}x}{x}$$

Question: Show that $F\left(\frac{1}{2},1,\frac{3}{2};-x^2\right)=\frac{\tan^{-1}x}{x}$

Solution: By the definition of Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=1+\frac{\alpha.\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!}+\ldots$$

Put α = 1/2 , β = 1 , γ = 3/2 , x = $-x^2$

$$F\left(\frac{1}{2},1,\frac{3}{2};-x^2\right)=1+\frac{\frac{1}{2}.1}{\frac{3}{2}}.\left(-x^2\right)+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right).1(1+1)}{\frac{3}{2}\left(\frac{3}{2}+1\right)}.\frac{\left(-x^2\right)^2}{2!}+\ldots$$

$$F\left(\frac{1}{2},1,\frac{3}{2};-x^2\right)=1-\frac{x^2}{3}+\frac{\frac{1}{2}.\frac{3}{2}.2}{\frac{3}{2}.\frac{5}{2}}.\frac{x^4}{2}+\ldots$$

$$F\left(\frac{1}{2},1,\frac{3}{2};-x^2\right)=1-\frac{x^2}{3}+\frac{x^4}{5}+\ldots \quad \_\_\_\_(i)$$

Now
$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}\ldots$$

$$\frac{\tan^{-1}x}{x}=1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}\ldots \quad \_\_\_\_(ii)$$

From (i) and (ii)

$$F\left(\frac{1}{2},1,\frac{3}{2};-x^2\right)=\frac{\tan^{-1}x}{x}$$

Question: Show that $F\left(\frac{1}{2},1,\frac{3}{2};x^2\right)=\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right)$

Solution: By the definition of Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x)=1+\frac{\alpha.\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!}+\ldots$$

Put $\alpha = 1/2$, $\beta = 1$, $\gamma = 3/2$, $x = x^2$

$$F\left(\frac{1}{2},1,\frac{3}{2};-x^2\right)=1+\frac{\frac{1}{2}.1}{\frac{3}{2}}.\left(x^2\right)+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right).1(1+1)}{\frac{3}{2}\left(\frac{3}{2}+1\right)}.\frac{\left(x^2\right)^2}{2!}+\ldots$$

$$F\left(\frac{1}{2},1,\frac{3}{2};x^2\right)=1+\frac{x^2}{3}+\frac{\frac{1}{2}.\frac{3}{2}.2}{\frac{3}{2}.\frac{5}{2}}.\frac{x^4}{2}+\ldots$$

$$F\left(\frac{1}{2},1,\frac{3}{2};x^2\right)=1+\frac{x^2}{3}+\frac{x^4}{5}+\frac{x^6}{7}+\ldots \quad \_\_\_\_(i)$$

Now by Taylor series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots \quad \_\_\_(ii)$$

$$f(x)=\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)$$

$$f'(x)=\frac{1}{1+x}+\frac{1}{1-x}=(1+x)^{-1}+(1-x)^{-1}$$

$$f''(x)=-(1+x)^{-2}+(1-x)^{-2}$$

$$f'''(x) = 2(1+x)^{-3} + 2(1-x)^{-3}$$

$$\begin{matrix} . & & . \\ . & & . \\ . & & . \end{matrix}$$

Now at x = 0

$$f(0)=0, f'(0)=2, f''(0)=0, f'''(0)=4 \; .....$$

Put in (ii)

$$\ln\left(\frac{1+x}{1-x}\right) = 0 + x(2) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(4) + ...$$

$$\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2x}\left\{2x + \frac{2}{3}x^3 + ...\right\}$$

$$\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) = 1 + \frac{x^2}{3} + \frac{x^4}{5} + ... \quad \_\_\_\_(iii)$$

From (i) and (iii)

$$F\left(\frac{1}{2}, 1, \frac{3}{2}; -x^2\right) = \frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right)$$

Question: Show that $\underset{n\to\infty}{Lim} F\left(1, \text{n}, 1; \frac{x}{n}\right) = e^x$

Solution: By the definition of Hypergeometric function

$$F(\alpha,\beta,\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{x^n}{n!}$$

$$F(\alpha,\beta,\gamma;x) = 1 + \frac{\alpha.\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)} \cdot \frac{x^2}{2!} + ...$$

Put $\alpha = 1$, $\beta = n$, $\gamma = 1$, x = x/n

$$F\left(1,\mathrm{n},1;\frac{x}{n}\right)=1+\frac{1.\mathrm{n}}{1}.\frac{x}{n}+\frac{1(1+1).\mathrm{n}(n+1)}{1(1+1)}.\frac{\left(\frac{x}{n}\right)^2}{2!}+\ldots$$

$$F\left(1,\mathrm{n},1;\frac{x}{n}\right)=1+x+\left(\frac{n+1}{n}\right)\frac{x^2}{2!}+\ldots\ldots$$

$$\lim_{n\to\infty}F\left(1,\mathrm{n},1;\frac{x}{n}\right)=\lim_{n\to\infty}\left\{1+x+\left(\frac{n+1}{n}\right)\frac{x^2}{2!}+\ldots\ldots\right\}$$

$$\lim_{n\to\infty}F\left(1,\mathrm{n},1;\frac{x}{n}\right)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \quad \_\_\_\_(i)$$

$$Now \quad e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \quad \_\_\_\_(ii)$$

From (i) and (ii)

$$\lim_{n\to\infty}F\left(1,\mathrm{n},1;\frac{x}{n}\right)=e^x$$

Exercise: Prove that $F(\alpha,\beta,\beta-\alpha+1;-1)=\dfrac{\Gamma(\beta-\alpha+1)\,.\,\Gamma\left(\frac{\beta}{2}+1\right)}{\Gamma(\beta+1)\,\Gamma\left(\frac{\beta}{2}-\alpha+1\right)}$

Proof: By the integral representation of Hypergeometric function

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$$

$$Put \quad \gamma=\beta-\alpha+1 \quad , \quad x=-1$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(\beta-\alpha+1-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\beta-\alpha+1-\beta-1}(1-(-1)t)^{-\alpha}\,dt$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(1-\alpha)}\int_0^1 t^{\beta-1}(1-t)^{-\alpha}(1+t)^{-\alpha}\,dt$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\overline{|\beta-\alpha+1}}{\overline{|\beta}\overline{|1-\alpha}}\int_0^1 t^{\beta-1}\left((1-t)(1+t)\right)^{-\alpha}dt$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\overline{|\beta-\alpha+1}}{\overline{|\beta}\overline{|1-\alpha}}\int_0^1 t^{\beta-1}\left(1-t^2\right)^{-\alpha}dt$$

$$Put\quad t^2=u\ \Rightarrow t=\sqrt{u}$$

$$2tdt=du\ \Rightarrow dt=\frac{du}{2t}=\frac{du}{2\sqrt{u}}$$

$$u\to 0\ \ at\ \ t\to 0$$

$$u\to 1\ \ at\ \ t\to 1$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\overline{|\beta-\alpha+1}}{\overline{|\beta}\overline{|1-\alpha}}\int_0^1\left(u^{1/2}\right)^{\beta-1}(1-u)^{-\alpha}\frac{1}{2\sqrt{u}}du$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\overline{|\beta-\alpha+1}}{2\overline{|\beta}.\overline{|1-\alpha}}\int_0^1\left(u^{1/2}\right)^{\beta-1-1}(1-u)^{-\alpha}du$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\overline{|\beta-\alpha+1}}{2\overline{|\beta}.\overline{|1-\alpha}}\int_0^1 u^{\frac{\beta-2}{2}}(1-u)^{1-\alpha-1}du$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\overline{|\beta-\alpha+1}}{2\overline{|\beta}.\overline{|1-\alpha}}\int_0^1 u^{\frac{\beta}{2}-1}(1-u)^{1-\alpha-1}du$$

$$\because\ \beta(m,\mathrm{n})=\int_0^1 t^{m-1}(1-t)^{n-1}dt$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\overline{|\beta-\alpha+1}}{2\overline{|\beta}.\overline{|1-\alpha}}.B\left(\frac{\beta}{2},1-\alpha\right)$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\overline{|\beta-\alpha+1}}{2\overline{|\beta}.\overline{|1-\alpha}}.\frac{\overline{|\frac{\beta}{2}}\overline{|1-\alpha}}{\overline{|\frac{\beta}{2}+1-\alpha}}\because\ \beta(m,\mathrm{n})=\frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$$

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\Gamma(\beta-\alpha+1)}{\beta\,\Gamma(\beta)}\cdot\frac{\frac{\beta}{2}\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}+1-\alpha\right)}$$

$\because$ *Multiplying and divide by* β

$$F(\alpha,\beta,\beta-\alpha+1;-1)=\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}\cdot\frac{\Gamma\left(\frac{\beta}{2}+1\right)}{\Gamma\left(\frac{\beta}{2}+1-\alpha\right)}$$

Hence proved.

Lecture # 05

Pochhammer symbol:

The Pochhammer symbol is defined as

$$(\alpha)_n=\alpha(\alpha+1)(\alpha+2).....(\alpha+n-1) \quad ; \; n=1,2,3....$$

Deductions:

Put $\alpha = 1$

$$(1)_n=1.2.3.......n=n!$$

Put n = n+1 $\Rightarrow$ $(\alpha)_{n+1}=\alpha(\alpha+1)(\alpha+2).....(\alpha+n)$

$$(\alpha)_{n+1}=\alpha(\alpha+1)(\alpha+2).....(\alpha+n+1-1)$$

$$(\alpha)_{n+1}=\alpha\{(\alpha+1)(\alpha+2).....(\alpha+1+(n-1))\}$$

$$(\alpha)_{n+1}=\alpha(\alpha+1)_n \quad \_\_\_(i)$$

$$(\alpha+n)(\alpha)_n=\alpha(\alpha+1)(\alpha+2).....(\alpha+n-1)(\alpha+n)$$

$$(\alpha+n)(\alpha)_n=(\alpha)_{n+1} \quad \_\_\_(ii)$$

Compare (2) and (3) $\Rightarrow$ $\alpha(\alpha+1)_n=(\alpha+n)(\alpha)_n$

Question: Show that

$$(\alpha-\beta)F(\alpha,\beta,\gamma;x)=\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x)$$

Solution: Taking R.H.S $\Rightarrow$ $\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x)$

By the definition of Hypergeometric function

$$\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x)=\alpha\sum_{n=0}^{\infty}\frac{(\alpha+1)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}-\beta\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta+1)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x)=\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n\beta(\beta+1)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$\because \alpha(\alpha+1)_n=(\alpha+n)(\alpha)_n$$

$$\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha+n)(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta+n)(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!}\{\alpha+n-\beta-n\}$$

$$\alpha F(\alpha+1,\beta,\gamma;x)-\beta F(\alpha,\beta+1,\gamma;x)=(\alpha-\beta)F(\alpha,\beta,\gamma;x)$$

Confluent Hypergeometric Differential Equation:

The 2nd order homogeneous linear differential equation

$$xy''+(\gamma-x)y'-\alpha y=0 \quad where\ \alpha\ \&\ \gamma\ are\ cons\tan ts$$

is called the Confluent Hypergeometric differential equation. The solution of this equation is called Confluent Hypergeometric function.

Solution of Confluent Hypergeometric function:

We shall solve the Confluent Hypergeometric differential equation

$$xy''+(\gamma-x)y'-\alpha y=0 \quad \_\_\_\_\_(i) \quad \text{by Frobenius method.}$$

Let $\quad y=\sum_{k=0}^{\infty}a_k x^{k+\beta} \quad \_\_\_\_(2)\ ;\ a_0\neq 0$

$$y'=\sum_{k=0}^{\infty}a_k(k+\beta)x^{k+\beta-1} \quad \_\_\_\_(3)$$

$$y''=\sum_{k=0}^{\infty}a_k(k+\beta)(k+\beta-1)x^{k+\beta-2} \quad \_\_\_\_(4)$$

Put the values of (2) , (3) & (4) in (1)

$$x\sum_{k=0}^{\infty}a_k(k+\beta)(k+\beta-1)x^{k+\beta-2}+(\gamma-x)\sum_{k=0}^{\infty}a_k(k+\beta)x^{k+\beta-1}-\alpha\sum_{k=0}^{\infty}a_kx^{k+\beta}=0$$

$$\sum_{k=0}^{\infty}a_k(k+\beta)(k+\beta-1)x^{k+\beta-1}+\gamma\sum_{k=0}^{\infty}a_k(k+\beta)x^{k+\beta-1}-\sum_{k=0}^{\infty}a_k(k+\beta)x^{k+\beta}-\alpha\sum_{k=0}^{\infty}a_kx^{k+\beta}=0$$

$$\sum_{k=0}^{\infty}\{(k+\beta)(k+\beta-1)+\gamma(k+\beta)\}a_k x^{k+\beta-1}-\sum_{k=0}^{\infty}\{(k+\beta)+\alpha\}a_k x^{k+\beta}=0$$

Shifting index 'k' by 'k–1' in the second term of the above equation.

$$\Rightarrow\sum_{k=0}^{\infty}\{(k+\beta)(k+\beta-1)+\gamma(k+\beta)\}a_k x^{k+\beta-1}-\sum_{k=0}^{\infty}\{(k-1+\beta)+\alpha\}a_{k-1}x^{k-1+\beta}=0$$

$$\sum_{k=0}^{\infty}\{(k+\beta)(k+\beta-1)+\gamma(k+\beta)\}a_k x^{k+\beta-1}-\sum_{k=0}^{\infty}\{(k-1+\beta)+\alpha\}a_{k-1}x^{k+\beta-1}=0$$

$$\sum_{k=0}^{\infty}\left[\{(k+\beta)(k+\beta-1)+\gamma(k+\beta)\}a_k-\{(k-1+\beta)+\alpha\}a_{k-1}\right]x^{k+\beta-1}=0 \quad \_\_\_\_(5)$$

The coefficient of the lowest degree $x^{\beta-1}$ is obtained by k = 0 in Eq (5)

$$\{\beta(\beta-1)+\gamma\beta\}a_0+0=0$$

$$\{\beta(\beta-1)+\gamma\beta\}a_0=0 \quad \_\_\_\_(\textit{Indicial equation})$$

$$\{\beta(\beta-1)+\gamma\beta\}=0 \quad , \because a_0\neq 0$$

$$\Rightarrow \beta=0 \quad and \quad \beta=1-\gamma$$

Recurrence relation of Confluent Hypergeometric function:

Equating zero to the coefficients of $x^{k+\beta-1}$ in equation (5)

$$\{(k+\beta)(k+\beta-1)+\gamma(k+\beta)\}a_k-\{(k-1+\beta)+\alpha\}a_{k-1}=0$$

$$\{(k+\beta)(k+\beta-1)+\gamma(k+\beta)\}a_k=\{(k-1+\beta)+\alpha\}a_{k-1}$$

$$a_k=\frac{\{(k-1+\beta)+\alpha\}a_{k-1}}{\{(k+\beta)(k+\beta-1)+\gamma(k+\beta)\}} \quad \_\_\_(6)$$

Case-I: For β = 0 equation (6) becomes

$$a_k=\frac{\{(k-1)+\alpha\}a_{k-1}}{(k)(k-1)+\gamma(k)}$$

For k = 1 $\quad a_1=\frac{(0+\alpha)}{\gamma}.a_0 \quad \Rightarrow \quad a_1=\frac{\alpha}{\gamma}.a_0$

For k = 2 , $$a_2 = \frac{\alpha+1}{2.1+2\gamma}.a_1$$

$$a_2 = \frac{\alpha+1}{2(\gamma+1)}.\frac{\alpha}{\gamma}.a_0 \Rightarrow a_2 = \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}.\frac{a_0}{2!}$$

For k = 3 , $$a_3 = \frac{(2+\alpha)}{3.2+3\gamma}.a_2$$

$$a_3 = \frac{(2+\alpha)}{3.2+3\gamma}.\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}.\frac{a_0}{2!}$$

$$a_3 = \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{a_0}{3!}$$

Similarly, $$a_3 = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\gamma(\gamma+1)(\gamma+2)(\gamma+3)}.\frac{a_0}{4!}$$

. .

. .

. .

$$a_n = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)...(\alpha+n-1)}{\gamma(\gamma+1)(\gamma+2)(\gamma+3)...(\gamma+n-1)}.\frac{a_0}{n!}$$

$$y = \sum_{k=0}^{\infty} a_k x^{k+0}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$y = a_0 + \frac{\alpha}{\gamma}.a_0.x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}.\frac{a_0}{2!}.x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{a_0}{3!}.x^3 + ...$$

$$y = a_0\left\{1 + \frac{\alpha}{\gamma}.x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{x^3}{3!} + ...\right\}$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n}.\frac{x^n}{n!} = AF(\alpha,\gamma\ ;x) \qquad \_\_(7) \quad \because A = a_0$$

If we put $a_0 = 0$ the above series is called Confluent Hypergeometric function.

