SPECIAL THEORY OF RELATIVITY

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Preface

"Try not to become a man of success. Rather become a man of value."

Albert Einstein

…………………………………………………..

This effort is dedicated to our dear teacher **"Sir Tahir Nazir"** who always blessed us and tell us the right path about every matter of life.

Few words for my working, as according to **Albert Einstein**

"Time and space are individually relative, spacetime as a whole is absolute. The faster you move through space the slower you move through time. Still both individual motions always add up to the speed of light."

Muhammad Usman Hamid

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"You're more Special than Relativity"

Muhammad Zeeshan Ahmad

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HISTORICAL BACKGROUND AND FUNDAMENTAL CONCEPTS OF SPECIAL THEORY OF RELATIVITY

Special theory of relativity developed by Einstein in 1905, after the failure of Newton's laws, when he was dealing with the relative motion in space. The theory of relativity deals with relations which exist between physical quantities (such as mass of a particle, length of a rod, electric field at a point etc.) as they appear to different observers in relative motion. The observers considered in this book are restricted to those in inertial frames of reference. The theory is then called **restricted theory** or **Special Theory of Relativity (STR).** When no such restriction is made, the theory is called the **General Theory of Relativity**. In 1915, Einstein developed General Theory of Relativity.

Relativity: The study of motion of one body with respect to another body.

Relative motion (Absolute motion)

A motion of particles relative to the reference point without any external source.

Law of Universal Gravity: This law was given by Newton, according "Every body attracts every other body with a force proportional to the product of their masses and inversely proportional to the square of the distance between them."

Frame of Reference

The system in which the clock and the meter scale used for the measurement are at rest. Such coordinate system is called a ***frame of reference***.

There are two types of frame of references

- Inertial frames of reference
- Non – inertial frames of reference

Newton's first law of motion (Galileo's Law)

An object continues its state of rest or of uniform motion in a straight line provided no net force or external force act on it. It is also called Law of inertia.

Inertial Frames of Reference

Inertial frame of reference is that in which the law of inertia (Newton's first law of motion) holds, that is a frame in which a body that is acted upon by zero net external force moves with a constant velocity.

The law of inertia holds in any frame of reference, which happens to move with a constant velocity relative to a given inertial frame. Therefore, any frame of reference, which moves with a constant velocity relative to an inertial frame, is also an inertial frame. These frames are non – accelerated. i.e. $\vec{a} = 0$

Examples

- A frame of reference fixed with respect to the stars is an inertial frame.
- A spaceship drifting in outer space without spinning and with its engines shut off would be an ideal inertial frame.
- However for all practical purposes, any frame of reference fixed to the earth such as a railway station or a laboratory can be taken as an inertial frame. Thus a railway station is an inertial frame and a train travelling at constant velocity with respect to the railway station is also an inertial frame.

Non – Inertial Frames of Reference

Non – Inertial frame of reference is that in which the law of inertia (Newton's first law of motion) does not holds, that is a frame in which a body that is acted upon by zero net external force does not moves with a constant velocity. i.e. velocity remains change. E.g. person sitting in a moving train.

The Principle of Relativity (Galilean Invariance)

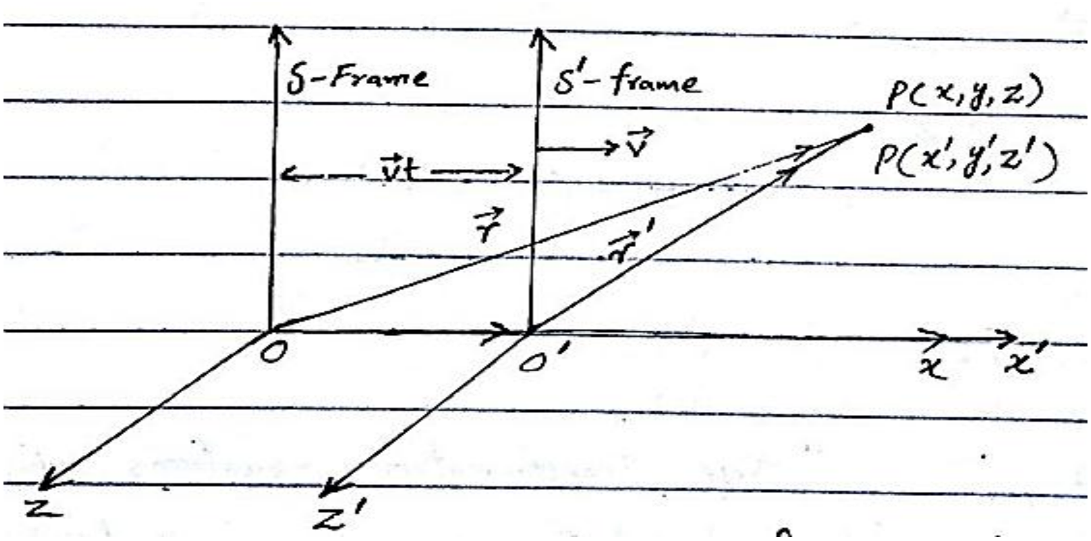
The Principle of Relativity (PR) applies to inertial frames of reference. This principle states that ***the laws of Physics take the same mathematical form in all inertial frames.***

Or the basic laws of Physics are identical in all frames of reference which are moving with uniform velocity (unaccelerated) relative to one another.

Or It is impossible by using any physical law to distinguish between inertial frames.

GALILEAN TRANSFORMATION (G.T) / NEWTONIAN TRANSFORMATION

This is the set of equations in classical physics that relate the space and time coordinates of two systems moving at a constant velocity relative to each other.



Consider two inertial frames S and S'. Frame S' is moving with a uniform velocity $\vec{v}$ in the $x-$ direction with respect to frame S while S is at rest. Originally both frame of references were at rest.

Suppose a particle is present at point P. Let the coordinates of P be denoted by $\overrightarrow{OP} = \vec{r} = (x, y, z)$ in frame S and by $\overrightarrow{O'P} = \vec{r}' = (x', y', z')$ in frame S'.

When both frames coincide then $t = t'$

Since S frame is at rest and S' is moving with a uniform velocity $\vec{v}$ then O' be the position of origin of S' at some time t, so $\overrightarrow{OO'} = \vec{v}t$

By using head to tail rule we get $\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$

$\Rightarrow \vec{r} = \vec{v}t + \vec{r}' \Rightarrow \vec{r}' = \vec{r} - \vec{v}t$

Since motion is only along $x-$ direction, therefore $\vec{v} = (v, 0,0)$

$\Rightarrow (x', y', z') = (x, y, z) - (v, 0,0)t = (x - vt, y, z)$

$\Rightarrow (x', y', z') = (x - vt, y, z)$

After comparing coordinates we get the transformation equations which relate the time and space coordinates in frames S and S' and are called Galilean Transformations (G.T.) as follows;

$x' = x - vt, \qquad y' = y, \qquad z' = z\,, \qquad t' = t$

Nature of time and Space: According to G.T.

- the concept of time is absolute (invariant) ($t' = t$)
- the concept of space that is the concept of distance or length is also absolute (invariant) ($L' = L$).

Absolute (Invariant) Space

Space that is not affected by what occupies it or occurs within it and that provides a standard for distinguishing inertial system from other frames of references. For example, Bob on Earth, sitting at his telescope, catches sight of Alice in her rocket ship streaking at 9/10 the speed of light right towards the sun.

Application of G.T. to Mechanics (all Results will be explained later)

On the basis of G.T., it is possible to obtain relations between physical quantities measured by two inertial observers in relative motion. Some of these are merely listed below:

(a) If $\vec{u}$ and $\vec{u}'$ are the velocities of a particle as observed from frames S and S' respectively, then $\vec{u}' = \vec{u} - \vec{v}$
Where $\vec{v}$ is the velocity of S' relative S. This is the familiar ***'common sense'*** formula of relative velocity.

(b) Acceleration of a particle as measured in S and S' is the same. That is say $\vec{a}' = \vec{a}$

(c) The mass of a particle has the same value in different inertial frames. If m' and m are the masses of a particle as determined in frames S' and S respectively, then $m' = m$.

Hence equation of motion such as $\vec{F} = m\vec{a}$ in frame S is transformed into $\vec{F}' = m'\vec{a}'$ in frame S'. Not only this equation but in fact Newtonian Mechanics has the same form in different inertial frames according to pre-Einstein relativity.

Application of G.T. to Electromagnetism

Fundamental laws of Electromagnetism can be expressed in a very elegant set of mathematical equations called Maxwell's Equations (M.E.). From these equations Maxwell deduced that electromagnetic waves (light, radio waves etc.) travel in empty space (and for all practical purposes through air) with a constant speed.

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8\, ms^{-1}$$

where ϵ_0 = permitivity of free space and μ_0 = permeability of free space.

On application of G.T., it is found that the form of M.E. changes, that is M.E. *are not form invariant* as required by the Principle of Relativity. This can be seen in a different but easier way by using the idea of relative velocity.

If $\vec{c}$ is the velocity of a light pulse as measured by an observer in S, then the velocity of the same light pulse as measured by an observer in S' is by Equation;

$$\vec{\boldsymbol{c}}' = \vec{\boldsymbol{c}} - \vec{\boldsymbol{v}}$$

where $\vec{v}$ is the velocity of S' relative to S.

It is obvious that magnitude of $\vec{c}'$ will in general be different from that of $\vec{c}$.

Einstein Twin Hypothesis / Einstein Twin Paradox / Einstein Twin Bases

Tow twins are born, one is put on a rocket ship and sent out into space at near the speed of light. The other lives on Earth. When the spaceship returns home, that twin as younger than his brother.

What is an Interval?

In the theory of relativity, a quantity that characterizes the relation between the spatial distance and the time interval that separate two events. It is the distance between two events in 4 – dimensional space – time.

Absolute Motion in Space

According to Newton "it is a motion of an object w.r.to the absolute space"

Newton thought that at any time every object has particular location in absolute space.

Absolute Motion Analysis

- A body is said to be at absolute rest when that object is in state of stationary.
- Absolute motion means a motion that does not depends on anything external to the moving object for its existence or specific nature.

Example

Let us take our universe in which earth is moving around sun, sun is moving around its bary system (moon and star system) and even galaxy also revolves and everything in universe is under motion. So there can be no absolute rest and absolute motion is possible.

Absolute Motion Vs Relative Motion

Absolute motion is nothing but the motion of a body from one absolute place to another. Where the Relative motion is defined as the motion of body from one relative place to another.

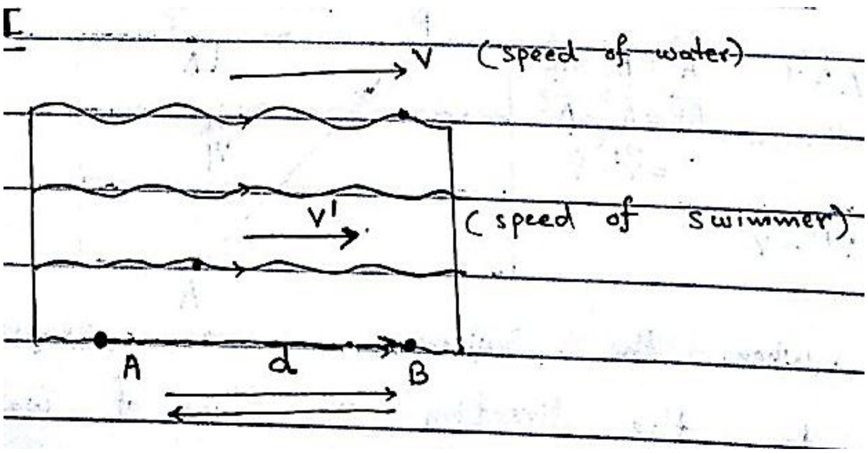
The Einstein velocity equation $v = c\sqrt{1 - \frac{1}{1+\left(\frac{E}{E_0}\right)^2}} = c\sqrt{1-\beta^2}$ where E is kinetic energy of mass m due to its relative velocity and $E_0 = m_0c^2$ is the rest mass energy of the object. This equation improved the Newton's Law of motion to the point where it can be precisely describe the behavior of with high relative velocities such as found in high energy particles accelerators.

The absolute velocity by Newton's equation relative to the rest of the universe is $v_{abs} = c\sqrt{\frac{GM_{uni}}{R_{uni}}} = c$

Effect of velocity of a medium on the motion of the particle

Consider a swimmer swimming in the water with speed v' and v be the uniform speed of water in a stream.

Case – I:



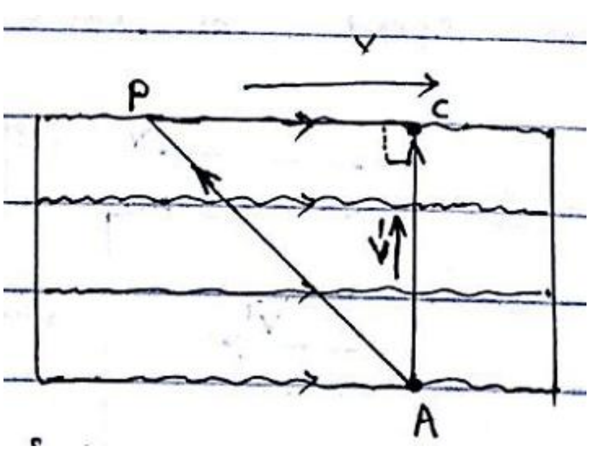
When swimmer moves towards the direction of flow of water, then we get relative speed $v' + v$. And if swimmer moves to the opposite direction of flow of water, then we get relative speed $v' - v$. Then total time from $A \to B$ and $B \to A$ is

$t_1 = \frac{d}{v'+v} + \frac{d}{v'-v}$ since $s = vt \Rightarrow t = \frac{s}{v}$

$t_1 = \frac{2dv'}{v'^2 - v^2}$

This is the time taken to complete one round trip between A and B.

Case – II:



In right triangle ACP by Pythagoras Theroem we have

$\overrightarrow{AC} = \overrightarrow{PC} + \overrightarrow{AP}$

$\overrightarrow{AP} = \overrightarrow{AC} - \overrightarrow{PC} = v' - v$

$|\overrightarrow{AP}| = \sqrt{v'^2 - v^2}$

When swimmer swims perpendicular to the direction of flow of water, then we get relative speed $\sqrt{v'^2 - v^2}$ in both up and downward motion. Then total time from $A \rightarrow C$ and $C \rightarrow A$ is

$t_2 = \frac{d}{\sqrt{v'^2 - v^2}} + \frac{d}{\sqrt{v'^2 - v^2}}$ since $s = vt \Rightarrow t = \frac{s}{v}$

$t_2 = \frac{2d}{\sqrt{v'^2 - v^2}}$

This is the time taken to complete one round trip between A and C.

We conclude that $t_1 \neq t_2$.

And the difference between two intervals is

$\Delta t = t_1 - t_2 = \frac{2dv'}{v'^2 - v^2} - \frac{2d}{\sqrt{v'^2 - v^2}}$

$\Rightarrow \Delta t = 2d\left(\frac{v'}{v'^2 - v^2} - \frac{1}{\sqrt{v'^2 - v^2}}\right)$

Covariant

Laws which remain same in all inertial frame of references are called covariant laws. e.g. Newton law $\vec{F} = m\vec{a}$ is covariant in all inertial frame of references.

Invariant (Absolute)

Quantities which remain same in all inertial frame of references are called invariant quantities. e.g. mass, length, time etc.

Example

Use G.T. to show that the distance measured is independent of the frame of reference. i.e. Distance (Length) is invariant.

Solution

Let (x_1', y_1', z_1') and (x_2', y_2', z_2') be the coordinates of some two points P_1 and P_2 respectively at some instant of time $t'(= t)$ as observed in S'. (P_1 and P_2 could be the end points of a rod.)

The distance between P_1 and P_2 as measured in S' is

$$L' = [(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2]^{\frac{1}{2}}$$

$$L' = \left[\left((x_2 - vt) - (x_1 - vt)\right)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{\frac{1}{2}} \qquad \text{by G.T.}$$

$$L' = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{\frac{1}{2}} = L \Rightarrow L' = L$$

where L = distance as measured in frame S.

The distance (Length) is thus independent of the frame of reference.

Example

Show that mass is invariant relative to Galilean Transformations (G.T.) between inertial frames.

Or The mass of a particle has the same value in different inertial frames. If m' and m are the masses of a particle as determined in frames S' and S respectively, then $m' = m$.

Solution

Consider a collision of two particles in an inertial frame S. Let $\vec{u}_1$ and $\vec{u}_2$ denote the velocities of the two particles of mass m_1 and m_2 before the collision. Let $\vec{v}_1$ and $\vec{v}_2$ denote velocities after the collision. Then from conservation of linear momentum,

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$\vec{u}_1 + \frac{m_2}{m_1}\vec{u}_2 = \vec{v}_1 + \frac{m_2}{m_1}\vec{v}_2$$

$$\frac{m_2}{m_1}(\vec{u}_2 - \vec{v}_2) = \vec{v}_1 - \vec{u}_1$$

$$\frac{m_2}{m_1} = \frac{|\vec{v}_1 - \vec{u}_1|}{|\vec{u}_2 - \vec{v}_2|}$$

This eqn. allows the ratio of masses to be determined from collision experiments. If one mass is chosen as unit, the mass of the other particle can be determined. Thus equation serves to provide a measurement of mass.

Consider the same collision as observed in frame S'. If $m_1', m_2', \vec{u}_1', \vec{u}_2', \vec{v}_1', \vec{v}_2'$ are the corresponding quantities in S', then

$$\frac{m_2'}{m_1'} = \frac{|\vec{v}_1' - \vec{u}_1'|}{|\vec{u}_2' - \vec{v}_2'|}$$

Implies $\frac{m_2}{m_1} = \frac{m_2'}{m_1'}$

If $m_1 = m_1' = 1$ by choice of unit mass. Then $m_2 = m_2'$

Thus $m = m'$

That is the mass of a particle has the same value in all inertial frames.

Hence mass is invariant relative to Galilean Transformations (G.T.).

Can Galilean Transformations be used in Special Relativity? Justify.

In general It cannot be used because it cannot satisfy the constancy of speed of light.

In special case Galilean Transformations can be used in Special Relativity. A G.T. between the coordinates of reference frames which differ only by constant of Newtonian Physics without the transformation in space and time. The group is the homogenous Galilean group.

POSTULATES OF SPECIAL THEORY OF RELATIVITY

a) The Principle of Relativity

All the laws of physics are identical (remain same) in all inertial frame of reference which are moving with uniform velocity (non – accelerated) relative to one another. This is called Galilean Invariance. (It is impossible by any physical measurements to trace an essential distinction between any two inertial frames which are in relative motion.)

b) The Principle of Constancy of Speed of Light ('c' is invariant)

The speed of light in free space has the same value in all inertial frames of reference (speed of light is constant for all observers).
It is denoted by 'c' and $c = 3 \times 10^8 ms^{-1}$

Relative Velocity Expression / Galilean Transformations in terms of velocity

If $\vec{u}$ and $\vec{u}'$ are the velocities of a particle as observed from frames S and S' respectively, then $\vec{\boldsymbol{u}}' = \vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}$ Where $\vec{v}$ is the velocity of S' relative S. This is the familiar ***'common sense'*** formula of relative velocity.

Or Show that if S is inertial frame then S' is also inertial frame using G.T.

Solution If $\vec{u}$ and $\vec{u}'$ are the velocities of a particle as observed from frames S and S' respectively, then according to Galilean Transformations

$$x' = x - vt \Rightarrow \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) \Rightarrow \frac{dx'}{dt'} = \frac{dx}{dt} - v\frac{dt}{dt} \qquad \text{in G.T. } t' = t$$

$$\Rightarrow u' = u - v \Rightarrow (u', 0,0) = (u, 0,0) - (v, 0,0) \Rightarrow \vec{\boldsymbol{u}}' = \vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}$$

$$\Rightarrow u'_x = u_x - v,\ u'_y = u_y - v,\ u'_z = u_z - v,\ \ t' = t$$

Question

Acceleration of a particle as measured in S and S' is the same. That is say $\vec{a}' = \vec{a}$

Solution If $\vec{u}$ and $\vec{u}'$ are the velocities of a particle as observed from frames S and S' respectively, then $\vec{u}' = \vec{u} - \vec{v}$ With $\vec{v} = Constant$ is the velocity of S' relative S.

$$\Rightarrow \frac{d\vec{u}'}{dt'} = \frac{d}{dt}(\vec{u} - \vec{v}) \Rightarrow \frac{d\vec{u}'}{dt'} = \frac{d\vec{u}}{dt} - \frac{d\vec{v}}{dt} = \frac{d\vec{u}}{dt} \Rightarrow \vec{\boldsymbol{a}}' = \vec{\boldsymbol{a}}$$

Question

On the basis of G.T. show that the force acting on a particle is independent of the inertial frame in which it is measured. i.e. $\vec{F}' = \vec{F}$

Or Show that Newton's 2nd Law of motion is Covariant.

Solution

If $\vec{u}$ and $\vec{u}'$ are the velocities of a particle as observed from frames S and S' respectively, then according to Galilean Transformations

$x' = x - vt \Rightarrow \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) \Rightarrow \frac{dx'}{dt'} = \frac{dx}{dt} - v\frac{dt}{dt}$ in G.T. $t' = t$

$\Rightarrow u' = u - v \Rightarrow (u', 0,0) = (u, 0,0) - (v, 0,0) \Rightarrow \vec{\boldsymbol{u}}' = \vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}$

$\Rightarrow \frac{d\vec{u}'}{dt'} = \frac{d}{dt}(\vec{u} - \vec{v}) \Rightarrow \frac{d\vec{u}'}{dt'} = \frac{d\vec{u}}{dt} - \frac{d\vec{v}}{dt} = \frac{d\vec{u}}{dt} \Rightarrow \vec{\boldsymbol{a}}' = \vec{\boldsymbol{a}}$

Multiplying m' on both sides we get $\Rightarrow m'\vec{a}' = m'\vec{a}$

$\Rightarrow m'\vec{a}' = m\vec{a}$ In inertial frame $m' = m$

$\Rightarrow \vec{\boldsymbol{F}}' = \vec{\boldsymbol{F}}$

Question

Why Galilean Transformations failed to satisfy 2nd Postulate of relativity?

Solution

Galilean Transformations failed to satisfy 2nd Postulate of relativity because speed of light does not remains same in S and S' frames. As by using G.T.

$x' = x - vt \Rightarrow \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) \Rightarrow \frac{dx'}{dt'} = \frac{dx}{dt} - v\frac{dt}{dt} \Rightarrow u' = u - v$

$\Rightarrow c' = c - v$ if we consider ray of light (Photon) instead of particles.

$\Rightarrow c' \neq c$ That is speed of light does not remains same in both frames.

Hence According to Galilean Transformations 2nd Postulate of relativity (Principle of Constancy of Speed of light) does not satisfy the invariant property.

Ether

Sound waves need a medium to carry them, so by analogy it was believed that a medium with mechanical properties (density, elasticity etc.) must exist to transmit light waves. That medium was called ether filled all space.

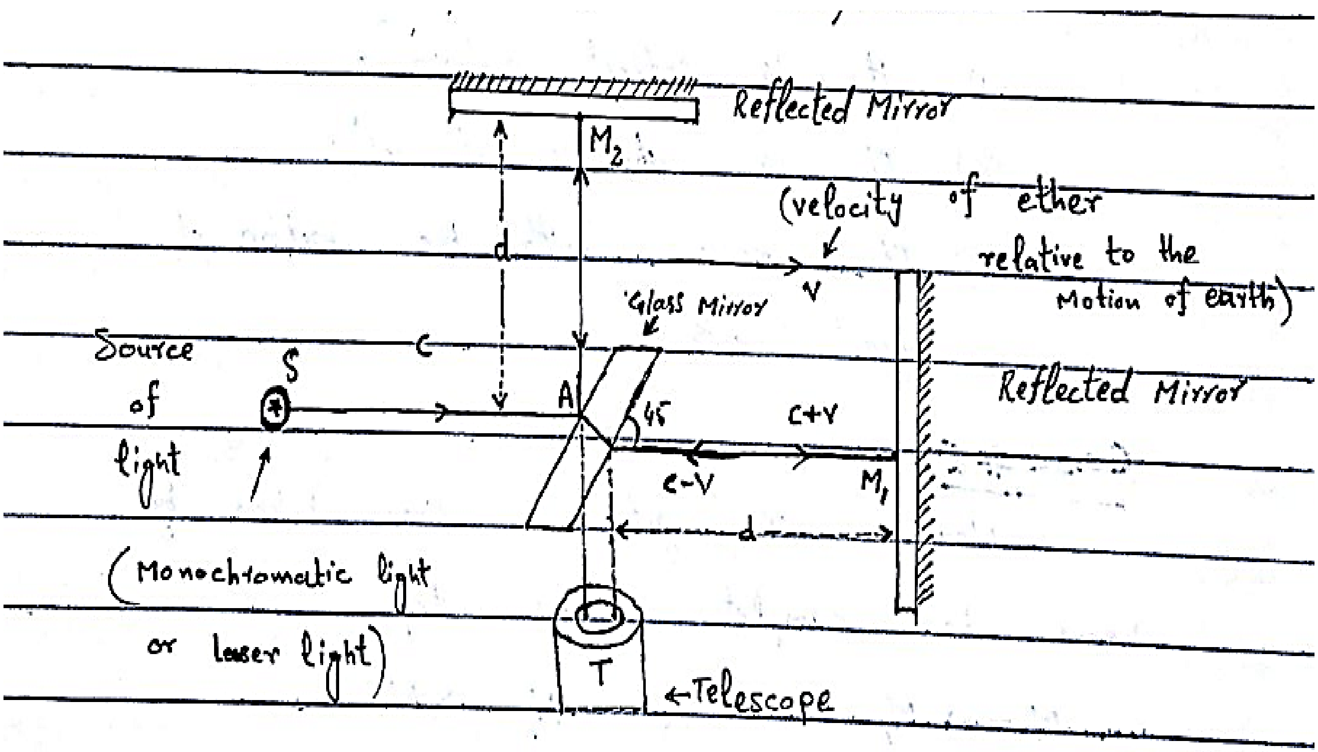
Its Characteristics are as follows;

- Ether was supposed to be present in vacuum, in the space between atoms, molecules and so on.
- Ether was thought to be the softest of all substances because matter can move through it without any friction or resistance.
- Earth and other planets travel through this medium year after year without any reduction in speed.
- Ether was actually at rest.
- Ether was move with the motion of object.
- Light travels through ether at such a fantastically high speed that extremely strong restoring forces are set up in ether when it is disturbed by propagation of light. This requires that elastic constants of ether must be the highest of all materials. In other words, ether must be the hardest of all materials.
- The hypothesis of ether is clearly self-contradictory and absurd. But the scientists did not give it up because they thought a wave cannot travel without a mechanical medium.

The concept of ether with its self-contradictory properties is obviously absurd. We now know that light consists of oscillating electric and magnetic fields that can travel in vacuum or free space with speed c and does not need ether or any mechanical medium whatsoever for its propagation. Ether is just a myth—in truth, what the scientists earlier called ether is just empty space. It may be added here that the null result of Michelson-Morley experiment does not by itself disprove the existence of ether—the null results means that the speed of light is always the same in all directions and is independent of the relative uniform motion of the observer and the source. In retrospect one may say, the concept of ether clouded the thinking of the scientists and a lot of scientific effort was wasted in retaining the concept. Scientific progress is not unimpeded!

MICHELSON MORLEY EXPERIMENT (1887)

It was done to conform the presence of hypothetical medium called Ether. If ether exists it should be possible to detect the motion of the earth through the ether and in particular to determine the speed of earth (v) relative to ether. This is what Michelson and Morley tried to find out using an instrument called **Michelson's interferometer**.



In this experiment a light beam from a source 'S' is split up into two parts by a glass mirror M which makes an angle of 45° with direction of light beam. These two parts of the light fall on the two fully reflecting mirrors M_1 and M_2 at right angle. After the reflection of light beam (rays) finally enter the telescope as shown in figure.

The mirrors M_1 and M_2 placed at the same distance 'd' from the glass mirror M. In such a way their planes are perpendicular to each other.

Let velocity of ether wind $= v$

And velocity of light $= v' = c = 3 \times 10^8 ms^{-1}$

Then, There are two cases appears;

Case – I:

We get a time taken by beam to complete one trip from $M \to M_1$ and $M_1 \to M$

$t_1 = \frac{d}{c+v} + \frac{d}{c-v} = \frac{2dc}{c^2-v^2}$ since $s = vt \Rightarrow t = \frac{s}{v}$

$$t_1 = \frac{2\frac{d}{c}}{1-\frac{v^2}{c^2}}$$

Case – II:

We get a time taken by beam to complete one trip from $M \to M_2$ and $M_2 \to M$

$t_2 = \frac{d}{\sqrt{c^2-v^2}} + \frac{d}{\sqrt{c^2-v^2}} = \frac{2d}{\sqrt{c^2-v^2}}$ since $s = vt \Rightarrow t = \frac{s}{v}$

$$t_2 = \frac{2\frac{d}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta t = t_1 - t_2 = \frac{2\frac{d}{c}}{1-\frac{v^2}{c^2}} - \frac{2\frac{d}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta t = 2\frac{d}{c}\left[\left(1-\frac{v^2}{c^2}\right)^{-1} - \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \Delta t = 2\frac{d}{c}\left[\left(1+\frac{v^2}{c^2} + higher\ terms\right) - \left(1+\frac{1}{2}\frac{v^2}{c^2} + higher\ terms\right)\right]$$

$$\Rightarrow \Delta t = 2\frac{d}{c}\left[1+\frac{v^2}{c^2} + higher\ terms - 1 - \frac{1}{2}\frac{v^2}{c^2} - higher\ terms\right]$$

Since $v \ll c$ implies $\frac{v}{c} \ll 1$ so we may neglect higher terms

$$\Rightarrow \Delta t = 2\frac{d}{c}\left[1+\frac{v^2}{c^2} - 1 - \frac{1}{2}\frac{v^2}{c^2}\right] = 2\frac{d}{c}\left[\frac{v^2}{2c^2}\right] \Rightarrow \boldsymbol{\Delta t = \frac{dv^2}{c^3}}$$

As d and c are known, so it was expected that by measuring time Δt we can find the value of v.

Negative (Null) Results of Michelson Morley Experiment

Michelson Morley Experiment failed by the following ways;

- **Ether Drag Hypothesis(The concept of ether to be stationary is wrong)**
  Ether drag hypothesis thus explains the null result of Michelson-Morley experiment and at the same time it retains the privileged either frame. This negative result was explained by assuming that earth drags the ether along the direction of its motion as it is always stationary with respect to ether.
  The speed of light is constant equal to c with respect to ether; but according to this hypothesis, a body moving through the ether drags or carries the ether in its neighbourhood alongwith it.
  The above explanation is contradicted by the phenomenon of aberration.
  Aberration is variation in the apparent position of a star due to the motion of the observer along with the earth.
- **Elastic Corpuscles Hypothesis (The speed of light does not depend on the motion of the source observer)**
  According to this hypothesis, light consists of extremely small elastic corpuscles emitted with speed c relative to the source of light. The velocity of these corpuscles is assumed to be independent of the state of motion of the medium (such as ether) transmitting the light. The light corpuscles are reflected from mirrors as per laws of reflection of elastic particles. By using pre-Einstein relativity (G.T.) it can be shown that speeds of such corpuscles of light would be the same along longitudinal and transverse paths in the Michelson-Morley experiment. The null result of the experiment is thus explained. And this is not true in actual, like ghost stars not exists.
- **The Lorentz – Fitzgerald Contraction Hypothesis**
  (The velocity of earth relative to ether is zero)
  The Michelson-Morley Experiment was explained by Lorentz and Fitzgerald who made the assumption that all material objects are contracted by a factor $\sqrt{c^2 - v^2}$. However this contraction cannot be calculated as it also applies to measuring rods.

Example

In the Michelson-Morley experiment, $(l_1 + l_2)$ was 22 m and the wavelength of light used was 6000 A°. They assumed that ether is fixed relative to the sun so that the earth and the interferometer move through the ether at a velocity $v = 3 \times 10^4\ ms^{-1}$ which is the orbital speed of the earth about the sun. Calculate the fringe shift they expected to observe.

Solution

$$\lambda = 6000 \times 10^{-10}\ \text{m} = 6 \times 10^{-7}\ \text{m}.$$

$$\frac{v}{c} = \frac{3 \times 10^4}{3 \times 10^8} = 10^{-4}$$

$$\Delta N = \left(\frac{l_1 + l_2}{\lambda}\right)\left(\frac{v^2}{c^2}\right) = \frac{22}{6 \times 10^{-7}}\ (10^{-4})^2 = 0.37$$

The experiment was sensitive enough to detect a shift as small as 0.01.

Question

Why we need to prove Lorentz Transformations over Galilean Transformations?