Case-II

For $\beta = 1-\gamma$ equation (6) becomes $a_k = \frac{\{(k+\beta-1)+\alpha\}}{\{(k+\beta)(k+\beta-1)+\gamma(k+\beta)\}}.a_{k-1}$

$$a_k = \frac{\{k+1-\gamma-1+\alpha\}}{\{(k+1-\gamma)(k+1-\gamma-1)+\gamma(k+1-\gamma)\}}.a_{k-1}$$

$$a_k = \frac{(k-\gamma+\alpha)}{\{(k+1-\gamma)(k-\gamma)+\gamma(k+1-\gamma)\}}.a_{k-1}$$

$$a_k = \frac{(k-\gamma+\alpha)}{(k+1-\gamma)(k)}.a_{k-1}$$

For k = 1 $\quad a_1 = \frac{(1-\gamma+\alpha)}{2-\gamma}.a_0$

For k = 2 $\quad a_2 = \frac{2-\gamma+\alpha}{2(3-\gamma)}.a_1 = \frac{2-\gamma+\alpha}{3-\gamma}.\frac{1-\gamma+\alpha}{2-\gamma}.\frac{a_0}{2!}$

$$a_2 = \frac{(1-\gamma+\alpha)(2-\gamma+\alpha)}{(2-\gamma)(3-\gamma)}.\frac{a_0}{2!}$$

For k = 3 $\quad a_3 = \frac{3-\gamma+\alpha}{3(4-\gamma)}.\frac{a_2}{2!} = \frac{(3-\gamma+\alpha)}{4-\gamma}.\frac{(1-\gamma+\alpha)(2-\gamma+\alpha)}{(2-\gamma)(3-\gamma)}.\frac{a_0}{3!}$

$$a_3 = \frac{(1-\gamma+\alpha)(2-\gamma+\alpha)(3-\gamma+\alpha)}{(2-\gamma)(3-\gamma)(4-\gamma)}.\frac{a_0}{3!}$$

For $\beta = 1-\gamma$ in equation (ii) $\Rightarrow$ $y = \sum_{k=0}^{\infty} a_k x^{k+1-r}$

$$y = a_0x^{1-r} + a_1x^{2-r} + a_2x^{3-r} + \ldots\ldots = x^{1-r}\left[a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\right]$$

Put the above values

$$y = x^{1-r}\left[a_0 + \frac{(1-\gamma+\alpha)}{2-\gamma}.a_0x + \frac{(1-\gamma+\alpha)(2-\gamma+\alpha)}{(2-\gamma)(3-\gamma)}.\frac{a_0}{2!}x^2 + \frac{(1-\gamma+\alpha)(2-\gamma+\alpha)(3-\gamma+\alpha)}{(2-\gamma)(3-\gamma)(4-\gamma)}.\frac{a_0}{3!}x^3 + \ldots\right]$$

$$y=x^{1-r}a_0\left[1+\frac{(1-\gamma+\alpha)}{2-\gamma}x+\frac{(1-\gamma+\alpha)(2-\gamma+\alpha)}{(2-\gamma)(3-\gamma)}.\frac{x^2}{2!}+\frac{(1-\gamma+\alpha)(2-\gamma+\alpha)(3-\gamma+\alpha)}{(2-\gamma)(3-\gamma)(4-\gamma)}.\frac{x^3}{3!}+...\right]$$

$$y=x^{1-r}a_0\sum_{n=0}^{\infty}\frac{(1-\gamma+\alpha)_n}{(2-\gamma)_n}.\frac{x^n}{n!}$$

$$y=B.x^{1-\gamma}F(1-\gamma+\alpha,2-\gamma\ ;x)\ \_\_\_\_(viii)$$

If we put $a_0 = 0$ the above series is called Confluent Hypergeometric function from (vii) and (viii) $y=AF(\alpha,\gamma\ ;x)+B.x^{1-\gamma}F(1-\gamma+\alpha,2-\gamma\ ;x)$ is the general solution of Confluent Hypergeometric differential equation.

Differentiation of Confluent Hypergeometric function:

Theorem: Prove that

(i) $$\frac{d}{dx}F(\alpha,\gamma;x)=\frac{\alpha}{\gamma}F(\alpha+1,\gamma+1;x)$$

(ii) $$\frac{d^2}{dx^2}F(\alpha,\gamma;x)=\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}F(\alpha+2,\gamma+2;x)$$

In general, $$\frac{d^n}{dx^n}F(\alpha,\gamma;x)=\frac{(\alpha)_n}{(\gamma)_n}F(\alpha+n,\gamma+n;x)$$

Proof:(i) As we know that $F(\alpha,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}.\frac{x^n}{n!}$

$$=1+\frac{\alpha}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}.\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{x^3}{3!}+...$$

Diff. w.r.t 'x'

$$\frac{d}{dx}F(\alpha,\gamma;x)=0+\frac{\alpha}{\gamma}+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}.\frac{2x}{2}+\frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}.\frac{3x^2}{3.2.1}+...$$

$$\frac{d}{dx}F(\alpha,\gamma;x)=\frac{\alpha}{\gamma}\left[1+\frac{(\alpha+1)}{(\gamma+1)}.x+\frac{(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}.\frac{x^2}{2.1}+...\right]\_\_\_(i)$$

$$\frac{d}{dx}F(\alpha,\gamma;x)=\frac{\alpha}{\gamma}F(\alpha+1,\gamma+1;x)$$

(ii) Again Diff. eq (i) w.r.t 'x'

$$\frac{d^2}{dx^2}F(\alpha,\gamma;x)=\frac{\alpha}{\gamma}\left[\frac{(\alpha+1)}{(\gamma+1)}+\frac{(\alpha+1)(\alpha+2)}{(\gamma+1)(\gamma+2)}.\frac{2x}{2.1}+...\right]$$

$$\frac{d^2}{dx^2}F(\alpha,\gamma;x)=\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\left[1+\frac{(\alpha+2)}{(\gamma+2)}.x+...\right]$$

$$\frac{d^2}{dx^2}F(\alpha,\gamma;x)=\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}F(\alpha+2,\gamma+2;x)$$

Similarly, in the same way we get the following results

$$\frac{d^3}{dx^3}F(\alpha,\gamma;x)=\frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}F(\alpha+3,\gamma+3;x)$$

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$$\frac{d^n}{dx^n}F(\alpha,\gamma;x)=\frac{(\alpha)_n}{(\gamma)_n}F(\alpha+n,\gamma+n;x)$$

Integral representation of Confluent Hypergeometric function:

Theorem: Prove that

$$F(\alpha,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}e^{xt}dt \text{ where } \gamma>\alpha>0$$

Proof: By the definition of Confluent Hypergeometric function

$$F(\alpha,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\gamma\;;x)=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma\alpha}.\frac{\Gamma\gamma}{\Gamma(\gamma+n)}\frac{x^n}{n!} \qquad \because (\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma\alpha}$$

$$F(\alpha,\gamma;x)=\frac{\Gamma\gamma}{\Gamma\alpha}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)}.\frac{x^n}{n!}$$

$$F(\alpha,\gamma;x)=\frac{\Gamma\gamma}{\Gamma(\gamma-\alpha).\Gamma\alpha}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n).\Gamma(\gamma-\alpha)}{\Gamma(\gamma+n)}.\frac{x^n}{n!}\qquad \because \textit{Multiplying and divide by } \Gamma(\gamma-\alpha)$$

$$F(\alpha,\gamma;x)=\frac{\Gamma\gamma}{\Gamma(\gamma-\alpha).\Gamma\alpha}\sum_{n=0}^{\infty}B(\alpha+n,\gamma-\alpha).\frac{x^n}{n!}\qquad \because \beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$F(\alpha,\gamma;x)=\frac{\Gamma\gamma}{\Gamma(\gamma-\alpha).\Gamma\alpha}\sum_{n=0}^{\infty}\left[\int_0^1 t^{\alpha+n-1}(1-t)^{\gamma-\alpha-1}\,dt\right].\frac{x^n}{n!}$$

$$\because \beta(m,\mathrm{n})=\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$F(\alpha,\gamma;x)=\frac{\Gamma\gamma}{\Gamma(\gamma-\alpha).\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right\}dt$$

$$\because e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$\Rightarrow e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Replace ‘x’ by ‘xt’

$$\Rightarrow e^{xt}=\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}$$

$$F(\alpha,\gamma;x)=\frac{\Gamma\gamma}{\Gamma(\gamma-\alpha).\Gamma\alpha}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}e^{xt}\,dt$$

Kummer's Relation:

Theorem: Prove that $F(\alpha,\gamma;x)=e^{x}F(\gamma-\alpha,\gamma;-x)$

Proof: By the integral representation of Confluent Hypergeometric function

$$F(\alpha,\gamma;x)=\frac{\lceil\gamma}{\lceil\gamma-\alpha\,.\lceil\alpha}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}e^{xt}dt$$

Replace α by α–γ & x by –x

$$F(\gamma-\alpha,\gamma;-x)=\frac{\lceil\gamma}{\lceil\gamma-\gamma+\alpha\,.\lceil\gamma-\alpha}\int_0^1 t^{\gamma-\alpha-1}(1-t)^{\gamma-\gamma+\alpha-1}e^{-xt}dt$$

$$F(\gamma-\alpha,\gamma;-x)=\frac{\lceil\gamma}{\lceil\alpha\,.\lceil\gamma-\alpha}\int_0^1 t^{\gamma-\alpha-1}(1-t)^{\alpha-1}e^{-xt}dt$$

Let u = 1–t ⇒ t = 1–u & du = –dt

$$u\to 1\ as\ t\to 0\ \&u\to 0\ as\ t\to 1$$

$$F(\gamma-\alpha,\gamma;-x)=\frac{\lceil\gamma}{\lceil\alpha\,.\lceil\gamma-\alpha}\int_1^0 (1-u)^{\gamma-\alpha-1}(u)^{\alpha-1}e^{-x(1-u)}(-du)$$

$$F(\gamma-\alpha,\gamma;-x)=\frac{\lceil\gamma}{\lceil\alpha\,.\lceil\gamma-\alpha}\int_0^1 (1-u)^{\gamma-\alpha-1}(u)^{\alpha-1}e^{-x}e^{xu}du$$

$$F(\gamma-\alpha,\gamma;-x)=e^{-x}F(\alpha,\gamma;x)$$

Lecture # 06

Application of Confluent Hypergeometric function:

Question: Prove that $e^x = F(\alpha,\alpha;x)$

Solution: R.H.S = $F(\alpha,\alpha;x)$

By the definition of Confluent Hypergeometric function

$$F(\alpha,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\alpha;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\alpha)_n}.\frac{x^n}{n!}$$

$$F(\alpha,\alpha;x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$F(\alpha,\alpha;x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+....$$

$$F(\alpha,\alpha;x)=e^x=\text{L.H.S}$$

Question: Prove that $e^x-1=xF(1,2;x)$

Solution: R.H.S = $xF(1,2;x)$

By the definition of Confluent Hypergeometric function

$$F(\alpha,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}.\frac{x^n}{n!}$$

$$xF(1,2;x)=x\sum_{n=0}^{\infty}\frac{(1)_n}{(2)_n}.\frac{x^n}{n!}$$

$$xF(1,2;x)=x\sum_{n=0}^{\infty}\frac{n!}{(n+1)!}.\frac{x^n}{n!} \qquad \because (\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},(1)_n=n!,(2)_n=(n+1)!$$

$$xF(1,2;x)=\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}$$

$$xF(1,2;x)=\left[\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right]$$

$$xF(1,2;x)=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots \quad \_\_\_(i)$$

L.H.S = e^x-1

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots.$$

$$e^x-1=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots. \quad \_\_\_(ii)$$

From (i) and (ii)

$$e^x-1=xF(1,2;x)$$

Question: Prove that $\left(1+\frac{x}{\alpha}\right)e^x=F(\alpha+1,\alpha;x)$

Solution: L.H.S = $\left(1+\frac{x}{\alpha}\right)e^x$

$$=\left(1+\frac{x}{\alpha}\right)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots.\right)$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x}{\alpha}+\frac{x^2}{\alpha}+\frac{x^3}{\alpha.2!}+\dots.$$

$$=1+\left(x+\frac{x}{\alpha}\right)+\left(\frac{x^2}{2!}+\frac{x^2}{\alpha}\right)+\left(\frac{x^3}{3!}+\frac{x^3}{\alpha.2!}\right)+\dots$$

$$\left(1+\frac{x}{\alpha}\right)e^x=1+\left(1+\frac{1}{\alpha}\right)x+\left(1+\frac{2}{\alpha}\right)\frac{x^2}{2!}+\left(1+\frac{3}{\alpha}\right)\frac{x^3}{3!}+\dots$$

R.H.S = $F\left(\alpha+1,\alpha;x\right)$

By the definition of Confluent Hypergeometric function

$$F\left(\alpha,\gamma;x\right)=\sum_{n=0}^{\infty}\frac{\left(\alpha\right)_n}{\left(\gamma\right)_n}.\frac{x^n}{n!}$$

$$F\left(\alpha+1,\alpha;x\right)=\sum_{n=0}^{\infty}\frac{\left(\alpha+1\right)_n}{\left(\alpha\right)_n}.\frac{x^n}{n!}$$

$$F\left(\alpha+1,\alpha;x\right)=\sum_{n=0}^{\infty}\frac{\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)\ldots\left(\alpha+n-1\right)\left(\alpha+n\right)}{\left(\alpha\right)\left(\alpha+1\right)\left(\alpha+2\right)\ldots\left(\alpha+n-1\right)}.\frac{x^n}{n!}$$

$$F\left(\alpha+1,\alpha;x\right)=\sum_{n=0}^{\infty}\frac{\left(\alpha+n\right)}{\left(\alpha\right)}.\frac{x^n}{n!}$$

$$F\left(\alpha+1,\alpha;x\right)=\sum_{n=0}^{\infty}\left(1+\frac{n}{\alpha}\right).\frac{x^n}{n!}$$

$$F\left(\alpha+1,\alpha;x\right)=1+\left(1+\frac{1}{\alpha}\right)x+\left(1+\frac{2}{\alpha}\right)\frac{x^2}{2!}+\left(1+\frac{3}{\alpha}\right)\frac{x^3}{3!}+\ldots$$

$\Rightarrow$ L.H.S = R.H.S

Hermite Function:

Hermite's Equation:

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2ny=0 \quad \_\_\_\_\_(i)$$

The solution of equation (i) is known as Hermite's Polynomial.

Solution of Hermite's equation:

We shall solve the equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2ny=0 \quad \_\_\_\_\_(i)$$

by Frobenius method.

Let $y=\sum_{k=0}^{\infty} a_k x^{m+k}$ ____(ii) ; $a_0 \neq 0$

$$Here\ a_{-1}=a_{-2}=....=a_{-k}=0$$

$$y'=\sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

$$y''=\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2}$$

Put all these values in (i)

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2} - 2x\sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1} + 2n\sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2} - 2\sum_{k=0}^{\infty} a_k (m+k) x^{m+k} + 2n\sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2} - 2\sum_{k=0}^{\infty} a_k (m+k-n) x^{m+k} = 0 \quad \_\_\_(iii)$$

To get the lowest degree term x^{m-2}

We put k = 0 in the first summation of (iii) and we cannot have x^{m-2} form in the second summation. Since $k \neq -2$. So, the coefficient of x^{m-2} is

$$a_0.\text{m}(m-1)=0$$

$$a_0 \neq 0\ ,\ m=0\ ,\ m-1=0$$

$$m=0\ ,\ m=1$$

This is the indicial equation.

Now equating the coefficient of next lowest degree term x^{m-1} to zero in (iii) we get by putting k = 1 in the first summation and we cannot have x^{m-1} from the second summation, since $k \neq -1$.

$$a_1(m+1).\text{m}=0$$

$\Rightarrow a_1$ may or may not be zero when m = 0 and $a_1 = 0$ when m = 1.

Recurrence relation of Hermite equation:

In equation (iii) replace k by k−2 in the second summation

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-2} - 2\sum_{k=2}^{\infty} a_{k-2}(m+k-2-n)x^{m+k-2} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-2} - 2\sum_{k=0}^{\infty} a_{k-2}(m+k-2-n)x^{m+k-2} = 0 \quad \_\_\_(iv)$$

Equate to zero the coefficient of x^{m+k-2} in (iv)

$$\Rightarrow a_k(m+k)(m+k-1) - 2a_{k-2}(m+k-2-n) = 0$$

$$\Rightarrow a_k(m+k)(m+k-1) = 2a_{k-2}(m+k-2-n)$$

$$\Rightarrow a_k = \frac{2(m+k-2-n)}{(m+k)(m+k-1)}.a_{k-2} \quad \_\_\_(v)$$

Case-I: When m = 0

Equation (v) ⇒ $a_k = \frac{2(k-2-n)}{k(k-1)}.a_{k-2}$

For k = 2 $a_2 = \frac{2(-n)}{2(1)}.a_0 = \frac{(-2)^1.n}{2!}.a_0$

For k = 3 $a_3 = \frac{2(1-n)}{3(2)}.a_1 = \frac{-2(n-1)}{3.2.1}.a_1$

$$a_3 = \frac{(-2)^1(n-1)}{3!}.a_1$$

For k = 4 $a_4 = \frac{2(2-n)}{4.3}.a_2 = \frac{(-2)(n-2)}{4.3}.\frac{(-2)n}{2!}.a_0$

$$a_4 = \frac{(-2)^2 n(n-2)}{4!}.a_0$$

For k = 5 $a_4 = \frac{2(3-n)}{5.4}.a_3 = \frac{(-2)(n-3)}{5.4}.\frac{(-2)(n-1)}{3!}.a_1$

$$a_5 = \frac{(-2)^2 (n-1)(n-3)}{5!}.a_1$$

For k = 6 $$a_6 = \frac{2(4-n)}{6.5}.a_4 = \frac{(-2)(n-4)}{6.5}.\frac{(-2)^2 n(n-2)}{4!}.a_0$$

$$a_6 = \frac{(-2)^3 n(n-2)(n-4)}{6!}.a_0$$

For k = 7 $$a_7 = \frac{2(5-n)}{7.6}.a_5 = \frac{(-2)(n-5)}{7.6}.\frac{(-2)^2 (n-1)(n-3)}{5!}.a_1$$

$$a_7 = \frac{(-2)^3 (n-1)(n-3)(n-5)}{7!}.a_1$$

Similarly, in general

$$a_{2r} = \frac{(-2)^r n(n-2)(n-4)...(n-(2r-2))}{(2r)!}.a_0$$

$$a_{2r+1} = \frac{(-2)^r (n-1)(n-3)...(n-(2r-1))}{(2r+1)!}.a_1$$

With the help of these values equation (ii) becomes when m = 0

$$y = \sum_{k=0}^{\infty} a_k x^k$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

$$y = \left\{a_0 + a_2 x^2 + a_4 x^4 + ...\right\} + \left\{a_1 x + a_3 x^3 + a_5 x^5 + ...\right\}$$

$$y = \left\{a_0 + \frac{(-2)(n)}{2!}a_0 x^2 + \frac{(-2)^2 (n-2)}{4!}a_0 x^4 + ...\right\} + \left\{a_1 x + \frac{(-2)(n-1)}{3!}a_1 x^3 + \frac{(-2)^2 (n-1)(n-3)}{5!}a_1 x^5 + ...\right\}$$

$$y = a_0\left\{1 + \frac{(-2)(n)}{2!}x^2 + \frac{(-2)^2 (n-2)}{4!}x^4 + ...\right\} + a_1\left\{x + \frac{(-2)(n-1)}{3!}x^3 + \frac{(-2)^2 (n-1)(n-3)}{5!}x^5 + ...\right\} \text{---}(vi)$$

Case-II: When m = 1

Equation (v) $\Rightarrow a_k = \dfrac{2(1+k-2-n)}{(1+k)(1+k-1)}.a_{k-2}$

$$\Rightarrow a_k = \frac{2(k-1-n)}{(1+k)k}.a_{k-2}$$

For k = 2 $a_2 = \dfrac{2(1-n)}{2.3}.a_0 = \dfrac{(-2)(n-1)}{3!}.a_0$

For k = 3 $a_3 = \dfrac{2(3-1-n)}{3(1+3)}.a_1 = \dfrac{(2)(2-n)}{4.3}.a_1 = \dfrac{2(-2)(n-2)}{4.3.2.1}.a_1$

$$a_3 = \frac{2(-2)(n-2)}{4!}.a_1$$

For k = 4 $a_4 = \dfrac{2(4-1-n)}{4.5}.a_2 = \dfrac{(2)(3-n)}{5.4}.\dfrac{(-2)(n-1)}{3!}.a_0$

$$a_4 = \frac{(-2)(n-3)(-2)(n-1)}{5.4.3!}.a_0 = \frac{(-2)^2(n-1)(n-3)}{5!}.a_0$$

For k = 5 $a_5 = \dfrac{2(5-1-n)}{5.6}.a_3 = \dfrac{(2)(4-n)}{6.5}.\dfrac{2(-2)(n-2)}{4!}.a_1$

$$a_5 = \frac{(-2)(n-4).2(-2)(n-2)}{6.5.4!}.a_1 = \frac{2(-2)^2(n-2)(n-4)}{6!}.a_1$$

For k = 6 $a_6 = \dfrac{2(6-1-n)}{6.7}.a_2 = \dfrac{(2)(5-n)}{7.6}.\dfrac{(-2)^2(n-1)(n-3)}{5!}.a_0$

$$a_6 = \frac{(-2)(n-5)(-2)^2(n-1)(n-3)}{7.6.5!}.a_0 = \frac{(-2)^2(n-1)(n-3)(n-5)}{7!}.a_0$$

For k = 7 $a_7 = \dfrac{2(7-1-n)}{7.8}.a_5 = \dfrac{2(6-n)}{8.7}.\dfrac{2(-2)^2(n-2)(n-4)}{6!}.a_1$

$$a_7 = \frac{2(-2)^3(n-2)(n-4)(n-6)}{8!}.a_1$$

Similarly, in general for even

$$a_{2r} = \frac{(-2)^r(n-1)(n-3)(n-5)...(n-(2r-1))}{(2r+1)!}.a_0$$

For odd

$$a_{2r+1} = \frac{2(-2)^r(n-2)(n-4)(n-6)..(n-2r)}{(2r+2)!}.a_1$$

With the help of these values equation (ii) becomes when m = 1

$$y = \sum_{k=0}^{\infty} a_k x^{1+k}$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

$$y = \{a_0 + a_2x^2 + a_4x^4 + ...\} + \{a_1x + a_3x^3 + a_5x^5 + ...\}$$

$$y = \left\{a_0x + \frac{(-2)(n-1)}{3!}a_0x^3 + \frac{(-2)^2(n-1)(n-3)}{5!}a_0x^5 + ...\right\}$$

$$+\left\{a_1x^2 + \frac{2(-2)(n-2)}{4!}a_1x^4 + \frac{2(-2)^2(n-2)(n-4)}{6!}a_1x^6 + ...\right\}$$

$$y = a_0\left\{x + \frac{(-2)(n-1)}{3!}x^3 + \frac{(-2)^2(n-1)(n-3)}{5!}x^5 + ...\right\}$$

$$+a_1\left\{x^2 + \frac{2(-2)(n-2)}{4!}x^4 + \frac{2(-2)^2(n-2)(n-4)}{6!}x^6 + ...\right\}$$

Lecture # 07

Generating function of Hermite polynomials:

Consider $e^{x^2}.\frac{\partial^2}{\partial t^n}.e^{-(t-x)^2} = H_n(x) + H_{n+1}(x).t + H_{n+2}(x).t^2 + ...\quad \_\_\_(i)$

Now differentiating $e^{-(t-x)^2}$ w.r.t 't'

$$\frac{\partial}{\partial t}.e^{-(t-x)^2} = e^{-(t-x)^2}\left\{-2(t-x)^1.(1-0)\right\}$$

$$\frac{\partial}{\partial t}.e^{-(t-x)^2} = -2(t-x)e^{-(t-x)^2}$$

Taking limit t → 0 on both side

$$\underset{t\to 0}{Lim}\frac{\partial}{\partial t}.e^{-(t-x)^2} = \underset{t\to 0}{Lim}\left\{-2(t-x)e^{-(t-x)^2}\right\}$$

$$\underset{t\to 0}{Lim}\frac{\partial}{\partial t}.e^{-(t-x)^2} = 2xe^{-x^2} \quad \_\_\_(ii)$$

Again differentiating $e^{-(t-x)^2}$ w.r.t 'x'

$$\frac{\partial}{\partial x}.e^{-(t-x)^2} = e^{-(t-x)^2}\left\{-2(t-x).(1-0)\right\}$$

$$\frac{\partial}{\partial t}.e^{-(t-x)^2} = 2(t-x)e^{-(t-x)^2}$$

Taking limit t → 0 on both side

$$\underset{t\to 0}{Lim}\frac{\partial}{\partial x}.e^{-(t-x)^2} = \underset{t\to 0}{Lim}\left\{2(t-x)e^{-(t-x)^2}\right\}$$

$$\underset{t\to 0}{Lim}\frac{\partial}{\partial x}.e^{-(t-x)^2} = -2xe^{-x^2} \quad \_\_\_(iii)$$

Compare (ii) and (iii)

$$\underset{t\to 0}{Lim}\frac{\partial}{\partial t}.e^{-(t-x)^2} = (-1)\underset{t\to 0}{Lim}\frac{\partial}{\partial x}.e^{-(t-x)^2}$$

Similarly,

$$\lim_{t\to 0}\frac{\partial^2}{\partial t^2}.e^{-(t-x)^2}=(-1)^2\lim_{t\to 0}\frac{\partial^2}{\partial x^2}.e^{-(t-x)^2}$$

$$\lim_{t\to 0}\frac{\partial^3}{\partial t^3}.e^{-(t-x)^2}=(-1)^3\lim_{t\to 0}\frac{\partial^3}{\partial x^3}.e^{-(t-x)^2}$$

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$$\lim_{t\to 0}\frac{\partial^n}{\partial t^n}.e^{-(t-x)^2}=(-1)^n\lim_{t\to 0}\frac{\partial^n}{\partial x^n}.e^{-(t-x)^2}$$

$$\lim_{t\to 0}\frac{\partial^n}{\partial t^n}.e^{-(t-x)^2}=(-1)^n.\frac{d^n}{dx^n}.e^{-x^2} \quad \_\_\_(iv)$$

$$eq(i)\Rightarrow e^{x^2}\frac{\partial^n}{\partial t^n}.e^{-(t-x)^2}=H_n(x)+H_{n+1}(x).t+H_{n+2}(x).t^2+....$$

Put t = 0 in (i)

$$\lim_{t\to 0} e^{x^2}\frac{\partial^n}{\partial t^n}.e^{-(t-x)^2}=H_n(x)$$

$$e^{x^2}\lim_{t\to 0}\frac{\partial^n}{\partial t^n}.e^{-(t-x)^2}=H_n(x) \quad \_\_\_(v)$$

Put (iv) in (v)

$$e^{x^2}(-1)^n.\frac{d^n}{dx^n}.e^{-x^2}=H_n(x)$$

$$\Rightarrow H_n(x)=e^{x^2}(-1)^n.\frac{d^n}{dx^n}.e^{-x^2} \quad \_\_\_\_(vi)$$

Put n = 0 in (vi)

$$H_0(x)=e^{x^2}.e^{-x^2}=e^{x^2-x^2}=e^0=1$$

$$\Rightarrow H_0(x)=1$$

is the zero-degree Hermite Polynomial.

Put n = 1 in (vi)

$$\Rightarrow\ H_1(x)=e^{x^2}(-1).\frac{d}{dx}.e^{-x^2}=-e^{x^2}.e^{-x^2}(-2x)$$

$$\Rightarrow\ H_1(x)=(-2x)$$

is the first degree Hermite polynomial.

Put n = 2 in (vi)

$$\Rightarrow\ H_2(x)=e^{x^2}\frac{d}{dx}.e^{-x^2}=e^{x^2}\left[-2\left\{e^{-x^2}+xe^{-x^2}(-2x)\right\}\right]$$

$$\Rightarrow\ H_2(x)=-2e^{-x^2}e^{x^2}+4x^2e^{-x^2}e^{x^2}$$

$$\Rightarrow\ H_2(x)=-2+4x^2$$

$$\Rightarrow\ H_2(x)=4x^2-2$$

is the second degree Hermite polynomial put n = 3 in (vi)

$$\Rightarrow\ H_3(x)=e^{x^2}(-1)\frac{d^3}{dx^3}.e^{-x^2}=-e^{x^2}\left[-2\left\{e^{-x^2}(-2x)-2\left(e^{-x^2}2x+x^2e^{-x^2}(-2x)\right)\right\}\right]$$

$$\Rightarrow\ H_3(x)=e^{x^2}\left[-2xe^{-x^2}+4x^3e^{-x^2}-4xe^{-x^2}\right]$$

$$\Rightarrow\ H_3(x)=-4xe^{x^2}.e^{-x^2}+8x^3e^{x^2}e^{-x^2}-8xe^{x^2}e^{-x^2}$$

$$\Rightarrow\ H_3(x)=-4x+8x^3-8x$$

$$\Rightarrow\ H_3(x)=8x^3-12x$$

is called third degree Hermite polynomial.

Similarly,

$$H_4(x)=16x^4-48x^2+12$$
$$H_5(x)=32x^5-160x^3+120x$$
$$H_6(x)=64x^6-480x^4+720x^2-120$$
$$H_7(x)=128x^7-1344x^5+3360x^3-1680x$$

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Question: Convert Hermite Polynomial into ordinary polynomial.

$$2H_4(x)+3H_3(x)-H_2(x)+5H_1(x)+6H_0(x)$$

Solution: Given that

$$2H_4(x)+3H_3(x)-H_2(x)+5H_1(x)+6H_0(x)$$
$$=2\left(16x^4-48x^2+12\right)+3\left(8x^3-12x\right)-\left(4x^2-2\right)+5(2x)+6(1)$$
$$=32x^4-96x^2+24+24x^3-36x-4x^2+2+10x+6$$
$$=32x^4+24x^3-100x^2-26x+32$$

Question: Convert ordinary polynomial into Hermite Polynomial.

$$64x^4+8x^3-32x^2+40x+10$$

Solution: Given that

$$64x^4+8x^3-32x^2+40x+10$$
$$64x^4+8x^3-32x^2+40x+10=AH_4(x)+BH_3(x)+CH_2(x)+DH_1(x)+EH_0(x) \quad \_\_\_(i)$$
$$=A\left(16x^4-48x^2+12\right)+B\left(8x^3-12x\right)-C\left(4x^2-2\right)+D(2x)+E(1)$$

Compare the respective coefficients

Comparing x^4

$$64 = 16A \quad \Rightarrow A = 4$$

Comparing x^3

$$8 = 8B \quad \Rightarrow B = 1$$

Comparing x^2

$$-32 = -48A+4C$$

$$= -48(4) + 4C = -192+4C$$

$$\Rightarrow 4C = 192-32 = 160$$

$$\Rightarrow C = 40$$

Comparing x

$$40 = -12B+2D$$

$$40 = -12(1)+2D$$

$$\Rightarrow 2D = 40+12 = 52$$

$$\Rightarrow D = 26$$

Comparing x^0

$$10 = 12A-2C+E$$

$$10 = 12(4) -2(40)+E = 48-80+e$$

$$10 = -32 +E$$

$$\Rightarrow E = 42$$

Put all these values in (i)

$$64x^4 + 8x^3 - 32x^2 + 40x + 10 = 4H_4(x) + H_3(x) + 40H_2(x) + 26H_1(x) + 42$$

Orthogonality Property:

The orthogonal property of Hermite's polynomial is

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x).H_n(x)dx = \begin{cases} 0 & ; if \;\; m \neq n \\ 2^n.n!\sqrt{\pi} & ; if \;\; m = n \end{cases}$$

Question: Find the value of $\int_{-\infty}^{\infty} e^{-x^2} H_2(x).H_3(x)\,dx$

Solution: Given that $\int_{-\infty}^{\infty} e^{-x^2} H_2(x).H_3(x)\,dx$

Here m = 2 and n = 3 $\Rightarrow$ m $\neq$ n

So, $\int_{-\infty}^{\infty} e^{-x^2} H_2(x).H_3(x)\,dx = 0$

Question: Find the value of $\int_{-\infty}^{\infty} e^{-x^2} \left[H_2(x)\right]^2 dx$

Solution: Given that $\int_{-\infty}^{\infty} e^{-x^2} \left[H_2(x)\right]^2 dx$

$$= \int_{-\infty}^{\infty} e^{-x^2} H_2(x).H_2(x)\,dx$$

Here m = 2 and n = 2 $\Rightarrow$ m = n

So, $\int_{-\infty}^{\infty} e^{-x^2} H_2(x).H_2(x)\,dx = 2^2.2!\sqrt{\pi} = 4.2.\sqrt{\pi}$

$$\int_{-\infty}^{\infty} e^{-x^2} \left[H_2(x)\right]^2 dx = 8\sqrt{\pi}$$

Lecture # 08

Recurrence formula for $H_n(x)$ of Hermite equation:

Prove that $$e^{2tx-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x).t^n}{n!}$$