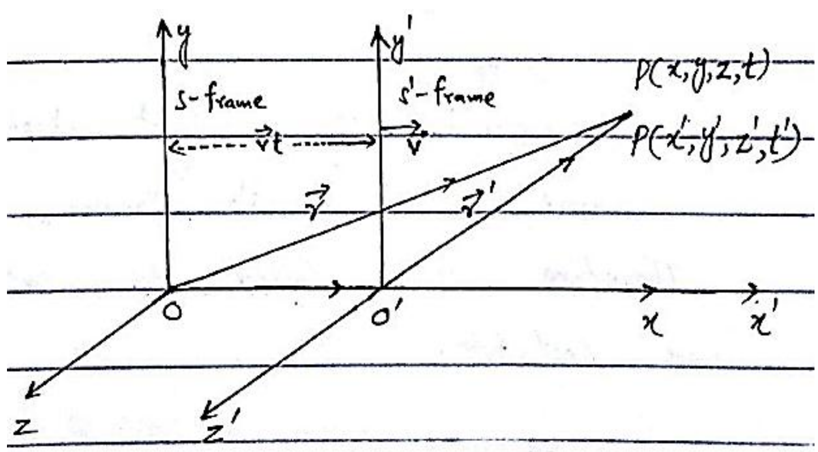
Answer

We need to prove Lorentz Transformations because Galilean Transformations failed to solve the problem of constancy of speed of light but Lorentz Transformations satisfy the 2^{nd} postulate of special theory of relativity.

LORENTZ TRANSFORMATIONS / RESTRICTED GALILEAN TRANSFORMATIONS /3D LORENTZ TRANSFORMATIONS

Lorentz Transformations is a set of observations made by two observers in different frame of references which preserve the constancy of speed of light in all system. It is the set of equations which relates coordinates of a single event in two different frames.

Consider two inertial frames S and S' as shown in Figure. Initially both frames coincide. i.e. $t = t'$. S' moves relative to S at a constant speed v along the x-axis. Both axis remains parallel to each other.



Since $S \parallel S'$ therefore

$x' = k(x - vt)$ ……..(i)   and $x = k(x' + vt')$ ……..(ii)

Now consider a ray of light instead of particle, then instead of $s = vt$ we use

$x = ct$ in $S -$ frame and $x' = ct'$ in $S' -$ frame and here we use $c = c'$ then

$ct' = k(ct - vt)$ and $ct = k(ct' + vt')$

Implies $ct' = k(c - v)t$ and $ct = k(c + v)t'$

Multiplying both $c^2tt' = k^2(c - v)(c + v)tt' \Rightarrow c^2 = k^2(c^2 - v^2)$

$\Rightarrow k^2 = \frac{c^2}{(c^2 - v^2)} \Rightarrow k^2 = \frac{c^2}{c^2\left(1 - \frac{v^2}{c^2}\right)} \Rightarrow k^2 = \frac{1}{1 - \frac{v^2}{c^2}}$

$\Rightarrow k = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$ (**Lorentz's factor**). Here $\sqrt{1 - \frac{v^2}{c^2}}$ is called **Clock Paradox**

If $\beta = \frac{v}{c}$ is a speed parameter then $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$

$(i) \Rightarrow \boldsymbol{x' = \gamma(x - vt)}$

Now for value of t' we solve (ii)

$x = k(x' + vt')$

$\Rightarrow \frac{x}{k} = (x' + vt') \Rightarrow \frac{x}{k} - x' = vt' \Rightarrow t' = \frac{x - kx'}{kv}$

$\Rightarrow t' = \frac{x - k\left(k(x - vt)\right)}{kv}$ since $x' = k(x - vt)$

$\Rightarrow t' = \frac{x - k^2x + k^2vt}{kv} \Rightarrow t' = k\left[\frac{x - k^2x + k^2vt}{k^2v}\right] \Rightarrow t' = k\left[\frac{k^2vt - (k^2 - 1)x}{k^2v}\right]$

$\Rightarrow t' = k\left[t - \left(\frac{k^2 - 1}{k^2}\right)\frac{x}{v}\right]$

$\Rightarrow t' = k\left[t - \left(\frac{v^2}{c^2}\right)\frac{x}{v}\right]$ since $k^2 = \frac{1}{1 - \frac{v^2}{c^2}}$ implies $\frac{k^2 - 1}{k^2} = \frac{v^2}{c^2}$

$\Rightarrow \boldsymbol{t' = \gamma\left(t - \frac{xv}{c^2}\right)}$

Hence the Lorentz Transformations are

$\boldsymbol{x' = \gamma(x - vt)}$, $\boldsymbol{y' = y}$, $\boldsymbol{z' = z}$

$\boldsymbol{t' = \gamma\left(t - \frac{xv}{c^2}\right)}$

INVERSE LORENTZ TRANSFORMATIONS

We know that $x' = \gamma(x - vt)$……..(i) and $t' = \gamma\left(t - \frac{xv}{c^2}\right)$ ……..(ii)

$(ii) \Rightarrow t' = \gamma\left(t - \frac{xv}{c^2}\right)$

$\Rightarrow \frac{t'}{\gamma} = t - \frac{xv}{c^2} \Rightarrow t = \frac{t'}{\gamma} + \frac{xv}{c^2}$ ……..(iii)

$(i) \Rightarrow x' = \gamma\left(x - v\left(\frac{t'}{\gamma} + \frac{xv}{c^2}\right)\right)$ using t

$\Rightarrow x' = \gamma x - vt' - \gamma\frac{xv}{c^2} \Rightarrow x' + vt' = \gamma\left(1 - \frac{v^2}{c^2}\right)x$

$\Rightarrow x' + vt' = \gamma\left(\frac{1}{\gamma^2}\right)x$ using $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$\Rightarrow \boldsymbol{x = \gamma(x' + vt')}$

$(iii) \Rightarrow t = \frac{t'}{\gamma} + \frac{v}{c^2}\left(\gamma(x' + vt')\right)$ using x

$\Rightarrow t = \frac{t'}{\gamma} + \gamma\frac{vx'}{c^2} + \gamma\frac{v^2t'}{c^2} \Rightarrow t = \gamma\left[\frac{t'}{\gamma^2} + \frac{vx'}{c^2} + \frac{v^2t'}{c^2}\right]$

$\Rightarrow t = \gamma\left[\left(1 - \frac{v^2}{c^2}\right)t' + \frac{vx'}{c^2} + \frac{v^2t'}{c^2}\right]$ using $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$\Rightarrow t = \gamma\left[\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right)t' + \frac{vx'}{c^2}\right]$

$\Rightarrow \boldsymbol{t = \gamma\left(t' + \frac{vx'}{c^2}\right)}$

Hence the inverse Lorentz Transformations are

$\boldsymbol{x = \gamma(x' + vt')}$

$\boldsymbol{y = y'}$, $\boldsymbol{z = z'}$

$\boldsymbol{t = \gamma\left(t' + \frac{vx'}{c^2}\right)}$

Remark:

(i) There is an appreciable quantitative **difference between** L.T. and G.T. only when $\frac{v}{c}$ is appreciable, that is only at high speeds.

(ii) In the limiting case $= \infty$, L.T. reduce to G.T. and in particular $t = t'$.

(iii) Equations, $x' = \gamma(x - vt), t' = \gamma\left(t - \frac{xv}{c^2}\right), x = \gamma(x' + vt')$ and $t = \gamma\left(t' + \frac{x'v}{c^2}\right)$ show that space and time get 'mixed up' in going from one inertial frame to another. Distinction between space and time is apparently less rigid or a little blurred in relativity.

(iv) If $v > c$, then γ is imaginary and we face the problem of imaginary spaces and imaginary times. It is impossible therefore to think of two inertial frames relative to each other (in real space and time) at a speed $v > c$.

Question

Show that Lorentz Transformations obey the postulate of constancy of speed of light.

Solution

According to Lorentz Transformations we have

$$x' = \gamma(x - vt) \Rightarrow \frac{dx'}{dt'} = \gamma\frac{d}{dt'}(x - vt) \Rightarrow \frac{dx'}{dt'} = \gamma\frac{d}{dt}(x - vt)\frac{dt}{dt'}$$

$$\Rightarrow \frac{dx'}{dt'} = \gamma\left(\frac{dx}{dt} - v\frac{dt}{dt}\right)\frac{dt}{dt'} \Rightarrow \frac{dx'}{dt'} = \gamma\left(\frac{dx}{dt} - v\right)\frac{dt}{dt'}$$

By considering the light ray instead of particle, we have

$x = ct$ and $x' = ct'$ then $\frac{dx}{dt} = c$ and $\frac{dx'}{dt'} = c'$

$$\Rightarrow c' = \gamma(c - v)\frac{dt}{dt'} \qquad \text{…………(i)}$$

Also using $t' = \gamma\left(t - \frac{xv}{c^2}\right)$

$$\Rightarrow \frac{dt'}{dt} = \gamma\frac{d}{dt}\left(t - \frac{xv}{c^2}\right) \Rightarrow \frac{dt'}{dt} = \gamma\left(\frac{dt}{dt} - \frac{v}{c^2}\frac{dx}{dt}\right) \Rightarrow \frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}.c\right)$$

$\Rightarrow \frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c}\right) \Rightarrow \frac{dt'}{dt} = \gamma\left(\frac{c-v}{c}\right) \Rightarrow \frac{dt}{dt'} = \frac{c}{\gamma(c-v)}$

$(i) \Rightarrow c' = \gamma(c-v)\left(\frac{c}{\gamma(c-v)}\right) \Rightarrow c' = c$

Hence the Lorentz Transformations satisfy the 2^{nd} postulate of STR that speed of light remains constant in all inertial frames.

Question

Show that a material object can never move with an equal or greater than the velocity of light. **Or** Speed of an object can never exceed the speed of light.

Solution

Since $x' = \gamma(x - vt), y' = y, z' = z, t' = \gamma\left(t - \frac{xv}{c^2}\right)$ where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

If $v = c$ then $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{v^2}}} = \frac{1}{\sqrt{1-1}} = \infty$

If $v > c$ then $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{v^2}}}$ =Complex Number

It shows that material object cannot move with an equal or greater than the velocity of light. Transformations should be in Real numbers, not complex and undefined.

Question

Show that G.T. is limiting case of L.T

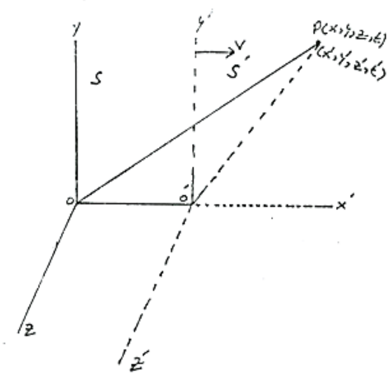
Solution

In the limiting case $c = \infty$ then $\frac{v}{c} = 0$ then

$x' = \gamma(x - vt) = (x - vt), y' = y, z' = z, t' = \gamma\left(t - \frac{xv}{c^2}\right) = (t - 0)$

Hence $t' = t$ (G.T. is limiting case of L.T)

Fundamental equation of Special Theory of Relativity



Consider two observers in two different inertial frames S and S'. Frame S is at rest and S' is moving with uniform velocity v along x-axis with respect to frame S. Suppose at $t = 0$, the origins of two frames coincide.

Both the observers observe the same event. The position and time of event observed by S is denoted by (x, y, z, t) and position and time of the event observed by S' is denoted by (x', y', z', t'). Consider a wave of light starts from O and O' at $t = 0$ with speed c. Let the wave reaches a point P after time t from O and takes the time t' to reach at P from point O'. Then the distance covered by light ray from point O to point P:

$$|OP| = ct$$
$$\Rightarrow \sqrt{x^2 + y^2 + z^2} = ct$$
$$\Rightarrow x^2 + y^2 + z^2 = c^2t^2$$
$$\Rightarrow x^2 + y^2 + z^2 - c^2t^2 = 0 \quad ------ \quad (1)$$

And the distance covered by light ray from point O' to point P:

$$|O'P| = ct'$$
$$\Rightarrow \sqrt{x'^2 + y'^2 + z'^2} = ct'$$
$$\Rightarrow x'^2 + y'^2 + z'^2 = c^2t'^2$$
$$\Rightarrow x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \quad ------ \quad (2)$$

Comparing these equations, we get:

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \quad ------ \quad (3)$$

This is the fundamental equation of special theory of relativity given by Einstein in 1905.

Galilean Transformations Doesn't Satisfy the Fundamental Equation of Relativity

Applying the values of x', y', z', t' from Galilean Transformation in Fundamental Equation of Relativity

$$x^2 + y^2 + z^2 - c^2t^2 = (x - vt)^2 + y^2 + z^2 - c^2t^2$$
$$\Rightarrow x^2 = (x - vt)^2$$

This is clearly impossible until $t = 0$. Hence Galilean Transformation fail to satisfy Fundamental Equation of Relativity.

Question L.T. Satisfy Fundamental equation of STR

Show that L.T. leaves the expression $x^2+y^2+z^2-c^2t^2$ invariant.

Solution using Lorentz Transformations

$x' = \gamma(x - vt), y' = y \ , \ z' = z, t' = \gamma\left(t - \frac{xv}{c^2}\right)$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \left(\gamma(x-vt)\right)^2 + (y)^2 + (z)^2 - c^2\left(\gamma\left(t - \frac{xv}{c^2}\right)\right)^2$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2(x-vt)^2 + (y)^2 + (z)^2 - c^2\gamma^2\left(t - \frac{xv}{c^2}\right)^2$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2(x^2 + v^2t^2 - 2xvt) + y^2 + z^2 - c^2\gamma^2\left(t^2 + \frac{x^2v^2}{c^4} - 2\frac{xtv}{c^2}\right)$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2x^2 + \gamma^2v^2t^2 - 2\gamma^2xvt + y^2 + z^2 - c^2\gamma^2t^2 - \gamma^2\frac{x^2v^2}{c^2} + 2\gamma^2xtv$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2x^2 + \gamma^2v^2t^2 + y^2 + z^2 - c^2\gamma^2t^2 - \gamma^2\frac{x^2v^2}{c^2}$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2\left(x^2 + v^2t^2 - c^2t^2 - \frac{x^2v^2}{c^2}\right) + y^2 + z^2$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2\left[x^2\left(1 - \frac{v^2}{c^2}\right) - c^2t^2\left(1 - \frac{v^2}{c^2}\right)\right] + y^2 + z^2$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2\left[\left(1 - \frac{v^2}{c^2}\right)(x^2 - c^2t^2)\right] + y^2 + z^2$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2\left[\left(\frac{1}{\gamma^2}\right)(x^2 - c^2t^2)\right] + y^2 + z^2$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$ proved.

Question When an observer records an event $x = 3.2 \times 10^8 m$ and $t = 2.5s$ in a rest frame S. find its respective coordinates in frame S' moving with velocity 0.38c

Solution using Lorentz Transformations $x' = \gamma(x - vt), t' = \gamma\left(t - \frac{xv}{c^2}\right)$

$x' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt) = \frac{1}{\sqrt{1-\frac{(0.38)^2c^2}{c^2}}}\left(3.2 \times 10^8 - 0.38 \times 3 \times 10^8 \times 2.5\right) = 3.5 \times 10^8$

$t' = \gamma\left(t - \frac{xv}{c^2}\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{xv}{c^2}\right) = \frac{1}{\sqrt{1-\frac{(0.38)^2c^2}{c^2}}}\left(2.5 - \frac{3.2\times10^8\times0.38c}{c^2}\right) = 2.3s$

Question (Length of 4 – Vector is invariant)

Show that $(ds)^2 = g_{\propto\beta} dx^{\propto} dx^{\beta}$ invariant under Lorentz Transformation.

Solution

Metric Tensor: Dot product of two tangent vectors is called metric tensor.

We have $(ds)^2 = g_{\propto\beta} dx^{\propto} dx^{\beta}$ where $g_{\propto\beta}$ is metric tensor

With $g_{11} = \frac{\partial\vec{r}}{\partial u_1} . \frac{\partial\vec{r}}{\partial u_1} = 1$ and $g_{12} = \frac{\partial\vec{r}}{\partial u_1} . \frac{\partial\vec{r}}{\partial u_2} = 0$ then

$(ds)^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + g_{44}(dx^4)^2$

using Lorentz Transformations $x' = \gamma(x - vt), y' = y \;\; , \;\; z' = z, t' = \gamma\left(t - \frac{xv}{c^2}\right)$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \left(\gamma(x - vt)\right)^2 + (y)^2 + (z)^2 - c^2\left(\gamma\left(t - \frac{xv}{c^2}\right)\right)^2$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2(x - vt)^2 + (y)^2 + (z)^2 - c^2\gamma^2\left(t - \frac{xv}{c^2}\right)^2$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2(x^2 + v^2t^2 - 2xvt) + y^2 + z^2 - c^2\gamma^2\left(t^2 + \frac{x^2v^2}{c^4} - 2\frac{xtv}{c^2}\right)$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2x^2 + \gamma^2v^2t^2 - 2\gamma^2xvt + y^2 + z^2 - c^2\gamma^2t^2 - \gamma^2\frac{x^2v^2}{c^2} + 2\gamma^2xtv$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2x^2 + \gamma^2v^2t^2 + y^2 + z^2 - c^2\gamma^2t^2 - \gamma^2\frac{x^2v^2}{c^2}$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2\left(x^2 + v^2t^2 - c^2t^2 - \frac{x^2v^2}{c^2}\right) + y^2 + z^2$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2\left[x^2\left(1 - \frac{v^2}{c^2}\right) - c^2t^2\left(1 - \frac{v^2}{c^2}\right)\right] + y^2 + z^2$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2\left[\left(1 - \frac{v^2}{c^2}\right)(x^2 - c^2t^2)\right] + y^2 + z^2$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \gamma^2\left[\left(\frac{1}{\gamma^2}\right)(x^2 - c^2t^2)\right] + y^2 + z^2$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$$

$$\Rightarrow (dx')^2 + (dy')^2 + (dz')^2 - c^2(dt')^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2(dt)^2$$

$$\Rightarrow \left(dx^{1'}\right)^2 + \left(dx^{2'}\right)^2 + \left(dx^{3'}\right)^2 - c^2\left(dx^{4'}\right)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2(dx^4)^2$$

$$\Rightarrow (ds')^2 = (ds)^2$$

Question

Two events occur simultaneously at points $(21,2,1)$ and $(1,0,0)$ of a frame S. determine the time interval between then in a frame S' moving with speed 0.6c relative to S along the direction of their common x – axis.

Solution

Given that $P = (21,2,1)$ and $Q = (1,0,0)$

Here $x_1 = 21$ and $x_2 = 1$ also $v = 0.6c$ or $\frac{v}{c} = 0.6$

using Lorentz Transformations

$$t' = \gamma\left(t - \frac{xv}{c^2}\right) \Rightarrow t' = \frac{\gamma}{c}\left(ct - \frac{xv}{c}\right)$$

$$\Rightarrow t_1{}' = \frac{\gamma}{c}\left(ct_1 - \frac{v}{c}x_1\right) \text{ and } t_2{}' = \frac{\gamma}{c}\left(ct_2 - \frac{v}{c}x_2\right)$$

$$\Rightarrow t_1{}' - t_2{}' = \frac{\gamma}{c}\left[c(t_1 - t_2) - \frac{v}{c}(x_1 - x_2)\right]$$

$$\Rightarrow t_1{}' - t_2{}' = \frac{\gamma}{c}\left[(ct_1 - ct_2) - \frac{v}{c}(x_1 - x_2)\right]$$

$$\Rightarrow t_1{}' - t_2{}' = \frac{\gamma}{c}\left[(x_1 - x_2) - \frac{v}{c}(x_1 - x_2)\right]$$

$$\Rightarrow t_1{}' - t_2{}' = \frac{\gamma}{c}\left(1 - \frac{v}{c}\right)(x_1 - x_2)$$

$$\Rightarrow t_1{}' - t_2{}' = \frac{1}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(1 - \frac{v}{c}\right)(x_1 - x_2) \Rightarrow t_1{}' - t_2{}' = \frac{1}{c}\frac{\sqrt{1-\frac{v}{c}}}{\sqrt{1+\frac{v}{c}}}(x_1 - x_2)$$

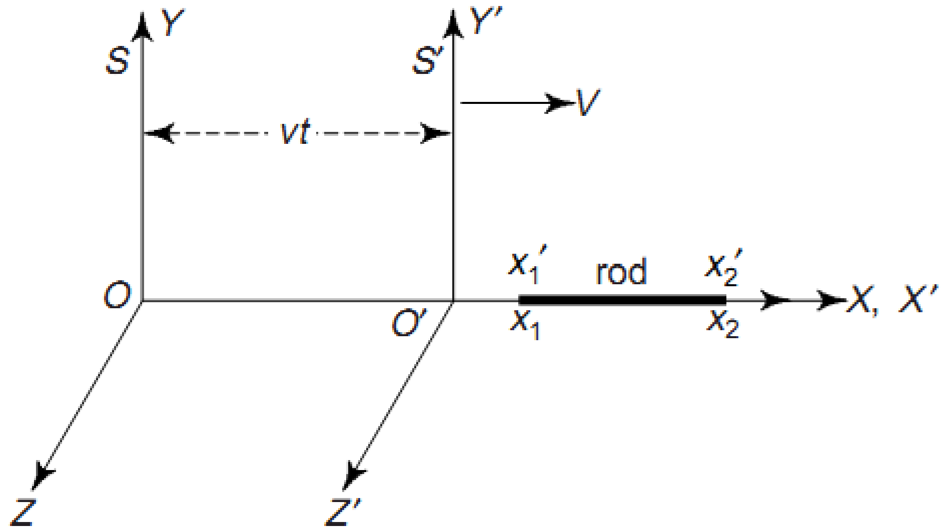
$$\Rightarrow t_1{}' - t_2{}' = \frac{1}{3\times10^8}\frac{\sqrt{1-0.6}}{\sqrt{1+0.6}}(21 - 1)$$

$$\Rightarrow t_1{}' - t_2{}' = 0.3 \times 10^{-7} s$$

LENGTH CONTRACTION /SPACE CONTRACTION /LORENTZ CONTRACTION

Consider two frames S and S'. S' is moving with uniform speed v in x – direction.

Let a rod place in a frame S' along x' – axis.



Let the two end of the rod be lebelled x_1' and x_2' in S' frame and x_1 and x_2 in S frame. Then the length or the rod in both frames is given by;

$L_0' = x_2' - x_1'$ …………….(1) (Rest Length)

$L_0 = x_2 - x_1$ …………….(2)

By using inverse Lorentz transformations,

$x = \gamma(x' + vt')$

At positions x_1 and x_2 we get

$x_1 = \gamma(x_1' + vt_1')$ …………….(3)

$x_2 = \gamma(x_2' + vt_2')$ …………….(4)

Subtracting (iii) and (iv) we have

$x_2 - x_1 = \gamma[(x_2' - x_1') + v(t_2' - t_1')]$ …………….(5)

Since observer is sitting in S' frame to measure the length of the rod at the same time so $t_2' = t_1' = t'$

$$(5) \Rightarrow x_2 - x_1 = \gamma[(x_2' - x_1') + v(t' - t')]$$

$$\Rightarrow x_2 - x_1 = \gamma(x_2' - x_1') \Rightarrow \boldsymbol{L_0 = \gamma L_0'}$$

$$\Rightarrow L_0 = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)L_0' \qquad \text{Since } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow L_0' = \left(\sqrt{1-\frac{v^2}{c^2}}\right)L_0$$

Suppose $v = \frac{\sqrt{3}}{2}c$ where $v < c$ then

$$\sqrt{1-\frac{v^2}{c^2}} = \sqrt{1-\frac{\left(\frac{\sqrt{3}}{2}c\right)^2}{c^2}} = \sqrt{1-\frac{3}{4}} = \frac{1}{2} < 1 \text{ implies } \sqrt{1-\frac{v^2}{c^2}} < 1$$

Due to this factor $\sqrt{1-\frac{v^2}{c^2}}$ we get $\boldsymbol{L_0' < L_0}$

This shortening of length for moving objects is called length contraction or space contraction or Lorentz contraction. It is the phenomenon that a moving object's length is measured to be shorter than its proper length.

Remarks

- In classical mechanics $v << c$ then $\frac{v}{c} << 1$ implies $\left(\frac{v}{c}\right)^2 \rightarrow 0$ using $\frac{v^2}{c^2} = 0$ so $\gamma = 1$ then $\boldsymbol{L_0' = L_0}$
- Length of an object is not absolute but depends upon the relative velocity of the object and the observer.
- **Proper Length** of an object is defined as its length measured in a reference frame in which the object is at rest.
- All observers who are moving with respect to the object (or with respect to whom the object is moving) will find that its length is shorter than its proper length.

Example

A 100-MeV electron, for which $v = 0.999987c$, moves along the axis of an evacuated tube that has a length of 2.86 m as measured by a laboratory observer S with respect to whom the tube is at rest. An observer S' moving with the electron, however, would see this tube moving past with speed v. What length would this observer measure for the tube?

Solution

Given $v = 0.99987c$ $L_0 = 2.86$ m $L=?$

Formula used $L = L_o\sqrt{1 - \frac{v^2}{c^2}} = (2.86)\sqrt{1-(0.999987)^2} = 1.4$ cm

Example

A rod lies parallel to the x axis of reference frame S, moving along this axis at a speed of $0.632c$. Its rest length is 1.68 m. What will be its measured length in frame S?

Solution

Given $v = 0.632c$ $L_0 = 1.68$ m $L=?$

Formula used $L = L_o\sqrt{1 - \frac{v^2}{c^2}} = (1.68)\sqrt{1-(0.632)^2} = 1.30$ m

Example

Calculate the speed with which a car move in order that length may be shortened to half of its proper length.

Solution

Proper Length $= L_0$

Given that $L = \frac{L_0}{2}$ ....(1)

By length contraction formula $L = \frac{L_0}{\gamma}$ ....(2)

From (1) and (2) $\frac{L_0}{2} = \frac{L_0}{\gamma} \Rightarrow \gamma = 2$

Since $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ therefore $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 2$

$$\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{4} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{4} \Rightarrow \frac{v^2}{c^2} = \frac{3}{4} \Rightarrow v^2 = \frac{3}{4}c^2$$

$\Rightarrow v = \frac{\sqrt{3}}{2}c$ required

Example

A car moves at a speed 160 kmh^{-1}. If the length of the car is 2.4m, calculate the decrease in length as noted by a stationary observer.

Solution

Speed of car = v = 160 $kmh^{-1} = \frac{160\times1000}{3600} ms^{-1} = \frac{400}{9} ms^{-1}$

length of the car = $l = 2.4$m and decrease in length = $l' = \frac{l}{\gamma}$

$$l' = \frac{l}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}\, l = \sqrt{1 - \frac{\left(\frac{400}{9}\right)^2}{(3\times10^8)^2}}(2.4) = (1)(2.4) = 2.4$$

$\Rightarrow \Delta l = l' - l = 2.4 - 2.4 = 0m$

Example

What is the velocity of a meter scale if its length is observed to be shortened by a centimeter?

Solution

Let original length $= 1m$

shortened length $= 1cm = 0.01m$

length difference $= \Delta L = 1 - 0.01 = 0.99m$

By using formula $L = \frac{L_0}{\gamma}$

$\Rightarrow \frac{L}{L_0} = \sqrt{1 - \frac{v^2}{c^2}} = 0.99 \Rightarrow 1 - \frac{v^2}{c^2} = (0.99)^2 \Rightarrow \frac{v^2}{c^2} = 1 - (0.99)^2$

$\Rightarrow \frac{v}{c} = \sqrt{1 - (0.99)^2} \Rightarrow v = \sqrt{1 - (0.99)^2}c \qquad \Rightarrow v = 0.141c$

Example

Find the velocity of a moving rod when its moving length is quarter of its proper length.

Solution

Proper Length $= L_0$

Given that $L = \frac{L_0}{4} \Rightarrow \frac{L}{L_0} = \frac{1}{4}$

By length contraction formula $L_0 = \gamma L$

$\Rightarrow \frac{L}{L_0} = \frac{1}{\gamma} = \frac{1}{4} \Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{4} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{16} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{16} \Rightarrow \frac{v^2}{c^2} = \frac{15}{16}$

$\Rightarrow v^2 = \frac{15}{16}c^2$

$\Rightarrow v = \frac{\sqrt{15}}{4}c \Rightarrow v = \frac{\sqrt{15}}{4} \times 3 \times 10^8 ms^{-1} \Rightarrow v = 2.905 \times 10^8 ms^{-1}$ required

Example

A rocket is moving at such a speed that its length as measured by an observer on the earth is only half of its proper length. How fast is the rocket moving relative to the earth?

Solution

Proper Length $= L_0$

Given that $L = \frac{L_0}{2}$ ……………(1)

By length contraction formula $L = \frac{L_0}{\gamma}$ ……………(2)

From (1) and (2) $\frac{L_0}{2} = \frac{L_0}{\gamma} \Rightarrow \gamma = 2$

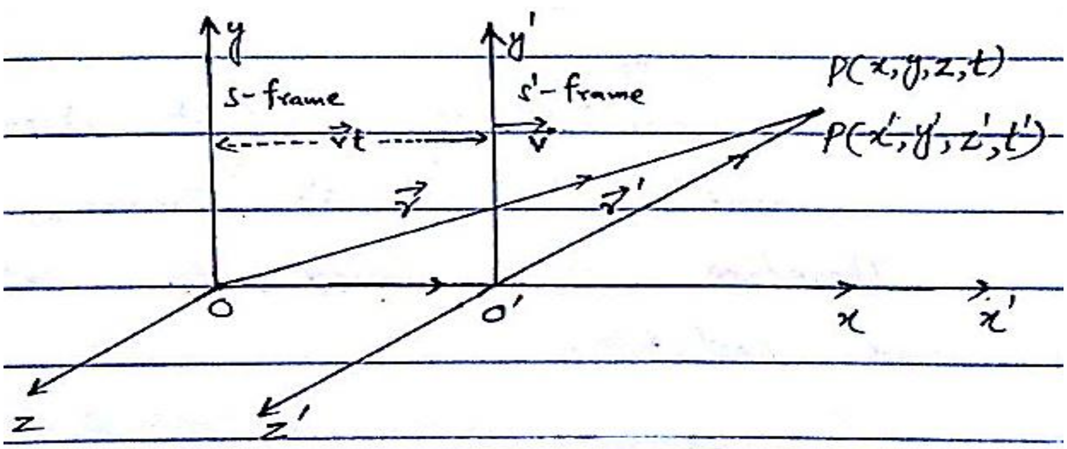
Since $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ therefore $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 2$

$\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{4} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{4} \Rightarrow \frac{v^2}{c^2} = \frac{3}{4} \Rightarrow v^2 = \frac{3}{4}c^2$

$\Rightarrow v = \frac{\sqrt{3}}{2}c = 0.866c$ required

TIME DILATION

Consider two inertial frames S and S'. S' moves relative to S with speed v along the positive x – direction.



Suppose a clock at rest at $(x', 0,0)$ in frame S' sends one light flash at time t_1' and the next one at a later time t_2'. In S', the time between the two events (consecutive flashes) is

$t_0' = t_2' - t_1'$ ……………(1)

If the observer is in $S -$ frame, then the time between the two events (consecutive flashes) is

$t_0 = t_2 - t_1$ ……………(2)

Since the observer is in $S -$ frame, then by using inverse Lorentz transformations

$t = \gamma \left(t' + \frac{x' v}{c^2} \right)$

$t_1 = \gamma \left(t_1' + \frac{x_1' v}{c^2} \right)$ ……………(3) At time t_1

$t_2 = \gamma \left(t_2' + \frac{x_2' v}{c^2} \right)$ ……………(4) At time t_2

Subtracting (3) and (4) we have

$t_0 = t_2 - t_1 = \gamma \left((t_2' - t_1') + (x_2' - x_1') \frac{v}{c^2} \right)$ ……………(5)

Since events take at the same place so $x_2' = x_1' = x'$

$(5) \Rightarrow t_0 = \gamma(t_2' - t_1')$

$\Rightarrow \boldsymbol{t_0 = \gamma t_0'}$

$$\Rightarrow t_0 = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) t_0' \Rightarrow t_0' = \left(\sqrt{1-\frac{v^2}{c^2}}\right) t_0 \qquad \text{Since } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Suppose $v = \frac{\sqrt{3}}{2}c$ where $v < c$ then

$$\sqrt{1-\frac{v^2}{c^2}} = \sqrt{1-\frac{\left(\frac{\sqrt{3}}{2}c\right)^2}{c^2}} = \sqrt{1-\frac{3}{4}} = \frac{1}{2} < 1 \text{ implies } \sqrt{1-\frac{v^2}{c^2}} < 1$$

Due to this factor $\sqrt{1-\frac{v^2}{c^2}}$ we get $\boldsymbol{t_0' < t_0}$

Time interval as recorded by an observer in S between two 'moving events' in S is longer than the time interval recorded by the observer in S' on a clock which is at rest with respect to where the events occur.

The smallest value for the time interval between two events is measured in the frame where the two events occur at the same location. All observers moving with respect to this frame will measure the time interval to be longer. This relativistic effect is called ***time dilation***.

According to this phenomenon clocks moving relative to an observer run more slowly compared to the clocks that are at rest relative to the observer.

A moving clock always appears to run more slowly than an identical clock at rest with respect to the observer. The time interval dilates, that is expands for a clock in motion. This effect is called time dilation.