Proof: As we know that

$$H_0(x)=1$$

$$H_1(x)=2x$$

$$H_2(x)=4x^2-2$$

$$\begin{matrix} . & . \\ . & . \\ . & . \end{matrix}$$

Also $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

Put $x=2tx-t^2$

$$e^{2tx-t^2}=1+\left(2tx-t^2\right)+\frac{\left(2tx-t^2\right)^2}{2!}+\ldots$$

$$e^{2tx-t^2}=1+2x.t-t^2+\frac{4t^2x^2+t^4-4t^3x}{2!}+\ldots$$

$$e^{2tx-t^2}=1+2x.t-t^2+\frac{4t^2x^2}{2!}+\frac{t^4}{2!}-\frac{4t^3x}{2!}\ldots$$

$$e^{2tx-t^2}=1+2x.t+\frac{t^2}{2!}\left(4x^2-2\right)+\ldots$$

$$e^{2tx-t^2}=H_0(x)+H_1(x).t+H_2(x).\frac{t^2}{2!}+\ldots=\sum_{n=0}^{\infty}H_n(x).\frac{t^n}{n!}$$

Recurrence Formula for $H_n(x)$ of Hermite Equation:

Four recurrence relations

(i) $2n.H_{n-1}(x) = H'_n(x)$

(ii) $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

(iii) $H'_n(x) = 2xH_n(x) - H_{n+1}(x)$

(iv) $H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$

Proof: (i) As we know that

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \quad \_\_\_\_(i)$$

Diff. (i) w.r.t 'x'

$$e^{2tx-t^2}.2t = \sum_{n=0}^{\infty} \frac{H'_n(x)t^n}{n!} \quad \_\_\_\_(ii)$$

$$2t.\sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = \sum_{n=0}^{\infty} \frac{H'_n(x)t^n}{n!} \quad \because from(i)$$

$$\Rightarrow 2\sum_{n=0}^{\infty} \frac{H_n(x)t^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{H'_n(x)t^n}{n!}$$

Replace 'n' by 'n−1' on the left-hand side of above expression.

$$2\sum_{n=1}^{\infty} \frac{H_{n-1}(x)t^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{H'_n(x)t^n}{n!}$$

$$2\sum_{n=1}^{\infty} \frac{H_{n-1}(x)n\, t^n}{n(n-1)!} = \sum_{n=0}^{\infty} \frac{H'_n(x)t^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{H_{n-1}(x).2n\, t^n}{n!} = \sum_{n=0}^{\infty} \frac{H'_n(x)t^n}{n!}$$

Comparing the coefficients of t^n on both side

$$\frac{H_{n-1}(x).2n}{n!} = \frac{H'_n(x)}{n!}$$

$$\Rightarrow \ 2n.H_{n-1}(x) = H'_n(x)$$

Proof: (ii) Diff. equation (i) w.r.t 't'

$$e^{2tx-t^2}(2x-2t) = \sum_{n=0}^{\infty}\frac{H_n(x)nt^{n-1}}{n!}$$

$$2(x-t).e^{2tx-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)nt^{n-1}}{n!}$$

$$2(x-t).\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!} = \sum_{n=0}^{\infty}\frac{H_n(x)nt^{n-1}}{n!} \quad \because \textit{from } (i)$$

$$\Rightarrow 2x\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!} - 2\sum_{n=0}^{\infty}\frac{H_n(x)t^{n+1}}{n!} = \sum_{n=0}^{\infty}\frac{H_n(x)t^{n-1}}{(n-1)!}$$

Equating the coefficients of t^n on both sides

$$\frac{2xH_n(x)}{n!} - \frac{2H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}(x)}{n!}$$

$$\frac{2xH_n(x)}{n!} - \frac{2nH_{n-1}(x)}{n(n-1)!} = \frac{H_{n+1}(x)}{n!}$$

$$\frac{2xH_n(x)}{n!} - \frac{2nH_{n-1}(x)}{n!} = \frac{H_{n+1}(x)}{n!}$$

$$\Rightarrow 2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)$$

$$\Rightarrow 2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x) \quad \text{proved}$$

Proof: (iii) First recurrence relation implies that

$$2nH_{n-1}(x) = H'_n(x) \quad \_\_\_(i)$$

Second recurrence relation implies that

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$$

$$2nH_{n-1}(x) = 2xH_n(x) - H_{n+1}(x) \quad \_\_\_\_(ii)$$

Compare (i) and (ii)

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x) \quad \text{proved}$$

Proof: (iv) Diff. third recurrence relation w.r.t. 'x'

$$H_n''(x) = 2\left[H_n(x) + xH_n'(x)\right] - H_{n+1}'(x)$$

$$H_n''(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x) \quad \_\_\_(*)$$

From first recurrence relation

$$2n.H_{n-1}(x) = H_n'(x)$$

Replace n by n+1

$$2(n+1).H_n(x) = H_{n+1}'(x)$$

Put this value in (*)

$$H_n''(x) = 2xH_n'(x) + 2H_n(x) - 2(n+1).H_n(x)$$

$$H_n''(x) = 2xH_n'(x) + 2H_n(x) - 2nH_n(x) - 2H_n(x)$$

$$\Rightarrow H_n''(x) = 2xH_n'(x) - 2nH_n(x)$$

$$\Rightarrow H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

Theorem: Prove that $\dfrac{d^m}{dx^m}\{H_n(x)\} = \dfrac{2^m.n!}{(n-m)!}.H_{n-m}(x)\,; m < n$

Proof: As we know that

$$H_n'(x) = 2n.H_{n-1}(x) \quad \_\_\_\_(i)$$

$$\Rightarrow \frac{d}{dx}\{H_n(x)\} = 2n.H_{n-1}(x)$$

Diff. w.r.t 'x'

$$\frac{d^2}{dx^2}\{H_n(x)\} = 2n.H'_{n-1}(x)$$

$$\Rightarrow \frac{d^2}{dx^2}\{H_n(x)\} = 2n.2(n-1)H_{n-2}(x)$$

$$\Rightarrow \frac{d^2}{dx^2}\{H_n(x)\} = 2^2 n(n-1)H_{n-2}(x)$$

From (i)

$H'_n(x) = 2n.H_{n-1}(x)$

Replace 'n' by 'n−1'

$H'_{n-1}(x) = 2(n-1).H_{n-2}(x)$

Replace 'n' by 'n−2'

$H'_{n-2}(x) = 2(n-2).H_{n-3}(x)$

Again Diff. w.r.t 'x'

$$\Rightarrow \frac{d^3}{dx^3}\{H_n(x)\} = 2^2 n(n-1)H'_{n-2}(x)$$

$$\Rightarrow \frac{d^3}{dx^3}\{H_n(x)\} = 2^2 n(n-1).2(n-2)H_{n-3}(x)$$

$$\frac{d^3}{dx^3}\{H_n(x)\} = 2^3 n(n-1)(n-2)H_{n-3}(x)$$

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$$\frac{d^m}{dx^m}\{H_n(x)\} = 2^m n(n-1)(n-2)\ldots(n-(m-1))H_{n-m}(x)$$

$$\frac{d^m}{dx^m}\{H_n(x)\} = \frac{2^m.n!}{(n-m)!}.H_{n-m}(x)\ ;\ m < n$$

Laguerre's Function:

The Laguerre's function is

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0 \quad \_\_\_\_\_\_ (*)$$

$$Or \qquad \frac{d^2y}{dx^2} + \frac{1-x}{x}\frac{dy}{dx} + \frac{n}{x}y = 0$$

Let $\quad y=\sum_{k=0}^{\infty} a_k x^{m+k}$ ____ (i) ; $a_0 \neq 0$

$$Here\ \ a_{-1}=a_{-2}=....=a_{-n}=0$$

$$y'=\sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

$$y''=\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2}$$

Put all these values in (*)

$$x\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2} + (1-x)\sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1} + n\sum_{k=0}^{\infty} a_k x^{m+k}=0$$

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-1} + \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1} - \sum_{k=0}^{\infty} a_k (m+k) x^{m+k} + n\sum_{k=0}^{\infty} a_k x^{m+k}=0$$

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1+1) x^{m+k-1} - \sum_{k=0}^{\infty} a_k (m+k-n) x^{m+k} =0$$

$$\sum_{k=0}^{\infty} a_k (m+k)^2 x^{m+k-1} - \sum_{k=0}^{\infty} a_k (m+k-n) x^{m+k} =0 \quad \_\_\_(ii)$$

Equating to zero the coefficients to lowest degree term x^{m-1}

$$a_0 .\mathrm{m}^2=0$$

$$a_0 \neq 0 \, , \, m^2=0$$

$$m=0$$

Again, equating to zero the coefficient of x^{m+k}

$$a_{k+1}(m+k+1)^2 - a_k (m+k-n)=0$$

$$\Rightarrow \ a_{k+1}=\frac{(m+k-n)}{(m+k+1)^2}.a_k$$

For m = 0

$$a_{k+1} = \frac{(k-n)}{(k+1)^2}.a_k$$

For k = 0
$$a_1 = \frac{0-n}{(0+1)^2}.a_0 = -na_0 = (-1)na_0$$

For k = 1
$$a_2 = \frac{(1-n)}{(1+1)^2}.a_1 = \frac{-(n-1)}{2^2}.(-1)na_0$$

$$a_2 = \frac{(-1)^2 n(n-1)a_0}{(2!)^2}$$

For k = 2
$$a_3 = \frac{(2-n)}{(2+1)^2}.a_2 = \frac{-(n-2)}{9}.\frac{(-1)^2 n(n-1)a_0}{(2!)^2}$$

$$a_3 = \frac{(-1)^3 n(n-1)(n-2)a_0}{36} = \frac{(-1)^3 n(n-1)(n-2)a_0}{(3!)^2}$$

For k = 3
$$a_4 = \frac{(3-n)}{(3+1)^2}.a_3 = \frac{-(n-3)}{16}.\frac{(-1)^3 n(n-1)(n-2)a_0}{36}$$

$$a_4 = \frac{(-1)^4 n(n-1)(n-2)(n-3)a_0}{576} = \frac{(-1)^4 n(n-1)(n-2)(n-3)a_0}{(4!)^2}$$

Similarly, in general

$$a_r = \frac{(-1)^r n(n-1)(n-2)(n-3)\ldots(n-(r-1))}{(r!)^2}.a_0$$

With the help of these values equation (i) becomes when m = 0

$$y=\sum_{k=0}^{\infty}a_k x^k$$

$$y=a_0+a_1x+a_2x^2+a_3x^3+.....$$

$$y=a_0+(-1)na_0x+\frac{(-1)^2 n(n-1)a_0}{(2!)^2}x^2+\frac{(-1)^3 n(n-1)(n-2)a_0}{(3!)^2}x^3+.....$$

$$y=a_0\left[1+(-1)nx+\frac{(-1)^2 n(n-1)}{(2!)^2}x^2+\frac{(-1)^3 n(n-1)(n-2)}{(3!)^2}x^3+.....\right]$$

Generating function for Laguerre polynomial

$$(1-t)\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=e^{\frac{-xt}{1-t}}$$

Proof: Here we have

$$(1-t)\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=e^{\frac{-xt}{1-t}}$$

$$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=\frac{1}{(1-t)}.e^{\frac{-xt}{1-t}}$$

$$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=\frac{1}{(1-t)}\left[1-\frac{xt}{(1-t)}+\frac{x^2t^2}{2!(1-t)^2}-...+\frac{(-1)^k x^k t^k}{\text{k}!(1-t)^k}+...\right]$$

$$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=\frac{1}{(1-t)}\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{\text{k}!(1-t)^k}$$

$$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{\text{k}!(1-t)^{k+1}}$$

$$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{\text{k}!}.(1-t)^{-(k+1)}$$

$$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{\text{k}!}\left[1+(k+1)t+\frac{(k+1)(k+2)}{2!}t^2+\ldots+\frac{(k+1)(k+2)\ldots(k+l)}{\text{l}!}t^l+\ldots\right]$$

$$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{\text{k}!}.\sum_{l=0}^{\infty}\frac{(k+1)_l}{(l!)}.t^l \quad \text{where } (k+1)_l=\frac{\lceil k+1+l}{\lceil k+1}$$

$$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}.t^n=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^k (k+1)_l .x^k.t^{k+l}}{\text{k}!.l!}$$

Equating the coefficient of t^n on both sides, we get on putting $l = n - k$,

$$\frac{L_n(x)}{n!}=\sum_{k=0}^{\infty}\frac{(-1)^k (k+1)_{n-k}}{\text{k}!.(n-k)!}.x^k \quad \underline{\qquad}(i)$$

Here $(k+1)_{n-k}=\frac{\lceil k+1+n-k}{\lceil k+1}=\frac{\lceil n+1}{\lceil k+1}=\frac{n!}{k!}$

And $\frac{(-1)^k}{(n-k)!}=\frac{(-1)^k.\text{n}(n-1)\ldots(n-k+1)}{\text{n}(n-1)\ldots(n-k+1)(n-k)!}$

$$\frac{(-1)^k}{(n-k)!}=\frac{(-\text{n})(-n+1)(-n+2)\ldots(-n+k-1)}{\text{n}!}$$

$$\frac{(-1)^k}{(n-k)!}=\frac{(-\text{n})_k}{\text{n}!}$$

Putting in (i)

$$\frac{L_n(x)}{n!}=\sum_{k=0}^{\infty}\frac{(-n)_k}{(k!)^2}.x^k$$

$$L_n(x)=n!\sum_{k=0}^{\infty}\frac{(-n)_k}{(k!)^2}.x^k$$

$$L_n(x) = n!\left[1 + \frac{(-n)}{(1!)^2}x + \frac{(-n)(-n+1)}{(2!)^2}x^2 + \frac{(-n)(-n+1)(-n+2)}{(3!)^2}x^3 + \ldots\right]$$

$$L_n(x) = n!\mathrm{F}(-n,1;x)$$

From which it follows that $L_n(x)$ is a polynomial of degree 'n' in 'x' and that the coefficient of x^n $is(-1)^n$.

Lecture # 09

Bessel's Function:

The differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0$$

is called the Bessel's Differential equation and parametric solutions of this D.E are called Bessel's function of order 'n'.

Solution of Bessel's D.E:

$$x^2y''+xy'+\left(x^2-n^2\right)y=0 \quad \_\_\_\_(i)$$

$$\text{Let}\quad y=\sum_{k=0}^{\infty}a_kx^{m+k} \quad \_\_\_\_(*)\ ;\ a_0\neq 0$$

$$Here\ \ a_{-1}=a_{-2}=....=a_{-n}=0$$

$$y'=\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k-1}$$

$$y''=\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}$$

Put all these values in (i)

$$x^2\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}+x\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k-1}+\left(x^2-n^2\right)\sum_{k=0}^{\infty}a_kx^{m+k}=0$$

$$\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k}+\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k}+\sum_{k=0}^{\infty}a_kx^{m+k+2}-n^2\sum_{k=0}^{\infty}a_kx^{m+k}=0$$

$$\sum_{k=0}^{\infty}a_k\left\{\left(m+k\right)\left(m+k-1\right)+\left(m+k\right)-n^2\right\}x^{m+k}+\sum_{k=0}^{\infty}a_kx^{m+k+2}=0$$

$$(m+k)(m+k-1)+(m+k)-n^2=(m+k)\{(m+k-1)\}-n^2=(m+k)(m+k)-n^2=(m+k)^2-n^2$$

$$\sum_{k=0}^{\infty}a_k\left\{\left(m+k\right)^2-n^2\right\}x^{m+k}+\sum_{k=0}^{\infty}a_kx^{m+k+2}=0 \quad \_\_\_(ii)$$

Equating to zero the coefficients to lowest degree term x^m in the expression (ii to zero by putting k = 0 in the first summation of (ii) we get the indicial equation

$$a_0\left(m^2-n^2\right)=0$$

$$\because a_0\neq 0\,,\, m^2-n^2=0$$

$$m=\pm n \qquad \_\_\_\_(iii)$$

Again, equating to zero the coefficient of x^{m+k+2} in expression (ii) to zero, we get the recurrence relation

$$a_{k+2}\left\{(m+k+2)^2-n^2\right\}+a_k=0$$

$$a_{k+2}\left\{(m+k+2)^2-n^2\right\}=-a_k$$

$$\Rightarrow\ a_{k+2}=\frac{-a_k}{(m+k+2)^2-n^2} \qquad \_\_\_\_(iv)$$

Equating the coefficients of x^{m+1} in expression (ii) to zero, we put k = 1 in the first summation of expression (ii)

$$a_1\left\{(m+1)^2-n^2\right\}=0$$

$$\Rightarrow\ a_1=0 \quad \because\ m+1\neq n$$

From equation (iv)

$$a_{k+2}=\frac{-a_k}{(m+k+2)^2-n^2}$$

For k = 0

$$a_2=\frac{-a_0}{(m+2)^2-n^2}$$

If k = 1 , $a_3 = 0$ $\because\ a_1=0$

For k = 2 $$a_4=\frac{-a_0}{\left\{(m+4)^2-n^2\right\}.\left\{(m+2)^2-n^2\right\}}$$

If k = 3 , $a_5 = 0 \qquad \because\ a_3 = 0$

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Put all of these coefficients in equation (*)

$$y=\sum_{k=0}^{\infty} a_k x^{m+k}$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots..$$

$$y = a_0x^m + a_2x^{m+2} + a_4x^{m+3} + \ldots.. \qquad \because\ \mathrm{a}_1 = a_3 = a_5 = \ldots = 0$$

$$y = a_0x^m - \frac{a_0}{(m+2)^2-n^2}x^{m+2} + \frac{a_0}{\left\{(m+4)^2-n^2\right\}\left\{(m+2)^2-n^2\right\}}x^{m+4} + \ldots$$

$$y = a_0x^m\left[1-\frac{x^2}{(m+2)^2-n^2}+\frac{x^4}{\left\{(m+4)^2-n^2\right\}\left\{(m+2)^2-n^2\right\}}+\ldots\right]$$

Case-I: If m = n

$$y = a_0x^n\left[1-\frac{x^2}{(n+2)^2-n^2}+\frac{x^4}{\left\{(n+4)^2-n^2\right\}\left\{(n+2)^2-n^2\right\}}+\ldots\right]$$

$$(n+2)^2-n^2 = n^2+4n+4-n^2 = 4(n+1) = 2^2(n+1).1!$$

$$(n+4)^2-n^2 = n^2+8n+16-n^2 = 8(n+2)$$

$$\left\{(n+4)^2-n^2\right\}\left\{(n+2)^2-n^2\right\} = 8(n+2).2^2(n+1) = 32(n+1)(n+2)$$

$$\left\{(n+4)^2-n^2\right\}\left\{(n+2)^2-n^2\right\} = 2^4.2!(n+1)(n+2)$$

$$y = a_0x^n\left[1-\frac{x^2}{2^2(n+1).1!}+\frac{x^4}{2^4.2!(n+1)(n+2)}+\ldots\right] \;\_\_\_\_\_(v)$$

Where a_0 is arbitrary constant.

Case-II: If m = −n

$$y = a_0 x^{-n}\left[1 - \frac{x^2}{2^2.1!(-n+1)} + \frac{x^4}{2^4.2!(-n+1)(-n+2)} + \ldots\right] \text{\_\_\_\_\_} (vi)$$

Where a_0 is arbitrary constant.

Bessel's Function $J_n(x)$:

The solution of Bessel's D.E from (v) is

$$y = a_0 x^n\left[1 - \frac{x^2}{2^2(n+1).1!} + \frac{x^4}{2^4.2!(n+1)(n+2)} + \ldots + (-1)^r \frac{x^{2r}}{2^{2r}.r!(n+1)(n+2)\ldots(n+r)} + \ldots\right]$$

$$y = a_0 x^n \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{2r}.r!(n+1)(n+2)\ldots(n+r)}$$

Where a_0 is an arbitrary constant. If $a_0 = \dfrac{1}{2^n \lceil n+1}$ then the above solution is called the Bessel's function of order 'n' and is denoted by $J_n(x)$.

$$J_n(x) = \frac{x^n}{2^n \lceil n+1} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r}.r!(n+1)(n+2)\ldots(n+r)}$$

$$J_n(x) = \left(\frac{x}{2}\right)^n . \frac{1}{\lceil n+1}\left[1 - \frac{x^2}{2^2(n+1).1!} + \frac{x^4}{2^4.2!(n+1)(n+2)} \ldots\right]$$

$$J_n(x) = \left(\frac{x}{2}\right)^n\left[\frac{1}{\lceil n+1} - \frac{1}{1!(n+1)\lceil n+1}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(n+1)\lceil n+1(n+2)}\left(\frac{x}{2}\right)^4 - \ldots\right]$$

$$J_n(x) = \left(\frac{x}{2}\right)^n\left[\frac{1}{\lceil n+1} - \frac{1}{1!\lceil n+2}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(n+2)\lceil n+2}\left(\frac{x}{2}\right)^4 - \ldots\right]$$

$$J_n(x) = \left(\frac{x}{2}\right)^n\left[\frac{1}{\lceil n+1} - \frac{1}{1!\lceil n+2}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\lceil n+3}\left(\frac{x}{2}\right)^4 - \ldots\right]$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\overline{\left|n+r+1\right.}} \left(\frac{x}{2}\right)^{2r}$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\overline{\left|n+r+1\right.}} \left(\frac{x}{2}\right)^{2r+n}$$

Or $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{2r+n}$$

This is called Bessel's function of order 'n'.

If n = 0

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(0+r)!} \left(\frac{x}{2}\right)^{2r+0} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,r!} \left(\frac{x}{2}\right)^{2r}$$

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \ldots$$

If n = 1 $$J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \ldots$$

Question: Show that the Bessel's function $J_n(x)$ is an even function when n is even and is odd function when n is odd.

Solution: As we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\overline{\left|n+r+1\right.}} \left(\frac{x}{2}\right)^{2r+n} \qquad \_\_\_\_(i)$$

Replace 'x' by '−x'

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\overline{\left|n+r+1\right.}} \left(\frac{-x}{2}\right)^{2r+n} \qquad \_\_\_\_(ii)$$

Case-I: If n is even then n+2r is also even. This implies that

$$\left(\frac{-x}{2}\right)^{2r+n} = \left(\frac{x}{2}\right)^{2r+n}$$

$$(ii) \Rightarrow J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil n+r+1} \left(\frac{x}{2}\right)^{2r+n}$$

$$J_n(-x) = J_n(x)$$

$\Rightarrow$ $J_n(x)$ is an even function when ‘n’ is even.

Case-II: If n is odd then n+2r is also odd. This implies that

$$\left(\frac{-x}{2}\right)^{2r+n} = -\left(\frac{x}{2}\right)^{2r+n}$$

$$(ii) \Rightarrow J_n(-x) = -\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil n+r+1} \left(\frac{x}{2}\right)^{2r+n}$$

$$J_n(-x) = -J_n(x)$$

$\Rightarrow$ $J_n(x)$ is an even function when ‘n’ is odd.

Question: Prove that $\underset{x\to 0}{Lim} \frac{J_n(x)}{x^n} = \frac{1}{2^n \lceil n+1}, (n > -1)$

Solution: As we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil n+r+1} \left(\frac{x}{2}\right)^{2r+n}$$

$$J_n(x) = \frac{x^n}{2^n \lceil n+1} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r}.r!(n+1)(n+2)...(n+r)}$$

Divide both side by x^n

$$\frac{J_n(x)}{x^n} = \frac{1}{2^n \lceil n+1} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r}.r!(n+1)(n+2)...(n+r)}$$

$$\frac{J_n(x)}{x^n} = \frac{1}{2^n \lceil n+1} \left[1 - \frac{x^2}{2^2(n+1).1!} + \frac{x^4}{2^4.2!(n+1)(n+2)} - ...\right]$$

Taking limit as x $\to$ 0

$$\lim_{x\to 0}\frac{J_n(x)}{x^n}=\frac{1}{2^n\,\overline{|n+1}}\lim_{x\to 0}\left[1-\frac{x^2}{2^2(n+1).1!}+\frac{x^4}{2^4.2!(n+1)(n+2)}-...\right]$$

$$\lim_{x\to 0}\frac{J_n(x)}{x^n}=\frac{1}{2^n\,\overline{|n+1}}[1-0+0-...]$$

$$\lim_{x\to 0}\frac{J_n(x)}{x^n}=\frac{1}{2^n\,\overline{|n+1}}$$

Lecture # 10

Question: Prove that

$$J_{-n}(x) = (-1)^n J_n(x)$$

Where n is a positive integer.

Solution: As we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

Replace 'n' by '−n' in above

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

$$J_{-n}(x) = \sum_{r=0}^{n-1} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} + \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

$$J_{-n}(x) = 0 + \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

Replace r = n + k in above summation

$$J_{-n}(x) = \sum_{n+k=n}^{\infty} \frac{(-1)^{n+k}}{(n+k)!\,\Gamma(-n+n+k+1)} \left(\frac{x}{2}\right)^{2n+2k-n}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^n(-1)^k}{(n+k)!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+n}$$

$$J_{-n}(x) = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+n} = (-1)^n J_n(x)$$

Question: Prove that

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}.\sin x$$

$$J_{\frac{-1}{2}}(x)=\sqrt{\frac{2}{\pi x}}.\cos x$$

Solution: As we know that

$$J_n(x)=\frac{x^n}{2^n\left\lceil n+1\right.}\left[1-\frac{x^2}{2^2(n+1).1!}+\frac{x^4}{2^4.2!(n+1)(n+2)}...\right]\text{——}(i)$$

Put n = ½ in above expression

$$J_{\frac{1}{2}}(x)=\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\left\lceil\frac{1}{2}+1\right.}\left[1-\frac{x^2}{2^2\left(\frac{1}{2}+1\right).1!}+\frac{x^4}{2^4.2!\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)}...\right]$$

$$J_{\frac{1}{2}}(x)=\frac{\sqrt{x}}{\sqrt{2}\left\lceil\frac{3}{2}\right.}\left[1-\frac{x^2}{2^2.\frac{3}{2}.1!}+\frac{x^4}{2^4.2!.\frac{3}{2}.\frac{5}{2}}-...\right]$$

$$J_{\frac{1}{2}}(x)=\frac{\sqrt{x}}{\sqrt{2}\left\lceil\frac{3}{2}\right.}\left[1-\frac{x^2}{6}+\frac{x^4}{120}-...\right]$$

$$J_{\frac{1}{2}}(x)=\frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}}.\frac{2}{x}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-...\right]$$

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

Put n = −½ in expression (i)

$$J_{\frac{-1}{2}}(x)=\frac{x^{\frac{-1}{2}}}{2^{\frac{-1}{2}}\left\lceil\frac{-1}{2}+1\right.}\left[1-\frac{x^2}{2^2\left(\frac{-1}{2}+1\right).1!}+\frac{x^4}{2^4.2!\left(\frac{-1}{2}+1\right)\left(\frac{-1}{2}+2\right)}\cdots\right]$$

$$J_{\frac{-1}{2}}(x)=\frac{\sqrt{2}}{\sqrt{x}\left\lceil\frac{1}{2}\right.}\left[1-\frac{x^2}{2^2.\frac{1}{2}}+\frac{x^4}{2^4.2.\frac{1}{2}.\frac{3}{2}}-\ldots\right]$$

$$J_{\frac{-1}{2}}(x)=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right]$$

$$J_{\frac{-1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

Exercise: Prove that

(i) $J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left\{\frac{\sin x}{x}-\cos x\right\}$

(ii) $J_{\frac{-3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left\{-\sin x-\frac{\cos x}{x}\right\}$

Solution: As we know that

$$J_n(x)=\frac{x^n}{2^n\left\lceil n+1\right.}\left[1-\frac{x^2}{2.2(n+1).1!}+\frac{x^4}{2.4.2^2(n+1)(n+2)}\cdots\right]\text{——}(A)$$

(i) Put n = 3/2 in (A)

$$J_{\frac{3}{2}}(x)=\frac{x^{\frac{3}{2}}}{2^{\frac{3}{2}}\left\lceil\frac{3}{2}+1\right.}\left[1-\frac{x^2}{2.2\left(\frac{3}{2}+1\right).1!}+\frac{x^4}{2.4.2^2\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)}\cdots\right]$$

$$J_{\frac{3}{2}}(x)=\frac{x\sqrt{x}}{2\sqrt{2}\,\Gamma\frac{5}{2}}\left[1-\frac{x^2}{2.2\frac{5}{2}}+\frac{x^4}{2.4.2^2.\frac{5}{2}.\frac{7}{2}}-...\right]$$

$$J_{\frac{3}{2}}(x)=\frac{x\sqrt{x}}{2\sqrt{2}\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}\left[1-\frac{x^2}{2.5}+\frac{x^4}{2.4.5.7}-...\right]$$

$$J_{\frac{3}{2}}(x)=\frac{x\sqrt{x}}{3\sqrt{2}\sqrt{\pi}}.\frac{x}{x}\left[1-\frac{x^2}{10}+\frac{x^4}{280}-...\right]$$

$$J_{\frac{3}{2}}(x)=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}.\frac{x^2}{3}\left[1-\frac{x^2}{10}+\frac{x^4}{280}-...\right]$$

$$J_{\frac{3}{2}}(x)=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[\frac{x^2}{3}-\frac{x^4}{30}+\frac{x^6}{840}-...\right]$$

$$J_{\frac{3}{2}}(x)=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[x^2\left(\frac{1}{2!}-\frac{1}{3!}\right)-x^4\left(\frac{1}{4!}-\frac{1}{5!}\right)+x^6\left(\frac{1}{6!}-\frac{1}{7!}\right)-...\right]$$

$$\because \frac{1}{2!}-\frac{1}{3!}=\frac{1}{3},\ \frac{1}{4!}-\frac{1}{5!}=\frac{1}{30},\ \frac{1}{6!}-\frac{1}{7!}=\frac{1}{840}$$

$$J_{\frac{3}{2}}(x)=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[\frac{x^2}{2!}-\frac{x^2}{3!}-\frac{x^4}{4!}+\frac{x^4}{5!}+\frac{x^6}{6!}-\frac{x^6}{7!}-...\right]$$

$$J_{\frac{3}{2}}(x)=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[1-1+\frac{x^2}{2!}-\frac{x^2}{3!}-\frac{x^4}{4!}+\frac{x^4}{5!}+\frac{x^6}{6!}-\frac{x^6}{7!}-...\right]$$

$$J_{\frac{3}{2}}(x)=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[\frac{1}{x}\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...\right\}-\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...\right\}\right]$$

$$J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x}-\cos x\right]$$

As we know that

$$J_n(x)=\frac{x^n}{2^n\overline{\left|n+1\right.}}\left[1-\frac{x^2}{2.2(n+1).1!}+\frac{x^4}{2.4.2^2(n+1)(n+2)}...\right]\text{\_\_\_\_}(A)$$

Multiplying and dividing by n+1

$$J_n(x)=\frac{x^n(n+1)}{2^n(n+1)\overline{\left|n+1\right.}}\left[1-\frac{x^2}{2.2(n+1)}+\frac{x^4}{2.4.2^2(n+1)(n+2)}-...\right]$$

$$J_n(x)=\frac{x^n(n+1)}{2^n\overline{\left|n+2\right.}}\left[1-\frac{x^2}{2.2(n+1)}+\frac{x^4}{2.4.2^2(n+1)(n+2)}-...\right]$$

Put n = −3/2

$$J_{\frac{-3}{2}}(x)=\frac{x^{\frac{-3}{2}}\left(\frac{-3}{2}+1\right)}{2^{\frac{-3}{2}}\overline{\left|\frac{-3}{2}+2\right.}}\left[1-\frac{x^2}{2.2\left(\frac{-3}{2}+1\right)}+\frac{x^4}{2.4.2^2\left(\frac{-3}{2}+1\right)\left(\frac{-3}{2}+2\right)}-...\right]$$

$$J_{\frac{-3}{2}}(x)=\frac{2\sqrt{2}\left(\frac{-1}{2}\right)}{x\sqrt{x}\sqrt{\pi}}\left[1-\frac{x^2}{2}+\frac{x^4}{8}-...\right]$$

$$J_{\frac{-3}{2}}(x)=\sqrt{\frac{2}{x\pi}}\left[-\frac{1}{x}\left\{1+x^2-\frac{x^2}{2!}-\frac{x^4}{3!}+\frac{x^4}{4!}...\right\}\right]$$

$$J_{\frac{-3}{2}}(x)=-\sqrt{\frac{2}{x\pi}}\left[-\frac{1}{x}\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-...\right\}-\frac{1}{x}\left\{x^2-\frac{x^4}{3!}+...\right\}\right]$$

$$J_{\frac{-3}{2}}(x)=\sqrt{\frac{2}{x\pi}}\left[-\frac{1}{x}\text{cosx}-\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}-...\right\}\right]$$

$$J_{\frac{-3}{2}}(x)=\sqrt{\frac{2}{x\pi}}\left[-\frac{\text{cosx}}{x}-\sin x\right]$$

Recurrence Formula:

(i) $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

(ii) $xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$

(iii) $2\,J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(iv) $2nJ_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)$

(v) $\dfrac{d}{dx}\left\{x^{-n}J_n(x)\right\} = -x^{-n}\,J_{n+1}(x)$

(vi) $\dfrac{d}{dx}\left\{x^{n}J_n(x)\right\} = x^{n}\,J_{n-1}(x)$

Proof: (i) As we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}$$

Differentiate w.r.t 'x'

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r(2r+n)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n-1}\cdot\frac{1}{2}$$

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r.2r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n-1}\cdot\frac{1}{2} + \sum_{r=0}^{\infty} \frac{(-1)^r.n}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n-1}\cdot\frac{1}{2}$$

Multiply both side by 'x'

$$xJ_n'(x) = x\sum_{r=0}^{\infty} \frac{(-1)^r.r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n-1} + n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n-1}\cdot\left(\frac{x}{2}\right)$$

$$xJ_n'(x) = x\sum_{r=1}^{\infty} \frac{(-1)^r.r}{r(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n-1} + n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}$$