Remember: In classical mechanics $v << c$ then $\frac{v}{c} << 1$ implies $\left(\frac{v}{c}\right)^2 \to 0$ using $\frac{v^2}{c^2} = 0$ so $\gamma = 1$ then $\boldsymbol{t_0' = t_0}$. <u>When watch is fixed in frame then we have in S frame $\boldsymbol{t_0' = \gamma t_0}$ and in S' frame $\boldsymbol{t_0' = (1/\gamma)\, t_0}$.</u>

Example

How fast must a space ship travel if a traveller in the space ship ages at only half the rate we are ageing on the earth?

Solution

Given that $t = \frac{t_0}{2}$ and we have to find v

By formula $t = \gamma t_0 \Rightarrow \frac{t_0}{2} = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \gamma = 2$

$\Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{4} \Rightarrow \frac{v^2}{c^2} = 1-\frac{1}{4} \Rightarrow \frac{v^2}{c^2} = \frac{3}{4} \Rightarrow v^2 = \frac{3}{4}c^2$

$\Rightarrow v = \frac{\sqrt{3}}{2}c = 0.866c$ required

Example

A particle moving at 0.8 c in a laboratory is observed to decay after travelling 3 m. What is its life span in its rest frame?

Solution

Given that $L_0 = 3m, v = 0.8c$ then we have to find t.

By formula $t = \frac{L}{v} = \frac{L_0/\gamma}{v} \Rightarrow t = \frac{3\times\sqrt{1-\frac{v^2}{c^2}}}{0.8c} \Rightarrow t = \frac{3\times\sqrt{1-(0.8)^2}}{0.8c}$

$\Rightarrow t = \frac{3\times\sqrt{1-0.64}}{0.8\times3\times10^8} \Rightarrow t = \frac{\sqrt{0.36}}{0.8}\times10^{-8} \Rightarrow t = \frac{0.6}{0.8}\times10^{-8}$

$\Rightarrow t = 0.75\times10^{-8}$

RECIPROCITY BETWEEN OBSERVERS

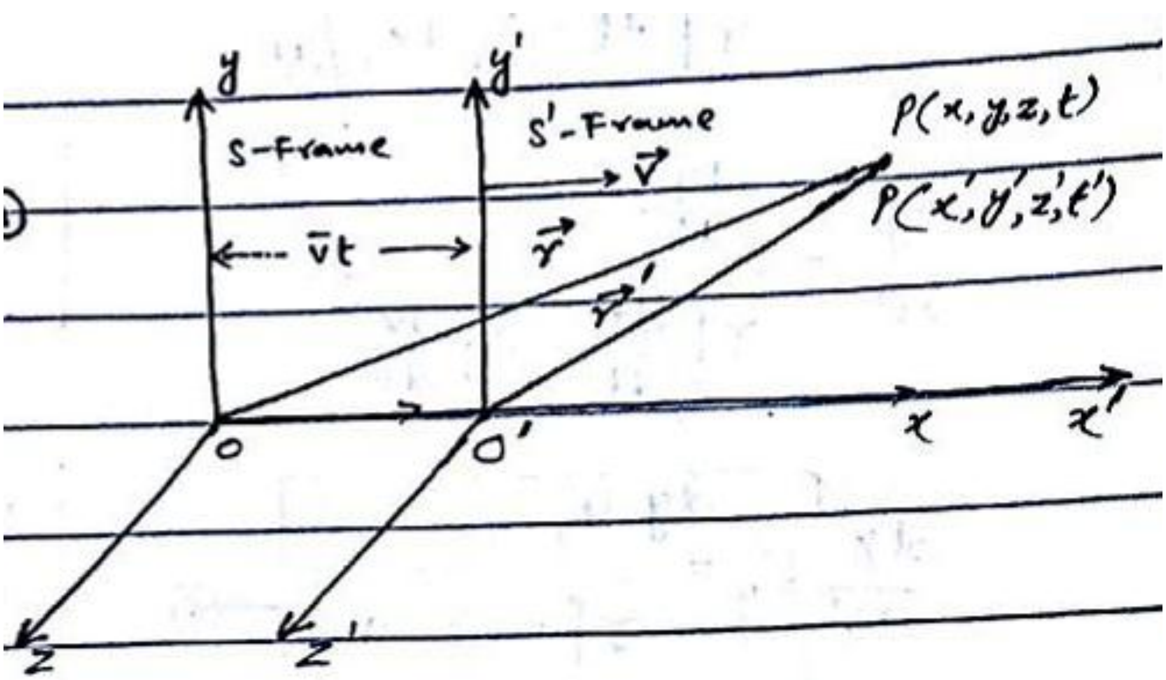
Just as a rod at rest in S' appears to be contracted to an observer in S, so also it can be shown that a rod at rest is frame S appears to be contracted by the same amount as measured by the observer in S'.

Similarly, to an observer in frame S, clocks at rest in S' appear to go slow and to an observer in S', clocks at rest in S appear to go slow. There is thus a complete reciprocity between the observers in frames S and S' as regards contraction in length and dilation of time.

COMPOSITION OF VELOCITIES /LORENTZ TRANSFORMATION LAW OF VELOCITIES / LORENTZ VELOCITIES

Consider an object moving with a constant velocity u' in an inertial frame S'. The object has velocity components u'_x, u'_y, u'_z in S'.

Let us find the velocity of the object as observed by an observer in frame S if S' is moving with a constant speed v along x – axis relative to frames S as shown in Figure. Where the space time coordinates are (x, y, z, t) and (x', y', z', t').



By using Lorentz Transformations

$x' = \gamma(x - vt)$

$y' = y, \quad z' = z$ ………………(A)

$t' = \gamma\left(t - \frac{xv}{c^2}\right)$

Taking differential of set of equations (A)

$dx' = \gamma(dx - vdt)$ ………………(1)

$dy' = dy$ ………………(2) and $dz' = dz$ ………………(3)

$dt' = \gamma\left(dt - dx\frac{v}{c^2}\right)$ ………………(4)

Dividing (1) and (4)

$$\frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma\left(dt - dx\frac{v}{c^2}\right)} = \frac{\gamma(dx - vdt)/_{dt}}{\gamma\left(dt - dx\frac{v}{c^2}\right)/_{dt}} = \frac{\gamma\left(\frac{dx}{dt} - v\frac{dt}{dt}\right)}{\gamma\left(\frac{dt}{dt} - \frac{dx}{dt}\frac{v}{c^2}\right)}$$

$\boldsymbol{u'_x} = \frac{\boldsymbol{u_x - v}}{\boldsymbol{1 - u_x\frac{v}{c^2}}}$ ………………(5)

Dividing (2) and (4)

$$\frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - dx\frac{v}{c^2}\right)} = \frac{dy/_{dt}}{\gamma\left(dt - dx\frac{v}{c^2}\right)/_{dt}} = \frac{\frac{dy}{dt}}{\gamma\left(\frac{dt}{dt} - \frac{dx}{dt}\frac{v}{c^2}\right)}$$

$\boldsymbol{u'_y} = \frac{\boldsymbol{u_y}}{\boldsymbol{\gamma\left(1 - u_x\frac{v}{c^2}\right)}}$ ………………(6)

Dividing (3) and (4)

$$\frac{dz'}{dt'} = \frac{dz}{\gamma\left(dt - dx\frac{v}{c^2}\right)} = \frac{dz/_{dt}}{\gamma\left(dt - dx\frac{v}{c^2}\right)\Big/_{dt}} = \frac{\frac{dz}{dt}}{\gamma\left(\frac{dt}{dt} - \frac{dx}{dt}\frac{v}{c^2}\right)}$$

$$\boldsymbol{u_z'} = \frac{\boldsymbol{u_z}}{\gamma\left(\boldsymbol{1} - \boldsymbol{u_x}\frac{\boldsymbol{v}}{\boldsymbol{c^2}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Thus u_x', u_y', u_z' are our required velocity transformations. In vector form, we can write as $\vec{u} = \left(u_x', u_y', u_z'\right)$.

Deduction (Lorentz Velocity Transformations under Non-Relativistic Limit)

If $v << c$ then $\frac{v}{c} << 1$ implies $\frac{v}{c^2} \to 0$ then we get

$$u_x' = u_x - v\,, \qquad u_y' = u_y\,, \qquad u_z' = u_z$$

Which give us the Galilean Transformations of velocity.

INVERSE LAW OF TRANSFORMATION OF VELOCITIES

By using inverse Lorentz Transformations

$$x = \gamma(x' + vt')$$

$$y = y'\ ,\ z = z' \qquad \ldots\ldots\ldots\ldots\ldots\ldots(B)$$

$$t = \gamma\left(t' + \frac{x'v}{c^2}\right)$$

Taking differential of set of equations (B)

$$dx = \gamma(dx' + vdt') \ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$dy = dy' \ \ldots\ldots\ldots\ldots\ldots\ldots(2) \quad \text{and} \quad dz = dz' \ \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$dt = \gamma\left(dt' + dx'\frac{v}{c^2}\right) \ \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Dividing (1) and (4)

$$\frac{dx}{dt} = \frac{\gamma(x' + vt')}{\gamma\left(dt' + dx'\frac{v}{c^2}\right)} = \frac{\gamma(x' + vt')\big/_{dt'}}{\gamma\left(dt' + dx'\frac{v}{c^2}\right)\Big/_{dt'}} = \frac{\gamma\left(\frac{dx'}{dt'} + v\frac{dt'}{dt'}\right)}{\gamma\left(\frac{dt'}{dt'} + \frac{dx'}{dt'}\frac{v}{c^2}\right)}$$

$$\boldsymbol{u_x} = \frac{\boldsymbol{u'_x + v}}{\boldsymbol{1 + u'_x\frac{v}{c^2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Dividing (2) and (4)

$$\frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + dx'\frac{v}{c^2}\right)} = \frac{dy'\big/_{dt'}}{\gamma\left(dt' + dx'\frac{v}{c^2}\right)\Big/_{dt'}} = \frac{\frac{dy'}{dt'}}{\gamma\left(\frac{dt'}{dt'} + \frac{dx'}{dt'}\frac{v}{c^2}\right)}$$

$$\boldsymbol{u_y} = \frac{\boldsymbol{u'_y}}{\boldsymbol{\gamma\left(1 + u'_x\frac{v}{c^2}\right)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Dividing (3) and (4)

$$\frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + dx'\frac{v}{c^2}\right)} = \frac{dz'\big/_{dt'}}{\gamma\left(dt' + dx'\frac{v}{c^2}\right)\Big/_{dt'}} = \frac{\frac{dz;}{dt'}}{\gamma\left(\frac{dt'}{dt'} + \frac{dx'}{dt'}\frac{v}{c^2}\right)}$$

$$\boldsymbol{u_z} = \frac{\boldsymbol{u'_z}}{\boldsymbol{\gamma\left(1 + u'_x\frac{v}{c^2}\right)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Thus u_x, u_y, u_z are our required inverse velocity transformations. In vector form, we can write as $\vec{u} = \left(u_x, u_y, u_z\right)$.

COMPOSITION OF ACCELERATION /LORENTZ TRANSFORMATION LAW OF ACCELERATION

By using Law of Transformation of velocities

$$u'_x = \frac{u_x - v}{1 - u_x\frac{v}{c^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$u'_y = \frac{u_y}{\gamma\left(1 - u_x\frac{v}{c^2}\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$u'_z = \frac{u_z}{\gamma\left(1 - u_x\frac{v}{c^2}\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$dt' = \gamma\left(dt - dx\,\frac{v}{c^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Taking differential of (1)

$$du'_x = \frac{\left(1 - u_x\frac{v}{c^2}\right)d(u_x - v) - (u_x - v)d\left(1 - u_x\frac{v}{c^2}\right)}{\left(1 - u_x\frac{v}{c^2}\right)^2} = \frac{\left(1 - u_x\frac{v}{c^2}\right)(du_x - 0) - (u_x - v)d\left(0 - du_x\frac{v}{c^2}\right)}{\left(1 - u_x\frac{v}{c^2}\right)^2}$$

$$du'_x = \frac{du_x\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - u_x\frac{v}{c^2}\right)^2}$$

$$\frac{du'_x}{dt'} = \frac{\dfrac{du_x\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - u_x\frac{v}{c^2}\right)^2}}{\gamma\left(dt - dx\,\frac{v}{c^2}\right)} = \frac{du_x\left(1 - \frac{v^2}{c^2}\right)\Big/dt}{\gamma\left(dt - dx\,\frac{v}{c^2}\right)\left(1 - u_x\,\frac{v}{c^2}\right)^2\Big/dt}$$

$$u'_{xx} = \frac{u_{xx}\left(\frac{1}{\gamma^2}\right)}{\gamma\left(\frac{dt}{dt} - \frac{dx}{dt}\frac{v}{c^2}\right)\left(1 - u_x\,\frac{v}{c^2}\right)^2} = \frac{u_{xx}\left(\frac{1}{\gamma^2}\right)}{\gamma\left(1 - u_x\,\frac{v}{c^2}\right)\left(1 - u_x\,\frac{v}{c^2}\right)^2}$$

$$\boldsymbol{a'_x = \frac{a_x}{\gamma^3\left(1 - u_x\frac{v}{c^2}\right)^3}}$$

Taking differential of (2)

$$du'_y = \frac{1}{\gamma}\left[\frac{\left(1-u_x\frac{v}{c^2}\right)d(u_y) - u_y d\left(1-u_x\frac{v}{c^2}\right)}{\left(1-u_x\frac{v}{c^2}\right)^2}\right] = \frac{1}{\gamma}\left[\frac{\left(1-u_x\frac{v}{c^2}\right)du_y - u_y\left(0-du_x\frac{v}{c^2}\right)}{\left(1-u_x\frac{v}{c^2}\right)^2}\right]$$

$$du'_y = \frac{1}{\gamma}\left[\frac{du_y}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_y du_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}\right]$$

$$\frac{du'_y}{dt'} = \frac{\frac{1}{\gamma}\left[\frac{du_y}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_y du_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}\right]}{\gamma\left(dt - dx\frac{v}{c^2}\right)}$$

$$\frac{du'_y}{dt'} = \frac{\frac{1}{\gamma}\left[\frac{du_y}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_y du_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}\right]\Big/dt}{\gamma\left(dt - dx\frac{v}{c^2}\right)\Big/dt}$$

$$u'_{yy} = \frac{\frac{\frac{du_y}{dt}}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_y\frac{du_x}{dt}\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}}{\gamma^2\left(\frac{dt}{dt} - \frac{dx}{dt}\frac{v}{c^2}\right)} = \frac{\frac{a_y}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_y a_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}}{\gamma^2\left(1-u_x\frac{v}{c^2}\right)}$$

$$\boldsymbol{a'_y = \frac{1}{\gamma^2\left(1-u_x\frac{v}{c^2}\right)^2}\left[a_y + \frac{u_y a_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)}\right]}$$

Taking differential of (3)

$$du'_z = \frac{1}{\gamma}\left[\frac{\left(1-u_x\frac{v}{c^2}\right)d(u_z)-u_z d\left(1-u_x\frac{v}{c^2}\right)}{\left(1-u_x\frac{v}{c^2}\right)^2}\right] = \frac{1}{\gamma}\left[\frac{\left(1-u_x\frac{v}{c^2}\right)du_z-u_z\left(0-du_x\frac{v}{c^2}\right)}{\left(1-u_x\frac{v}{c^2}\right)^2}\right]$$

$$du'_z = \frac{1}{\gamma}\left[\frac{du_z}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_z du_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}\right]$$

$$\frac{du'_z}{dt'} = \frac{\frac{1}{\gamma}\left[\frac{du_z}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_z du_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}\right]}{\gamma\left(dt-dx\frac{v}{c^2}\right)}$$

$$\frac{du'_z}{dt'} = \frac{\frac{1}{\gamma}\left[\frac{du_z}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_z du_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}\right]\Big/dt}{\gamma\left(dt-dx\frac{v}{c^2}\right)\Big/dt}$$

$$u'_{zz} = \frac{\frac{\frac{du_z}{dt}}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_z\frac{du_x}{dt}\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}}{\gamma^2\left(\frac{dt}{dt}-\frac{dx}{dt}\frac{v}{c^2}\right)} = \frac{\frac{a_z}{\left(1-u_x\frac{v}{c^2}\right)} + \frac{u_z a_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)^2}}{\gamma^2\left(1-u_x\frac{v}{c^2}\right)}$$

$$\boldsymbol{a'_z = \frac{1}{\gamma^2\left(1-u_x\frac{v}{c^2}\right)^2}\left[a_z + \frac{u_z a_x\frac{v}{c^2}}{\left(1-u_x\frac{v}{c^2}\right)}\right]}$$

Thus a'_x, a'_y, a'_z are our required acceleration transformations. In vector form, we can write as $\vec{a} = \left(a'_x, a'_y, a'_z\right)$.

INVERSE LORENTZ TRANSFORMATION LAW OF ACCELERATION

By using inverse Lorentz Transformations of velocities

$$u_x = \frac{u'_x + v}{1 + u'_x \frac{v}{c^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$u_y = \frac{u'_y}{\gamma\left(1 + u'_x \frac{v}{c^2}\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$u_z = \frac{u'_z}{\gamma\left(1 + u'_x \frac{v}{c^2}\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$t = \gamma\left(t' + \frac{x'v}{c^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$dt = \gamma\left(dt' + dx'\frac{v}{c^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Taking differential of (1), (2) and (3)

$$du_x = \frac{\left(1 + u'_x \frac{v}{c^2}\right) d(u'_x + v) - (u'_x + v)\, d\left(1 + u'_x \frac{v}{c^2}\right)}{\left(1 + u'_x \frac{v}{c^2}\right)^2} = \frac{\left(1 + u'_x \frac{v}{c^2}\right)(du'_x + 0) - (u'_x + v)\left(0 + du'_x \frac{v}{c^2}\right)}{\left(1 + u'_x \frac{v}{c^2}\right)^2}$$

$$du_x = \frac{du'_x + du'_x u'_x \frac{v}{c^2} - du'_x u'_x \frac{v}{c^2} - v\, du'_x \frac{v}{c^2}}{\left(1 + u'_x \frac{v}{c^2}\right)^2} = \frac{\left(1 - \frac{v^2}{c^2}\right) du'_x}{\left(1 + u'_x \frac{v}{c^2}\right)^2}$$

$$du_x = \frac{du'_x}{\gamma^2\left(1 + u'_x \frac{v}{c^2}\right)^2}$$

$$\frac{du_x}{dt} = \frac{\frac{du'_x}{\gamma^2\left(1 + u'_x \frac{v}{c^2}\right)^2}}{\gamma\left(dt' + dx'\frac{v}{c^2}\right)} \qquad \text{dividing by (5)}$$

$$\frac{du_x}{dt} = \frac{1}{\gamma^3}\ \frac{\frac{du'_x}{\left(1 + u'_x \frac{v}{c^2}\right)^2} \Big/ dt'}{\left(dt' + dx'\frac{v}{c^2}\right) \Big/ dt'} = \frac{1}{\gamma^3}\ \frac{\frac{du'_x / dt'}{\left(1 + u'_x \frac{v}{c^2}\right)^2}}{\left(\frac{dt'}{dt'} + \frac{dx'}{dt'}\frac{v}{c^2}\right)} = \frac{1}{\gamma^3}\ \frac{u''_x}{\left(1 + u'_x \frac{v}{c^2}\right)\left(1 + u'_x \frac{v}{c^2}\right)^2}$$

$$\boldsymbol{a_x} = \frac{\boldsymbol{a'_x}}{\boldsymbol{\gamma^3\left(1+u'_x\frac{v}{c^2}\right)^3}}$$

$$du_y = \frac{1}{\gamma}\left[\frac{\left(1+u'_x\frac{v}{c^2}\right)du'_y - u'_y d\left(1+u'_x\frac{v}{c^2}\right)}{\left(1+u'_x\frac{v}{c^2}\right)^2}\right] = \frac{1}{\gamma}\left[\frac{\left(1+u'_x\frac{v}{c^2}\right)du'_y - u'_y\left(0+du'_x\frac{v}{c^2}\right)}{\left(1+u'_x\frac{v}{c^2}\right)^2}\right]$$

$$du_y = \frac{1}{\gamma}\left[\frac{du'_y}{\left(1+u'_x\frac{v}{c^2}\right)} - \frac{u'_y du'_x\frac{v}{c^2}}{\left(1+u'_x\frac{v}{c^2}\right)^2}\right]$$

$$\frac{du_y}{dt} = \frac{\frac{1}{\gamma}\left[\frac{du'_y}{\left(1+u'_x\frac{v}{c^2}\right)} - \frac{u'_y du'_x\frac{v}{c^2}}{\left(1+u'_x\frac{v}{c^2}\right)^2}\right]}{\gamma\left(dt'+dx'\frac{v}{c^2}\right)}$$ dividing by (5)

$$\frac{du_y}{dt} = \frac{1}{\gamma^2\left(1+u'_x\frac{v}{c^2}\right)}\frac{\left[du'_y - \frac{u'_y du'_x\frac{v}{c^2}}{\left(1+u'_x\frac{v}{c^2}\right)}\right]}{\left(dt'+dx'\frac{v}{c^2}\right)} = \frac{1}{\gamma^2\left(1+u'_x\frac{v}{c^2}\right)}\frac{\left[\frac{du'_y}{dt'} - \frac{u'_y\frac{du'_x}{dt}\frac{v}{c^2}}{\left(1+u'_x\frac{v}{c^2}\right)}\right]}{\left(\frac{dt'}{dt'}+\frac{dx'}{dt'}\frac{v}{c^2}\right)}$$

$$a_y = \frac{1}{\gamma^2\left(1+u'_x\frac{v}{c^2}\right)}\frac{\left[a'_y - \frac{u'_y a'_x\frac{v}{c^2}}{\left(1+u'_x\frac{v}{c^2}\right)}\right]}{\left(1+u'_x\frac{v}{c^2}\right)}$$

$$\boldsymbol{a_y} = \frac{\boldsymbol{1}}{\boldsymbol{\gamma^2\left(1+u'_x\frac{v}{c^2}\right)^2}}\left[\boldsymbol{a'_y} - \frac{\boldsymbol{u'_y a'_x\frac{v}{c^2}}}{\boldsymbol{\left(1+u'_x\frac{v}{c^2}\right)}}\right]$$

Similarly

$$\boldsymbol{a_z} = \frac{\boldsymbol{1}}{\boldsymbol{\gamma^2\left(1+u'_x\frac{v}{c^2}\right)^2}}\left[\boldsymbol{a'_z} - \frac{\boldsymbol{u'_z a'_x\frac{v}{c^2}}}{\boldsymbol{\left(1+u'_x\frac{v}{c^2}\right)}}\right]$$

Remember: In classical mechanics $v << c$ then $\frac{v}{c} << 1$ implies $\left(\frac{v}{c}\right)^2 \to 0$ using $\frac{v^2}{c^2} = 0$ so $\gamma = 1$ then $\boldsymbol{a'_x = a_x, a'_y = a_y, a'_z = a_z}$

Acceleration remains same in all inertial frames.

Question

Show that the speed of light c is the same in all directions in all inertial frames.

Solution

Consider a light signal travelling with speed c along the x' −axis in frame S'.
According to Galilean theory the speed of light as observed in S would be $(c+v)$.
Now using formula $u_x = \frac{u'_x + v}{1 + u'_x \frac{v}{c^2}}$

$$\Rightarrow u_x = \frac{c+v}{1+c\frac{v}{c^2}} = \frac{c+v}{1+\frac{v}{c}} = \frac{c+v}{\frac{1}{c}(c+v)} \Rightarrow u_x = c$$

Consider next the case of observers moving at right angles to the direction of light propagation.

Suppose a ray of light travels along $O'Y'$ in frames S'. Then

$$u'_x = 0, u'_y = c, u'_z = 0$$

Now $u_x = \frac{u'_x + v}{1 + u'_x \frac{v}{c^2}} = v$ since $u'_x = 0$

$$u_y = \frac{u'_y}{\gamma\left(1 + u'_x \frac{v}{c^2}\right)} = \frac{c}{\gamma} \qquad \text{since } u'_x = 0, u'_y = c$$

$$u_z = \frac{u'_z}{\gamma\left(1 + u'_x \frac{v}{c^2}\right)} = 0 \qquad \text{since } u'_z = 0$$

$$\Rightarrow u = \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{u_x^2 + u_y^2} = \sqrt{v^2 + \frac{c^2}{\gamma^2}}$$

$$\Rightarrow u = \sqrt{v^2 + c^2\left(1 - \frac{v^2}{c^2}\right)} = \sqrt{v^2 + c^2 - v^2}$$

$$\Rightarrow u = c$$

Question

Show that $\boldsymbol{c} + \boldsymbol{c} = \boldsymbol{c}!$

Solution

Using formula $u_x = \frac{u'_x + v}{1 + u'_x \frac{v}{c^2}}$

$u_x = \frac{c+c}{1 + c\frac{c}{c^2}}$ putting $u'_x = c, v = c$

$u_x = c$

Thus relative velocity of two objects or two frames or an object and a frame cannot exceed c. i.e. $c + c = c!$

> $c!$
>
> is actually the values which are less or equal to c

Example

Calculate $\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$ when (a) $v = 10^{-2}c$ and (b) $v = 0.9998c$.

Solution

According to the binomial expansion

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots \approx 1 + nx \text{ when } x \ll 1$$

(a) Setting $x = -\frac{v^2}{c^2} = 10^{-4}$ and $n = \frac{1}{2}$, we have

$$\left(1 - \frac{v^2}{c^2}\right)^{1/2} = (1 - 10^{-4})^{1/2} \approx 1 + \frac{1}{2}(-10^{-4}) = 1 - 0.00005 = 0.99995$$

(b) $\left(1 - \frac{v^2}{c^2}\right)^{1/2} = (1 - (0.9998)^2)^{1/2} = [1 - (1 - 0.0002)^2]^{1/2}$

Now $(1 - 0.0002)^2 \approx 1 + 2(-0.0002) = 1 - 0.0004$

$\therefore$ $\left(1 - \frac{v^2}{c^2}\right)^{1/2} = [1 - (1 - 0.0004)]^{1/2} = (0.0004)^{1/2} = 0.02.$

Example

What is the velocity of a meter scale if its length is observed to be 0.99 m?

Solution

$$L = L_0\left(1-\frac{v^2}{c^2}\right)^{1/2} \quad \text{or} \quad \left(1-\frac{v^2}{c^2}\right)=\left(\frac{L}{L_0}\right)^2 = (0.99)^2$$

$$\therefore \qquad \frac{v^2}{c^2} = 1-(0.99)^2 = 1-(1-0.01)^2 \approx 1-[1-2\,(0.01)] = 0.02$$

$$\therefore \qquad v = 0.141c$$

Example

How long does it take for a meter scale to pass you if it is travelling with a speed of 0.6c relative to you along the direction of its length?

Solution

The time required is $t = \frac{\text{length}}{\text{velocity}}$, where length L is the contracted length $L = L_0\left(1-\frac{v^2}{c^2}\right)^{1/2}$

$$\therefore \qquad t = \frac{L_0\left(1-\frac{v^2}{c^2}\right)^{1/2}}{v} = \frac{1[1-(0.6)^2]^{1/2}}{0.6\times 3\times 10^8} = \frac{0.8}{1.8\times 10^8} = 4.44\times 10^{-9}\text{ s.}$$

Example

Two particles approach one another. Calculate their relative speed if each has a speed of 0.9 c with respect to the laboratory.

Solution

$v_1 = 0.9c, v_2 = 0.9c$ then we have to find v.

Using formula $v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$

$$v = \frac{0.9c+0.9c}{1+\frac{(0.9c)(0.9c)}{c^2}} = \frac{2(0.9c)}{1+\frac{(0.9c)^2 c^2}{c^2}} = \frac{2(0.9c)}{1+0.81} = \frac{1.8}{1.81}c$$

$$v = 0.99447c$$

Example

Two bodies are moving in opposite directions with speeds c relative to an inertial frame. Show that their relative velocity is c.

Solution

$v_1 = c, v_2 = c$ then we have to find v.

Using formula $v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$

$$v = \frac{c+c}{1+\frac{(c)(c)}{c^2}} = \frac{2c}{1+\frac{c^2}{c^2}} = \frac{2c}{1+1} = \frac{2}{2}c$$

$$v = c$$

Example

An astronaut is travelling in a space vehicle with velocity 0.6c relative to the earth. The astronaut measures his pulse rate to be 75 per minute. Signals generated by astronaut's pulse are radioed to earth when the space vehicle is moving perpendicular to a line that connects the vehicle with an earth observer. What is the pulse rate as measured by the observer on the earth? Hence comment on the life span of the astronaut from point of view of the earth observer.

Solution

The time interval between two consecutive pulses as measured by the astronaut is $T_0 = \frac{60}{75} = 0.8s$. The interval between the pulses as measured by the earth observer is

$$T = \frac{T_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \frac{0.80}{(0.64)^{\frac{1}{2}}} = 1.0s$$

The earth observer measures a pulse rate of 60/1.0 = 60 pulses per minute. The pulse rate measured by the earth observer is less than that measured by the astronaut. Hence the life span of the astronaut determined by the total number of his heartbeats is longer as measured by the earth clock than the life span measured by a clock aboard the space vehicle.

Note that beating of the heart is a kind of clock mechanism. The repetitive radio signal from the space vehicle is subject to time dilation as well as Doppler Effect. We have eliminated the usual Doppler Effect by choosing to calculate the pulse rate at the instant when the vehicle is travelling perpendicular (transverse) to the line connecting the vehicle and the earth observer. The time dilation still brings about a shift in the frequency. Hence time dilation brings about a transverse Doppler shift.

Example

Show that the velocities U' and U measured in frames S' and S are related by

$$\sqrt{1-\frac{U'^2}{c^2}} = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{U^2}{c^2}\right)}{\left(1-\frac{vUx}{c^2}\right)^2}}$$

where v is the velocity of S' relative to S. What is the inverse relation?

$$\vec{U}' = (U'_x, U'_y, U'_z)$$

$$U'^2 = U'^2_x + U'^2_y + U'^2_z \quad \text{——} (1)$$

Using Law of Transformation of velocity

$$U'_x = \frac{U_x - v}{1 - \frac{vU_x}{c^2}}$$

$$U'_y = \frac{U_y}{\gamma\left(1 - \frac{vU_x}{c^2}\right)} \quad \& \quad U'_z = \frac{U_z}{\gamma\left(1 - \frac{vU_x}{c^2}\right)}$$

in eq (1)

$$U'^2 = \left(\frac{U_x - v}{1 - \frac{vU_x}{c^2}}\right)^2 + \left(\frac{U_y \gamma^{-1}}{1 - \frac{vU_x}{c^2}}\right)^2 + \left(\frac{U_z \gamma^{-1}}{1 - \frac{vU_x}{c^2}}\right)^2$$

$$U'^2 = \frac{(U_x - v)^2 + (U_y^2 + U_z^2)\gamma^{-2}}{\left(1 - \frac{vU_x}{c^2}\right)^2} \quad \left[\begin{array}{l} \because U^2 = U_x^2 + U_y^2 + U_z^2 \\ U_y^2 + U_z^2 = U^2 - U_x^2 \end{array}\right.$$

$$\frac{u'^2}{c^2} = \frac{\left(\frac{u_x}{c} - \frac{v}{c}\right)^2 + \frac{1}{c^2}\left(u^2 - u_x^2\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v u_x}{c^2}\right)^2}$$

$$\frac{u'^2}{c^2} = \frac{\left(\frac{u_x}{c} - \frac{v}{c}\right)^2 + \left(\frac{u^2}{c^2} - \frac{u_x^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v u_x}{c^2}\right)^2}$$

$$\left[\because \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \gamma^{-2} = \left(1 - \frac{v^2}{c^2}\right)\right.$$

$$\frac{u'^2}{c^2} = \frac{\frac{u_x^2}{c^2} + \frac{v^2}{c^2} - \frac{2 v u_x}{c^2} + \frac{u^2}{c^2} - \frac{u^2 v^2}{c^4} - \frac{u_x^2}{c^2} + \frac{v^2 u_x^2}{c^4}}{\left(1 - \frac{v u_x}{c^2}\right)^2}$$

$$1 - \frac{u'^2}{c^2} = 1 - \frac{\frac{u_x^2}{c^2} + \frac{v^2}{c^2} - \frac{2 v u_x}{c^2} + \frac{u^2}{c^2} - \frac{u^2 v^2}{c^4} - \frac{u_x^2}{c^2} + \frac{v^2 u_x^2}{c^4}}{\left(1 - \frac{v u_x}{c^2}\right)^2}$$

$$1 - \frac{u_1^2}{c^2} = \frac{\left(1 - \frac{v u_x}{c^2}\right)^2 - \frac{u_x^2}{c^2} - \frac{v^2}{c^2} + \frac{2 v u_x}{c^2} - \frac{u^2}{c^2} + \frac{u^2 v^2}{c^4} + \frac{u_x^2}{c^2} - \frac{v^2 u_x^2}{c^4}}{\left(1 - \frac{v u_x}{c^2}\right)^2}$$

$$1 - \frac{u_1^2}{c^2} = \frac{1 + \frac{v^2 u_x^2}{c^4} - \frac{2 v u_x}{c^2} - \frac{u_x^2}{c^2} - \frac{v^2}{c^2} + \frac{2 v u_x}{c^2} - \frac{u^2}{c^2} + \frac{u^2 v^2}{c^4} + \frac{u_x^2}{c^2} - \frac{v^2 u_x^2}{c^4}}{\left(1 - \frac{v u_x}{c^2}\right)^2}$$

$$= \frac{1 - \frac{v^2}{c^2} - \frac{u^2}{c^2} + \frac{u^2 v^2}{c^4}}{\left(1 - \frac{v u_x}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{1\left(1-\frac{v^2}{c^2}\right)-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{vu_x}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1-v^2/c^2\right)\left(1-\frac{u^2}{c^2}\right)}{\left(1-\frac{vu_x}{c^2}\right)^2}.$$

Taking Squar-root

$$\sqrt{1-\frac{u_1^2}{c^2}} = \sqrt{\frac{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}{\left(1-\frac{vu_x}{c^2}\right)^2}}$$

Then Inverse relation can be written by changing the sign of v.

$$\sqrt{1-\frac{u_1^2}{c^2}} = \sqrt{\frac{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}{\left(1+\frac{vu_x}{c^2}\right)}}$$

Question

A rocket is moving at such a speed that its length as measured by an observer on the earth is only half of its proper length. How fast is the rocket moving relative to the earth?

Solution

proper length $= L_0$

Given that $L = \frac{1}{2} L_0$ —— ①

where L is the measured by the observer on the earth.

$v = ?$

By using Length Contraction

$L_0 = \gamma L \Rightarrow L_0 = \gamma \left(\frac{1}{2} L_0\right)$

$\Rightarrow 1 = \gamma \left(\frac{1}{2}\right)$

$\Rightarrow \gamma = 2$

$$\Rightarrow \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 2$$

(by Inverting) $\Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2}$

(Squaring) $\Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{4}$

$$\Rightarrow 1 - \frac{1}{4} = \frac{v^2}{c^2}$$

$$\Rightarrow \frac{4-1}{4} = \frac{v^2}{c^2}$$

$$\Rightarrow \frac{3}{4} = \frac{v^2}{c^2}$$

$$\Rightarrow v^2 = \frac{3}{4}c^2 \Rightarrow v^2 = 0.75c^2$$

$$\Rightarrow \sqrt{v^2} = \sqrt{0.75c^2} \Rightarrow \boxed{v = 0.866c}$$

Question

What is the velocity of a metre scale if its length is observed to be shortened by a centimeter?

Solution: let

proper Length $= L_0 = 1m = 100\,cm$

Given that

$L = (100-1)\,cm = 99\,cm$ $\qquad v = ?$

By using Length Contraction

$$L_0 = \gamma L \Rightarrow \frac{1}{\gamma} = \frac{L}{L_0} = \frac{99}{100}$$

$$\sqrt{\left(1 - \frac{v^2}{c^2}\right)} = 0.99$$

Taking square → $1 - \frac{v^2}{c^2} = (0.99)^2$

$1 - (0.99)^2 = v^2/c^2$

$\frac{v^2}{c^2} = 1 - 0.9801$

$\frac{v^2}{c^2} = 0.0199$

$\sqrt{v^2} = \sqrt{0.0199\, c^2}$

$\boxed{v = 0.1411\, c}$

Question

A space ship is observed to cover 90 m in $5 \times 10^{-7} s$. What is the distance travelled and time taken as measured by an observer in the space ship?

Solution: Given $L_o = 90\,m$ $t_o = 5 \times 10^{-7}\,s$

$L = ?$ & $t = ?$

$v = L_o/t_o$ $(\because S = vt \Rightarrow v = S/t)$

$v = \frac{90m}{5 \times 10^{-7} s} = \frac{1.8 \times 10^{8}}{3 \times 10^{8}} c \Rightarrow \boxed{v = 0.6\, c}$

Now By using Length Contraction

$L_o = \gamma L \Rightarrow L = \frac{1}{\gamma} L_o$

$L = \sqrt{1 - v^2/c^2}\, L_o$

$L = \sqrt{1 - \frac{0.6^2 c^2}{c^2}}\,(90)$

$$L = 90 \times \sqrt{1-(0.6)^2}$$

$$\boxed{L = 72\ m}$$

Now by using time Dialation

$$t = \gamma t_o = \frac{1}{\sqrt{1 - v^2/c^2}}\, t_o$$

$$t = \frac{1}{\sqrt{1-(0.6)^2}} \left(5 \times 10^{-7}\, s\right)$$

$$t = \frac{5}{4} \times 5 \times 10^{-7}\, s$$

Example

It takes 10^5 years for light to reach us from the farthest part of our galaxy. Is it possible for a man to travel out to that part of our galaxy at a constant speed in a reasonable time of say 50 years?

Solution The distance travelled by light in 10^5 years is according to an observer on earth given by $L_0 = 10^5$ Nc, where c is the speed of light (3×10^8 m/s) and N is the number of seconds in one year.

For the traveller who observes the farthest part of the galaxy approaching him at a speed say v, the distance is

$$L = L_0\left(1 - \frac{v^2}{c^2}\right)^{1/2} = 10^5\,\text{Nc}\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

Total time available for the journey is 50 N seconds. Hence the speed necessary is given by

$$v = \frac{L}{50\,\text{N}} = \frac{10^5\,\text{Nc}\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{50\,\text{N}}$$

$$\therefore \qquad \frac{v}{c} = 2 \times 10^3\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$\therefore \qquad \frac{v^2}{c^2} = 4 \times 10^6\left(1 - \frac{v^2}{c^2}\right)$$

$$\therefore \qquad \frac{v^2}{c^2} = \frac{4 \times 10^6}{1 + 4 \times 10^6} \quad \text{or} \quad \frac{v}{c} = \left(\frac{1}{1 + 2.5 \times 10^{-7}}\right)^{1/2}$$

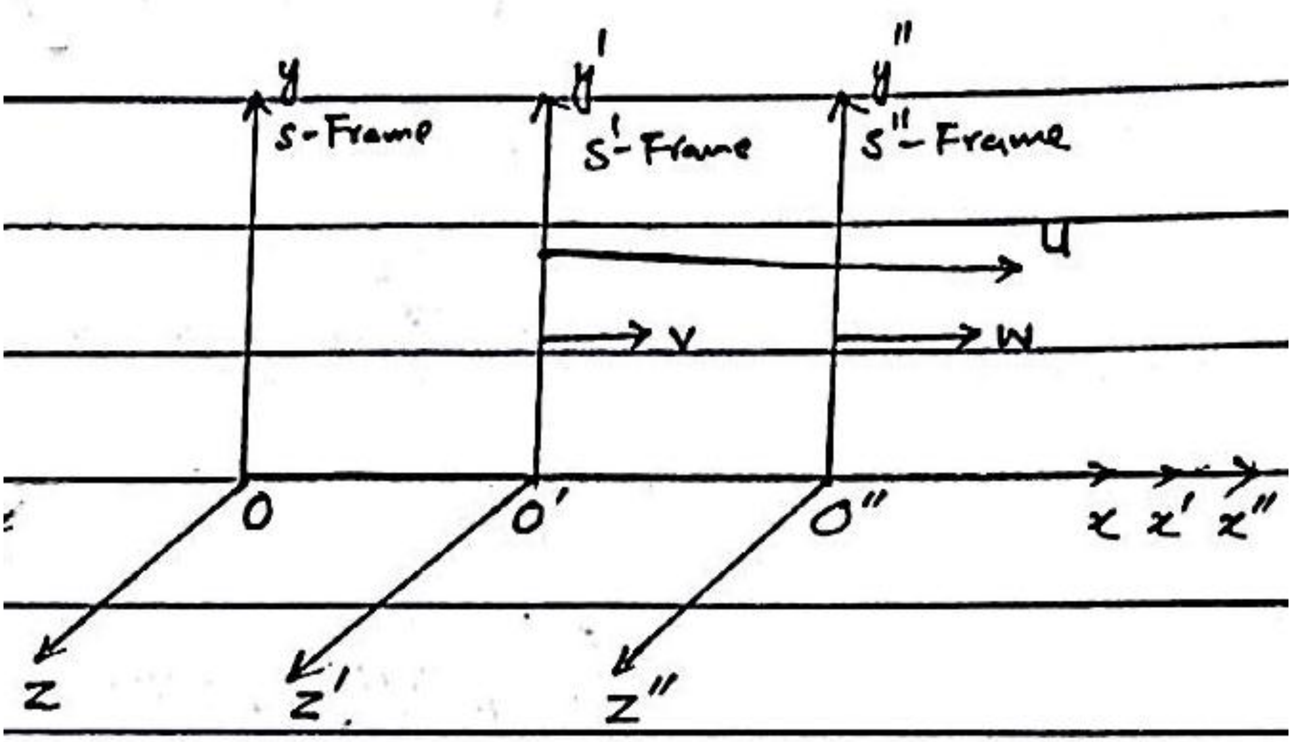
$$= (1 + 2.5 \times 10^{-7})^{-1/2} \approx 1 - 1.25 \times 10^{-7} \text{ using binomial theorem.}$$

$$\therefore \qquad v = 0.99999975c.$$

A traveller travelling at above speed will be able to complete the trip in 50 years. The traveller ages by 50 years in the course of the journey. The time for the journey as measured on the earth will be more than 10^5 years!

RELATIVISTIC LAW OF ADDTION OF VELOCITIES / VELOCITY ADDITION FORMULA / COMPOSITION OF TWO LORENTZ TRANSFORMATIONS IS AGAIN LORENTZ TRANSFORMATION

Consider a frame of reference S' moving with uniform velocity v with respect to a frame S. Also suppose that there is another frame S'' moving with uniform velocity w with respect to a frame S'.



Then

Lorentz Transformations of S' with respect to S is

$x' = \gamma_1(x - vt)$

$y' = y, \quad z' = z$ ....(1)

$t' = \gamma_1\left(t - \frac{xv}{c^2}\right)$

Where $\gamma_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

And Lorentz Transformations of S'' with respect to S' is

$x'' = \gamma_2(x' - wt')$

$y'' = y', \quad z'' = z'$ ………………(2)

$t'' = \gamma_2\left(t' - \frac{x'w}{c^2}\right)$

Where $\gamma_2 = \frac{1}{\sqrt{1-\frac{w^2}{c^2}}}$

Using Set of equation (1) in (2)

$\Rightarrow x'' = \gamma_2(x' - wt')$

$\Rightarrow x'' = \gamma_2\left[\gamma_1(x - vt) - w\gamma_1\left(t - \frac{xv}{c^2}\right)\right]$

$\Rightarrow x'' = \gamma_1\gamma_2\left[x - vt - wt + \frac{xwv}{c^2}\right]$

$\Rightarrow x'' = \gamma_1\gamma_2\left[x\left(1 + \frac{wv}{c^2}\right) - (v + w)t\right]$

$\Rightarrow x'' = \gamma_1\gamma_2\left(1 + \frac{wv}{c^2}\right)\left[x - \left(\frac{v+w}{1+\frac{wv}{c^2}}\right)t\right]$ ………………(3)

$$\gamma_1\gamma_2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\sqrt{1-\frac{w^2}{c^2}}} = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{w^2}{c^2}\right)}} = \frac{1}{\sqrt{1-\frac{w^2}{c^2}-\frac{v^2}{c^2}+\frac{v^2w^2}{c^4}}}$$

$$\gamma_1\gamma_2 = \frac{1}{\sqrt{1+\frac{v^2w^2}{c^4}+2\frac{vw}{c^2}-2\frac{vw}{c^2}-\frac{w^2}{c^2}-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1+\frac{v^2w^2}{c^4}+2\frac{vw}{c^2}-\left(\frac{w^2}{c^2}+\frac{v^2}{c^2}+2\frac{vw}{c^2}\right)}}$$

$$\gamma_1\gamma_2 = \frac{1}{\sqrt{\left(1+\frac{vw}{c^2}\right)^2-\left(\frac{v}{c}+\frac{w}{c}\right)^2}} = \frac{1}{\sqrt{\left(1+\frac{vw}{c^2}\right)^2-\frac{1}{c^2}(v+w)^2}} = \frac{1}{\left(1+\frac{vw}{c^2}\right)\sqrt{1-\frac{1}{c^2}\frac{(v+w)^2}{\left(1+\frac{vw}{c^2}\right)^2}}}$$

$$\gamma_1\gamma_2 = \frac{1}{\left(1+\frac{vw}{c^2}\right)\sqrt{1-\frac{1}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)^2}}$$ ………………(4)

$$(3) \Rightarrow x'' = \frac{1}{\left(1+\frac{vw}{c^2}\right)\sqrt{1-\frac{1}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)^2}}\left(1+\frac{wv}{c^2}\right)\left[x-\left(\frac{v+w}{1+\frac{wv}{c^2}}\right)t\right]$$

$$\Rightarrow x'' = \frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)^2}}\left[x-\left(\frac{v+w}{1+\frac{wv}{c^2}}\right)t\right]$$

$$\Rightarrow x'' = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}(x-ut) \qquad \text{using } u = \frac{v+w}{1+\frac{wv}{c^2}}$$

$$\Rightarrow x'' = \gamma(x-ut) \qquad \text{with } \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

Also $y'' = y' = y, \quad z'' = z' = z$

Finally

$$\Rightarrow t'' = \gamma_2\left(t' - \frac{x'w}{c^2}\right)$$

$$\Rightarrow t'' = \gamma_2\left[\gamma_1\left(t-\frac{xv}{c^2}\right) - \gamma_1(x-vt)\frac{w}{c^2}\right]$$

$$\Rightarrow t'' = \gamma_1\gamma_2\left[t-\frac{xv}{c^2}-\frac{xw}{c^2}+\frac{vw}{c^2}t\right]$$

$$\Rightarrow t'' = \gamma_1\gamma_2\left[\left(1+\frac{vw}{c^2}\right)t-\left(\frac{v}{c^2}+\frac{w}{c^2}\right)x\right]$$

$$\Rightarrow t'' = \gamma_1\gamma_2\left[\left(1+\frac{vw}{c^2}\right)t-\frac{x}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Using (4) in (5)

$$\Rightarrow t'' = \frac{1}{\left(1+\frac{vw}{c^2}\right)\sqrt{1-\frac{1}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)^2}}\left[\left(1+\frac{vw}{c^2}\right)t-\frac{x}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)\right]$$

$$\Rightarrow t'' = \frac{\left(1+\frac{vw}{c^2}\right)}{\left(1+\frac{vw}{c^2}\right)\sqrt{1-\frac{1}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)^2}}\left[t - \frac{x}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)\right]$$

$$\Rightarrow t'' = \frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)^2}}\left[t - \frac{x}{c^2}\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)\right]$$

$$\Rightarrow t'' = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(t - \frac{xu}{c^2}\right) \qquad \text{using } u = \frac{v+w}{1+\frac{wv}{c^2}}$$

$$\Rightarrow t'' = \gamma\left(t - \frac{xu}{c^2}\right) \qquad \text{with } \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

Hence

$$x'' = \gamma(x - ut)$$

$$y'' = y' = y, \quad z'' = z' = z$$

$$t'' = \gamma\left(t - \frac{xu}{c^2}\right)$$

with $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ and $u = \frac{v+w}{1+\frac{wv}{c^2}}$

this implies that Composition of two Lorentz Transformation is again Lorentz Transformation and the frame of reference S'' moves with the uniform velocity $u = \frac{v+w}{1+\frac{wv}{c^2}}$ relative to S. It is the reason that the law is called relativistic law of velocities.

Note: u is not an algebraic sum of velocities due to relative motion.

World Line

The world line of an object is the path that an object traces in 4 – dimensional spacetime. It is distinguished from the concepts such as an “orbit” or a trajectory by the time dimension.

EVENT

An event is defined as an occurrence which takes place at some point (x, y, z) at some instant of time t.

Examples: Arrival of a particle at a point (x, y, z) at time t, a bulb located at (x, y, z) flashing at time t, a gun at (x, y, z) firing at time t are examples of events. An event is thus described by the set of four numbers (x, y, z, t).

Remark

- An event has a meaning in an inertial frame but the numbers describing its position and time (x, y, z, t) are different in different inertial frames of reference. Their transformations are given by L.T.
- If two events happen at the same place and at the same time, they are called **coincident events**.
- If two events happen at the same place but not necessarily at the same time, they are called **colocal events**.
- If two events happen at the same time but not necessarily at the same place, they are called **simultaneous events**.

Question (coincident events are not relative)

Show that if two incidents are coincident in one inertial frame, they are coincident in every inertial frame.

Solution

Consider two events (x_1', y_1', z_1', t_1') and (x_2', y_2', z_2', t_2') are coincident in frame S'.
Then $\Delta x' = (x_2' - x_1') = 0$, similarly $\Delta y' = \Delta z' = 0$ (same place).
Also $\Delta t' = (t_2' - t_1') = 0$ (same time).

In frame S, the above two events are described by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2)
From L.T. we get $\Delta x = (x_2 - x_1) = \gamma(\Delta x' + v\Delta t') = 0$. Also $\Delta y = \Delta z = 0$.
Similarly $\Delta t = (t_2 - t_1) = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) = 0$. Thus the events are coincident in frame S also. In other words if two incidents are coincident in one inertial frame, they are coincident in every inertial frame. The statement that two events are coincident is thus true in all inertial frames, that is, it is universally true.

Question (colocality is relative)

Show that if two incidents are colocal in one inertial frame, they are not colocal in every inertial frame.

Solution

Consider two events (x_1', y_1', z_1', t_1') and (x_2', y_2', z_2', t_2') are colocal in frame S'.
Then $\Delta x' = (x_2' - x_1') = 0$, similarly $\Delta y' = \Delta z' = 0$.
But $\Delta t' = (t_2' - t_1') \neq 0$.

In frame S, the above two events are described by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2)
From L.T. we get $\Delta x = (x_2 - x_1) = \gamma(\Delta x' + v\Delta t') = \gamma v \Delta t'$. Also $\Delta y = \Delta z = 0$.

And $\Delta t = (t_2 - t_1) = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) = \gamma \Delta t'$.

We see that the two events which happened at the same point in S', do not happen at the same point in frame S .They happen a distance $\gamma v \Delta t'$ apart.

Question (simultaneity is relative)

Show that the two events are not simultaneous. Or show that two events which are simultaneous to an observer may not be simultaneous to another observer.

Solution

Consider two events (x_1', y_1', z_1', t_1') and (x_2', y_2', z_2', t_2') are simultaneous in frame S'. Then $\Delta x' = (x_2' - x_1') \neq 0$, similarly $\Delta y' = \Delta z' = 0$.
and $\Delta t' = (t_2' - t_1') = 0$.

In frame S, the above two events are described by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2)
From L.T. we get $\Delta x = (x_2 - x_1) = \gamma(\Delta x' + v\Delta t') = \gamma \Delta x'$. Also $\Delta y = \Delta z = 0$.

And $\Delta t = (t_2 - t_1) = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) = \gamma \frac{v}{c^2}\Delta x'$.

Thus simultaneity is not an absolute property of a pair of events. It depends upon the state of motion of the observer.

Example 1 A frame S' is moving uniformly relative to an inertial frame S along their common X-direction with a speed of $v = 0.5$ c. Identical clocks at both origins are set to zero when the origins coincide. Two simultaneous light flashes are observed in S at ($x_1 = 100$ m, $y_1 = 20$ m, $z_1 = 20$ m, $t_1 = 10^{-6}$ s) and ($x_2 = 200$ m, $y_2 = 30$ m, $z_2 = 30$ m, $t_2 = 10^{-6}$ s). At what space time coordinates are these observed in S?

Solution Since $v = 0.5$ c, $v/c = 0.5$ and $v = 0.5 \times 3 \times 10^8 = 1.5 \times 10^8$ m/s.

Also
$$\Gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{[1-(0.5)^2]^{1/2}} = \frac{1}{(0.75)^{1/2}} = 1.155$$

By L.T.,
$$x_1' = \Gamma(x_1 - vt_1) = 1.155\,[100 - (1.5 \times 10^8)10^{-6}]$$
$$= 1.155\,(-50) = -57.75 \text{ m.}$$
$$y_1' = y_1 = 20 \text{ m}; \quad z_1' = z_1 = 20 \text{ m};$$
$$t_1' = \Gamma(t_1 - vx_1/c^2) = 1.155\left[10^{-6} - \frac{1.5\times10^8\times100}{9\times10^{16}}\right]$$
$$= 1.155\left(1 - \frac{1.5}{9}\right)10^{-6} = 1.155\,(1 - 0.166)10^{-6} = 0.962 \times 10^{-6}\text{s.}$$

Similarly,
$$x_2' = 57.75 \text{ m}, \quad y_2' = 30 \text{ m}, \quad z_2' = 30 \text{ m}, \quad t_2' = 0.770 \times 10^{-6} \text{ s}$$

Therefore in S' the two flashes are observed at (–57.75 m, 20 m, 20 m, 0.962×10^{-6} s) and (57.75 m, 30 m, 30 m, 0.770×10^{-6} s). The flashes are simultaneous in S but NOT simultaneous in S'.

Example 2 In an inertial frame one event occurs at $x_1 = 0$ at $t_1 = 0$ and another event occurs at $x_2 = 12$ (3×10^8) m at $t_2 = 20$ s. Show that it is possible to find an inertial frame S' in which the above two events are observed at the same place. What is the time interval between the two events in S'?

Solution In frame S, $ds^2 = (x_2 - x_1)^2 - c^2dt^2 = [(12)^2 - (20)^2]c^2 = -256c^2$. The interval between the two events is timelike. It is therefore possible to find a frame S' in which the two events occur at the same place.

Let S' move with speed v along the X-axis relative to frame S. Then

$$x_1' = \Gamma(x_1 - vt_1); \quad x_2' = \Gamma(x_2 - vt_2)$$

$$\therefore \quad x_2' - x_1' = \Gamma[(x_2 - x_1) - v(t_2 - t_1)]$$

For events to occur at the same place in S', we must have $x_1' = x_2'$ or $\left(x_1' - x_2'\right) = 0$. This will happen when

$$v = \frac{x_2 - x_1}{t_2 - t_1} = \frac{12\,\text{c}}{20} = 0.6 \text{ c}$$

Thus in frame S' moving with speed $v = 0.6$ c along the X-axis relative to frame S, the two events occur at the same place.

The factor
$$\Gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{[1-(0.6)^2]^{1/2}} = 1.25$$

Then $$x_1' = \Gamma(x_1 - vt_1) = 0, \quad x_2' = x_1' = 0$$

$$t_1' = \Gamma(t_1 - vx_1/c^2) = 0$$

$$t_2' = \Gamma(t_2 - vx_2/c^2) = 1.25\left(20 - \frac{0.6\,c \times 12\,c}{c^2}\right) = 1.25\ (20 - 7.2) = 16\ \text{s}$$

In S', the events are (0, 0) and (0, 16 s). In S', the time interval between the two events is 16s. This result could have been derived from time dilation or from invariance of ds^2.

Example 3 In an inertial frame S event A occurs at $x = 0$ at $t = 0$ and event B occurs at $x = 20$ c and $t = 12$ s. Find a frame S' in which the two events are simultaneous.

Solution In frame S, $ds^2 = (x_2 - x_1)^2 - c^2(t_2 - t_1)^2 = (20\ c)^2 = c^2(12)^2 = 256\ c^2$. The interval $ds = 16$ c is spacelike. Let v denote the velocity of S' relative to S. In S'

$(t_2' - t_1') = \Gamma\lfloor(t_2 - t_1) - v(x_2 - x_1)/c^2\rfloor$ and the two events will be simultaneous if $(t_2' - t_1') = 0$

That is if $$v = \frac{c^2\,(t_2 - t_1)}{x_2 - x_1} = \frac{c^2\,(12)}{20\,c} = 0.6\ c$$

Thus in frame S' moving with velocity $v = 0.6$ c relative to frame S, the two events A and B are simultaneous.

Example 4 In an inertial frame S event A occurs at $x = 0$ at $t = 0$ and event B occurs at $x = 20$ c at $t = 12$ s. In a frame S' moving with uniform velocity $v = 4/5$ c relative to frame S, show that event B occurs earlier to event A.

Solution The factor $$\Gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{[1-(4/5)^2]^{1/2}} = \frac{5}{3}.$$

In S', the events A and B occur respectively at $t_1' = \Gamma\,(t_1 - vx_1/c^2) = 0$ and $t_2' = \Gamma\,(t_2 - vx_2/c^2)$

$$= \frac{5}{3}\left(12 - \frac{4c}{5}\cdot\frac{20c}{c^2}\right) = \frac{5}{3}(-4) = -\frac{20}{3}\ \text{s}.$$

Thus $t_2' < t_1'$. Hence event B occurs earlier to event A in frame S'.

Example 5 An observer A is situated on the X-axis of frame S at $x = a$ and an observer B is situated on the X'-axis of S' at $x' = a$. Show that in both frames, the events (i) O passes O' and (ii) A passes B are separated by a time $\frac{a}{v}\left[1 - \left(1 - v^2/c^2\right)^{1/2}\right]$ but that the occurrence of the two events is different.

Solution Since the clocks are set (as usual in this book) so that $t = t' = 0$ when O passes O', the events are described as

(O, O) and $(x = a, t)$ in frame S

and (O, O) and $(x' = a, t')$ in frame S'.

Here t is the time noted by observer A when the observer B passes him and t' is the time noted by observer B when observer A passes him.

From L.T., $x' = a = \Gamma(x - vt) = \Gamma(a - vt)$

Or $$a = \Gamma(a - vt) \qquad \text{... (1)}$$

Similarly $$t' = \left(t - vx/c^2\right)\Gamma = \Gamma\left(t - va/c^2\right) \qquad \text{... (2)}$$

From eqn. (1), $\frac{a}{\Gamma} = a - vt$

$$\therefore \quad t = \frac{a}{v}\left(1 - \frac{1}{\Gamma}\right) = \frac{a}{v}\left[1 - \left(1 - v^2/c^2\right)^{1/2}\right] \quad \text{... (3)}$$

Substituting this value of t in eqn. (2) and simplifying we get

$$t' = -\frac{a}{v}\left[1 - \left(1 - v^2/c^2\right)^{1/2}\right]$$

Thus though the time intervals are equal, the orders of occurrence of the two events are different. That is in frame S, the event O passes O' occurs earlier than the event A passes B because t is positive, whereas in frame S', the event A passes B occurs earlier to the event O passes O'.

Example 6 An observer notes that two events are separated in space and time by 3.6×10^8 m and 2 s. What is the proper time interval between these events?

Solution Proper time interval between two events is the time interval measured by an observer in frame say S' for whom the two events are colocal.

By L.T., $\Delta x' = \Gamma(\Delta x - v\Delta t)$ where $\Delta x = 3.6 \times 10^8$ m and $\Delta t = 2$ s

Since $\Delta x' = 0$, $v = \frac{\Delta x}{\Delta t} = \frac{3.6 \times 10^8}{2} = 1.8 \times 10^8$ m/s.

$$\therefore \quad \Gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{[1 - (.6)^2]^{1/2}} = 5/4$$

Proper time $\Delta t' = \Gamma\left(\Delta t - vx/c^2\right)$

$$= \frac{5}{4}\left[2 - \frac{1.8 \times 10^8 \times 3.6 \times 10^8}{9 \times 10^{16}}\right] = \frac{5}{4}[2 - 0.72] = 1.6 \text{ s}$$

Question

Observer A notes that two simultaneous events occur 40 m apart. What is the time separation of this pair of events as observed by B who finds that the events occurred 50 m apart? Find his velocity relative to the observer A.

Solution

$\Delta x = 40m\,,\;\; \Delta x' = 50m,\;\; \Delta t = 0\,,\;\; v = ?$

From L.T. we get $\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma\Delta x$

$$\gamma = \frac{\Delta x'}{\Delta x} \Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{50}{40} \Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{5}{4} \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{4}{5} \Rightarrow 1-\frac{v^2}{c^2} = \frac{16}{25}$$

$$\Rightarrow \frac{v^2}{c^2} = \frac{9}{25} \Rightarrow v^2 = \frac{9}{25}c^2 \Rightarrow v = \frac{3}{5}c \Rightarrow v = 0.6c$$

And $\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = -\gamma\frac{v}{c^2}\Delta x = -\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{v}{c^2}\Delta x\right)$

$$\Delta t' = -\frac{1}{\sqrt{1-\frac{(0.6c)^2}{c^2}}}\left(\frac{0.6c}{c^2}\times 40\right) = -10^{-7}s$$

$\Delta t' = 10^{-7}s$ Since time is non – negative scalar quantity.

Question

Frame S' travels along the common X-X'- axis with speed v = 0.8 c relative to the frame S. Clocks are so set that $t = t' = 0$ when $x = x' = 0$. Two events A and B as described in frame S are: A(60 m,10^{-7} s) and B(10 m,10^{-7}s). What is the time interval between the events in frame S'?

Solution

$\Delta x = (x_2 - x_1) = 60 - 10 = 50m$. Also $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}} = \frac{1}{0.6}$

And $\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \frac{1}{0.6}\left(10^{-7} - \frac{0.8c}{c^2}\times 40\right) = \frac{35}{9}\times 10^{-7}s$

Question

Two events are separated in space and time by 600 m and $8 \times 10^{-7} s$ as observed by an observer in frame S. Find the velocity of the frame S' relative to the frame S if the two events are simultaneous in S'.

Solution

$\Delta x = 600m\,,\; \Delta t' = 0,\; \Delta t = 8 \times 10^{-7} s\,,\; v = ?$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

$$0 = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \Rightarrow \Delta t = \frac{v}{c^2}\Delta x \Rightarrow v = \frac{\Delta t c^2}{\Delta x} \Rightarrow v = 0.4c$$

Question

Two colocal events in frame S are separated by a time interval of 4 seconds. What is the spatial separation between these events in frame S'in which the two events are separated by a time interval of 6 seconds?

Solution

$\Delta x = 0m\,,\; \Delta t' = 6s,\; \Delta t = 4s\,,\; \Delta x' = ?$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \gamma \Delta t \Rightarrow 6 = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 6 = \frac{4}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{4}{6} \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{6}{4} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{16}{36} \Rightarrow \frac{v^2}{c^2} = \frac{20}{36} \Rightarrow v^2 = \frac{20}{36}c^2 \Rightarrow v = \frac{\sqrt{5}}{3}c$$

$$\Delta x' = \gamma(\Delta x - v\Delta t) = -\gamma v \Delta t$$

$$\Delta x' = -\frac{v\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} = 1.3416 \times 10^9 m$$

Question

Two simultaneous events in frame S are separated by a distance of 1 km along the X-axis. What is the time interval between these two events as measured in frame S' in which the spatial separation between the two events is observed to be 2 km?

Solution

$\Delta x = 1000m = 1km\ ,\ \ \Delta t' = ?,\ \ \Delta t = 0s\ ,\ \ \Delta x' = 2km = 2000m$

$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma\Delta x \Rightarrow \Delta x' = \gamma\Delta x \Rightarrow \gamma = 2$

$\Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 2 \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{4} \Rightarrow \frac{v^2}{c^2} = \frac{3}{4} \Rightarrow v^2 = \frac{3}{4}c^2 \Rightarrow v = \frac{\sqrt{3}}{2}c$

$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = -\gamma\frac{v}{c^2}\Delta x \Rightarrow \Delta t' = 5.77 \times 10^{-6}s$

Question

Two light bulbs in frame S situated at $x_1 = 0$ and $x_2 = 10$ km. flash simultaneously at t = 0. An observer in S' travelling with speed 0.6 c relative to frame S in the positive X-direction also observes the flashes. What is the time interval between the flashes according to the observer in S'? Which bulb flashes first according to him?

Solution

$x_1 = 0$ and $x_2 = 10$ km = 10^4m

$\Delta x = 10^4m,\ \ \Delta t' = ?,\ \ \Delta t = 0s\ ,\ \ v = 0.6c$

$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = -\gamma\frac{v}{c^2}\Delta x \Rightarrow \Delta t' = 2.5 \times 10^{-8}s$

Where the bulb $x_2 = 10$ km flashes first according to him because it is near.

PROPER TIME INTERVAL / PROPER TIME / WORLD TIME

The time shown by a clock which is moving with body is called proper time. It is denoted by $\boldsymbol{\tau}$.

For example a clock is fixed in a train and moving with the train, then time measured with this clock is called proper time.

Explanation

Consider a pair of events such as the arrival of a particle in motion at two neighbouring points P and P'. These two points consider as two event of emission of light in space time.