Put r – 1 = s ⇒ r = 1+s

$$xJ_n'(x) = x\sum_{1+s=1}^{\infty} \frac{(-1)^{1+s}}{s!\,\Gamma(n+1+s+1)}\left(\frac{x}{2}\right)^{2+2s+n-1} + nJ_n(x)$$

$$xJ_n'(x) = x\sum_{s=0}^{\infty}\frac{(-1)^1.(-1)^s}{s!\overline{|n+s+2}}\left(\frac{x}{2}\right)^{2s+n+1} + nJ_n(x)$$

$$xJ_n'(x) = -x\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\overline{|s+(n+1)+1}}\left(\frac{x}{2}\right)^{2s+(n+1)} + nJ_n(x)$$

$$xJ_n'(x) = -xJ_{n+1}(x) + nJ_n(x)$$

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Proof: (ii) As we know that

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\overline{|n+r+1}}\left(\frac{x}{2}\right)^{2r+n}$$

Differentiate w.r.t 'x'

$$J_n'(x) = \sum_{r=0}^{\infty}\frac{(-1)^r(2r+n)}{r!\overline{|n+r+1}}\left(\frac{x}{2}\right)^{2r+n-1}.\frac{1}{2}$$

Multiply both side by 'x'

$$xJ_n'(x) = \sum_{r=0}^{\infty}\frac{(-1)^r(2r+2n-n)}{r!\overline{|n+r+1}}\left(\frac{x}{2}\right)^{2r+n-1}.\left(\frac{x}{2}\right)$$

$$xJ_n'(x) = \sum_{r=0}^{\infty}\frac{(-1)^r(2r+2n)}{r!\overline{|n+r+1}}\left(\frac{x}{2}\right)^{2r+n} - \sum_{r=0}^{\infty}\frac{(-1)^r n}{r!\overline{|n+r+1}}\left(\frac{x}{2}\right)^{2r+n}$$

$$xJ_n'(x) = \sum_{r=0}^{\infty}\frac{(-1)^r 2(n+r)}{r!(n+r)\overline{|n+r}}\left(\frac{x}{2}\right)^{2r+n} - n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\overline{|n+r+1}}\left(\frac{x}{2}\right)^{2r+n}$$

$$xJ_n'(x) = \sum_{r=0}^{\infty}\frac{(-1)^r 2}{r!\overline{|n+r}}\left(\frac{x}{2}\right)^{2r+n} - nJ_n(x)$$

$$xJ_n'(x) = x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\overline{|n+r}}\left(\frac{x}{2}\right)^{2r+n}\left(\frac{2}{x}\right) - nJ_n(x)$$

$$xJ_n'(x) = x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\left|\overline{n+r}\right.}\left(\frac{x}{2}\right)^{2r+n-1} - nJ_n(x)$$

$$xJ_n'(x) = x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\left|\overline{(n-1)+r+1}\right.}\left(\frac{x}{2}\right)^{2r+n-1} - nJ_n(x)$$

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$$

$$\Rightarrow\ xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$$

Proof: (iii) From the (i) and (ii) Recurrence formula

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \quad \_\_\_\_\_(i)$$

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \quad \_\_\_\_\_(ii)$$

Adding (i) and (ii)

$$2xJ_n'(x) = -xJ_{n+1}(x) + xJ_{n-1}(x)$$

Divide by 'x'

$$2J_n'(x) = -J_{n+1}(x) + J_{n-1}(x)$$

Proof: (iv) From the (i) and (ii) Recurrence formula

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \quad \_\_\_\_\_(i)$$

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \quad \_\_\_\_\_(ii)$$

Subtract (ii) from (i)

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

$$\pm xJ_n'(x) = \mp nJ_n(x) \pm xJ_{n-1}(x)$$

$$0 = 2nJ_n(x) - x\left(J_{n+1}(x) + J_{n-1}(x)\right)$$

$$2nJ_n(x) = x\left(J_{n+1}(x) + J_{n-1}(x)\right)$$

Proof: (v) As we know that

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Multiplying by x^{-n-1} both side

$$x^{-n-1}.xJ_n'(x) = x^{-n-1}.nJ_n(x) - x^{-n-1}.xJ_{n+1}(x)$$

$$x^{-n}J_n'(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

$$\frac{d}{dx}\left(x^{-n}J_n(x)\right) = -x^{-n}J_{n+1}(x)$$

Proof: (vi) As we know that

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Multiplying by x^{n-1} both side

$$x^{n-1}.xJ_n'(x) = x^{n-1}.nJ_n(x) - x^{n-1}.xJ_{n+1}(x)$$

$$x^{n}J_n'(x) - nx^{n-1}J_n(x) = -x^{n}J_{n+1}(x)$$

$$\frac{d}{dx}\left(x^{n}J_n(x)\right) = -x^{n}J_{n+1}(x)$$

Question: Find the value of $J_{-1}(x) + J_1(x)$

Solution: From the (iv) Recurrence formula

$$2nJ_n(x) = x\left(J_{n+1}(x) + J_{n-1}(x)\right)$$

Put n = 0

$$0 = x\left(J_1(x) + J_{-1}(x)\right)$$

$$J_1(x) + J_{-1}(x) = 0$$

Question: Express $J_5(x)$ in term of J_1 & J_2

Solution: From the (iv) Recurrence formula

$$2nJ_n(x) = x\left(J_{n+1}(x) + J_{n-1}(x)\right) \quad \_\_\_\_\_\_(*)$$

Put n = 4 in (*)

$$8J_4(x) = x\left(J_5(x) + J_3(x)\right)$$

$$J_5(x) = \frac{8}{x}J_4(x) - J_3(x) \qquad \_\_\_\_(i)$$

Put n = 3 in (*)

$$6J_3(x) = x\left(J_4(x) + J_2(x)\right)$$

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) \qquad \_\_\_\_(ii)$$

Put (ii) in (i) $\Rightarrow$ $$J_5(x) = \frac{8}{x}\left(\frac{6}{x}J_3(x) - J_2(x)\right) - J_3(x)$$

$$J_5(x) = \left(\frac{48}{x^2} - 1\right)J_3(x) - \frac{8}{x}J_2(x) \qquad \_\_\_\_(iii)$$

Put n = 2 in (*)

$$4J_2(x) = x\left(J_3(x) + J_1(x)\right)$$

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x) \qquad \_\_\_\_(iv)$$

Put (iv) in (iii) $$J_5(x) = \left(\frac{48}{x^2} - 1\right)\left[\frac{4}{x}J_2(x) - J_1(x)\right] - \frac{8}{x}J_2(x)$$

$$J_5(x) = \left(\frac{48}{x^2} - 1\right)\frac{4}{x}J_2(x) - \left(\frac{48}{x^2} - 1\right)J_1(x) - \frac{8}{x}J_2(x)$$

$$J_5(x) = \frac{4}{x}\left(\frac{48}{x^2} - 1 - 2\right)J_2(x) - \left(\frac{48}{x^2} - 1\right)J_1(x)$$

$$J_5(x) = \frac{4}{x}\left(\frac{48}{x^2} - 3\right)J_2(x) - \left(\frac{48}{x^2} - 1\right)J_1(x)$$

$$J_5(x) = \left(\frac{192}{x^3} - \frac{12}{x}\right)J_2(x) + \left(1 - \frac{48}{x^2}\right)J_1(x)$$

Question: Express $J_6(x)$ in term of J_0 & J_1

Solution: From the (iv) Recurrence formula

$$2nJ_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)$$

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \quad \_\_\_\_(*)$$

Put n = 5 in (*)

$$J_6(x) = \frac{10}{x}J_5(x) - J_4(x) \quad \_\_\_\_(i)$$

Put n = 4 in (*)

$$J_5(x) = \frac{8}{x}J_4(x) - J_3(x)$$

Put (i) $\Rightarrow$ $$J_6(x) = \frac{10}{x}\left[\frac{8}{x}J_4(x) - J_3(x)\right] - J_4(x)$$

$$J_6(x) = \left(\frac{80}{x^2} - 1\right)J_4(x) - \frac{10}{x}J_3(x) \quad \_\_\_\_\_(ii)$$

Put n = 3 in (*)

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x)$$

Put in (ii) $\Rightarrow$ $$J_6(x) = \left(\frac{80}{x^2} - 1\right)\left[\frac{6}{x}J_3(x) - J_2(x)\right] - \frac{10}{x}J_3(x)$$

$$J_6(x) = \left(\frac{480}{x^3} - \frac{16}{x}\right)J_3(x) - \left(\frac{80}{x^2} - 1\right)J_2(x) \quad \_\_\_\_(iii)$$

Put n = 2 in (*)

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x)$$

Put in (iii) $$J_6(x) = \left(\frac{480}{x^3} - \frac{16}{x}\right)\left[\frac{4}{x}J_2(x) - J_1(x)\right] - \left(\frac{80}{x^2} - 1\right)J_2(x)$$

$$J_6(x)=\left(\frac{1920}{x^4}-\frac{64}{x^2}-\frac{80}{x^2}+1\right)J_2(x)-\left(\frac{480}{x^3}-\frac{16}{x}\right)J_1(x)$$

$$J_6(x)=\left(\frac{1920}{x^4}-\frac{144}{x^2}+1\right)J_2(x)-\left(\frac{480}{x^3}-\frac{16}{x}\right)J_1(x) \quad \_\_\_\_\_\_(iv)$$

Put n = 1 in (*) $\Rightarrow$ $J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$

Put in (iv) $J_6(x)=\left(\frac{1920}{x^4}-\frac{144}{x^2}+1\right)\left(\frac{2}{x}J_1(x)-J_0(x)\right)-\left(\frac{480}{x^3}-\frac{16}{x}\right)J_1(x)$

$$J_6(x)=\left(\frac{3840}{x^5}-\frac{288}{x^3}+\frac{2}{x}-\frac{480}{x^3}+\frac{16}{x}\right)J_1(x)-\left(\frac{1920}{x^4}-\frac{144}{x^2}+1\right)J_0(x)$$

$$J_6(x)=\left(\frac{3840}{x^5}-\frac{768}{x^3}+\frac{18}{x}\right)J_1(x)-\left(\frac{1920}{x^4}-\frac{144}{x^2}+1\right)J_0(x)$$

Question: If n > − 1 , show that $\int_0^x x^{-n}J_{n+1}(x)\,dx=\frac{1}{2^n\overline{|n+1}}-x^{-n}J_n(x)$

Solution: As we know that $\frac{d}{dx}\left(x^{-n}J_n(x)\right)=-x^{-n}J_{n+1}(x)$

$$d\left(x^{-n}J_n(x)\right)=-x^{-n}J_{n+1}(x)\,dx$$

Integrate both side from 0 to x

$$\int_0^x d\left(x^{-n}J_n(x)\right)=-\int_0^x x^{-n}J_{n+1}(x)\,dx$$

$$\int_0^\infty x^{-n}J_{n+1}(x)\,dx=-\left[x^{-n}J_n(x)\right]_0^x$$

$$\int_0^\infty x^{-n}J_{n+1}(x)\,dx=-\left[x^{-n}J_n(x)-\lim_{x\to\infty}\frac{J_n(x)}{x^n}\right]$$

$$\int_0^\infty x^{-n}J_{n+1}(x)\,dx=-x^{-n}J_n(x)+\frac{1}{2^n\overline{|n+1}}$$

Lecture # 11

Question: Prove that $\int xJ_0^2(x)dx = \frac{1}{2}x^2\left[J_0^2(x)+J_1^2(x)\right]+c$

Solution: L.H.S = $\int xJ_0^2(x)dx = \frac{x^2}{2}J_0^2(x) - \int \frac{x^2}{2}.2J_0(x)J_0'(x)dx$

$$= \frac{x^2}{2}J_0^2(x) - \int x^2 J_0(x)\left(-J_1(x)\right)dx \quad \because \frac{d}{dx}J_0(x) = -J_1(x)$$

$$= \frac{x^2}{2}J_0^2(x) + \int xJ_0(x).xJ_1(x)dx \quad \_\_\_\_(i)$$

As $\frac{d}{dx}x^n J_n(x) = x^n J_{n-1}(x)$

Put n = 1

$$\frac{d}{dx}xJ_1(x) = xJ_0(x)$$

Put in (i)

$$= \frac{x^2}{2}J_0^2(x) + \int xJ_1(x).\frac{d}{dx}\left(xJ_1(x)\right)dx$$

$$= \frac{x^2}{2}J_0^2(x) + \frac{\left(xJ_1(x)\right)^2}{2} + c$$

$$= \frac{x^2}{2}J_0^2(x) + \frac{x^2}{2}J_1^2(x) + c$$

$$= \frac{x^2}{2}\left[J_0^2(x)+J_1^2(x)\right] + c = R.H.S$$

Question: Show that

(a) $$J_{n+3}(x)+J_{n+5}(x)=\frac{2}{x}(n+4)J_{n+4}(x)$$

(b) Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$

Solution: (a) As we know that

$$2nJ_n(x)=x\left(J_{n+1}(x)+J_{n-1}(x)\right)$$

$$\Rightarrow\ J_{n+1}(x)+J_{n-1}(x)=\frac{2}{x}nJ_n(x)$$

Put n = n+4

$$\Rightarrow\ J_{n+5}(x)+J_{n+3}(x)=\frac{2}{x}(n+4)J_{n+4}(x)$$

Solution: (b) As we know that $J_{n+1}(x)+J_{n-1}(x)=\frac{2}{x}nJ_n(x)$

$$J_{n+1}(x)=\frac{2}{x}nJ_n(x)-J_{n-1}(x)\quad \_\_\_\_(*)$$

Put n = 3 in (*) $\Rightarrow\ J_4(x)=\frac{6}{x}J_3(x)-J_2(x)\ \_\_\_\_(i)$

Put n = 2 in (*) $\Rightarrow\ J_3(x)=\frac{4}{x}J_2(x)-J_1(x)$

Put in (i) $\Rightarrow\ J_4(x)=\frac{6}{x}\left[\frac{4}{x}J_2(x)-J_1(x)\right]-J_2(x)$

$$J_4(x)=\frac{24}{x^2}J_2(x)-\frac{6}{x}J_1(x)-J_2(x)$$

$$J_4(x)=\left(\frac{24}{x^2}-1\right)J_2(x)-\frac{6}{x}J_1(x)\quad \_\_\_\_(ii)$$

Put n = 1 in (*) $\Rightarrow\ J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$

Put in (ii) $\Rightarrow$ $$J_4(x)=\left(\frac{24}{x^2}-1\right)\left[\frac{2}{x}J_1(x)-J_0(x)\right]-\frac{6}{x}J_1(x)$$

$$J_4(x)=\left(\frac{48}{x^3}-\frac{2}{x}\right)J_1(x)-\left(\frac{24}{x^2}-1\right)J_0(x)-\frac{6}{x}J_1(x)$$

$$J_4(x)=\left(\frac{48}{x^3}-\frac{2}{x}-\frac{6}{x}\right)J_1(x)+\left(1-\frac{24}{x^2}\right)J_0(x)$$

$$J_4(x)=\left(\frac{48}{x^3}-\frac{8}{x}\right)J_1(x)+\left(1-\frac{24}{x^2}\right)J_0(x)$$

Question: Prove that $J_2'(x)=\left(1-\frac{4}{x^2}\right)J_1(x)+\frac{2}{x}J_0(x)$

Solution: As we know that $xJ_n'(x)=-nJ_n(x)+xJ_{n-1}(x)$ $\because$ *by Recurrence II*

$$J_2'(x)=-\frac{n}{x}J_n(x)+J_{n-1}(x)$$

Put n = 2 $\Rightarrow$ $$J_2'(x)=-\frac{2}{x}J_2(x)+J_1(x) \quad \_\_\_\_\_(i)$$

Also, by Recurrence formula IV $\Rightarrow$ $J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x)$

$$J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$$

Put n = 1 $\Rightarrow$ $$J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$$

Put in (i) $\Rightarrow$ $$J_2'(x)=-\frac{2}{x}\left[\frac{2}{x}J_1(x)-J_0(x)\right]+J_1(x)$$

$$J_2'(x)=-\frac{4}{x^2}J_1(x)+\frac{2}{x}J_0(x)+J_1(x)$$

$$J_2'(x)=\left(1-\frac{4}{x^2}\right)J_1(x)+\frac{2}{x}J_0(x)$$

Lecture # 12

Rodrigues Formula:

$$P_n(x)=\frac{1}{2^n.n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \quad ; n=0,1,2,...$$

Put n = 0 ⇒ $$P_0(x)=\frac{1}{1.1}\left(x^2-1\right)^0=1$$

Put n = 1 ⇒ $$P_1(x)=\frac{1}{2^1.1}\frac{d}{dx}\left(x^2-1\right)^1=\frac{1}{2}.(2x)=x$$