Let the space – time coordinates of the two events be (x, y, z, ict) and $(x + dx, y + dy, z + dz, ic(t + dt))$ in an inertial frame S. Let the space – time coordinates of the pair be $(x', y', z', ic\ t')$ and $(x' + dx', y' + dy', z' + dz', ic(t' + dt'))$ in frame S'. Consider interval of two events (ds) which is the distance between two points is given as follows;

$$(ds)^2 = dx^2 + dy^2 + dz^2 + (icdt)^2$$

$$(ds)^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \qquad S - \tau - \text{frame}$$

$$(ds')^2 = dx'^2 + dy'^2 + dz'^2 - c^2d't^2 \qquad S' - \tau - \text{frame}$$

As $x^2 + y^2 + z^2 - c^2t^2$ is invariant then

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$$

$$\Rightarrow dx'^2 + dy'^2 + dz'^2 - c^2d't^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

$$\Rightarrow (ds')^2 = (ds)^2 \qquad \text{Length is invariant.}$$

Suppose $-k^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$

$$\Rightarrow \frac{k^2}{c^2} = -\frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2} + dt^2$$

$$\Rightarrow \frac{k^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$$

$$\Rightarrow \frac{k^2}{c^2} = dt^2\left[1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)\right]$$

$$\Rightarrow \frac{k^2}{c^2} = dt^2\left[1 - \frac{1}{c^2}(u_x^2 + u_y^2 + u_z^2)\right]$$

$$\Rightarrow \frac{k^2}{c^2} = dt^2\left[1 - \frac{1}{c^2}(u^2)\right]$$

$$\Rightarrow \boldsymbol{\frac{k^2}{c^2} = dt^2\left(1 - \frac{u^2}{c^2}\right)} \qquad \boldsymbol{S - \tau - \text{frame}}$$

$$\Rightarrow \boldsymbol{\frac{k^2}{c^2} = dt'^2\left(1 - \frac{u'^2}{c^2}\right)} \qquad \boldsymbol{S' - \tau - \text{frame}}$$

$$\Rightarrow dt'^2\left(1 - \frac{u'^2}{c^2}\right) = dt^2\left(1 - \frac{u^2}{c^2}\right) \qquad \text{equating both}$$

$$\Rightarrow d\tau^2 = dt^2\left(1 - \frac{u^2}{c^2}\right) \qquad \text{for proper time } u' = 0, t' = \tau$$

$$\Rightarrow d\tau = dt\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}} \quad \text{or} \quad \Rightarrow d\tau = dt'\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}} \text{ (May use)}$$

$$\Rightarrow \boldsymbol{\tau = \int\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}} dt} \qquad \text{required formula for proper time.}$$

From Equation $d\tau = dt\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}} = dt'\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}}$ we see that the quantity $d\tau$ has the same value in all inertial frames. It is therefore an invariant for all inertial observers. This invariant is called proper time interval.

It is seen that $d\tau = dT_0 =$ time interval measured in the frame in which the particle is at rest, that is $u = 0$. Therefore proper time interval can be defined to be the time shown by a clock which is moving with the body.

Remember

- From formula $d\tau = dt\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}}$ we may use later $\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(u)$
- When $u << c$ then $\frac{u}{c} << 1$ implies $\left(\frac{u}{c}\right)^2 \to 0$ using $\frac{u^2}{c^2} = 0$ then $\tau = t$, under this condition proper time become ordinary.

Question

A particle travels at speed v for a time interval Δt as observed by an observer in the inertial frame S. Show that the proper time elapsed in the particle frame is $\tau = \frac{\Delta t}{\gamma}$. Hence find the proper time elapsed for a particle travelling at 0.99 c for a time of $\sqrt{2} \times 10^{-6} s$.

Solution

Since $d\tau = dt\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \Rightarrow d\tau = \Delta t\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$

$\Rightarrow d\tau = \sqrt{2} \times 10^{-6}\left(1 - \frac{(0.99c)^2}{c^2}\right)^{\frac{1}{2}} = 2 \times 10^{-7} s$

Question

Space-time coordinates of a pair of events in frame S are: event $A(a, o, o, a/c)$, event $B(2a, o, o, a/2c)$. Find the speed of frame S' in which the two events are observed to be simultaneous. When do these events occur according to the observer in S'?

Solution

$\Delta x = a,\ \ \Delta t' = 0,\ \ \Delta t = -\frac{a}{2c}\ ,\ \ v = ?\ , t_1' = ?\ , t_2' = ?$

$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \Rightarrow 0 = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \Rightarrow \Delta t = \frac{v}{c^2}\Delta x$

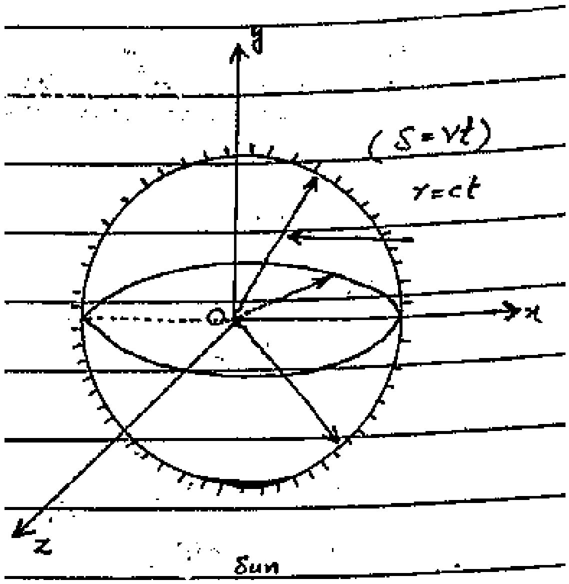
$\Rightarrow v = \frac{-\frac{a}{2c}c^2}{a} \Rightarrow v = -\frac{c}{2}$

Also $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2}{\sqrt{3}}$

$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) \Rightarrow t_1' = \frac{2}{\sqrt{3}}\left(\frac{a}{c} - \frac{ac}{2c^2}\right) \Rightarrow t_1' = \frac{2}{\sqrt{3}}\left(\frac{a}{c} - \frac{a}{2c}\right) \Rightarrow t_1' = \sqrt{3}\left(\frac{a}{c}\right)$

$t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) \Rightarrow t_2' = \frac{2}{\sqrt{3}}\left(\frac{a}{2c} - \frac{2ac}{2c^2}\right) \Rightarrow t_2' = \frac{2}{\sqrt{3}}\left(\frac{a}{2c} - \frac{a}{c}\right) \Rightarrow t_2' = \sqrt{3}\left(\frac{a}{c}\right)$

SPACE – TIME CONTINUUM (MINKOWSKI SPACE)



Minkowski Space concept was given by Minkowski in 1908.

Consider a sphere of radius r and origin O, then the equation of sphere is

$x^2 + y^2 + z^2 = r^2$ ………..(1)

If this sphere is a sun then the rays emitted from origin O with the speed of light c. therefore we take $r = ct$ using $s = vt$, then(1) becomes

$x^2 + y^2 + z^2 = c^2t^2$

$x^2 + y^2 + z^2 - c^2t^2 = 0 \Rightarrow x^2 + y^2 + z^2 + i^2c^2t^2 = 0$

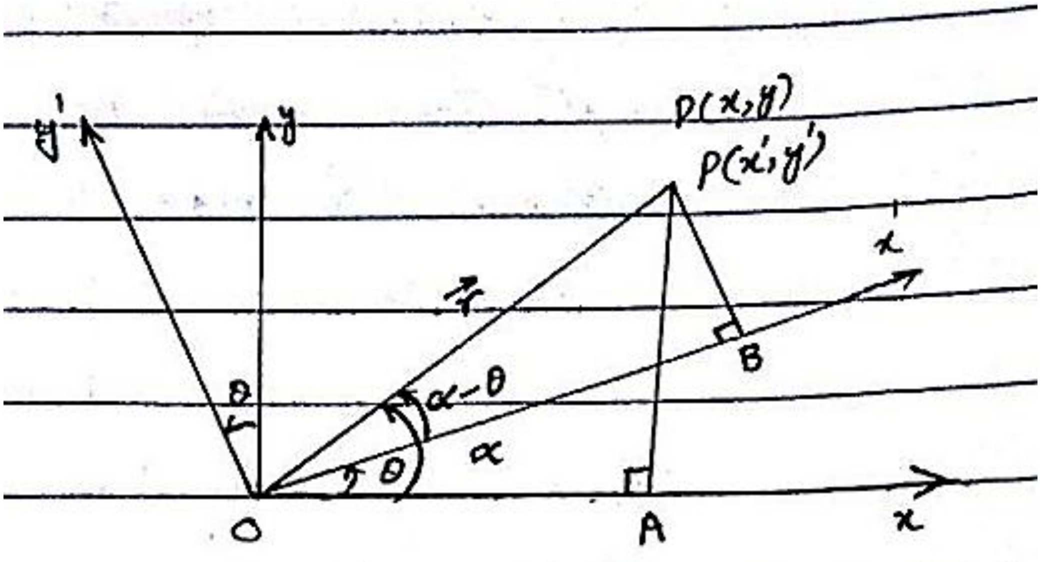
$x^2 + y^2 + z^2 + (ict)^2 = 0$ ………..(2)

Now naming these coordinates we get $x_1 = x\,, x_2 = y\,, x_3 = z\,, x_4 = ict$

$(2) \Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$

Then $x_\mu = (x_1\,, x_2\,, x_3\,, x_4)$ is called Minkowski Space.

ROTATION OF AXES



In xy – plane ΔOAP is a right angle triangle at A. So

$x = rCos \propto$; $y = rSin \propto$ ……………….(1)

After rotating xy – plane we get new plane, $x'y'$ – plane which makes an angle θ with xy – plane.

In $x'y'$ – plane ΔOBP is a right angle triangle at B. So

$x' = rCos(\propto - \theta)$; $y' = rSin(\propto - \theta)$ ……………….(2)

$(2) \Rightarrow x' = rCos \propto Cos\theta + rSin \propto Sin\theta$

$\Rightarrow x' = xCos\theta + ySin\theta$

$(2) \Rightarrow y' = rSin \propto Cos\theta - rCos \propto Sin\theta$

$\Rightarrow y' = yCos\theta - xSin\theta$

Hence following equations are called the rotation of the axe;

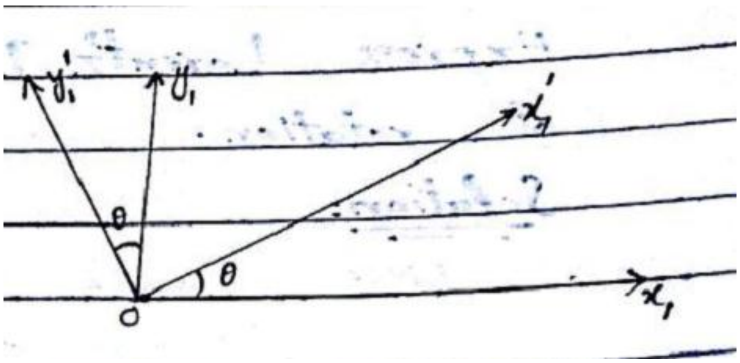
$\boldsymbol{x' = xCos\theta + ySin\theta}$

$\boldsymbol{y' = yCos\theta - xSin\theta}$

Question

In $x_1 x_4 -$ plane show that by using Lorentz transformations $\boldsymbol{Tan\theta = \frac{iv}{c}}$

Solution



$\boldsymbol{Tanh} \propto = \frac{v}{c}$

Since $Tan\theta = \frac{iv}{c}$

$\Rightarrow Tan(i \propto) = \frac{iv}{c}$

$\Rightarrow iTanh \propto = \frac{iv}{c}$

$\Rightarrow Tanh \propto = \frac{v}{c}$

In Lorentz transformations x_2, x_3 remains same and just x_1, x_4 changed w.r.to angle θ, so $x_1 x_4 -$ plane used as a rotating axes plane.

By using rotating axes

$x' = xCos\theta + ySin\theta$

$y' = yCos\theta - xSin\theta$

Changing first of above equation we have $\quad x_1' = x_1 Cos\theta + x_4 Sin\theta$

Using Minkowski space coordinates $x_1 = x_1$ and $x_4 = ict$

$x_1' = xCos\theta + ictSin\theta \quad$ …………(1)

By using Lorentz transformations

$x_1' = \gamma(x - vt) = \gamma x - \gamma vt \quad$ …………(2)

Comparing (1) and (2) we gave

$Cos\theta = \gamma \quad$ and $\quad icSin\theta = -\gamma v \Rightarrow Sin\theta = \frac{-\gamma v}{ic} = \frac{i^2 \gamma v}{ic} \Rightarrow Sin\theta = \frac{i\gamma v}{c}$

$\Rightarrow \frac{Sin\theta}{Cos\theta} = \frac{i\gamma v}{c\gamma}$

$\Rightarrow \boldsymbol{Tan\theta = \frac{iv}{c}}$

Question

Express Lorentz Transformations in terms of Minkowski space coordinates.

Or Express Lorentz Transformations in terms of rotation.

Solution

Since from Lorentz Transformations

$x' = \gamma(x - vt)$

$y' = y, \quad z' = z$ ………………(1)

$t' = \gamma\left(t - \frac{xv}{c^2}\right)$

By using Minkowski Space coordinates

$x_1 = x \, , x_2 = y \, , x_3 = z$

$x_4 = ict \Rightarrow t = \frac{x_4}{ic}$ and $t' = \frac{x_4'}{ic'} = \frac{x_4'}{ic}$ in L.T. we have $c' = c$

Then we get

$x_1' = \gamma(x - vt) = \gamma\left(x_1 - v\frac{x_4}{ic}\right) = \gamma x_1 + \gamma v\frac{i^2}{ic}x_4 = \gamma x_1 + \gamma\frac{iv}{c}x_4$

$\boldsymbol{x_1' = Cos\theta x_1 + Sin\theta x_4}$ also $\boldsymbol{x_2 = x_2'; \ x_3 = x_3'}$

As $t' = \gamma\left(t - \frac{xv}{c^2}\right)$

$\Rightarrow \frac{x_4'}{ic} = \gamma\left(\frac{x_4}{ic} - \frac{v}{c^2}x_1\right) \Rightarrow x_4' = \gamma x_4 - \frac{i\gamma v}{c}x_1 \Rightarrow \boldsymbol{x_4' = Cos\theta x_4 - Sin\theta x_1}$

Hence

$\boldsymbol{x_1' = Cos\theta x_1 + Sin\theta x_4}$ also $\boldsymbol{x_2 = x_2'; \ x_3 = x_3'}$

$\boldsymbol{x_4' = Cos\theta x_4 - Sin\theta x_1}$

Are called Lorentz Transformations in terms of Minkowski space coordinates or in terms of rotation.

In matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} Cos\theta & 0 & 0 & Sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -Sin\theta & 0 & 0 & Cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Question

Show that $AA^t = I$ where $A = \begin{bmatrix} Cos\theta & 0 & 0 & Sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -Sin\theta & 0 & 0 & Cos\theta \end{bmatrix}$

Or Show that set of Lorentz transformation equations in terms of Minkowski space system is orthogonal.

Solution

Given that $A = \begin{bmatrix} Cos\theta & 0 & 0 & Sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -Sin\theta & 0 & 0 & Cos\theta \end{bmatrix}$ then

$$A^t = \begin{bmatrix} Cos\theta & 0 & 0 & Sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -Sin\theta & 0 & 0 & Cos\theta \end{bmatrix}^t = \begin{bmatrix} Cos\theta & 0 & 0 & -Sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ Sin\theta & 0 & 0 & Cos\theta \end{bmatrix}$$

$$AA^t = \begin{bmatrix} Cos\theta & 0 & 0 & Sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -Sin\theta & 0 & 0 & Cos\theta \end{bmatrix} \begin{bmatrix} Cos\theta & 0 & 0 & -Sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ Sin\theta & 0 & 0 & Cos\theta \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $AA^t = I$

Question

Prove that Lorentz transformation in terms of rotation leaves the expression $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is invariant.

Solution

For this we have to prove

$x'^2_1 + x'^2_2 + x'^2_3 + x'^2_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2$

$L.H.S = x'^2_1 + x'^2_2 + x'^2_3 + x'^2_4$

$= (Cos\theta x_1 + Sin\theta x_4)^2 + x_2^2 + x_3^2 + (Cos\theta x_4 - Sin\theta x_1)^2$

$= Cos^2\theta x_1^2 + Sin^2\theta x_4^2 + 2x_1x_4 Sin\theta Cos\theta + x_2^2 + x_3^2 + Cos^2\theta x_4^2 + Sin^2\theta x_1^2 - 2x_1x_4 Sin\theta Cos\theta$

$= (Cos^2\theta + Sin^2\theta)x_1^2 + x_2^2 + x_3^2 + (Cos^2\theta + Sin^2\theta)x_4^2$

$= x_1^2 + x_2^2 + x_3^2 + x_4^2 = R.H.S$

$x'^2_1 + x'^2_2 + x'^2_3 + x'^2_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ proved.

Question

Define a parameter $\propto$ such that $Tanh \propto = \frac{v}{c}$ and find Lorentz and inverse Lorentz transformations in terms of hyperbolic funcitons.

Solution

Since $Tan\theta = \frac{iv}{c} \Rightarrow Tan(i\propto) = \frac{iv}{c} \Rightarrow iTanh \propto = \frac{iv}{c} \Rightarrow Tanh \propto = \frac{v}{c}$

Since $Sech \propto = \sqrt{1 - Tanh^2 \propto} = \sqrt{1 - \frac{v^2}{c^2}} \Rightarrow Cosh \propto = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \boldsymbol{Cosh \propto = \gamma}$

Also $Cosh^2 \propto - Sinh^2 \propto = 1 \Rightarrow Sinh^2 \propto = Cosh^2 \propto - 1 \Rightarrow Sinh \propto = \sqrt{Cosh^2 \propto - 1}$

$\Rightarrow Sinh \propto = \sqrt{\frac{1}{1-\frac{v^2}{c^2}} - 1} \Rightarrow Sinh \propto = \frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \boldsymbol{Sinh \propto = \gamma \frac{v}{c}}$

Since from Lorentz Transformations

$x' = \gamma(x - vt)$

$y' = y, \quad z' = z$

$t' = \gamma\left(t - \frac{xv}{c^2}\right)$

Let $\quad x' = \gamma(x - vt) = \gamma x - \gamma vt = \gamma x - \gamma\frac{v}{c}ct$

$x' = xCosh \propto -ctSinh \propto$ also $y' = y, \quad z' = z$

Now $t' = \gamma\left(t - \frac{xv}{c^2}\right) = \gamma t - \gamma\frac{v}{c}\frac{x}{c}$

$t' = tCosh \propto -\frac{x}{c}Sinh \propto$

By using inverse Lorentz Transformation

$x = \gamma(x' + vt')$

$y = y', \ z = z'$

$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$

Now $x = \gamma(x' + vt') = \gamma x' + \gamma vt' = \gamma x' + \gamma\frac{v}{c}ct'$

$x = x'Cosh \propto +ct'Sinh \propto$ also $y' = y, \quad z' = z$

Now $t = \gamma\left(t' + \frac{vx'}{c^2}\right) = \gamma t' + \gamma\frac{vx'}{c^2} = \gamma t' + \gamma\frac{v}{c}\frac{x'}{c}$

$t = t'Cosh \propto +\frac{x'}{c}Sinh \propto$

Hence required transformations in hyperbolic function are

$x' = xCosh \propto -ctSinh \propto$ also $y' = y, \quad z' = z \ , t' = tCosh \propto -\frac{x}{c}Sinh \propto$

$x = x'Cosh \propto +ct'Sinh \propto$ also $y' = y, \quad z' = z \ , t = t'Cosh \propto +\frac{x'}{c}Sinh \propto$

Question

Prove that resultant of two velocities each of which is less than c is itself less than c. **Or** Is it possible to obtain velocity of light by adding velocities which are less than c?

Solution

Consider S, S' and S'' are three inertial frames. S' frame is moving with velocity 'v' and with respect to S frame and S'' frame is moving with velocity 'w' and with respect to S' frame.

Given that $v < c$ and $w < c$

In STR the resultant of velocities is defined as $u = \frac{v+w}{1+\frac{vw}{c^2}}$

We have to show $u < c$ or $c - u < 0$

Consider $c - u = c - \frac{v+w}{1+\frac{vw}{c^2}}$

$$c - u = \frac{c - \frac{vw}{c} - v - w}{1+\frac{vw}{c^2}} = \frac{c}{1+\frac{vw}{c^2}}\left[1 + \frac{vw}{c^2} - \frac{v}{c} - \frac{w}{c}\right] = \frac{c}{1+\frac{vw}{c^2}}\left[\left(1 - \frac{v}{c}\right) - \frac{w}{c}\left(1 - \frac{v}{c}\right)\right]$$

$$c - u = \frac{c}{1+\frac{vw}{c^2}}\left[\left(1 - \frac{v}{c}\right)\left(1 - \frac{w}{c}\right)\right] \qquad \text{................(1)}$$

Since $v < c \Rightarrow \frac{v}{c} < 1 \Rightarrow 1 - \frac{v}{c} > 0$ also $w < c \Rightarrow \frac{w}{c} < 1 \Rightarrow 1 - \frac{w}{c} > 0$

$$\Rightarrow \left(1 - \frac{v}{c}\right)\left(1 - \frac{w}{c}\right) > 0$$

$$\Rightarrow c - u = \frac{c}{1+\frac{vw}{c^2}}\left[\left(1 - \frac{v}{c}\right)\left(1 - \frac{w}{c}\right)\right] > 0 \Rightarrow c - u > 0 \Rightarrow u < c$$

Hence

Resultant of two velocities each of which is less than c is itself less than c.

Or

It is possible to obtain velocity of light by adding velocities which are less than c.

RAPIDLY OR PSEUDO VELOCITY

According to STR the resultant of two velocities v and w cannot be added algebraically in relative motion. i.e. $u \neq v + w$. In STR the resultant of velocities can be expressed as $u = \frac{v+w}{1+\frac{vw}{c^2}}$. To overcome this deficiency , we define a term rapidly or pseudo velocity as

$$Tanh \propto = \frac{v}{c}$$

$$\propto = Tanh^{-1}\left(\frac{v}{c}\right)$$

$$y_v = cTanh^{-1}\left(\frac{v}{c}\right)$$

$$\frac{y_v}{c} = Tanh^{-1}\left(\frac{v}{c}\right)$$

$$\frac{v}{c} = Tanh\left(\frac{y_v}{c}\right)$$

Similarly

$$\frac{w}{c} = Tanh\left(\frac{y_w}{c}\right)$$

Now we have $\frac{u}{c} = \frac{\frac{v}{c}+\frac{w}{c}}{1+\frac{vw}{c\,c}}$

$$\Rightarrow Tanh\left(\frac{y_u}{c}\right) = \frac{Tanh\left(\frac{y_v}{c}\right)+Tanh\left(\frac{y_w}{c}\right)}{1+Tanh\left(\frac{y_v}{c}\right)Tanh\left(\frac{y_w}{c}\right)}$$

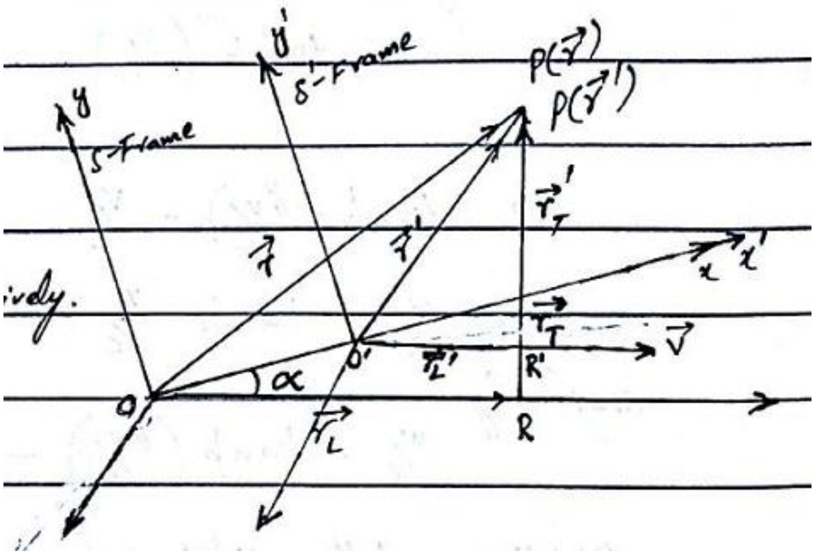
$$\Rightarrow Tanh\left(\frac{y_u}{c}\right) = Tanh\left(\frac{y_v}{c} + \frac{y_w}{c}\right)$$

$$\Rightarrow \frac{y_u}{c} = \frac{y_v}{c} + \frac{y_w}{c}$$

$$\Rightarrow y_u = y_v + y_w$$

This is the algebraic sum of velocities.

GENERAL LORENTZ TRANSFORMATIONS FIXED WITHOUT ROTATION / RESTRICTED LORENTZ TRANSFORMATIONS



Consider a system of two inertial frames S and S' coincident at origin O, which makes an angle $\propto$ with horizontal and moving with uniform velocity along a fixed horizontal direction.

Consider $\vec{r}$ and $\vec{r}'$ are two position vectors at same point P with respect to S and S' frames respectively.

Resolve $\vec{r}$ into components $\overrightarrow{r_T}$ (Transversal component) perpendicular to $\vec{v}$ and $\overrightarrow{r_L}$ (Horizontal component) parallel to $\vec{v}$ or in the direction of $\vec{v}$. Then by head to tail rule using for $\vec{r} = (x, y, z), \vec{r}' = (x', y', z')$

$\vec{r} = \overrightarrow{r_T} + \overrightarrow{r_L}$ ……………(1) and $\vec{r}' = \overrightarrow{r_T}' + \overrightarrow{r_L}'$ ……………(2)

By using Lorentz Transformations in Vector form

$\vec{r}' = \gamma(\vec{r} - \vec{v}t)$

$\Rightarrow \overrightarrow{r_L}' = \gamma(\overrightarrow{r_L} - \vec{v}t)$ ……………(3)

For $\overrightarrow{r_L}$ using ΔORP

$$Cos\theta = \frac{base}{hyp} = \frac{r_L}{r} \Rightarrow r_L = rCos\theta = \frac{vrCos\theta}{v} = \frac{\vec{v}.\vec{r}}{v} \Rightarrow r_L\hat{v} = \frac{\vec{v}.\vec{r}}{v}\hat{v} = \frac{\vec{v}.\vec{r}}{v}\frac{\vec{v}}{v}$$

$\Rightarrow \overrightarrow{r_L} = \frac{\vec{v}.\vec{r}}{v^2}\vec{v}$ ……………(4)

$(3) \Rightarrow \overrightarrow{r_L}' = \gamma(\overrightarrow{r_L} - \vec{v}t) = \gamma\left(\frac{\vec{v}.\vec{r}}{v^2}\vec{v} - \vec{v}t\right)$

$\Rightarrow \overrightarrow{r_L}' = \gamma\left(\frac{\vec{v}.\vec{r}}{v^2} - t\right)\vec{v}$ ……………(5)

Since from figure $\overrightarrow{r_T} = \overrightarrow{r_T}'$ (Transversal component is perpendicular to the line) where $\overrightarrow{r_T}, \overrightarrow{r_T}'$ are same transversal vectors to the horizontal line. Then

$(1) \Rightarrow \vec{r} = \overrightarrow{r_T} + \overrightarrow{r_L} \Rightarrow \overrightarrow{r_T} = \vec{r} - \overrightarrow{r_L}$

$\Rightarrow \overrightarrow{r_T}' = \vec{r} - \frac{\vec{v}.\vec{r}}{v^2}\vec{v}$ ……………(6) using (4) and $\overrightarrow{r_T} = \overrightarrow{r_T}'$

Using (5) and (6) in (2)

$\Rightarrow \vec{r}' = \overrightarrow{r_T}' + \overrightarrow{r_L} = \vec{r} - \frac{\vec{v}.\vec{r}}{v^2}\vec{v} + \gamma\left(\frac{\vec{v}.\vec{r}}{v^2} - t\right)\vec{v} = \vec{r} - \frac{\vec{v}.\vec{r}}{v^2}\vec{v} + \gamma\frac{\vec{v}.\vec{r}}{v^2}\vec{v} - \gamma t\vec{v}$

$\Rightarrow \vec{r}' = \vec{r} + (\gamma - 1)\frac{\vec{v}.\vec{r}}{v^2}\vec{v} - \gamma t\vec{v}$

$\Rightarrow \boldsymbol{\vec{r}' = \vec{r} + \left[(\gamma - 1)\frac{\vec{v}.\vec{r}}{v^2} - \gamma t\right]\vec{v}}$ ……………(A)

Now for t' in vector notation we have

$t' = \gamma\left(t - \frac{\vec{v}}{c^2}.x\right)$

$\Rightarrow t' = \gamma\left(t - \frac{\vec{v}}{c^2}.\overrightarrow{r_L}\right)$

$\Rightarrow t' = \gamma\left(t - \frac{\vec{v}.(\vec{r} - \overrightarrow{r_T})}{c^2}\right)$

$\Rightarrow t' = \gamma\left(t - \frac{\vec{v}.\vec{r}}{c^2} + \frac{\vec{v}.\overrightarrow{r_T}}{c^2}\right) = \gamma\left(t - \frac{\vec{v}.\vec{r}}{c^2}\right)$

$\Rightarrow \boldsymbol{t' = \gamma\left(t - \frac{\vec{v}.\vec{r}}{c^2}\right)}$ ……………(B)

In vector notation

$x = \overrightarrow{r_L}$

Also

$(1) \Rightarrow \vec{r} = \overrightarrow{r_T} + \overrightarrow{r_L}$

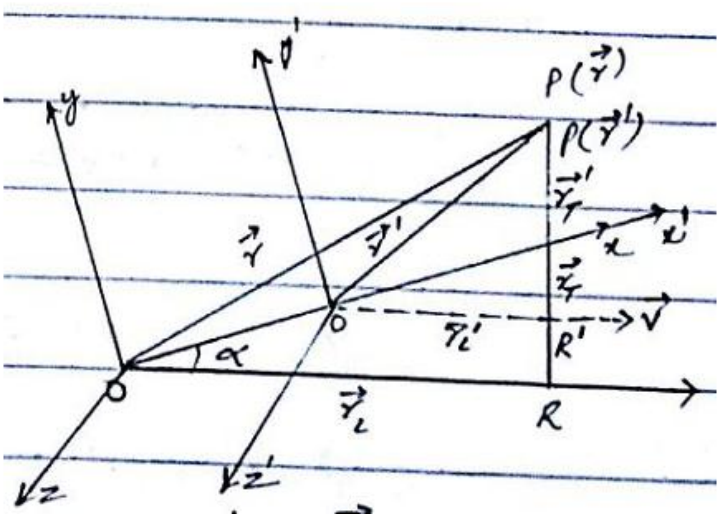
$\Rightarrow \overrightarrow{r_L} = \vec{r} - \overrightarrow{r_T}$

And $\vec{v} \perp \overrightarrow{r_T}$

$\Rightarrow \vec{v}.\overrightarrow{r_T} = 0$

(A) and (B) represent the general Lorentz Transformations.