Put n = 2 ⇒ $$P_2(x)=\frac{1}{4.2}\frac{d^2}{dx^2}\left(x^2-1\right)^2=\frac{1}{8}\frac{d}{dx}.2\left(x^2-1\right)(2x)$$

$$P_2(x)=\frac{1}{2}\frac{d}{dx}\left(x^3-x\right)=\frac{1}{2}\left(3x^2-1\right)$$

Proof: Let $v=\left(x^2-1\right)^n$ ____ (i)

Diff. w.r.t 'x'

$$\frac{dv}{dx}=n\left(x^2-1\right)^{n-1}.2x$$

$$\frac{dv}{dx}=n\left(x^2-1\right)^n.\left(x^2-1\right)^{-1}.2x=\frac{2nx\left(x^2-1\right)^n}{\left(x^2-1\right)}$$

$$\left(x^2-1\right)\frac{dv}{dx}=2nx\left(x^2-1\right)^n$$

$$\left(x^2-1\right)\frac{dv}{dx}=2nx.v \quad \text{\_\_\_\_}(ii) \quad \because by(i)$$

Leibnitz Rule $$\frac{d^n}{dx^n}(u.v)=u^n.\text{v}+\text{n}\text{u}^{n-1}.v'+\frac{n(n-1)}{2!}\text{u}^{n-2}.v''+...+u.v^n$$

Now diff. (ii) 'n+1" times by Leibnitz Rule

$$\left(x^2-1\right)\frac{d^{n+2}(v)}{dx^{n+2}}+(n+1)\frac{d^{n+1}(v)}{dx^{n+1}}.2x+\frac{(n+1).n}{2!}.\frac{d^n(v)}{dx^n}.2=2n\left\{x\frac{d^{n+1}(v)}{dx^{n+1}}+(n+1)\frac{d^n(v)}{dx^n}.1\right\}$$

$$\left(x^2-1\right)\frac{d^{n+2}(v)}{dx^{n+2}}+(n+1)\frac{d^{n+1}(v)}{dx^{n+1}}.2x-2nx\frac{d^{n+1}(v)}{dx^{n+1}}+(n+1).n\frac{d^n(v)}{dx^n}-2n(n+1)\frac{d^n(v)}{dx^n}=0$$

$$\left(x^2-1\right)\frac{d^{n+2}(v)}{dx^{n+2}}+(2nx+2x-2nx)\frac{d^{n+1}(v)}{dx^{n+1}}-n(n+1)\frac{d^n(v)}{dx^n}=0$$

$$\left(x^2-1\right)\frac{d^{n+2}(v)}{dx^{n+2}}+2x\frac{d^{n+1}(v)}{dx^{n+1}}-n(n+1)\frac{d^n(v)}{dx^n}=0$$

$$Put\ \frac{d^n(v)}{dx^n}=y$$

$$\left(x^2-1\right)y''+2xy'-n(n+1)y=0$$

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0 \quad \_\_\_(iii)$$

The solution of equation (iii) is called the Legendre polynomial. Therefore,

$$cy=P_n(x)$$

$$c\frac{d^n(v)}{dx^n}=P_n(x) \quad \_\_\_\_(iv)$$

Where c is constant. But $v=\left(x^2-1\right)^n$

$$v=\left((x+1)(x-1)\right)^n$$

$$v=(x+1)^n(x-1)^n$$

Now diff. n times by Leibnitz Rule.

$$\frac{d^n(v)}{dx^n}=(x+1)^n.\frac{d^n}{dx^n}(x-1)^n+n\left[n(x+1)^{n-1}.\frac{d^{n-1}}{dx^{n-1}}(x-1)^n+\ldots+(x-1)^n\frac{d^n}{dx^n}(x+1)^n\right]$$

____(v)

Now $$\frac{d}{dx}(x-1)^n = n(x-1)^{n-1}$$

$$\frac{d^2}{dx^2}(x-1)^n = n(n-1)(x-1)^{n-2}$$

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$$\frac{d^n}{dx^n}(x-1)^n = n(n-1)(n-2)...3.2.1 = n!$$

$$eq(v) \Rightarrow \frac{d^n(v)}{dx^n} = (x+1)^n.n! + n\left[n(x+1)^{n-1}.\frac{d^{n-1}}{dx^{n-1}}(x-1)^n + ... + (x-1)^n\frac{d^n}{dx^n}(x+1)^n\right]$$

Put x = 1 $\Rightarrow \left.\frac{d^n(v)}{dx^n}\right]_{x=1} = (1+1)^n.n! + 0 + 0 + ... + 0 = 2^n.n!$

Put x = 1 in (iv) $\Rightarrow \left. c.\frac{d^n(v)}{dx^n}\right]_{x=1} = P_n(1)$

$$c.2^n.n! = P_n(1) \quad \because \left.\frac{d^n(v)}{dx^n}\right]_{x=1} = 2^n.n!$$

$$c.2^n.n! = 1$$

$$c = \frac{1}{2^n.n!}$$

Put in (iv) $\Rightarrow \frac{1}{2^n.n!}.\frac{d^n(v)}{dx^n} = P_n(x)$

$$P_n(x) = \frac{1}{2^n.n!}.\frac{d^n}{dx^n}(x^2-1)^n \quad \because v = (x^2-1)^n$$

Question: Show that

(i) $\int_{-1}^{+1} P_n(x)dx = 0 \quad , \; n \neq 0$

(ii) $\int_{-1}^{+1} P_n(x)dx = 2 \quad , \; n = 0$

Solution: (i) By using the Rodrigues formula

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

$$\int_{-1}^{+1} P_n(x)dx = \int_{-1}^{+1} \frac{1}{2^n.n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \; dx$$

$$\int_{-1}^{+1} P_n(x)dx = \frac{1}{2^n.n!}\left[\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\right]_{-1}^{+1}$$

$$\int_{-1}^{+1} P_n(x)dx = \frac{1}{2^n.n!}\left[0-0\right] = 0$$

Solution: (ii) By using the Rodrigues formula

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

$$\text{For n} = 0 \Rightarrow P_0(x) = 1$$

$$\int_{-1}^{+1} P_0(x)dx = \int_{-1}^{+1} 1 dx = \left[x\right]_{-1}^{+1} = 1-(-1) = 2$$

Question: Let $P_n(x)$ be the Legendre polynomial of degree n. Show that for any function f(x), for which the nth derivative is continuous

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)^n}{2^n.n!}\int_{-1}^{1} f^n(x)(x^2-1)^n dx$$

Solution: By using the Rodrigues formula

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\int_{-1}^{1} f(x)P_n(x)dx = \int_{-1}^{1} f(x)\frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n\ dx$$

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n.n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}(x^2-1)^n\ dx$$

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n.n!}\left[f(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\Big|_{-1}^{1} - \int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\ dx\right]$$

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n.n!}\left[0 - \int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\ dx\right]$$

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)}{2^n.n!}\left[f'(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\Big|_{-1}^{1} - \int_{-1}^{1} f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\ dx\right]$$

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)}{2^n.n!}\left[0 - \int_{-1}^{1} f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\ dx\right]$$

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)^2}{2^n.n!}\left[\int_{-1}^{1} f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\ dx\right]$$

Similarly integrate n times by parts we get

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)^n}{2^n.n!}\int_{-1}^{1} f^{\text{n}}(x)(x^2-1)^n dx$$

Question: Given that $P_0(x)=1$ *and* $P_1(x)=x$. *Then*

Show that (i) $x^2=\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)$

(ii) $x^3=\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)$

Solution: (i) As we know that

$$P_2(x)=\frac{1}{2}\left(3x^2-1\right)$$

$$2P_2(x)=3x^2-1$$

$$3x^2=2P_2(x)+P_0(x) \qquad \because P_0(x)=1$$

$$x^2=\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)$$

Solution: (ii) As we know that

$$P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$$

$$2P_3(x)=\left(5x^3-3x\right)$$

$$5x^3=2P_3(x)+3x=2P_3(x)+3P_1(x) \qquad \because P_1(x)=x$$

$$x^3=\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)$$

Question: Express in terms of Legendre polynomial.

$$f(x)=4x^3-2x^2-3x+8$$

Solution: As we know that

$$x^3=\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)\ ,\ \ x^2=\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)$$

$$x=P_1(x)\,,\,1=P_0(x)$$

$$4x^3-2x^2-3x+8=4\left(\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)\right)-2\left(\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)\right)-3P_1(x)+8P_0(x)$$

$$4x^3-2x^2-3x+8=\frac{8}{5}P_3(x)+\frac{12}{5}P_1(x)-\frac{4}{3}P_2(x)-\frac{2}{3}\left(+\frac{3}{5}\right)-2P_0(x)-3P_1(x)+8P_0(x)$$

$$4x^3-2x^2-3x+8=\frac{8}{5}P_3(x)-\frac{4}{3}P_2(x)+\left(\frac{12}{5}-3\right)P_1(x)+\left(8-\frac{2}{3}\right)P_0(x)$$

$$4x^3-2x^2-3x+8=\frac{8}{5}P_3(x)-\frac{4}{3}P_2(x)-\frac{3}{5}P_1(x)+\frac{22}{3}P_0(x)$$

Question: Express in terms of Legendre polynomial.

$$f(x)=4x^3-6x^2-7x+2$$

Solution: As we know that

$$x^3=\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)\ ,\ \ x^2=\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)$$

$$x=P_1(x)\ ,\ 1=P_0(x)$$

$$4x^3-6x^2-7x+2=4\left(\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)\right)+6\left(\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)\right)+7P_1(x)+2P_0(x)$$

$$4x^3-6x^2-7x+2=\frac{8}{5}P_3(x)+\frac{12}{5}P_1(x)+\frac{12}{3}P_2(x)+2P_0(x)+7P_1(x)+2P_0(x)$$

$$4x^3-6x^2-7x+2=\frac{8}{5}P_3(x)+4P_2(x)+\left(\frac{12}{5}+7\right)P_1(x)+4P_0(x)$$

$$4x^3-6x^2-7x+2=\frac{8}{5}P_3(x)+4P_2(x)+\frac{47}{5}P_1(x)+4P_0(x)$$

Special results involving Legendre Polynomial:

Theorem: Prove that

(i) $P_n(1)=1$

(ii) $P_n(-1)=(-1)^n$

(iii) $P_n(-x)=(-1)^n P_n(x)$

Proof: (i) As we know that the generating function for Legendre polynomial is

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty}P_n(x).z^n$$

Put x = 1

$$\frac{1}{\sqrt{1-2z+z^2}}=\sum_{n=0}^{\infty}P_n(1).z^n$$

$$(1-2z+z^2)^{\frac{-1}{2}}=\sum_{n=0}^{\infty}P_n(1).z^n$$

$$\left((1-z)^2\right)^{\frac{-1}{2}}=\sum_{n=0}^{\infty}P_n(1).z^n$$

$$(1-z)^{-1}=\sum_{n=0}^{\infty}P_n(1).z^n$$

$$1+z+z^2+z^3+\ldots=\sum_{n=0}^{\infty}P_n(1).z^n$$

$$\sum_{n=0}^{\infty}z^n=\sum_{n=0}^{\infty}P_n(1).z^n$$

Compare the coefficient of z^n

$$P_n(1)=1$$

Proof: (ii) As we know that the generating function for Legendre polynomial is

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty}P_n(x).z^n$$

Put x = −1

$$\frac{1}{\sqrt{1+2z+z^2}}=\sum_{n=0}^{\infty}P_n(-1).z^n$$

$$\left(1+2z+z^2\right)^{\frac{-1}{2}}=\sum_{n=0}^{\infty}P_n(-1).z^n$$

$$\left(\left(1+z\right)^2\right)^{\frac{-1}{2}}=\sum_{n=0}^{\infty}P_n(-1).z^n$$

$$\left(1+z\right)^{-1}=\sum_{n=0}^{\infty}P_n(-1).z^n$$

$$1-z+z^2-z^3+\ldots=\sum_{n=0}^{\infty}P_n(-1).z^n$$

$$\sum_{n=0}^{\infty}z^n=\sum_{n=0}^{\infty}P_n(1).z^n$$

Compare the coefficient of z^n

$$P_n(-1)=(-1)^n$$

Proof: (iii) As we know that the generating function for Legendre polynomial is

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty}P_n(x).z^n \quad \_\_\_\_\_(i)$$

Put x = −x

$$\frac{1}{\sqrt{1+2xz+z^2}}=\sum_{n=0}^{\infty}P_n(-x).z^n \qquad \_\_\_\_(ii)$$

Put z = − z in (i)

$$\frac{1}{\sqrt{1+2xz+z^2}}=\sum_{n=0}^{\infty}P_n(x).(-z)^n=\sum_{n=0}^{\infty}(-1)^n P_n(x).z^n \qquad \_\_\_\_(iii)$$

Compare (ii) and (iii)

$$\sum_{n=0}^{\infty}P_n(-x).z^n=\sum_{n=0}^{\infty}(-1)^n P_n(x).z^n$$

Compare the coefficient of z^n

$$P_n(-x)=(-1)^n P_n(x)$$

Question: Prove that

(i) $\quad P_n'(1)=\frac{1}{2}n(n+1)$

(ii) $\quad P_n'(-1)=(-1)^{n+1}.\frac{1}{2}n(n+1)$

Proof: (i) The Legendre equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

$$(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0 \qquad \_\_\_\_(i)$$

Put x = 1 in (i)

$$(1-1)P_n''(1)-2P_n'(1)+n(n+1)P_n(1)=0$$

$$-2P_n'(1)+n(n+1)P_n(1)=0$$

$$P_n'(1)=\frac{1}{2}n(n+1) \qquad \because P_n(1)=1$$

Proof: (ii) The Legendre equation is

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0$$

$$\left(1-x^2\right)P_n''\left(x\right)-2xP_n'\left(x\right)+n\left(n+1\right)P_n\left(x\right)=0 \quad \_\_\_\_(i)$$

Put x = −1 in (i)

$$\left(1-1\right)P_n''\left(-1\right)+2P_n'\left(-1\right)+n\left(n+1\right)P_n\left(-1\right)=0$$

$$2P_n'\left(-1\right)+n\left(n+1\right)P_n\left(-1\right)=0$$

$$P_n'\left(-1\right)=\frac{-1}{2}n\left(n+1\right).\left(-1\right)^n \qquad \because P_n\left(-1\right)=\left(-1\right)^n$$

Question: Prove that $\sum_{n=0}^{\infty}P_n\left(x\right)=\frac{1}{\sqrt{2-2x}}$

Proof: As we know that

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty}P_n\left(x\right).z^n$$

Put z = 1

$$\frac{1}{\sqrt{1-2x+1}}=\sum_{n=0}^{\infty}P_n\left(x\right).\left(1\right)^n$$

$$\sum_{n=0}^{\infty}P_n\left(x\right)=\frac{1}{\sqrt{2-2x}}$$

Question: Prove that $P_n'\left(-x\right)=\left(-1\right)^{n+1}.P_n'\left(x\right)$

Proof: As we know that

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty}P_n\left(x\right).z^n$$

$$\sum_{n=0}^{\infty} P_n(x).z^n = \left(1-2xz+z^2\right)^{\frac{-1}{2}}$$

Diff. w.r.t ‘x’ we get

$$\sum_{n=0}^{\infty} P_n'(x).z^n = \frac{-1}{2}\left(1-2xz+z^2\right)^{\frac{-3}{2}}(-2z)$$

$$\sum_{n=0}^{\infty} P_n'(x).z^n = z\left(1-2xz+z^2\right)^{\frac{-3}{2}} \qquad \_\_\_\_(i)$$

Replace ‘x’ by ‘−x’ in (i)

$$\sum_{n=0}^{\infty} P_n'(-x).z^n = z\left(1+2xz+z^2\right)^{\frac{-3}{2}}$$

Replace ‘z’ by ‘−z’ in (i)

$$\sum_{n=0}^{\infty} P_n{}'(x).(-z)^n = -z\left(1+2xz+z^2\right)^{\frac{-3}{2}}$$

$$\sum_{n=0}^{\infty} (-1)^n P_n'(x).z^n = -z\left(1+2xz+z^2\right)^{\frac{-3}{2}}$$

Multiplying by (−1)

$$\sum_{n=0}^{\infty} (-1)^{n+1} P_n'(x).z^n = z\left(1+2xz+z^2\right)^{\frac{-3}{2}} \qquad \_\_\_\_(iii)$$

Compare (ii) and (iii)

$$\sum_{n=0}^{\infty} P_n'(-x).z^n = \sum_{n=0}^{\infty} (-1)^{n+1} P_n'(x).z$$

Compare the coefficient of z^n

$$P_n'(-x) = (-1)^{n+1}.P_n'(x)$$

Question: Prove that

(i) $P_{2n+1}(0)=0$

(ii) $P_{2n}(0)=(-1)^n.\dfrac{2n!}{2^{2n}(n!)^2}$

(iii) $P'_{2n}(0)=0$

(iv) $P'_{2n+1}(0)=(-1)^n.\dfrac{(2n+1)!}{2^{2n}(n!)^2}$

Proof: (i) As we know that the generating function for Legendre polynomial is

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty}P_n(x).z^n \quad \_\_\_\_(*)$$