GENERAL INVERSE LORENTZ TRANSFORMATIONS FIXED WITHOUT ROTATION / RESTRICTED INVERSE LORENTZ TRANSFORMATIONS



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Resolve $\vec{r}$ into components $\overrightarrow{r_T}$ (Transversal component) perpendicular to $\vec{v}$ and $\overrightarrow{r_L}$ (Horizontal component) parallel to $\vec{v}$ or in the direction of $\vec{v}$. Then by head to tail rule using for $\vec{r} = (x, y, z), \vec{r}' = (x', y', z')$

$\vec{r} = \overrightarrow{r_T} + \overrightarrow{r_L}$ ……………(1) and $\vec{r}' = \overrightarrow{r_T}' + \overrightarrow{r_L}'$ ……………(2)

By using inverse Lorentz Transformations in Vector form

$\vec{r} = \gamma(\vec{r}' + \vec{v}t')$

$\Rightarrow \overrightarrow{r_L} = \gamma(\overrightarrow{r_L}' + \vec{v}t')$ ……………(3)

For $\overrightarrow{r_L}'$ using $\Delta O'R'P$

$$Cos\theta = \frac{base}{hyp} = \frac{r_L'}{r'} \Rightarrow r_L' = r'Cos\theta = \frac{vr'Cos\theta}{v} = \frac{\vec{v}.\vec{r}'}{v} \Rightarrow r_L'\hat{v} = \frac{\vec{v}.\vec{r}'}{v}\hat{v} = \frac{\vec{v}.\vec{r}'}{v}\frac{\vec{v}}{v}$$

$\Rightarrow \overrightarrow{r_L}' = \frac{\vec{v}.\vec{r}'}{v^2}\vec{v}$ ……………(4)

(3) $\Rightarrow \overrightarrow{r_L} = \gamma\left(\overrightarrow{r_L}' + \vec{v}t'\right) = \gamma\left(\frac{\vec{v}.\vec{r}'}{v^2}\vec{v} + \vec{v}t'\right)$

$\Rightarrow \overrightarrow{r_L} = \gamma\left(\frac{\vec{v}.\vec{r}'}{v^2} + t'\right)\vec{v}$ ……………(5)

Since from figure $\overrightarrow{r_T} = \overrightarrow{r_T}'$ (Transversal component is perpendicular to the line) where $\overrightarrow{r_T}, \overrightarrow{r_T}'$ are same transversal vectors to the horizontal line. Then

(2) $\Rightarrow \vec{r}' = \overrightarrow{r_T}' + \overrightarrow{r_L}' \Rightarrow \overrightarrow{r_T}' = \vec{r}' - \overrightarrow{r_L}'$

$\Rightarrow \overrightarrow{r_T} = \vec{r}' - \frac{\vec{v}.\vec{r}'}{v^2}\vec{v}$ ……………(6) using (4) and $\overrightarrow{r_T} = \overrightarrow{r_T}'$

Using (5) and (6) in (1)

$\Rightarrow \vec{r} = \overrightarrow{r_T} + \overrightarrow{r_L} = \vec{r}' - \frac{\vec{v}.\vec{r}'}{v^2}\vec{v} + \gamma\left(\frac{\vec{v}.\vec{r}'}{v^2} + t'\right)\vec{v} = \vec{r}' - \frac{\vec{v}.\vec{r}'}{v^2}\vec{v} + \gamma\frac{\vec{v}.\vec{r}'}{v^2}\vec{v} + \gamma t'\vec{v}$

$\Rightarrow \vec{r} = \vec{r}' + (\gamma - 1)\frac{\vec{v}.\vec{r}'}{v^2}\vec{v} + \gamma t'\vec{v}$

$\Rightarrow \boldsymbol{\vec{r} = \vec{r}' + \left[(\gamma - 1)\frac{\vec{v}.\vec{r}'}{v^2} + \gamma t'\right]\vec{v}}$ ……………(A)

Now for t in vector notation we have

$t = \gamma\left(t' + \frac{\vec{v}}{c^2}.x'\right)$

$\Rightarrow t = \gamma\left(t' + \frac{\vec{v}}{c^2}.\overrightarrow{r_L}'\right)$

$\Rightarrow t = \gamma\left(t' - \frac{\vec{v}.(\vec{r}' - \overrightarrow{r_T}')}{c^2}\right)$

$\Rightarrow t = \gamma\left(t' - \frac{\vec{v}.\vec{r}'}{c^2} + \frac{\vec{v}.\overrightarrow{r_T}'}{c^2}\right) = \gamma\left(t' - \frac{\vec{v}.\vec{r}'}{c^2}\right)$

$\Rightarrow \boldsymbol{t = \gamma\left(t' - \frac{\vec{v}.\vec{r}'}{c^2}\right)}$ ……………(B)

In vector notation

$x' = \overrightarrow{r_L}'$

Also

(2) $\Rightarrow \vec{r}' = \overrightarrow{r_T}' + \overrightarrow{r_L}'$

$\Rightarrow \overrightarrow{r_L}' = \vec{r}' - \overrightarrow{r_T}'$

And $\vec{v} \perp \overrightarrow{r_T}'$

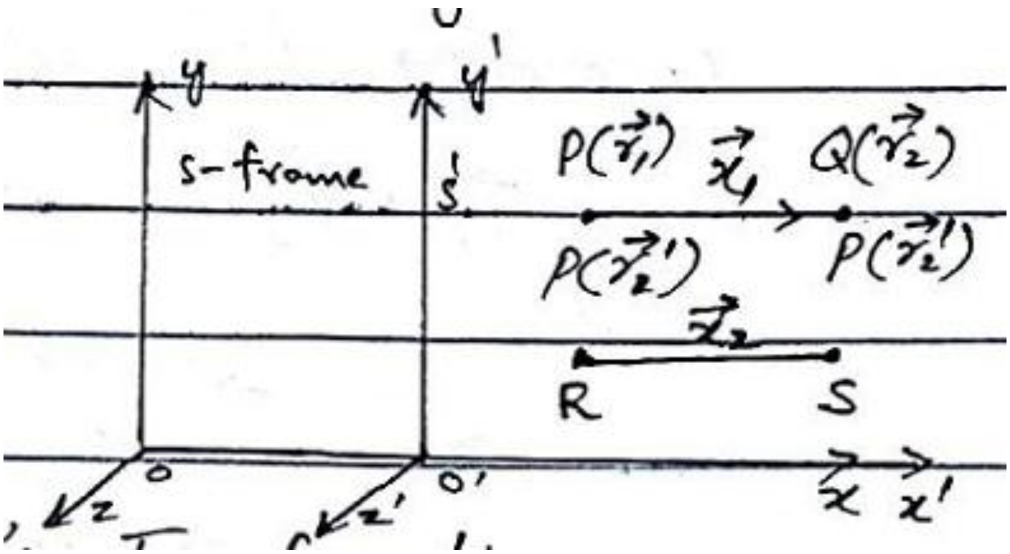
$\Rightarrow \vec{v}.\overrightarrow{r_T}' = 0$

(A) and (B) represent the general inverse Lorentz Transformations.

Question

Prove that if two vectors perpendicular in frame S then these two vectors are not perpendicular in frame S'. Under what conditions these two vectors are perpendicular in frame S'?

Solution



Let $\vec{r}_1$ and $\vec{r}_1{}'$ be the position vectors at a point P with respect to S and S' frames respectively and $\vec{r}_2$ and $\vec{r}_2{}'$ be the position vectors at a point Q with respect to S and S' frames respectively. Then suppose

$\vec{r}_2 - \vec{r}_1 = \vec{x} = \overrightarrow{PQ}$ (S – frame)

$\vec{r}_2{}' - \vec{r}_1{}' = \vec{x}' = \overrightarrow{PQ}$ (S' – frame)

By using general Lorentz Transformations

$$\vec{r}_1{}' = \vec{r}_1 + \left[(\gamma - 1)\frac{\vec{v}.\vec{r}_1}{v^2} - \gamma t\right]\vec{v} \;\ldots\ldots\ldots\ldots\ldots(1)$$

$$\vec{r}_2{}' = \vec{r}_2 + \left[(\gamma - 1)\frac{\vec{v}.\vec{r}_2}{v^2} - \gamma t\right]\vec{v} \;\ldots\ldots\ldots\ldots\ldots(2)$$

$$\Rightarrow \vec{r}_2^{\,'} - \vec{r}_1^{\,'} = \vec{r}_2 - \vec{r}_1 + \left[(\gamma - 1)\frac{\vec{v}.(\vec{r}_2 - \vec{r}_1)}{v^2} - 0\right]\vec{v}$$ Subtraction (1) from (2)

$$\Rightarrow \vec{x}' = \vec{x} + (\gamma - 1)\frac{\vec{v}.\vec{x}}{v^2}\vec{v}$$

Let $\vec{x}_1$ and $\vec{x}_2$ are two vectors then

$\Rightarrow \vec{x}_1{}' = \vec{x}_1 + (\gamma - 1)\frac{\vec{v}.\vec{x}_1}{v^2}\vec{v}$ ……………(3)

$\Rightarrow \vec{x}_2{}' = \vec{x}_2 + (\gamma - 1)\frac{\vec{v}.\vec{x}_2}{v^2}\vec{v}$ ……………(4)

Taking dot product of (3) and (4)

$$\vec{x}_1'.\vec{x}_2' = \vec{x}_1.\vec{x}_2 + (\gamma - 1)\frac{(\vec{v}.\vec{x}_1)(\vec{v}.\vec{x}_2)}{v^2} + (\gamma - 1)\frac{(\vec{v}.\vec{x}_1)(\vec{v}.\vec{x}_2)}{v^2} + (\gamma - 1)^2\frac{(\vec{v}.\vec{x}_1)}{v^2}\frac{(\vec{v}.\vec{x}_2)}{v^2}(\vec{v}.\vec{v})$$

$$\vec{x}_1'.\vec{x}_2' = \vec{x}_1.\vec{x}_2 + 2(\gamma - 1)\frac{(\vec{v}.\vec{x}_1)(\vec{v}.\vec{x}_2)}{v^2} + (\gamma - 1)^2\frac{(\vec{v}.\vec{x}_1)}{v^2}\frac{(\vec{v}.\vec{x}_2)}{v^2}(v^2)$$

$$\vec{x}_1'.\vec{x}_2' = \vec{x}_1.\vec{x}_2 + 2(\gamma - 1)\frac{(\vec{v}.\vec{x}_1)(\vec{v}.\vec{x}_2)}{v^2} + (\gamma - 1)^2\frac{(\vec{v}.\vec{x}_1)(\vec{v}.\vec{x}_2)}{v^2}$$

Since in S – frame $\vec{x}_1 \perp \vec{x}_2$ therefore $\vec{x}_1.\vec{x}_2 = 0$

But in S' – frame $\vec{x}_1{}' \not\perp \vec{x}_2{}'$ therefore $\vec{x}_1{}'.\vec{x}_2{}' \neq 0$

Under the following conditions $\vec{x}_1{}'$ and $\vec{x}_2{}'$ becomes perpendicular in in S' – frame

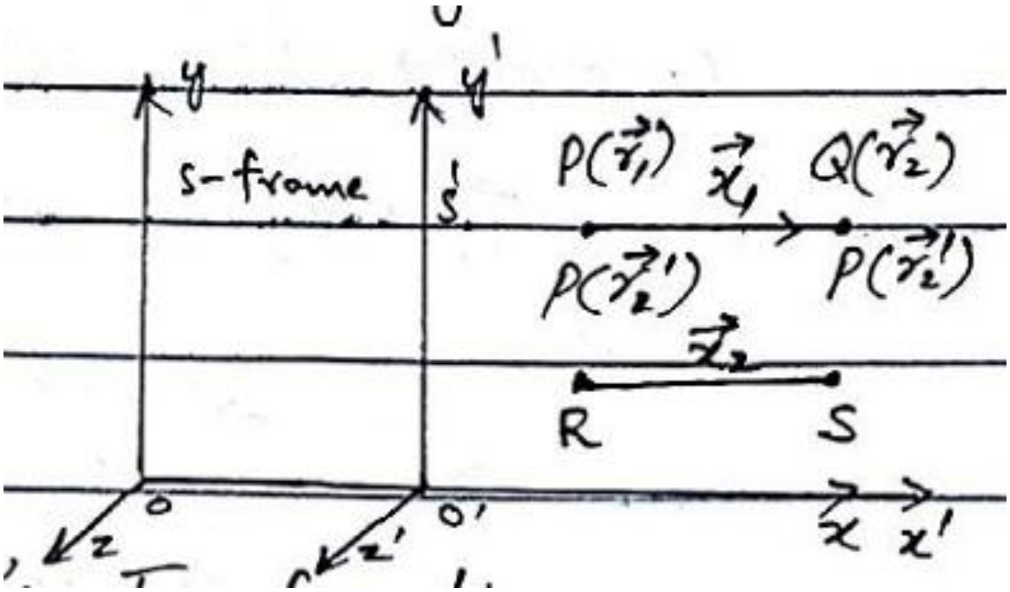
- If $\gamma - 1 = 0 \Rightarrow \gamma = 1$ (Classical Mechanics)
- If $\vec{x}_1 \perp \vec{v} \Rightarrow \vec{x}_1.\vec{v} = 0$
- If $\vec{x}_2 \perp \vec{v} \Rightarrow \vec{x}_2.\vec{v} = 0$

In all cases $\vec{x}_1'.\vec{x}_2' = 0$ and we get $\vec{x}_1{}'$ and $\vec{x}_2{}'$ perpendicular in in S' – frame

Question

Prove that if two vectors parallel in frame S then these two vectors are not parallel in frame S'. Under what conditions these two vectors are parallel in frame S'?

Solution



Let $\vec{r}_1$ and $\vec{r}_1{}'$ be the position vectors at a point P with respect to S and S' frames respectively and $\vec{r}_2$ and $\vec{r}_2{}'$ be the position vectors at a point Q with respect to S and S' frames respectively. Then suppose

$\vec{r}_2 - \vec{r}_1 = \vec{x} = \overrightarrow{PQ}$ (S – frame)

$\vec{r}_2{}' - \vec{r}_1{}' = \vec{x}' = \overrightarrow{PQ}$ (S' – frame)

By using general Lorentz Transformations

$$\vec{r}_1{}' = \vec{r}_1 + \left[(\gamma - 1)\frac{\vec{v}.\vec{r}_1}{v^2} - \gamma t\right]\vec{v} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\vec{r}_2{}' = \vec{r}_2 + \left[(\gamma - 1)\frac{\vec{v}.\vec{r}_2}{v^2} - \gamma t\right]\vec{v} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$\Rightarrow \vec{r}_2{}' - \vec{r}_1{}' = \vec{r}_2 - \vec{r}_1 + \left[(\gamma - 1)\frac{\vec{v}.(\vec{r}_2 - \vec{r}_1)}{v^2} - 0\right]\vec{v}$$ Subtraction (1) from (2)

$$\Rightarrow \vec{x}' = \vec{x} + (\gamma - 1)\frac{\vec{v}.\vec{x}}{v^2}\vec{v}$$

Let $\vec{x}_1$ and $\vec{x}_2$ are two vectors then

$\Rightarrow \vec{x}_1{}' = \vec{x}_1 + (\gamma - 1)\frac{\vec{v}.\vec{x}_1}{v^2}\vec{v}$ ……………(3)

$\Rightarrow \vec{x}_2{}' = \vec{x}_2 + (\gamma - 1)\frac{\vec{v}.\vec{x}_2}{v^2}\vec{v}$ ……………(4)

Taking cross product of (3) and (4)

$$\vec{x}_1' \times \vec{x}_2' = \vec{x}_1 \times \vec{x}_2 + (\gamma - 1)\frac{(\vec{v}.\vec{x}_2)}{v^2}(\vec{v} \times \vec{x}_1) + (\gamma - 1)\frac{(\vec{v}.\vec{x}_1)}{v^2}(\vec{v} \times \vec{x}_2) + (\gamma - 1)^2\frac{(\vec{v}.\vec{x}_1)}{v^2}\frac{(\vec{v}.\vec{x}_2)}{v^2}(\vec{v} \times \vec{v})$$

$$\vec{x}_1' \times \vec{x}_2' = \vec{x}_1 \times \vec{x}_2 + (\gamma - 1)\frac{(\vec{v}.\vec{x}_2)}{v^2}(\vec{v} \times \vec{x}_1) + (\gamma - 1)\frac{(\vec{v}.\vec{x}_1)}{v^2}(\vec{v} \times \vec{x}_2)$$

Since in S – frame $\vec{x}_1 \parallel \vec{x}_2$ therefore $\vec{x}_1 \times \vec{x}_2 = 0$

But in S' – frame $\vec{x}_1{}' \nparallel \vec{x}_2{}'$ therefore $\vec{x}_1{}' \times \vec{x}_2{}' \neq 0$

Under the following conditions $\vec{x}_1{}'$ and $\vec{x}_2{}'$ becomes parallel in in S' – frame

- If $\gamma - 1 = 0 \Rightarrow \gamma = 1$ (Classical Mechanics)
- If $\vec{x}_1 \perp \vec{v} \Rightarrow \vec{x}_1.\vec{v} = 0$ and $\vec{x}_2 \perp \vec{v} \Rightarrow \vec{x}_2.\vec{v} = 0$
- If $\vec{x}_1 \parallel \vec{v} \Rightarrow \vec{x}_1 \times \vec{v} = 0$ and $\vec{x}_2 \parallel \vec{v} \Rightarrow \vec{x}_2 \times \vec{v} = 0$

In all cases $\vec{x}_1' \times \vec{x}_2' = 0$ and we get $\vec{x}_1{}'$ and $\vec{x}_2{}'$ parallel in in S' – frame

FOUR – VECTOR

A four – vector A_μ is a set of four quantities (A_1, A_2, A_3, A_4) which transform under a Lorentz Transformation in the same way as the x_1, x_2, x_3, x_4 coordinates of a point in the four dimensional space-time continuum.

Thus $\quad A_1' = \gamma\left(A_1 + \frac{iv}{c}A_4\right), A_2' = A_2, A_3' = A_3, A_4' = \gamma\left(A_4 - \frac{iv}{c}A_1\right)$

The first three components (A_1, A_2, A_3) are the components of an ordinary three – dimensional vector.

Position four – vector

The four-vector x with components $x_1 = x, x_2 = y, x_3 = z, x_4 = ict$ is called position four-vector.

FOUR – VELOCITY / MINKOWSKI VELOCITY

In Minkowski space we have $x_\mu = (x_1, x_2, x_3, x_4)$

Where $x_1 = x, x_2 = y, x_3 = z, x_4 = ict$

$\Rightarrow \frac{dx_\mu}{d\tau} = \frac{d}{d\tau}(x_1, x_2, x_3, x_4)$ ` $\qquad$ diff.w.r.to proper time

$\Rightarrow \frac{dx_\mu}{d\tau} = \frac{d}{dt}(x_1, x_2, x_3, x_4)\frac{dt}{d\tau}$

$\Rightarrow \frac{dx_\mu}{d\tau} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}\right)\frac{dt}{d\tau} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, ic\frac{dt}{dt}\right)\frac{dt}{d\tau}$

$\Rightarrow U_\mu = (u_x, u_y, u_z, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = (\vec{u}, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ $\qquad$ Since $\frac{dt}{d\tau} = \sqrt{1 - \frac{u^2}{c^2}}$

Comparing with 4 – velocity $U_\mu = (u_1, u_2, u_3, u_4)$ we have

$u_1 = \frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}, u_2 = \frac{u_y}{\sqrt{1-\frac{u^2}{c^2}}}, u_3 = \frac{u_z}{\sqrt{1-\frac{u^2}{c^2}}}, u_4 = \frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}$

TRANSFORMATION LAW OF 4 – VELOCITY

By using law of transformation of 4 – vectors $A_\mu = (A_1, A_2, A_3, A_4)$ where

$A'_1 = \gamma\left(A_1 + \frac{iv}{c}A_4\right), A'_2 = A_2, A'_3 = A_3, A'_4 = \gamma\left(A_4 - \frac{iv}{c}A_1\right)$ …………(1)

Change these terms in term of 4 – velocity vector $U_\mu = (u_1, u_2, u_3, u_4)$

$u'_1 = \gamma\left(u_1 + \frac{iv}{c}u_4\right)$ …………(2)

$u'_2 = u_2$ …………(3) $u'_3 = u_3$ …………(4)

$u'_4 = \gamma\left(u_4 - \frac{iv}{c}u_1\right)$ …………(5)

(2) $\Rightarrow u'_1 = \gamma\left(u_1 + \frac{iv}{c}u_4\right)$

$$\Rightarrow \frac{u'_x}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma\left(\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}} + \frac{iv}{c}\frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}\right) \Rightarrow u'_x = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{v}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow u'_x = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}(u_x - v)$$

$\Rightarrow u'_x = \lambda(u_x - v)$ …………(6) where we use $\lambda = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}$

(2) $\Rightarrow u'_2 = u_2 \Rightarrow \frac{u'_y}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{u_y}{\sqrt{1-\frac{u^2}{c^2}}}$

$$\Rightarrow u'_y = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}}u_y \Rightarrow u'_y = \frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}u_y \Rightarrow u'_y = \frac{\lambda}{\gamma}u_y$$ …………(7)

$$(3) \Rightarrow u_3' = u_3 \Rightarrow \frac{u'_z}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{u_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\Rightarrow u'_z = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}} u_z \Rightarrow u'_z = \frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}} u_z \Rightarrow u'_z = \frac{\lambda}{\gamma} u_z \qquad \ldots\ldots\ldots\ldots(8)$$

$$(4) \Rightarrow u_4' = \gamma\left(u_4 - \frac{iv}{c} u_1\right)$$

$$\Rightarrow \frac{ic}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma\left(\frac{ic}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{ic}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow ic = \frac{ic.\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{v}{c^2}.\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right) \Rightarrow 1 = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{v}{c^2}.\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow 1 = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}\left(1 - \frac{v}{c^2} u_x\right) \Rightarrow 1 = \lambda\left(1 - \frac{v}{c^2} u_x\right)$$

$$\Rightarrow \boldsymbol{\lambda = \frac{1}{1-\frac{v}{c^2}u_x}}$$

Using λ in (6), (7), (8) we have

$$u'_x = \frac{u_x - v}{1-\frac{v}{c^2}u_x}, u'_y = \frac{u_y}{\gamma\left(1-\frac{v}{c^2}u_x\right)}, u'_z = \frac{u_z}{\gamma\left(1-\frac{v}{c^2}u_x\right)} \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

Inverse transformation law of 4 – velocity become

$$u_x = \frac{u'_x + v}{1+\frac{v}{c^2}u'_x}, u_y = \frac{u'_y}{\gamma\left(1+\frac{v}{c^2}u'_x\right)}, u_z = \frac{u'_z}{\gamma\left(1+\frac{v}{c^2}u'_x\right)} \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

FOUR – MOMENTUM / MINKOWSKI MOMENTUM

Consider a particle is moving with 4 – velocity U_μ. If mass of particle is m_0 then 4 – momentum is defined as $P_\mu = m_0 U_\mu$

$\Rightarrow P_\mu = m_0(\vec{u}, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ where $\vec{u} = (u_x, u_y, u_z)$

$\Rightarrow P_\mu = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}(u_x, u_y, u_z, ic)$

Comparing with 4 – velocity $P_\mu = (P_1, P_2, P_3, P_4)$ we have

$P_1 = \frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}} = m u_x = P_x,\; P_2 = \frac{m_0 u_y}{\sqrt{1-\frac{u^2}{c^2}}} = m u_y = P_y,\; P_3 = \frac{m_0 u_z}{\sqrt{1-\frac{u^2}{c^2}}} = m u_z = P_z$

$\boldsymbol{P_4 = \frac{m_0 ic}{\sqrt{1-\frac{u^2}{c^2}}} = mic = \frac{mc^2}{c} i = \frac{E}{c} i}$ where $m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}},\; E = mc^2$

TRANSFORMATION LAW OF 4 – MOMENTUM

By using law of transformation of 4 – vectors $A_\mu = (A_1, A_2, A_3, A_4)$ where

$A_1' = \gamma\left(A_1 + \frac{iv}{c}A_4\right), A_2' = A_2, A_3' = A_3, A_4' = \gamma\left(A_4 - \frac{iv}{c}A_1\right)$ …………(1)

Change these terms in term of 4 – momentum vector $P_\mu = (P_1, P_2, P_3, P_4)$

$P_1' = \gamma\left(P_1 + \frac{iv}{c}P_4\right)$ …………(2)

$P_2' = P_2$ …………(3) $P_3' = P_3$ …………(4)

$P_4' = \gamma\left(P_4 - \frac{iv}{c}P_1\right)$ …………(5)

Using 4 – momentum components

$P_1 = \frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}},\; P_2 = \frac{m_0 u_y}{\sqrt{1-\frac{u^2}{c^2}}},\; P_3 = \frac{m_0 u_z}{\sqrt{1-\frac{u^2}{c^2}}},\; P_4 = \frac{m_0 ic}{\sqrt{1-\frac{u^2}{c^2}}}$

$(2) \Rightarrow P_1^{'} = \gamma\left(P_1 + \frac{iv}{c}P_4\right)$

$$\Rightarrow \frac{m_0 u'_x}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma\left(\frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}} + \frac{iv}{c}\frac{m_0 ic}{\sqrt{1-\frac{u^2}{c^2}}}\right) \Rightarrow m_0 u'_x = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{m_0 v}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow P_x{'} = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}\left(P_x - m_0 c^2\frac{v}{c^2}\right)$$

$$\Rightarrow P_x{'} = \lambda\left(P_x - E_0\frac{v}{c^2}\right) \quad \ldots\ldots\ldots\ldots(6) \qquad \text{where we use } \lambda = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}$$

$$(2) \Rightarrow P_2^{'} = P_2 \Rightarrow \frac{m_0 u_y^{'}}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{m_0 u_y}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\Rightarrow P_y{'} = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}}P_y \Rightarrow P_y{'} = \frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}P_y \Rightarrow P_y{'} = \frac{\lambda}{\gamma}P_y \qquad \ldots\ldots\ldots\ldots(7)$$

$$(3) \Rightarrow P_3^{'} = P_3 \Rightarrow \frac{m_0 u_z^{'}}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{m_0 u_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\Rightarrow P_z{'} = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}}P_z \Rightarrow P_z{'} = \frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}P_z \Rightarrow P_z{'} = \frac{\lambda}{\gamma}P_z \qquad \ldots\ldots\ldots\ldots(8)$$

$(4) \Rightarrow P_4^{'} = \gamma\left(P_4 - \frac{iv}{c}P_1\right)$

$$\Rightarrow \frac{m_0 ic}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma\left(\frac{m_0 ic}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{m_0 ic}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow m_0ic = \frac{m_0ic.\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{v}{c^2}.\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right) \Rightarrow 1 = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{v}{c^2}.\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow 1 = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}\left(1-\frac{v}{c^2}u_x\right) \Rightarrow 1 = \lambda\left(1-\frac{v}{c^2}u_x\right) \Rightarrow \boldsymbol{\lambda} = \frac{\mathbf{1}}{\mathbf{1}-\frac{v}{c^2}\boldsymbol{u_x}}$$

Using λ in (6), (7), (8) we have

$$P_x{}' = \frac{P_x - E_0\frac{v}{c^2}}{1-\frac{v}{c^2}u_x}, P_y{}' = \frac{P_y}{\gamma\left(1-\frac{v}{c^2}u_x\right)}, P_z{}' = \frac{P_z}{\gamma\left(1-\frac{v}{c^2}u_x\right)} \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

Inverse transformation law of 4 – velocity become

$$P_x = \frac{P_x' + E_0\frac{v}{c^2}}{1+\frac{v}{c^2}u'_x}, P_y = \frac{P_y{}'}{\gamma\left(1+\frac{v}{c^2}u'_x\right)}, P_z = \frac{P_z{}'}{\gamma\left(1+\frac{v}{c^2}u'_x\right)} \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

Transformation law of energy by using 4th component of momentum

Since $P_4' = \gamma\left(P_4 - \frac{iv}{c}P_1\right) \Rightarrow im'c = \gamma\left(imc - \frac{iv}{c}P_x\right)$

$$\Rightarrow \frac{im'c^2}{c} = \gamma\left(\frac{imc^2}{c} - \frac{iv}{c}P_x\right) \Rightarrow \frac{i}{c}E' = \gamma\left(\frac{i}{c}E - \frac{iv}{c}P_x\right)$$

$\Rightarrow E' = \gamma(E - vP_x)$ Transformation law of energy

$\Rightarrow E = \gamma\left(E' + vP_x\right)$ inverse Transformation law of energy

This equation tells us that what is called as energy in frame S' is a sort of admixture of energy and momentum in frame S. These equations show that momentum and energy of a system of particles transform in the same way as in case of individual particles.

Application of 4 – momentum

Law of transformation of energy and Aberration of light.

Question

Show that $P_\mu = \left(\vec{P}, i\frac{E}{c}\right)$ and deduce that $P^2 - \frac{E^2}{c^2}$ is an invariant $-m_0^2c^2$ with respect to a Lorentz transformation.

Solution

Consider a particle is moving with 4 – velocity U_μ. If mass of particle is m_0 then 4 – momentum is defined as $P_\mu = m_0 U_\mu$

$\Rightarrow P_\mu = m_0(\vec{u}, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ where $\vec{u} = (u_x, u_y, u_z)$

$\Rightarrow P_\mu = \left(\frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}, i\frac{m_0 c}{\sqrt{1-\frac{u^2}{c^2}}}\right) \Rightarrow P_\mu = (m\vec{u}, imc)$

$\Rightarrow P_\mu = \left(\vec{P}, imc\right)$ where $m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$

$P_\mu = \left(\vec{P}, i\frac{E}{c}\right)$ where we use $E = mc^2 \Rightarrow mc = \frac{E}{c}$

Since the length of a four-vector is invariant we must have $p^2 + \left(\frac{iE}{c}\right)^2$ = constant.

$\therefore \quad p^2 - \frac{E^2}{c^2}$ = constant.

When $p = 0$, $E = m_o c^2$

$\therefore \quad \text{Constant} = -\frac{(m_o c^2)^2}{c^2} = -m_o^2 c^2$

$\therefore \quad p^2 - \frac{E^2}{c^2} = -m_o^2 c^2$

or $\quad E^2 = p^2c^2 + m_o^2 c^4$

Now to show that invariance of $P^2 - \frac{E^2}{c^2}$ we have the following result;

$$P'^2 - \frac{E'^2}{c^2} = P_x'^2 + P_y'^2 + P_z'^2 - \frac{E'^2}{c^2}$$

Now using the following relations

$$P_x' = P_x - \frac{Ev}{c^2}, P_y' = P_y, P_z' = P_z, E' = \gamma(E - vP_x)$$

$$P'^2 - \frac{E'^2}{c^2} = \gamma^2 \left(P_x - \frac{Ev}{c^2}\right)^2 + P_y^2 + P_z^2 - \frac{\gamma^2}{c^2}(E - vP_x)$$

$$P'^2 - \frac{E'^2}{c^2} = P_x^2\gamma^2 \left(1 - \frac{v^2}{c^2}\right) + P_y^2 + P_z^2 - \frac{E^2\gamma^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)$$

$$P'^2 - \frac{E'^2}{c^2} = P_x^2 \frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left(1 - \frac{v^2}{c^2}\right) + P_y^2 + P_z^2 - \frac{E^2}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left(1 - \frac{v^2}{c^2}\right)$$

$$P'^2 - \frac{E'^2}{c^2} = P_x^2 + P_y^2 + P_z^2 - \frac{E^2}{c^2}$$

$$P'^2 - \frac{E'^2}{c^2} = P^2 - \frac{E^2}{c^2}$$

Law of Transformation of Mass

$$P_4' = \gamma\left(P_4 - \frac{iv}{c}P_1\right)$$

$$\Rightarrow \frac{m_0 ic}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma\left(\frac{m_0 ic}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{m_0 ic}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow im'c = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(imc - \frac{iv}{c}mu_x\right) \Rightarrow m' = \gamma\left(m - \frac{v}{c^2}mu_x\right)$$

$$\Rightarrow \boldsymbol{m' = \gamma\left(1 - \frac{v}{c^2}u_x\right)m}$$

FOUR – FORCE / MINKOWSKI FORCE

If $\vec{P}$ is momentum of the moving particle then the force acting on it can be defined as $\vec{f} = \frac{d\vec{P}}{dt} = \dot{\vec{P}}$. Then 4 – force is defined as

"the rate of change of 4 – momentum w.r.to proper time"

$$F_\mu = \frac{dP_\mu}{d\tau} = \frac{d}{d\tau}\left(\vec{P}, imc\right) = \frac{d}{dt}\left(\vec{P}, imc\right)\frac{dt}{d\tau} = \left(\frac{d\vec{P}}{dt}, ic\frac{dm}{dt}\right)\frac{dt}{d\tau}$$

$$\Rightarrow F_\mu = \left(\dot{\vec{P}}, i\dot{m}c\right)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \qquad \text{where } \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\Rightarrow F_\mu = \left(\vec{f}, i\dot{m}c\right)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\Rightarrow F_\mu = \left(f_x, f_y, f_z, i\dot{m}c\right)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \qquad \text{where } \vec{f} = \left(f_x, f_y, f_z\right)$$

Comparing with 4 – force $F_\mu = (F_1, F_2, F_3, F_4)$ we get ***relative equations of motion***

$$F_1 = \frac{f_x}{\sqrt{1-\frac{u^2}{c^2}}},\ F_2 = \frac{f_y}{\sqrt{1-\frac{u^2}{c^2}}},\ F_3 = \frac{f_z}{\sqrt{1-\frac{u^2}{c^2}}},\ F_4 = \frac{i\dot{m}c}{\sqrt{1-\frac{u^2}{c^2}}}$$

Question

If $\vec{U}$ is a 4 – velocity then prove that $\vec{U}.\vec{U} = U^2 = -c^2$. Show that it is of constant magnitude ic. The four-velocity is then said to be a time like vector.

Solution

$$\vec{U}.\vec{U} = (\vec{u}, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}.(\vec{u}, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\left(\sqrt{1-\frac{u^2}{c^2}}\right)^2}(\vec{u}, ic).(\vec{u}, ic) = \frac{1}{\left(1-\frac{u^2}{c^2}\right)}(\vec{u}.\vec{u} + i^2c^2)$$

$$\vec{U}.\vec{U} = \frac{u^2-c^2}{\left(1-\frac{u^2}{c^2}\right)} = -c^2.\frac{u^2-c^2}{u^2-c^2} \Rightarrow \vec{U}.\vec{U} = U^2 = -c^2$$

$$\Rightarrow \left\|\vec{U}\right\| = \sqrt{\vec{U}.\vec{U}} = \sqrt{-c^2} = ic$$

Question

If $\vec{U}$ is a 4 – velocity and $\vec{F}$ is a 4 – force then prove that $\vec{U}.\vec{F} = 0$

Solution

As $\vec{U}.\vec{U} = -c^2$

$\frac{d}{d\tau}\left(\vec{U}.\vec{U}\right) = \frac{d}{d\tau}(-c^2)$ differentiating w.r.to τ

$\vec{U}.\frac{d\vec{U}}{d\tau} + \vec{U}.\frac{d\vec{U}}{d\tau} = 0 \Rightarrow 2\vec{U}.\frac{d\vec{U}}{d\tau} = 0 \Rightarrow \vec{U}.\frac{d}{d\tau}\left(m_0\vec{U}\right) = 0$ multiplying with $\frac{m_0}{2}$

$\vec{U}.\frac{d\vec{P}}{d\tau} = 0 \Rightarrow \vec{U}.\vec{F} = 0$

Question

If $\vec{u}$ is a velocity and $\vec{f}$ is a force then prove that $\vec{u}.\vec{f} = \dot{m}c^2$

Solution

As $\vec{U}.\vec{F} = 0$

$$(\vec{u}, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}.\left(\vec{f}, i\dot{m}c\right)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 0$$

$$\frac{1}{1-\frac{u^2}{c^2}}(\vec{u}, ic).\left(\vec{f}, i\dot{m}c\right) = 0 \Rightarrow \vec{u}.\vec{f} + i^2\dot{m}c^2 = 0 \Rightarrow \vec{u}.\vec{f} - \dot{m}c^2 = 0 \Rightarrow \vec{u}.\vec{f} = \dot{m}c^2$$

Question

Show that $F = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}\left(\vec{f}, \frac{i}{c}\vec{f}.\vec{u}\right)$

Solution

Since $F = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(\vec{f}, i\dot{m}c\right) = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}\left(\vec{f}, i\dot{m}c\right) = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}\left(\vec{f}, \frac{i}{c}\dot{m}c^2\right)$

$F = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}\left(\vec{f}, \frac{i}{c}\vec{f}.\vec{u}\right)$ where we use the fact $\vec{f}.\vec{u} = \dot{m}c^2$

Question

Show that $F_1 = m_0\gamma(v)\frac{d}{dt}(\gamma(v)v_x)$

Solution

$$F_\mu = \frac{dP_\mu}{d\tau} = \frac{dP_\mu}{dt}\frac{dt}{d\tau}$$

$$F_\mu = \frac{d}{dt}\left[m_0(\vec{u}, ic)\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right]\frac{dt}{d\tau}$$

$$F_\mu = \frac{d}{dt}\left[(m_0u_x, m_0u_y, m_0u_z, im_0c)\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right]\frac{dt}{d\tau}$$

$$F_\mu = \frac{d}{dt}\left[(p_x, p_y, p_z, im_0c)\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right]\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{where } \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$F_\mu = \frac{d}{dt}\left[(p_x, p_y, p_z, im_0c)\gamma(v)\right]\gamma(v)$$

$$F_\mu = \left(\frac{d}{dt}(p_x\gamma(v)), \frac{d}{dt}\left(p_y\gamma(v)\right), \frac{d}{dt}(p_z\gamma(v)), \frac{d}{dt}\left(im_0c\gamma(v)\right)\right)\gamma(v)$$

$$\Rightarrow F_1 = \left(\frac{d}{dt}\left(p_x\gamma(v)\right)\right)\gamma(v)$$

$$\Rightarrow F_2 = \left(\frac{d}{dt}\left(p_y\gamma(v)\right)\right)\gamma(v)$$

$$\Rightarrow F_3 = \left(\frac{d}{dt}\left(p_z\gamma(v)\right)\right)\gamma(v)$$

$$\Rightarrow F_4 = \left(\frac{d}{dt}\left(im_0c\gamma(v)\right)\right)\gamma(v)$$

TRANSFORMATION LAW OF 4 – MOMENTUM

By using law of transformation of 4 – vectors $A_\mu = (A_1, A_2, A_3, A_4)$ where

$A_1' = \gamma\left(A_1 + \frac{iv}{c}A_4\right), A_2' = A_2, A_3' = A_3, A_4' = \gamma\left(A_4 - \frac{iv}{c}A_1\right)$ …………(1)

Change these terms in term of 4 – force vector $F_\mu = (F_1, F_2, F_3, F_4)$

$F_1' = \gamma\left(F_1 + \frac{iv}{c}F_4\right)$ …………(2)

$F_2' = F_2$ …………(3) $F_3' = F_3$ …………(4)

$F_4' = \gamma\left(F_4 - \frac{iv}{c}F_1\right)$ …………(5)

Using 4 – force components

$$F_1 = \frac{f_x}{\sqrt{1-\frac{u^2}{c^2}}},\ F_2 = \frac{f_y}{\sqrt{1-\frac{u^2}{c^2}}},\ F_3 = \frac{f_z}{\sqrt{1-\frac{u^2}{c^2}}},\ F_4 = \frac{i\dot{m}c}{\sqrt{1-\frac{u^2}{c^2}}}$$

(2) $\Rightarrow F_1' = \gamma\left(F_1 + \frac{iv}{c}F_4\right)$

$$\Rightarrow \frac{f_x'}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma\left(\frac{f_x}{\sqrt{1-\frac{u^2}{c^2}}} + \frac{iv}{c}\frac{i\dot{m}c}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow f_x' = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{f_x}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{v}{c^2}\frac{\dot{m}c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow f_x' = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}\left(f_x - \frac{v}{c^2}\dot{m}c^2\right)$$

$\Rightarrow f_x' = \lambda\left(f_x - \left(\frac{v}{c^2}\right)\vec{u}.\vec{f}\right)$ where we use $\lambda = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}$

$\Rightarrow f_x^{'} = \frac{f_x - \left(\frac{v}{c^2}\right)\vec{u}.\vec{f}}{1-\frac{v}{c^2}u_x}$ where we use $\lambda = \frac{1}{1-\frac{v}{c^2}u_x}$

$\Rightarrow f_x^{'} = \frac{f_x - \left(\frac{v}{c^2}\right)\left(u_x f_x + u_y f_y + u_z f_z\right)}{1-\frac{v}{c^2}u_x}$ since $\vec{u}.\vec{f} = u_x f_x + u_y f_y + u_z f_z$

$$\Rightarrow f_x^{'} = \frac{\left(1-\frac{v}{c^2}u_x\right)f_x - \left(\frac{v}{c^2}\right)\left(u_y f_y + u_z f_z\right)}{1-\frac{v}{c^2}u_x}$$

$$\Rightarrow \boldsymbol{f_x^{'} = f_x - \frac{\left(\frac{v}{c^2}\right)\left(u_y f_y + u_z f_z\right)}{1-\frac{v}{c^2}u_x}}$$

$$(2) \Rightarrow F_2^{'} = F_2 \Rightarrow \frac{f_y{}'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{f_y}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\Rightarrow f_y^{'} = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}} f_y$$

$$\Rightarrow f_y^{'} = \frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}} f_y$$

$\Rightarrow \boldsymbol{f_y^{'} = \frac{f_y}{\gamma\left(1-\frac{v}{c^2}u_x\right)}}$ where we use $\lambda = \frac{1}{1-\frac{v}{c^2}u_x}$

$$(3) \Rightarrow F_3^{'} = F_3 \Rightarrow \frac{f_z{}'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{f_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\Rightarrow f_z^{'} = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}} f_z$$

$$\Rightarrow f_z{}' = \frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}} f_z \Rightarrow f_z{}' = \frac{\lambda}{\gamma} f_z$$

$\Rightarrow \boldsymbol{f_z}{}' = \frac{\boldsymbol{f_z}}{\boldsymbol{\gamma}\left(\mathbf{1}-\frac{\boldsymbol{v}}{\boldsymbol{c^2}}\boldsymbol{u_x}\right)}$ where we use $\lambda = \frac{1}{1-\frac{v}{c^2}u_x}$

$$(4) \Rightarrow F_4' = \gamma\left(F_4 - \frac{iv}{c}F_1\right)$$

$$\Rightarrow \frac{i\dot{m}'c}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma\left(\frac{i\dot{m}c}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{f_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow \frac{i\dot{m}'c}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{i\dot{m}c}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{f_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow i\dot{m}'c = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{i\dot{m}c}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{iv}{c}.\frac{f_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\Rightarrow i\dot{m}'c = \frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}\left(i\dot{m}c - \frac{iv}{c}f_x\right)$$

$$\Rightarrow i\dot{m}'c = \lambda\left(i\dot{m}c - \frac{iv}{c}f_x\right)$$

$$\Rightarrow \frac{i}{c}\dot{m}'c^2 = \frac{i}{c}\lambda(\dot{m}c^2 - vf_x)$$

$$\Rightarrow \dot{m}'c^2 = \lambda(\dot{m}c^2 - vf_x)$$

$$\Rightarrow \vec{u}'.\vec{f}' = \lambda\left(\vec{u}.\vec{f} - vf_x\right)$$

$\Rightarrow \vec{\boldsymbol{u}}'.\vec{\boldsymbol{f}}' = \frac{\vec{\boldsymbol{u}}.\vec{\boldsymbol{f}}-\boldsymbol{v}\boldsymbol{f_x}}{\mathbf{1}-\frac{\boldsymbol{v}}{\boldsymbol{c^2}}\boldsymbol{u_x}}$ where we use $\lambda = \frac{1}{1-\frac{v}{c^2}u_x}$

Example

Show that four-momentum and four-force are mutually perpendicular, that is show that the scalar product of P and F is zero. Hence show that the rate of change of energy of a particle is equal to the rate at which the force does work on the particle.

Solution

$$P.P = \sum_{\mu} P_{\mu}^2 = m_0^2 \sum_{\mu} U_{\mu}^2 = -m_0^2 c^2 = \text{constant} \qquad \ldots (1)$$

since $\sum_{\mu} U_{\mu}^2 = -c^2$

Differentiating eqn. (1) with respect to τ we get

$$P \cdot \frac{dP}{d\tau} = P.F. = 0 \qquad \ldots (2)$$

Thus the scalar product of P and F is zero.

From Eqn. (2) we see that $P_1F_1 + P_2F_2 + P_3F_3 + P_4F_4 = 0$

$$\therefore \qquad P_4F_4 = -(P_1F_1 + P_2F_2 + P_3F_3) = -\vec{p} \cdot \frac{\vec{f}}{\sqrt{1 - u^2/c^2}}$$

$$\therefore \quad \left(\frac{im_0 c}{\sqrt{1-u^2/c^2}}\right)\left(\frac{i}{c\sqrt{1-u^2/c^2}}\frac{dE}{dt}\right) = -\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right) \cdot \frac{\vec{f}}{\sqrt{1-u^2/c^2}}$$

$$\therefore \qquad \frac{dE}{dt} = \vec{u} \cdot \vec{f}$$

Hence the rate of change of energy of a particle is equal to the rate at which the force does work on the particle.

Example

A photon of energy E collides with a stationary electron of rest mass m_0. As a result of the collision, the photon energy is reduced to $E' < E$ and the photon is observed to be deflected through an angle θ. Using appropriate four-vectors, show that the increase in photon wavelength is given by

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta).$$

Solution

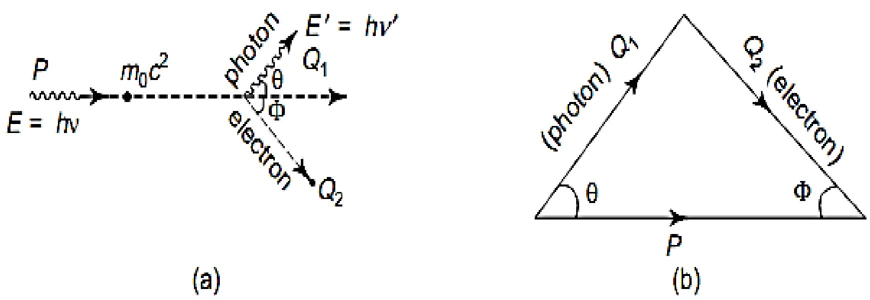


Fig. 8.2 For Illustrative Example 5

Let $P = \left[\frac{E}{c}\vec{a}_1, i\left(\frac{E + m_0c^2}{c}\right)\right]$ be the four-momentum of the system before collision. Let $Q_1 = \left[\frac{E'}{c}\vec{a}_2, \frac{iE'}{c}\right]$ be the four-momentum of the photon deflected through angle θ. Note that $\vec{a}_1$ is a unit vector in the direction of the incident photon and $\vec{a}_2$ is a unit vector in the direction of the photon deflected through angle θ, i.e. $\vec{a}_1 \cdot \vec{a}_2 = \cos\theta$. Let $Q_2 = \left[p_e\vec{a}_3, \frac{iE_e}{c}\right]$ be the four-momentum of the recoiling electron.

Then $P = Q_1 + Q_2$. This is shown in Fig. 8.2(b). Since $Q_2 = P - Q_1$

$$Q_2^2 = P^2 + Q_1^2 - 2PQ_1$$

$$\therefore \left(p_e\vec{a}_3, \frac{iE_e}{c}\right)^2 = \left[\frac{E}{c}\vec{a}_1, i\left(\frac{E + m_oc^2}{c}\right)\right]^2 + \left(\frac{E'}{c}\vec{a}_2, \frac{iE'}{c}\right)^2 - 2\left[\frac{E}{c}\vec{a}_1, i\left(\frac{E + m_oc^2}{c}\right)\right]\cdot\left(\frac{E'}{c}\vec{a}_2, \frac{iE'}{c}\right)$$

$$\therefore \quad \left(p_e^2 - \frac{E_e^2}{c^2}\right) = \frac{E^2}{c^2} - \left(\frac{E^2}{c^2} + m_0^2 c^2 + 2Em_0\right) + \left(\frac{E'^2}{c^2} - \frac{E'^2}{c^2}\right) + \frac{2EE'}{c^2} + 2m_0 E' - \frac{2EE'}{c^2}\cos\theta$$

Multiplying throughout by c^2 and using

$$(p_e^2 c^2 - E_e^2) = -m_0^2 c^4 \text{ we get}$$

$$\therefore \quad -m_0^2 c^4 = -m_0^2 c^4 - 2Em_0c^2 - 2EE' \cos\theta + 2EE' + 2m_0c^2E'$$

$$\therefore \quad EE'(1 - \cos\theta) = m_0c^2(E - E')$$

$$\therefore \quad E - E' = \frac{EE'(1-\cos\theta)}{m_0 c^2}$$

Let $E = \frac{hc}{\lambda}$ and $E' = \frac{hc}{\lambda'}$; λ and λ' being the wavelengths of the incident and scattered photon respectively. Then

$$hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h^2c^2(1-\cos\theta)}{\lambda\lambda'(m_0c^2)}$$

$$\therefore \quad \frac{\lambda' - \lambda}{\lambda\lambda'} = \frac{h}{m_0 c}\frac{(1-\cos\theta)}{\lambda\lambda'}$$

$\therefore$ Increase in wavelength $\lambda' - \lambda = \Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta)$

Remember

$$E'^2 - P'^2c^2 = E'^2 - \left(P_x'^2 + P_y'^2 + P_z'^2\right)c^2$$

$$= \Gamma^2 (E^2 - 2EvP_x + v^2P_x^2) - c^2\left[\Gamma^2\left(P_x^2 - 2\frac{P_x vE}{c^2} + \frac{v^2E^2}{c^4}\right) + P_y^2 + P_z^2\right]$$

$$= \Gamma^2E^2\left(1 - \frac{v^2}{c^2}\right) - c^2\left[\Gamma^2P_x^2\left(1 - \frac{v^2}{c^2}\right) + P_y^2 + P_z^2\right]$$

$$= E^2 - c^2\left(P_x^2 + P_y^2 + P_z^2\right) = E^2 - P^2c^2$$

Thus $\quad E'^2 - P'^2c^2 = E^2 - P^2c^2 \quad \ldots (7)$

MASS ENERGY RELATIONSHIP IN TERMS OF STR (E = T + E_0)

As $\vec{U}.\vec{F} = 0$

$$(\vec{u}, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}.\left(\vec{f}, i\dot{m}c\right)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 0 \Rightarrow \frac{1}{1-\frac{u^2}{c^2}}(\vec{u}, ic).\left(\vec{f}, i\dot{m}c\right) = 0$$

$$\Rightarrow \vec{u}.\vec{f} + i^2\dot{m}c^2 = 0 \Rightarrow \vec{u}.\vec{f} - \dot{m}c^2 = 0 \Rightarrow \vec{u}.\vec{f} = \dot{m}c^2 \Rightarrow \vec{u}.\frac{d\vec{P}}{dt} = \frac{dm}{dt}c^2$$

$$\Rightarrow \int \vec{u}.\frac{d\vec{P}}{dt}dt = \int \frac{dm}{dt}c^2\,dt \Rightarrow \int \vec{u}.\frac{d(m\vec{u})}{dt}dt = \int dmc^2$$

$$\Rightarrow m\int \vec{u}.\frac{d\vec{u}}{dt}dt = \int dmc^2 \Rightarrow m\int \vec{u}.d\vec{u} = \int dmc^2$$

$$\Rightarrow m\int udu = \int dmc^2$$

$$\Rightarrow m\frac{u^2}{2} = mc^2 + A \quad \ldots\ldots\ldots..(1)$$

$$\vec{u}.\vec{u} = u^2$$

$$\vec{u}.\frac{d\vec{u}}{dt} + \vec{u}.\frac{d\vec{u}}{dt} = 2u\frac{du}{dt}$$

$$2\vec{u}.\frac{d\vec{u}}{dt} = 2u\frac{du}{dt}$$

$$\vec{u}.\frac{d\vec{u}}{dt} = u\frac{du}{dt}$$

Initially using $u = 0$ and $m = m_0$

$(1) \Rightarrow 0 = m_0c^2 + A \Rightarrow A = -m_0c^2$

$$(1) \Rightarrow \frac{1}{2}mu^2 = mc^2 - m_0c^2 \quad \ldots\ldots\ldots..(2)$$

$\Rightarrow \boldsymbol{T = E - E_0}$

$T = \frac{1}{2}mu^2$ is kinetic energy.

$E = mc^2$ is Mass energy equation

$E_0 = m_0c^2$ is rest Mass energy equation

EQUIVALENCE OF MASS AND ENERGY (Or PROOF Of $E = mc^2$)

Suppose a force F acts on a body and as the result of this force, the body covers a distance dx in direction of force. The work done by this force is:

$$dW = \vec{F}.\vec{dx}$$

$$\Rightarrow dW = F\,dx \quad ---- \quad (1) \qquad \because \theta = 0°$$

By Work-Energy Theorem, the work done dW on a body result in increase of its kinetic energy dK:

$$dW = dK \quad ---- \quad (2)$$

Equating (1) and (2), we have:

$$dK = F\,dx \quad ---- \quad (3)$$

By Newton's 2nd Law of Motion, the time rate of change of linear momentum of body is equal applied force:

$$F = \frac{dp}{dt}$$

$$\Rightarrow F = \frac{d}{dt}(mv)$$

$$\Rightarrow F = m\,\frac{dv}{dt} + v\,\frac{dm}{dt}$$

Equation (3) becomes:

$$dK = \left(m\,\frac{dv}{dt} + v\,\frac{dm}{dt}\right)\,dx$$

$$\Rightarrow dK = m\,\frac{dv}{dt}\,dx \; + v\,\frac{dm}{dt}dx$$

$$\Rightarrow dK = m\,dv\frac{dx}{dt} \; + v\,dm\frac{dx}{dt}$$

$$\Rightarrow dK = mv\,dv \; + v^2\,dm \quad ---- \quad (4)$$

From relativistic mechanics,

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow m = m_o\left(1 - \frac{v^2}{c^2}\right)^{\frac{-1}{2}}$$

$$\Rightarrow dm = m_o\left(\frac{-1}{2}\right)\left(1 - \frac{v^2}{c^2}\right)^{\frac{-3}{2}}\left(\frac{-2v}{c^2}\right)dv$$

$$\Rightarrow dm = \frac{m_o}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}\ \frac{v}{c^2}dv$$

$$\Rightarrow dm = \frac{m_o}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}\ \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}\ \frac{v}{c^2}dv$$

$$\Rightarrow dm = m\ \frac{1}{\left(\frac{c^2 - v^2}{c^2}\right)}\ \frac{v}{c^2}dv$$

$$\Rightarrow dm = \frac{mv\,dv}{c^2 - v^2}$$

$$\Rightarrow mv\,dv = (c^2 - v^2)dm$$

Putting values in equation (4), we get:

$$dK = (c^2 - v^2)dm\ \ + v^2\,dm$$

$$\Rightarrow dK = (c^2 - v^2)dm\ \ + v^2\,dm$$

$$\Rightarrow dK = (c^2 - v^2 + v^2)dm$$

$$\Rightarrow dK = c^2\,dm$$

Integrating bother sides:

$$K = c^2\,m + A\ \ ----\ \ (5)$$

where A is constant of integration.

At $t = 0, m = m_o$, and K = 0, equation (5) $\Longrightarrow A = -m_o c^2$

Now the equation (5) becomes:

$$K = mc^2 - m_o c^2$$

$$\Longrightarrow K + m_o c^2 = mc^2 \quad ---- \quad (6)$$

This equation shows that when $K = 0$, the body still possess some energy equal to $m_o c^2$, called rest mass energy. Here $K + m_o c^2 = E$ is called total energy. Equation (6) takes the form:

$$E = mc^2$$

This equation is called Einstein's Energy-Mass Relationship.

RELATIVISTIC ENERGY /MOMENTUM ENERGY RELATION

From Einstein's Energy-Mass Relationship:

$$E = mc^2$$

$$\Longrightarrow E = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} c^2 \qquad \because m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Longrightarrow E^2 = \frac{m_o^2 c^4}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$\Longrightarrow \frac{E^2}{c^2} = \frac{m_o^2 c^2}{\left(1 - \frac{v^2}{c^2}\right)} \quad ---- \quad (1)$$

The linear momentum p a particle having mass m moving with velocity v is described as:

$$p = mv$$

$$\Longrightarrow p = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Longrightarrow p^2 = \frac{m_o^2 v^2}{\left(1 - \frac{v^2}{c^2}\right)} \quad ---- \quad (2)$$

Subtracting equation (1) and (2):

$$\frac{E^2}{c^2} - p^2 = \frac{m_o^2 c^2}{\left(1 - \frac{v^2}{c^2}\right)} - \frac{m_o^2 v^2}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$\Rightarrow \frac{E^2}{c^2} - p^2 = \frac{m_o^2}{\left(1 - \frac{v^2}{c^2}\right)}(c^2 - v^2)$$

$$\Rightarrow \frac{E^2}{c^2} - p^2 = \frac{m_o^2}{\left(\frac{c^2 - v^2}{c^2}\right)}(c^2 - v^2)$$

$$\Rightarrow \frac{E^2}{c^2} - p^2 = m_o^2 c^2$$

$$\Rightarrow E^2 = p^2 c^2 + m_o^2 c^4$$

$$\Rightarrow E = \sqrt{p^2 c^2 + m_o^2 c^4}$$

This is the expression of relativistic energy.

Question

At what fraction of the speed of light does a particle travel if its K.E. is twice its rest mass energy?

Solution: Rest Mass energy od the particle is $E_0 = m_0 c^2$ and $E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$ is the total energy. Then $E_{Total} = E_{K.E} + E_{Rest} \Rightarrow E_{Total} = 2E_{Rest} + E_{Rest} = 3E_{Rest}$

$$\Rightarrow \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = 3m_0 c^2 \Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{3} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{9} \Rightarrow \frac{v^2}{c^2} = \frac{8}{9} \Rightarrow \frac{v}{c} = \sqrt{\frac{8}{9}} \Rightarrow \frac{v}{c} = 0.943$$

MASS ENERGY RELATIONSHIP IN TERMS OF CLASSICAL MECHANICS (T = ½ m_0u^2)

In classical mechanics $v \ll c$

Since we have $T = E - E_0$ and $m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$

$\Rightarrow T = mc^2 - m_0c^2 \Rightarrow T = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}c^2 - m_0c^2 \Rightarrow T = m_0c^2\left(\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}} - 1\right)$

$\Rightarrow T = m_0c^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + neglecting - 1\right) \Rightarrow \boldsymbol{T = \frac{1}{2}m_0u^2}$ required

MASS ENERGY RELATIONSHIP IN TERMS OF $\boldsymbol{\gamma(u)}$

As $m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$ then $m = m_0\gamma(u)$ where $\gamma(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

Since we have $T = E - E_0$

$\Rightarrow T = mc^2 - m_0c^2 \Rightarrow T = m_0\gamma(u)c^2 - m_0c^2 \Rightarrow \boldsymbol{T = (\gamma(u) - 1)m_0c^2}$

Question

Show that $T = \frac{\gamma^2m_0u^2}{\gamma+1}$ where $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

Solution

Since we have $T = E - E_0$

$\Rightarrow T = mc^2 - m_0c^2 \Rightarrow T = m_0\gamma c^2 - m_0c^2 \Rightarrow T = (\gamma - 1)m_0c^2$ ....(1)

Now $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \Rightarrow \gamma^2 = \frac{1}{1-\frac{u^2}{c^2}} \Rightarrow \frac{1}{\gamma^2} = 1 - \frac{u^2}{c^2} \Rightarrow \frac{u^2}{c^2} = 1 - \frac{1}{\gamma^2} \Rightarrow c^2 = \frac{u^2\gamma^2}{\gamma^2-1}$

(1) $\Rightarrow T = (\gamma - 1)m_0\frac{u^2\gamma^2}{\gamma^2-1}) \Rightarrow T = (\gamma - 1)m_0\frac{u^2\gamma^2}{(\gamma-1)(\gamma+1)} \Rightarrow \boldsymbol{T = \frac{\gamma^2m_0u^2}{\gamma+1}}$

RELATIVITY OF MASS / MASS DILATION PROBLEM $\boldsymbol{m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}}$

As $\vec{U}.\vec{F} = 0$

$(\vec{u}, ic)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}.\left(\vec{f}, i\dot{m}c\right)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 0 \Rightarrow \frac{1}{1-\frac{u^2}{c^2}}(\vec{u}, ic).\left(\vec{f}, i\dot{m}c\right) = 0$

$\Rightarrow \vec{u}.\vec{f} + i^2\dot{m}c^2 = 0 \Rightarrow \vec{u}.\vec{f} - \dot{m}c^2 = 0 \Rightarrow \vec{u}.\vec{f} = \dot{m}c^2$

$\Rightarrow ufCos0° = \dot{m}c^2$ since $\vec{u} \parallel \vec{f} \Rightarrow \theta = 0°$

$\Rightarrow uf = \dot{m}c^2 \Rightarrow \dot{m}c^2 = uf$

$\Rightarrow \frac{dm}{dt}c^2 = u\frac{dP}{dt} \Rightarrow \frac{dm}{dt}c^2 = u\frac{d(mu)}{dt} \Rightarrow \frac{dm}{dt}c^2 = u^2\frac{dm}{dt} \Rightarrow 2m\frac{dm}{dt}c^2 = 2mu^2\frac{dm}{dt}$

$\Rightarrow \frac{d}{dt}(m^2c^2) = 2u^2m\frac{dm}{dt} \Rightarrow \int \frac{d}{dt}(m^2c^2)\,dt = 2u^2\int m\frac{dm}{dt}dt$ where u is constant

$\Rightarrow \int d(m^2c^2) = 2u^2\int m\,dm \Rightarrow m^2c^2 = 2u^2\frac{m^2}{2} + A$

$\Rightarrow m^2c^2 = m^2u^2 + A$ …………..(1)

Initially using $u = 0$ and $m = m_0$

$(1) \Rightarrow m_0^2c^2 = 0 + A \Rightarrow A = m_0^2c^2$

$(1) \Rightarrow m^2c^2 = m^2u^2 + m_0^2c^2 \Rightarrow m^2c^2 - m^2u^2 = m_0^2c^2 \Rightarrow m^2(c^2 - u^2) = m_0^2c^2$

$\Rightarrow m^2 = \frac{m_0^2c^2}{c^2-u^2} \Rightarrow m^2 = \frac{m_0^2c^2}{c^2\left(1-\frac{u^2}{c^2}\right)} \Rightarrow m^2 = \frac{m_0^2}{\left(1-\frac{u^2}{c^2}\right)}$

Hence $\boldsymbol{m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}}$

Invariant Mass

A quantity is said to be invariant if it is the same in different inertial frames. Rest mass of a particle is an example of an invariant quantity.

Question

Find the ratio of mass of a particle in frame S' to S when the particle is moving with velocity $(0{,}25c, 0{,}0)$ where S' is moving with relative velocity 0.75c.

Solution

As we know that $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Case – I: When particle move from S' to S

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\left(\frac{0.25c}{c}\right)^2}} = \frac{1}{\sqrt{1-0.00625}} = \frac{1}{0.968} = 1.0328$$

Case – II: When S' is moving with relative velocity 0.75c

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\left(\frac{0.75c}{c}\right)^2}} = \frac{1}{\sqrt{1-0.5625}} = 1.512$$

Question

Calculate the rest mass of an electron.

Solution

Rest mass energy $= E = m_0c^2 = m_ec^2$

$$E = 9.1 \times 10^{-3}kg \times (3 \times 10^8 ms^{-1})^2$$

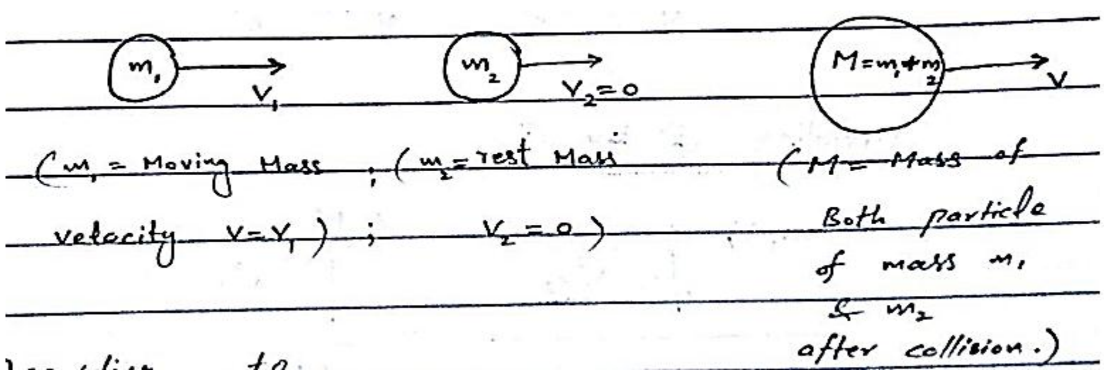
$$E = 8.199 \times 10^{-14}J \times \left(\frac{1eV}{1.602\times10^{-19}J}\right)$$

$$E = 5.118 \times 10^5 eV = 0.512MeV$$

Question

A particle of rest mass m_1 with speed v_1 collides with another particle of rest mass m_2 which is at rest. After an elastic collision the two particles combine to form a single particle of mass M. Then prove that $M^2 = m_1^2 + m_2^2 + \frac{2m_1m_2}{\sqrt{1-\frac{v_1^2}{c^2}}}$

Solution



According to the law of conservation of mass in STR

$$\frac{m_1}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2}{\sqrt{1-\frac{v_2^2}{c^2}}} = \frac{M}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{m_1}{\sqrt{1-\frac{v_1^2}{c^2}}} + m_2 = \frac{M}{\sqrt{1-\frac{v^2}{c^2}}}$$ ……….(1) since $v_2 = 0$ therfore $\sqrt{1-\frac{v_2^2}{c^2}} = 1$

Now according to the law of conservation of momentum

$$P_1 + P_2 = P$$

$$\frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2v_2}{\sqrt{1-\frac{v_2^2}{c^2}}} = \frac{Mv}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}} = \frac{Mv}{\sqrt{1-\frac{v^2}{c^2}}}$$ ……….(2) since $v_2 = 0$

Dividing (2) by (1) we have

$$\frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}} \div \frac{m_1}{\sqrt{1-\frac{v_1^2}{c^2}}} + m_2 = \frac{Mv}{\sqrt{1-\frac{v^2}{c^2}}} \div \frac{M}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}} \div \frac{m_1+m_2\sqrt{1-\frac{v_1^2}{c^2}}}{\sqrt{1-\frac{v_1^2}{c^2}}} = \frac{Mv}{\sqrt{1-\frac{v^2}{c^2}}} \div \frac{M}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{m_1v_1}{\sqrt{1-\frac{v_1^2}{c^2}}} \times \frac{\sqrt{1-\frac{v_1^2}{c^2}}}{m_1+m_2\sqrt{1-\frac{v_1^2}{c^2}}} = \frac{Mv}{\sqrt{1-\frac{v^2}{c^2}}} \times \frac{\sqrt{1-\frac{v^2}{c^2}}}{M}$$

$$v = \frac{m_1v_1}{m_1+m_2\sqrt{1-\frac{v_1^2}{c^2}}} \qquad \ldots\ldots\ldots.(3)$$

Now squaring equation (2)

$$\frac{m_1^2v_1^2}{1-\frac{v_1^2}{c^2}} = \frac{M^2v^2}{1-\frac{v^2}{c^2}} \Rightarrow M^2 = \frac{m_1^2v_1^2}{1-\frac{v_1^2}{c^2}} \times \frac{1-\frac{v^2}{c^2}}{v^2} \Rightarrow M^2 = \frac{m_1^2v_1^2}{1-\frac{v_1^2}{c^2}}\left(\frac{1}{v^2}-\frac{1}{c^2}\right)$$

$$\Rightarrow M^2 = \frac{m_1^2v_1^2}{1-\frac{v_1^2}{c^2}}\left(\left(\frac{m_1+m_2\sqrt{1-\frac{v_1^2}{c^2}}}{m_1v_1}\right)^2 - \frac{1}{c^2}\right) \qquad \text{using (3)}$$

$$\Rightarrow M^2 = \frac{m_1^2v_1^2}{1-\frac{v_1^2}{c^2}}\left(\frac{m_1^2+m_2^2\left(1-\frac{v_1^2}{c^2}\right)+2m_1m_2\sqrt{1-\frac{v_1^2}{c^2}}}{m_1^2v_1^2} - \frac{1}{c^2}\right)$$

$$\Rightarrow M^2 = \left(\frac{m_1^2+m_2^2\left(1-\frac{v_1^2}{c^2}\right)+2m_1m_2\sqrt{1-\frac{v_1^2}{c^2}}}{1-\frac{v_1^2}{c^2}} - \frac{m_1^2v_1^2}{1-\frac{v_1^2}{c^2}}\frac{1}{c^2}\right)$$

$$\Rightarrow M^2 = \frac{m_1^2}{1-\frac{v_1^2}{c^2}} + m_2^2 + 2m_1m_2\frac{\sqrt{1-\frac{v_1^2}{c^2}}}{1-\frac{v_1^2}{c^2}} - \frac{m_1^2}{1-\frac{v_1^2}{c^2}}\frac{v_1^2}{c^2}$$

$$\Rightarrow M^2 = \frac{m_1^2}{1-\frac{v_1^2}{c^2}}\left(1-\frac{v_1^2}{c^2}\right) + m_2^2 + 2m_1m_2\frac{\sqrt{1-\frac{v_1^2}{c^2}}}{1-\frac{v_1^2}{c^2}}$$

$$\Rightarrow \boldsymbol{M^2 = m_1^2 + m_2^2 + 2m_1m_2\frac{\sqrt{1-\frac{v_1^2}{c^2}}}{1-\frac{v_1^2}{c^2}}}$$

Question

Show that the acceleration in three dimension is not parallel to the force, under what condition $\vec{f}, \vec{a}, \vec{v}$ are parallel?