Put x = 0

$$\frac{1}{\sqrt{1+z^2}}=\sum_{n=0}^{\infty}P_n(0).z^n$$

$$\sum_{n=0}^{\infty}P_n(0).z^n=\left(1+z^2\right)^{\frac{-1}{2}}=\left(1-\left(-z^2\right)\right)^{\frac{-1}{2}}$$

$$\left(1-\left(-z^2\right)\right)^{\frac{-1}{2}}=1+\frac{1}{2}\left(-z^2\right)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}\left(-z^2\right)^2-\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}\left(-z^2\right)^3+...$$

$$\sum_{n=0}^{\infty}P_n(0).z^n=1-\frac{1}{2}z^2+\frac{\frac{1}{2}.\frac{3}{2}}{2}z^4-\frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{6}z^6+...$$

$$\sum_{n=0}^{\infty}P_n(0).z^n=1-\frac{1}{2}z^2+\frac{1.3}{2.4}z^4-\frac{1.3.5}{2.4.6}z^6+...+(-1)^n\frac{1.3.5...(2n-1)}{2.4.6...2n}z^{2n}+... \quad \_\_\_(i)$$

Since the R.H.S of above equation consist of even power of z. So, equate the coefficient of z^{2n+1}

$$P_{2n+1}(0)=0 \quad \text{i.e. no odd power in R.H.S}$$

Proof: (ii) Now equating the coefficient of z^{2n} on both side of equation (i)

$$P_{2n}=(-1)^n\frac{1.3.5...(2n-1)}{2.4.6...2n}$$

$$P_{2n}=(-1)^n\frac{1.3.5...(2n-1)(2.4.6...2n)}{2.4.6...2n(2.4.6...2n)}$$

$$P_{2n}=(-1)^n\frac{2n!}{(2.4.6...2n)^2}$$

$$P_{2n}=(-1)^n\frac{2n!}{2^{2n}(1.2.3...n)^2}$$

$$P_{2n}=(-1)^n\frac{2n!}{2^{2n}(n!)^2}$$

Proof: (iii) Diff. eq(*) w.r.t 'x'

$$\sum_{n=0}^{\infty}P_n'(x).z^n=\frac{-1}{2}\left(1-2xz+z^2\right)^{\frac{-3}{2}}(-2z)$$

$$\sum_{n=0}^{\infty}P_n'(x).z^n=z\left(1-2xz+z^2\right)^{\frac{-3}{2}}$$

Put x = 0

$$\sum_{n=0}^{\infty}P_{2n}'(0).z^n=z\left(1+z^2\right)^{\frac{-3}{2}}=z\left(1-\left(-z^2\right)\right)^{\frac{-3}{2}}$$

$$\sum_{n=0}^{\infty}P_{2n}'(0).z^n=z\left[1+\frac{3}{2}\left(-z^2\right)+\frac{\left(\frac{-3}{2}\right)\left(\frac{-3}{2}-1\right)}{2!}\left(-z^2\right)^2-\frac{\left(\frac{-3}{2}\right)\left(\frac{-3}{2}-1\right)\left(\frac{-3}{2}-2\right)}{3!}\left(-z^2\right)^3+...\right]$$

$$\sum_{n=0}^{\infty} P'_{2n}(0).z^n = z\left[1-\frac{3}{2}z^2+\frac{\frac{3}{2}.\frac{5}{2}}{2}z^4-\frac{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}{2}z^6+\ldots\right]$$

$$\sum_{n=0}^{\infty} P'_{2n}(0).z^n = \left[z-\frac{3}{2}z^3+\frac{3.5}{2.4}z^5-\frac{3.5.7}{2.4.6}z^7+\ldots+(-1)^n\frac{3.5.7\ldots(2n+1)}{2.4.6\ldots2\text{n}}z^{2n+1}+\ldots\right] \quad (ii)$$

Since R.H.S of above equation consist of odd power of z. So, equate the coefficient of z^{2n}

$$P'_{2n}(0)=0 \text{ i.e. no even power}$$

Proof: (iv) Now equating the coefficient of z^{2n+1} on both side of equation (ii)

$$P'_{2n+1}(0)=(-1)^n\frac{3.5.7\ldots(2n+1)}{2.4.6\ldots2n}$$

$$P'_{2n+1}(0)=(-1)^n\frac{\left[3.5.7\ldots(2n+1)\right](2.4.6\ldots2n)}{(2.4.6\ldots2n)(2.4.6\ldots2n)}$$

$$P'_{2n+1}(0)=(-1)^n\frac{(2n+1)2n\ldots7.6.5.4.3.2.1}{(2.4.6\ldots2n)^2}$$

$$P'_{2n+1}(0)=(-1)^n\frac{(2n+1)!}{2^{2n}(1.2.3\ldots n)^2}$$

$$P'_{2n+1}(0)=(-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2}$$

Legendre's Equation:

The differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0 \quad \_\_\_\_(i)$$

is called the Legendre's Differential equation. The above equation can also be written as

$$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dy}{dx}\right\}+n\left(n+1\right)y=0$$

Solution of Legendre's Equation:

$$\text{Let} \quad y=\sum_{k=0}^{\infty}a_k x^{m+k} \quad \_\_\_\_(*) \; ; a_0 \neq 0$$

$$Here \;\; a_{-1}=a_{-2}=....=a_{-n}=0$$

$$y'=\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k-1}$$

$$y''=\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}$$

Put all these values in (i)

$$\left(1-x^2\right)\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}-2x\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k-1}+n\left(n+1\right)\sum_{k=0}^{\infty}a_k x^{m+k}=0$$

$$\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}-\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k}$$

$$-2\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k}+n\left(n+1\right)\sum_{k=0}^{\infty}a_k x^{m+k}=0$$

$$\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}-\sum_{k=0}^{\infty}a_k\left[\left(m+k\right)\left(m+k-1\right)+2\left(m+k\right)-n\left(n+1\right)\right]x^{m+k}=0$$

Let

$$(m+k)(m+k-1)+2(m+k)-n(n+1)=(m+k)(m+k+1)-n(n+1)$$

$$\sum_{k=0}^{\infty}a_k(m+k)(m+k-1)x^{m+k-2}-\sum_{k=0}^{\infty}a_k\left[(m+k)(m+k+1)-n(n+1)\right]x^{m+k}=0$$

Equating to zero the coefficients to lowest degree term x^{m-2} by putting k = 0 in the first summation

$$a_0m(m-1)=0$$

$$\because a_0\neq 0\ ,\ m(m-1)=0$$

$$m=0\ ,\ m=1$$

Now equate to zero the coefficient of x^{m+k-2} for recurrence relation

$$a_k(m+k)(m+k-1)-a_{k-2}(m+k-2)(m+k-2+1)-n(n+1)=0$$

$$a_k(m+k)(m+k-1)-a_{k-2}\left\{(m+k-2)^2+(m+k-2)-n(n+1)\right\}=0$$

$$a_k=\frac{(m+k-2)^2+(m+k-2)-n(n+1)}{(m+k)(m+k-1)}a_{k-2}$$

Equating the coefficients of x^{m-1} for a_1 by putting r = 1 in first summation of expression

$$a_1(m+1)(m+1-1)=0$$

$$a_1m(m+1)=0$$

a_1 may or may not zero when m = 0 and a_1 is zero when m = 1

Case-I: When m = 0

$$a_k=\frac{(k-2)^2+(k-2)-n(n+1)}{(k)(k-1)}a_{k-2}$$

$$a_{-1}=a_{-2}=....=a_{-n}=0$$

For k = 2
$$a_2 = \frac{(2-2)^2 + (2-2) - n(n+1)}{(2)(2-1)} a_{2-2}$$

$$a_2 = \frac{-n(n+1)}{2!} a_0$$

For k = 3
$$a_3 = \frac{(3-2)^2 + (3-2) - n(n+1)}{(2)(3-1)} a_{3-2}$$

$$a_3 = \frac{1+1-n(n+1)}{3.2} a_1$$

$$a_3 = \frac{2-n(n+1)}{3!} a_1$$

$$a_3 = -\frac{n^2+2n-n-2}{3!} a_1 = -\frac{n(n+2)-1(n+2)}{3!} a_1$$

$$a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

For k = 4
$$a_4 = \frac{4+2-n(n+1)}{4.3} a_2$$

$$a_4 = -\frac{n(n+1)-6}{4.3} a_2 = -\frac{n^2+n-6}{4.3} . \frac{-n(n+1)}{2!} a_0$$

$$a_4 = \frac{n^2+3n-2n-6}{4.3.2!} n(n+1) a_0$$

$$a_4 = \frac{n(n-2)(n+1)(n+3)}{4!} a_0$$

For k = 5
$$a_5 = \frac{9+3-n(n+1)}{5.4} a_3$$

$$a_5 = -\frac{n(n+1)-12}{5.4} . -\frac{(n-1)(n+2)}{3!} a_1$$

$$a_5 = \frac{\left(n^2+n-12\right)(n-1)(n+2)}{5.4.3!}a_1$$

$$a_5 = \frac{\left(n^2+4n-3n-12\right)(n-1)(n+2)}{5.4.3!}a_1$$

$$a_5 = \frac{\left(n(n+4)-3(n+4)\right)(n-1)(n+2)}{5!}a_1$$

$$a_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1$$

From equation (ii) when m = 0

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Put the value of a_1, a_2 ,...

$$y = a_0 + a_1 x + \left(\frac{-n(n+1)}{2!}a_0\right)x^2 + \left(-\frac{(n-1)(n+2)}{3!}a_1\right)x^3 + \left(\frac{n(n-2)(n+1)(n+3)}{4!}a_0\right)x^4$$

$$+\left(\frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1\right)x^5 + \ldots$$

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}a_0 x^4 + \ldots\right]$$

$$+a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \ldots\right]$$

Thus, for any integer $n \geq 0$ the above equation has a polynomial solution. These polynomials are multiplied by some constants called Legendre's polynomial of order n and denoted by $P_n(x)$.

Case-I: When m = 1 $a_k = \dfrac{(1+k-2)^2 + (1+k-2) - n(n+1)}{(1+k)(1+k-1)} a_{k-2}$

$$a_k = \frac{(k-1)^2 + (k-1) - n(n+1)}{k(1+k)} a_{k-2}$$

$$a_{-1} = a_{-2} = ... = a_{-n} = 0$$

For k = 2 $a_2 = \dfrac{1+1-n(n+1)}{2.3} a_0 = -\dfrac{n^2+n-2}{3.2} a_0$

$$a_2 = \frac{-(n-1)(n+2)}{3!} a_0$$

For k = 4 $a_4 = \dfrac{9+3-n(n+1)}{4.5} a_2$

$$a_4 = -\frac{n^2+n-12}{5.4} . \frac{-(n-1)(n+2)}{3!} a_0$$

$$a_4 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_0$$ And so on.

From equation (ii) when m = 1 $y = \sum_{k=0}^{\infty} a_k x^{1+k} = \sum_{k=0}^{\infty} a_k x . x^k$

$$y = x \sum_{k=0}^{\infty} a_k x^k \qquad \Rightarrow \; y = x\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right]$$

$$y = x\left[a_0 + a_2 x^2 + a_4 x^4 + ...\right] \qquad \because a_1 = a_3 = a_5 = ... = 0$$

$$y = x\left[a_0 + \left(\frac{-(n-1)(n+2)}{3!} a_0\right)x^2 + \left(\frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_0\right)x^4 + ...\right]$$

$$y = a_0 x\left[1 - \left(\frac{(n-1)(n+2)}{3!}\right)x^2 + \left(\frac{(n-1)(n-3)(n+2)(n+4)}{5!}\right)x^4 + ...\right]$$

Lecture # 13

Some Modern Special functions:

- Wright Function
- Mittag-Leffler Function
- Dini Function
- Struve Function
- Lommel Function
- Hyper-Bessel Function

(i) Wright Function

The Wright function is

$$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n+\mu)}\ ,\ \lambda>-1,\mu\in\mathbb{C}$$

Normalization:

Any function f(x) is said to be Normalized if

$$f(0)=0\ \ and\ \ f^{'}(0)=1$$

e.g.

$$K(z)=\frac{z}{z-1}$$

$$K(z)=z.\frac{1}{z-1}=z\left\{1+z+z^2+z^3+\ldots\right\}$$

$$K(z)=z+z^2+z^3+\ldots$$

$$K(0)=0$$

$$\text{Now }\ K^{'}(z)=1+2z+3z^2+\ldots$$

$$K^{'}(0)=1+0$$

$$K^{'}(0)=1$$

Hence, $K(z)=\dfrac{z}{z-1}$ is normalized.

Normalized form of Wright Function:

As $$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n+\mu)}\ ,\ \lambda>-1,\mu\in\mathbb{C}$$

$$W_{\lambda,\mu}(z)=\frac{1}{\Gamma\mu}+\sum_{n=1}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n+\mu)}$$

Multiplying by $\Gamma\mu\ z$

$$z\Gamma\mu\ W_{\lambda,\mu}(z)=z+\sum_{n=1}^{\infty}\frac{z\Gamma\mu z^n}{n!\,\Gamma(\lambda n+\mu)}$$

$$W_{\lambda,\mu}(z)=z+\sum_{n=1}^{\infty}\frac{\Gamma\mu z^{n+1}}{n!\,\Gamma(\lambda n+\mu)}\quad where\ \ W_{\lambda,\mu}(z)=z\Gamma\mu$$

Deduction:

If we put $\lambda=1\ and\ \mu=v+1\ \&\ z=\frac{-z^2}{4}$ multiplying by $\left(\frac{z}{2}\right)^2$ then the Wright function converted into Bessel function

Put $\lambda=1\ and\ \mu=v+1\ \&\ z=\frac{-z^2}{4}$

$$W_{1,v+1}\left(\frac{-z^2}{4}\right)=\sum_{n=0}^{\infty}\frac{\left(\frac{-z^2}{4}\right)^n}{n!\,\Gamma(n+v+1)}$$

$$W_{1,v+1}\left(\frac{-z^2}{4}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+v+1)}\left[\left(\frac{z}{4}\right)^2\right]^n$$

multiplying by $\left(\frac{z}{2}\right)^2$

$$W_{1,v+1}\left(\frac{-z^2}{4}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+v+1)}\left(\frac{z}{4}\right)^{2n}\cdot\left(\frac{z}{2}\right)^{2}$$

$$W_{1,v+1}\left(\frac{-z^2}{4}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+v+1)}\left(\frac{z}{4}\right)^{2n+2}$$

Which is the Bessel Function.

(ii) Mittag-Leffler Function:

In 1903 the Mittag-Leffler function $E_\alpha(z)$ is define as

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}\ ;\alpha\in\mathbb{C},R_e(\alpha)>0$$

In 1905 Wiman generalize this function

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\ ;\alpha,\beta\in\mathbb{C},R_e(\alpha)>0,R_e(\beta)>0$$

In 1997 Perbhakker generalize this function

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}.\frac{z^n}{n!}\ ;\alpha,\beta\in\mathbb{C}\,\&R_e(\alpha)>0,R_e(\beta)>0$$

Normalized form of Mittag-Leffler function:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$$

$$E_{\alpha,\beta}(z)=\frac{1}{\Gamma\beta}+\sum_{n=1}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$$

Multiplying by $\Gamma\beta.z$

$$\Gamma\beta.zE_{\alpha,\beta}(z)=z+\sum_{n=1}^{\infty}\frac{\Gamma\beta.z.z^n}{\Gamma(\alpha n+\beta)}$$

$$E_{\alpha,\beta}(z)=z+\sum_{n=1}^{\infty}\frac{\Gamma\beta\, z^{n+1}}{\Gamma(\alpha n+\beta)} \quad where\ E_{\alpha,\beta}(z)=\Gamma\beta . zE_{\alpha,\beta}(z)$$

is the normalized form of Mittag-Leffler function.

Question: Show that $E_{1,1}(z)=ze^{z}$

Solution: As we know that $E_{\alpha,\beta}(z)=z+\sum_{n=1}^{\infty}\frac{\Gamma\beta\, z^{n+1}}{\Gamma(\alpha n+\beta)}$

$$Put\ \ \alpha=\beta=1$$

$$E_{1,1}(z)=z+\sum_{n=1}^{\infty}\frac{\Gamma 1 . z^{n+1}}{\Gamma(n+1)}$$

$$E_{1,1}(z)=z+\sum_{n=1}^{\infty}\frac{z^{n+1}}{n!}$$

$$E_{1,1}(z)=z+z^{2}+\frac{z^{3}}{2!}+\frac{z^{4}}{3!}+\ldots$$

$$E_{1,1}(z)=z\left[1+z+\frac{z^{2}}{2!}+\frac{z^{3}}{3!}+\ldots\right]$$

$$E_{1,1}(z)=ze^{z}$$