Solution

Consider $\vec{f} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$ where m is moving mass

$\Rightarrow \vec{f} = \left(\frac{\vec{f}.\vec{v}}{c^2}\right)\vec{v} + m\vec{a}$ ……………(1) since $\vec{f}.\vec{v} = \dot{m}c^2 \Rightarrow \frac{\vec{f}.\vec{v}}{c^2} = \frac{dm}{dt}$

This shows that the acceleration in three dimension is not parallel to the force.

Under the following conditions $\vec{f}, \vec{a}, \vec{v}$ are parallel;

Condition – I: If $\vec{f}$ and $\vec{v}$ are perpendicular then

$\vec{f}.\vec{v} = 0$

$\Rightarrow \vec{f} = m\vec{a} \Rightarrow \vec{f} \parallel \vec{a}$

Condition – II: If $\vec{f} = f\hat{n}, \vec{a} = a\hat{n}$ and $\vec{v} = v\hat{n}$ then

$\vec{f}.\vec{v} = 0$

$(1) \Rightarrow \vec{f} = \left(\frac{\vec{f}.\vec{v}}{c^2}\right)\vec{v} + m\vec{a} \Rightarrow m\vec{a} = \vec{f} - \left(\frac{\vec{f}.\vec{v}}{c^2}\right)\vec{v} \Rightarrow ma\hat{n} = f\hat{n} - \left(\frac{\vec{f}.\vec{v}}{c^2}\right)v\hat{n}$

$\Rightarrow a\hat{n} = \frac{1}{m}\left[f\hat{n} - \left(\frac{\vec{f}.\vec{v}}{c^2}\right)v\right]\hat{n}$

$\Rightarrow \vec{a} = (Scalar)\hat{n}$

This shows that $\vec{a}$ is in the direction of $\vec{f}$ and $\vec{v}$.

Thus $\vec{f}, \vec{a}, \vec{v}$ are parallel if they are in the same direction. i.e. Straight line.

Question

If $\vec{f}, \vec{a}, \vec{v}$ are parallel and perpendicular then what is the value of moving mass?

Solution Consider $\vec{f} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v})$

$\vec{f} = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$ ……………(1) where m is moving mass

$$\vec{v}.\vec{v} = v^2$$

$$2\vec{v}.\frac{d\vec{v}}{dt} = 2v\frac{dv}{dt} \Rightarrow \vec{v}.\frac{d\vec{v}}{dt} = v\frac{dv}{dt}$$

Since $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\Rightarrow \frac{dm}{dt} = \frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{m_0}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}} v\frac{dv}{dt} \Rightarrow \frac{dm}{dt} = \frac{m_0}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left(\vec{v}.\frac{d\vec{v}}{dt}\right)$$

$\Rightarrow \frac{dm}{dt} = \frac{m_0}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}(\vec{v}.\vec{a})$ ……………(2) where $\vec{a} = \frac{d\vec{v}}{dt}$

$(1) \Rightarrow \vec{f} = \frac{m_0}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}(\vec{v}.\vec{a})\vec{v} + \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\vec{a}$ ……………(A)

Condition – I: If $\vec{a}$ is parallel to $\vec{f}$ and $\vec{v}$. i.e. $\vec{f} = f\hat{n}, \vec{a} = a\hat{n}$ and $\vec{v} = v\hat{n}$ then

$$(A) \Rightarrow f\hat{n} = \frac{m_0}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}(v\hat{n}.a\hat{n})v\hat{n} + \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}a\hat{n}$$

$\Rightarrow f = \frac{m_0}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}(v.a)(\hat{n}.\hat{n})v + \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}a$ ÷ing by $\hat{n}$

$$\Rightarrow f = \frac{m_0}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}(v.a)v + \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}a \Rightarrow f = \frac{m_0 a}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}\right] \Rightarrow f = \frac{m_0 a}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$\Rightarrow \vec{a} = m_L\vec{a}$ where $m_L = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$ is called longitudinal mass.

Condition – II: If $\vec{a}$ is parallel to $\vec{f}$ but $\vec{a}$ and $\vec{f}$ are perpendicular to $\vec{v}$.

i.e. $\vec{v}.\vec{f} = 0$ and $\vec{v}.\vec{a} = 0$ then

$$(A) \Rightarrow \vec{f} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\vec{a}$$

$\Rightarrow \vec{f} = m_T\vec{a}$ where $m_T = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ is called transverse mass.

Question

Discuss the motion of particle with variable mass.

Or show that the quantity $c^2\left(1-\frac{u^2}{c^2}\right)\frac{dm_0}{d\tau}$ be the energy taken from the external source. m_0 is variable mass of the moving particle with velocity u.

Solution

As mass varies due to external source, so that the 4 – force acting on the particle is

$$\vec{F} = \frac{d\vec{P}}{d\tau} = \frac{d}{d\tau}\left(m_0\vec{U}\right)$$

$$\Rightarrow \vec{F} = \frac{dm_0}{d\tau}\vec{U} + m_0\frac{d\vec{U}}{d\tau} \Rightarrow \vec{F}.\vec{U} = \frac{dm_0}{d\tau}\vec{U}.\vec{U} + m_0\vec{U}.\frac{d\vec{U}}{d\tau} \Rightarrow \vec{F}.\vec{U} = \frac{dm_0}{d\tau}\vec{U}.\vec{U} + m_0\vec{U}.\frac{d\vec{U}}{d\tau}$$

$\Rightarrow \frac{1}{\left(1-\frac{u^2}{c^2}\right)}\left(\vec{f}.\vec{u} - \dot{m}c^2\right) = -\frac{dm_0}{d\tau}c^2$ Since $\boldsymbol{\vec{F}.\vec{U} = \frac{1}{\left(1-\frac{u^2}{c^2}\right)}\left(\vec{f}.\vec{u} - \dot{m}c^2\right), \vec{U}.\vec{U} = -c^2, \vec{U}.\frac{d\vec{U}}{d\tau} = 0}$

$$\Rightarrow \vec{f}.\vec{u} - \dot{m}c^2 = -\frac{dm_0}{d\tau}c^2\left(1-\frac{u^2}{c^2}\right)$$

$$\Rightarrow \dot{m}c^2 = \vec{f}.\vec{u} + c^2\left(1-\frac{u^2}{c^2}\right)\frac{dm_0}{d\tau} \qquad \ldots\ldots\ldots\ldots..(1)$$

Since $E = mc^2$ then $\frac{dE}{dt} = \dot{m}c^2$ also $W = \vec{f}.\vec{r}$ then $\frac{dW}{dt} = \vec{f}.\vec{u}$

$$(1) \Rightarrow \frac{dE}{dt} = \frac{dW}{dt} + c^2\left(1-\frac{u^2}{c^2}\right)\frac{dm_0}{d\tau}$$

This expression shows the rate of increase in energy due to external source.

CONSERVATION OF ENERGY

We come across two types of forces in our daily life:

- Conservative forces
- Non-conservative forces

Conservative Forces

If the work done by the force on the body depends upon the initial and final locations and is independent of path taken by the body between the two points, then such a force is a conservative force.

Or

If the net work done by the force on the body along the close path is zero, then such a force is called conservative force.

Conservative forces are also distinguished by the ability to store energy only due configuration of the system. This stored energy is called potential energy.

Examples of Conservative Forces

The Spring Force

The Force of Gravity

Example 1 For an extremely relativistic particle of rest energy $E_0 = m_0c^2$, show that the momentum p is given by $pc = E\left[1 - \frac{1}{2}\left(\frac{E_0}{E}\right)^2\right]$ to a good approximation.

Solution

$$E^2 = p^2c^2 + m_0^2c^4 = p^2c^2 + E_0^2$$

$$\therefore \quad p^2c^2 = E^2 - E_0^2 \text{ or } pc = (E^2 - E_0^2)^{1/2}$$

$$= E\left[1 - \left(\frac{E_0}{E}\right)^2\right]^{1/2} \approx E\left[1 - \frac{1}{2}\left(\frac{E_0}{E}\right)^2\right] \text{ when } \frac{E_0}{E} \ll 1.$$

Example 2 At what fraction of the speed of light does a particle travel if its kinetic energy is twice its rest mass energy?

Solution

$$\text{K.E.} = (m - m_0)c^2 = m_0c^2\,(\Gamma - 1) = 2\,m_0c^2$$

$$\text{Or} \quad \Gamma = 3 \text{ or } \frac{1}{1-\beta^2} = 9$$

$$\therefore \quad 1 = 9 - 9\beta^2 \text{ or } 9\beta^2 = 8 \text{ or } \beta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

$$\therefore \quad \beta = \frac{2(1.414)}{3} = \frac{2.828}{3} = 0.943.$$

Example 3 Calculate the momentum of a neutron (rest mass 940 MeV) whose kinetic energy is 200 MeV.

Solution

$$E^2 = (m_0c^2 + \text{K.E.})^2 = p^2c^2 + m_0^2c^4$$

$$\therefore \quad p^2c^2 = (m_0c^2 + \text{K.E.})^2 - m_0^2c^4$$

$$= (940 + 200)^2 - (940)^2$$

$$= (1140)^2 - (940)^2 = (2080)\,(200) = 416000$$

$$\therefore \quad pc = (416000)^{1/2} = 644.9 \text{ MeV}.$$

$$\therefore \quad p = 644.9 \text{ MeV/C}.$$

Example 4 What is the ratio of the relativistic mass to the rest mass for (a) an electron (b) a proton when it is accelerated from rest through a pd. of 15 megavolts. Take m_e = 0.5 MeV, m_p = 1000 MeV.

Solution

By definition of electron volt, the kinetic energy of each particle is 15 MeV.

$$\frac{m}{m_0} = \frac{mc^2}{m_0c^2} = \frac{m_0c^2 + \text{K.E.}}{m_0c^2}$$

(a) $\frac{m}{m_0} = \frac{0.5+15}{0.5} = 31$ for electron

(b) $\frac{m}{m_0} = \frac{1000+15.}{1000} = 1.015$ for proton.

Example 5 Prove that the velocity of a particle can be written as $\vec{v} = \frac{c^2}{E}\vec{p}$ and its magnitude as $v = \frac{dE}{dp}$.

Solution

$$\vec{p} = m\vec{v} \text{ or } \vec{v} = \frac{\vec{p}}{m} = \frac{\vec{p}c^2}{mc^2} = \frac{c^2}{E}\vec{p}$$

Since $E^2 = p^2c^2 + m_0^2c^4$

$$2E\frac{dE}{dp} = 2pc^2 \text{ or } \frac{dE}{dp} = \frac{pc^2}{E} = \text{magnitude of } v.$$

Example 6 Calculate the speed of an electron (rest mass 0.5 MeV) that has been accelerated through a potential difference of 2×10^6 V (a) classically (b) relativistically. Calculate the electron mass in case (b).

(a) Kinetic energy = $\frac{1}{2}m_0v^2 = \frac{1}{2}m_0\beta^2c^2 = Ve$

$$\therefore \quad \beta^2 = \frac{2Ve}{m_0c^2} = \frac{(2)(2\times10^6)\,\text{eV}}{(0.5)10^6\,\text{eV}} = 8 \text{ or } \beta = 2\sqrt{2}$$

$$\therefore \quad v = \beta c = 2\sqrt{2}\,c = 2(1.414)\,3 \times 10^8$$

$$\therefore \quad v = 8.484 \times 10^8 \text{ m/s. (This is obviously impossible).}$$

(b) $T = m_0c^2\ (\Gamma - 1) = 2 \times 10^6$ eV = 2 MeV

$$\therefore \quad (\Gamma - 1) = \frac{T}{m_0 c^2} \quad \text{or} \quad \Gamma = 1 + \frac{T}{m_0 c^2}$$

$$= 1 + \frac{2}{0.5} = 5 \quad \text{or} \quad \frac{1}{(1-\beta^2)^{1/2}} = 5$$

$$\therefore \quad 1 - \beta^2 = \frac{1}{25} = 0.04$$

$$\therefore \quad \beta^2 = (1 - 0.04) \quad \text{or} \quad \beta = (1 - 0.04)^{1/2}$$

$$\therefore \quad \beta \approx 1 - \frac{1}{2}(0.04) = 0.98$$

$$\therefore \quad v = \beta c = (0.98)(3 \times 10^8) = 2.94 \times 10^8 \text{ m/s.}$$

Total electron energy = $E = mc^2 = m_0c^2 + T$

$$= (0.5 + 2) \text{ MeV} = 2.5 \text{ MeV} = 5\ m_0c^2 = mc^2$$

$$\therefore \quad m = 5\ m_0$$

Example 7 A particle has a total energy of 5 GeV and a momentum of 3 GeV/c in a certain frame of reference. (a) Find its energy in a frame in which its momentum is 4 GeV/c. (b) Calculate the rest mass of the particle.

Solution

(a) $m_0^2 c^4 = E^2 - p^2c^2 = \text{constant}$

$$\therefore \quad E_2^2 - p_2^2 c^2 = E_1^2 - p_1^2 c^2$$

$$\therefore \quad E_2^2 = p_2^2 c^2 + E_1^2 - p_1^2 c^2 = (4)^2 + (5)^2 - (3)^2 = 32$$

$$\therefore \quad E_2 = \sqrt{32} \text{ GeV} = 5.656 \text{ GeV.}$$

(b) $m_0^2 c^4 = E_1^2 - p_1^2 c^2 = (5)^2 - (3)^2 = 16$

$$\therefore \quad m_0c^2 = 4 \text{ GeV} = 4000 \text{ MeV}$$

$$\therefore \quad m_0 = \frac{4000}{931.5} = 4.3 \text{ u.}$$

Example 8 An electron moves in a circle of 0.4 m diameter in a uniform magnetic field of 0.03 T. Obtain the speed and kinetic energy of the electron. Take $m_0c^2 = 0.511$ MeV.

Solution

$$p = \text{BeR or } pc = \text{BeRc (J)} = \text{BRc (eV)}$$

$$pc = (3 \times 10^{-2} \times 0.2 \times 3 \times 10^{8}) = 1.8 \times 10^{6} \text{ eV} = 1.8 \text{ MeV.}$$

$$\therefore \quad E^2 = (m_0c^2 + T)^2 = p^2c^2 + m_0^2c^4$$

$$\therefore \quad (0.511 + T)^2 = (1.8)^2 + (0.511)^2$$

Solving this; kinetic energy $T = 1.360$ MeV.

Total energy $E = (1.360 + 0.511) = 1.871 \text{ MeV} = \Gamma m_0c^2$

$$\therefore \quad \Gamma = \frac{1.871}{0.511} = 3.66$$

$$\therefore \quad \Gamma^2 = \frac{1}{1-\beta^2} = (3.66)^2$$

Solving this, $\beta = v/c = 0.9618$

Or $v = 0.9618$ C.

Example 9 Calculate the radius of curvature of a proton of velocity 0.1 c in a magnetic field of 1 Wb/m^2. (For proton $m_0c^2 = 1000$ MeV).

Solution

$$R = \frac{mv}{Be} = \frac{\Gamma m_0 v}{Be} = \frac{\Gamma m_0 c^2 v}{Bec^2}$$

$$\Gamma^2 = \frac{1}{1-\beta^2} = \frac{1}{1-0.01} \quad \text{or} \quad \Gamma = 1 + \frac{1}{2}(0.01) = 1.005$$

$$m_0c^2 = 1000 \text{ MeV} = 1000 \times 10^6 \text{ eV} = 10^9 \text{ eV} = 10^9 \times 1.6 \times 10^{-19} \text{ J}$$

$$\therefore \quad m_0c^2 = 1.6 \times 10^{-10} \text{ J.}$$

Substituting these values

$$R = (1.005)\ \frac{1.6\times10^{-10}\times0.1\times3\times10^{8}}{(1)(1.6\times10^{-19})(9\times10^{16})} = (1.005)\ \frac{1.6\times3\times10^{-3}}{1.6\times9\times10^{-3}} = 0.335 \text{ m.}$$

Example 10 Show that the speed v of an extremely relativistic particle differs from the speed of light c by $\Delta v = c - v \approx \frac{c}{2}\left(\frac{m_0 c^2}{E}\right)^2$. Find Δv for an electron of kinetic energy (a) 100 MeV, (b) 25 GeV. Take $m_0 c^2 = 0.5$ MeV.

Solution

$$E^2 = m_0^2 c^4 + p^2 c^2 = m_0^2 c^4 + \frac{m_0^2 v^2 c^2}{1 - v^2/c^2} = m_0^2 c^4 + E^2 v^2/c^2$$

$$\therefore \quad E^2 (1 - v^2/c^2) = m_0^2 c^4$$

$$\therefore \quad E^2 (c^2 - v^2) = m_0^2 c^6 = E^2 (c + v)(c - v)$$

$$\therefore \quad \Delta v = c - v = \frac{m_0^2 c^6}{E^2 (c + v)} = \left(\frac{m_0 c^2}{E}\right)^2 \frac{c^2}{c + v}$$

$$\approx \frac{c}{2}\left(\frac{m_0 c^2}{E}\right)^2 \quad \text{when } v \approx c.$$

(a) $$\Delta v = \frac{c}{2}\left(\frac{m_0 c^2}{E}\right)^2 = \frac{c}{2}\left(\frac{m_0 c^2}{T + m_0 c^2}\right) = \frac{3 \times 10^8}{2}\left(\frac{0.5}{100.5}\right)^2$$

$$\approx \frac{3 \times 10^8}{2}\left(\frac{0.5}{100}\right)^2 = 1.5 \times 10^8 (5 \times 10^{-3})^2$$

$$= 3.75 \times 10^3 \text{ m/s.}$$

(b) $$\Delta v = \frac{3 \times 10^8}{2}\left(\frac{0.5}{25 \times 10^3 + 0.5}\right)^2 = 1.5 \times 10^8 \left(\frac{0.5}{25 \times 10^3}\right)^2$$

$$= 1.5 \times 10^8 (2 \times 10^{-5})^2 = 6 \times 10^{-2} = 6 \text{ cm/s.}$$

Example 11 A neutral pion moving with velocity v decays into two photons; one photon of energy E_1 travelling in the original direction of the parent pion and the other photon of energy E_2 in the exactly opposite direction. If $E_1 = 2E_2$, find v.

Solution

From conservation of energy principle,

$$\Gamma m_0 c^2 = E_1 + E_2 = 2E_2 + E_2 = 3E_2 \quad \text{... (1)}$$

From conservation of linear momentum,

$$\Gamma m_0 v = \frac{E_1}{c} - \frac{E_2}{c} = \frac{2E_2}{c} - \frac{E_2}{c} = \frac{E_2}{c} \quad \text{... (2)}$$

Dividing Eqn. (2) by Eqn. (1) we get

$$\frac{v}{c^2} = \frac{E_2/c}{3E_2} = \frac{1}{3c}$$

$$\therefore \quad v = c^2/3c = \frac{c}{3} = 10^8 \text{ m/s.}$$

Example

Momentum of a particle is given to be m_0c. What is (i) its speed. (ii) its mass (iii) its kinetic energy?

Solution

(i) $m_0\Gamma v = m_0 c$ or $v = \dfrac{c}{\Gamma}$ or $v^2 = \dfrac{c^2}{\Gamma^2}$

$$\therefore \quad v^2 = c^2(1 - v^2/c^2) = c^2 - v^2 \quad \text{or} \quad 2v^2 = c^2 \quad \text{or} \quad v = \frac{c}{\sqrt{2}}$$

$$\therefore \quad v = \frac{c\sqrt{2}}{2} = \frac{c(1.414)}{2} = 0.707 \text{ C.}$$

(ii) Mass $m = m_0\Gamma = m_0 \dfrac{c}{v} = \dfrac{m_0 c\sqrt{2}}{c} = 1.414\ m_0$

(iii) Kinetic energy $T = (\Gamma - 1)\ m_0c^2 = (1.414 - 1)\ m_0c^2$
$= 0.414\ m_0c^2.$

Example

What is the radius of curvature of a 100 MeV electron in a magnetic field of 10000 Gauss? For electron $m_0c^2 = 0.51$ MeV.

Solution

$$B = 10000 \text{ G} = 1 \text{ T.}$$

$$P^2c^2 = E^2 - m_0^2c^4 = 100^2 - 0.51^2 \approx 100^2$$

Or

$$pc \approx E \approx 100 \text{ MeV}$$

$$R(m) = \frac{pc(\text{MeV})}{300\,\text{nB}} = \frac{100}{300(1)(1)} = 0.33 \text{ m.}$$

Example

Compute the radius of curvature of a proton of velocity 0.1 c in a magnetic field of 1T. Take $m_0c^2 = 10^3$MeV.

Solution

$$\Gamma = (1 - 0.01)^{-1/2} = 1 + \frac{1}{2}(0.01) = 1.005$$

$$R = \frac{pc}{300\,\text{nB}} = \frac{\Gamma m_0 vc}{300\,\text{nB}} = \frac{(1.005)\,m_0c^2(0.1)}{300\,\text{nB}}$$

$$= \frac{(1.005)(10^3)(0.1)}{300(1)(1)} = 0.335 \text{ m.}$$

Question

Show that it is impossible for a photon to transfer all its energy to a free electron.

Solution

According to the conservation of linear momentum

$\frac{h}{\lambda} = P_e$ …………(1)   where P_e is the momentum of electron after collision.

If K is K.E. of the electron after the collision and m_0 is its rest mass then conservation of energy is

$hf + m_0c^2 = m_0c^2 + K$

$hf = K$ or $K = \frac{hc}{\lambda}$

$\Rightarrow K = P_e c$ using (1)

Therefore the total energy of electron can be found using the relativistic expression

$E_e = \sqrt{(m_0c^2)^2 + (Pc)^2}$

$(m_0c^2 + K)^2 = (m_0c^2)^2 + (Pc)^2$

$(m_0c^2)^2 + K^2 + 2Km_0c^2 = (m_0c^2)^2 + (Pc)^2$

$K^2 + 2Km_0c^2 = (Pc)^2$

$P_e^2c^2 + 2P_e c m_0 c^2 = P_e^2 c^2$ since $K = P_e c$

$2P_e c m_0 c^2 = 0$

$P_e = 0$

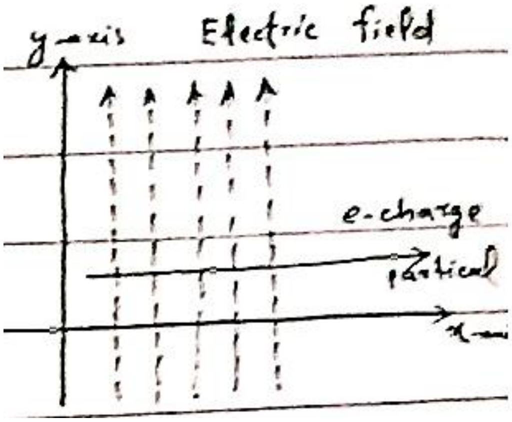
This is not possible because it corresponds to $\lambda = \infty$

Therefore it is proved that it is impossible for a photon to transfer all its energy to a free electron.

Question

Discuss the motion of a charged particle in a uniform transverse electric field.

Solution



Consider a charged particle 'e' moving along x – axis and electric field is along y – axis.

Now we have $\frac{d\vec{P}}{dt} = \vec{F}$

$\Rightarrow \frac{d}{dt}\left(P_x\hat{\imath} + P_y\hat{\jmath} + P_z\hat{k}\right) = e\varepsilon\hat{\jmath}$ where $\vec{F} =$ (charge)(Electric field along y – axis)

$\Rightarrow \left(\frac{dP_x}{dt}\hat{\imath} + \frac{dP_y}{dt}\hat{\jmath} + \frac{dP_z}{dt}\hat{k}\right) = e\varepsilon\hat{\jmath}$

$\Rightarrow \frac{dP_x}{dt} = 0,\ \frac{dP_y}{dt} = e\varepsilon,\ \frac{dP_z}{dt} = 0 \Rightarrow P_x = A,\ P_y = e\varepsilon t + B,\ P_z = D$

Initially $t = 0, P_x = P_0, P_y = 0,\ P_z = 0$ then $A = P_0, B = 0,\ D = 0$

$\Rightarrow P_x = P_0,\ P_y = e\varepsilon t,\ P_z = 0 \Rightarrow mv_x = P_0,\ mv_y = e\varepsilon,\ mv_z = 0$

$\Rightarrow v_x = \frac{P_0}{m},\ v_y = \frac{e\varepsilon t}{m},\ v_z = 0$

Since we know that $E = mc^2$ then $m = \frac{E}{c^2}$

$\Rightarrow v_x = \frac{P_0}{\frac{E}{c^2}},\ v_y = \frac{e\varepsilon t}{\frac{E}{c^2}},\ v_z = 0 \Rightarrow \frac{dx}{dt} = \frac{P_0c^2}{E},\ \frac{dy}{dt} = \frac{e\varepsilon t c^2}{E},\ \frac{dz}{dt} = 0$

As $E^2 = m_0^2c^4 + P^2c^2$

$\Rightarrow E^2 = m_0^2c^4 + (P_x^2 + P_y^2 + P_z^2)c^2$

$\Rightarrow E^2 = m_0^2c^4 + (P_0^2 + e^2\varepsilon^2t^2 + 0)c^2$

$\Rightarrow E^2 = m_0^2c^4 + P_0^2c^2 + e^2\varepsilon^2t^2c^2$

$\Rightarrow E^2 = E_0^2 + e^2\varepsilon^2t^2c^2$ since $E_0^2 = m_0^2c^4 + P_0^2c^2$

$\Rightarrow E = \sqrt{E_0^2 + e^2\varepsilon^2t^2c^2}$

$\Rightarrow E = e\varepsilon c\sqrt{\left(\frac{E_0}{e\varepsilon c}\right)^2 + t^2} \Rightarrow E = e\varepsilon c\sqrt{a^2 + t^2}$ with $a = \frac{E_0}{e\varepsilon c}$

Now For x :-

Consider

$$\frac{dx}{dt} = \frac{P_0 c^2}{ec\varepsilon\sqrt{a^2+t^2}}$$

$$\int dx = \frac{P_0 c}{e\varepsilon}\int\frac{1}{\sqrt{a^2+t^2}}\,dt \qquad \because \int\frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}(x/a)$$

$$x = \frac{P_0 c}{e\varepsilon}\sinh^{-1}(t/a) + c_1$$

Initially at $t=0$ $x=0$ we get $c_1 = 0$

then

$$x = \frac{P_0 c}{e\varepsilon}\sinh^{-1}\left(\frac{t}{(E_0/ec\varepsilon)}\right)$$

$$\boxed{x = \frac{P_0 c}{e\varepsilon}\sinh^{-1}\left(\frac{ec\varepsilon t}{E_0}\right)} \longrightarrow (2)$$

Now For y:

Consider
$$\frac{dy}{dt} = \frac{e\mathcal{E} t c^2}{e c \mathcal{E}\sqrt{(a)^2+t^2}}$$

$$\int dy = \frac{c}{2}\int (a^2+t^2)^{-1/2}(2t)\, dt$$

$$y = \frac{c}{2}\,\frac{(a^2+t^2)^{1/2}}{1/2} + C_2$$

Initialy $t=0$ $y=0$ then $C_2 = -ac$

then

$$y = c\sqrt{a^2+t^2} - ac$$

$$y = c\sqrt{\frac{E_0^2}{e^2c^2\mathcal{E}^2}+t^2} - \frac{E_0}{ec\mathcal{E}}c$$

$$y = c\,\frac{1}{ec\mathcal{E}}\sqrt{E_0^2+e^2c^2\mathcal{E}^2t^2} - \frac{E_0}{e\mathcal{E}}$$

$$\boxed{y = \frac{1}{e\mathcal{E}}\left[\left(E_0^2+e^2c^2\mathcal{E}^2t^2\right)^{1/2} - E_0\right]} \quad (3)$$

For equation of trajectory for charge particle we consider eq (2) & solve for t.

$$\frac{e\mathcal{E}x}{P_0c} = \sinh^{-1}\left(\frac{ec\mathcal{E}t}{E_0}\right) \Rightarrow t = \frac{E_0}{ec\mathcal{E}}\sinh\left(\frac{e\mathcal{E}x}{P_0c}\right)$$

using in eq (3)

$$y = \frac{1}{e\mathcal{E}}\left[\left(E_0^2 + e^2c^2\mathcal{E}^2\,\frac{E_0^2}{e^2c^2\mathcal{E}^2}\sinh^2\left(\frac{e\mathcal{E}x}{P_0c}\right)\right)^{1/2} - E_0\right]$$

$$y = \frac{E_0}{e\mathcal{E}}\left[\left(1+\sinh^2\left(\frac{e\mathcal{E}x}{P_0c}\right)\right)^{1/2} - 1\right]$$

$$y = \frac{E_0}{e\mathcal{E}}\left[\left\{\cosh^2\left(\frac{e\mathcal{E}x}{P_0c}\right)\right\}^{1/2} - 1\right]$$

$$\boxed{y = \frac{E_0}{e\mathcal{E}}\left[\cosh\left(\frac{e\mathcal{E}x}{P_0c}\right) - 1\right]}$$

Question

Discuss the motion of a simple particle under a constant force.

Solution: Consider a simple particle of mass "m" moving under a constant force

i.e $\vec{F} = \text{constant}$

Since $\frac{d\vec{p}}{dt} = \vec{F}$

On Integrating, we get

$$\vec{p} = \vec{F}t + c_1$$

Initially at $t=0$ & $p=0$ Then $c_1 = 0$

$$\Rightarrow p = Ft$$

$$\because p = mv$$

$$mv = Ft$$

Now converting it into the charge particle

$$\frac{m_0 v}{\sqrt{1 - v^2/c^2}} = Ft \qquad \left(\because m = \frac{m_0}{\sqrt{1 - v^2/c^2}}\right)$$

$$m_0 v = \sqrt{1 - v^2/c^2}\; Ft$$

Squaring on both sides

$$m_0^2 v^2 = \left(1 - v^2/c^2\right) F^2 t^2$$

$$m_0^2 v^2 = F^2 t^2 - \frac{v^2}{c^2} F^2 t^2$$

$$\left(m_0^2 + \frac{F^2}{c^2} t^2\right) v^2 = F^2 t^2$$

$$\Rightarrow v^2 = \frac{F^2 t^2}{m_0^2 + \frac{F^2}{c^2} t^2} \Rightarrow v = \frac{Ft}{\sqrt{m_0^2 + \frac{F^2}{c^2} t^2}}$$

$$V = \left(m_0^2 + \frac{F^2}{c^2} t^2\right)^{-1/2} F t$$

$$= m_0^{-1}\left(1 + \frac{F^2}{m_0^2}\frac{1}{c^2}t^2\right)^{-1/2} F t$$

$$V = \left(1 + \left(\frac{F}{m_0}\right)^2 \frac{1}{c^2} t^2\right)^{-1/2} \frac{F}{m_0} t \qquad \because F = m_0 a$$

$$\frac{dx}{dt} = \left(1 + \frac{a^2}{c^2} t^2\right)^{-1/2} a t$$

$$\int dx = \frac{c^2}{2a}\int \left(1 + \frac{a^2}{c^2} t^2\right)^{-1/2} \left(\frac{2a^2}{c^2} t\right) dt$$

$$x = \frac{c^2}{2a} \frac{\left(1 + \frac{a^2}{c^2} t^2\right)^{1/2}}{1/2} + C_2$$

$$x = \frac{c^2}{a}\left(1 + \frac{a^2}{c^2} t^2\right)^{1/2} + C_2$$

Initialy at $t = 0$, $x = 0$

$$\Rightarrow C_2 = -\frac{c^2}{a}$$

&

$$x = \frac{c^2}{a}\left(1 + \frac{a^2}{c^2} t^2\right)^{1/2} - \frac{c^2}{a} \quad \text{--- (1)}$$

For equation of trajectory

$$x+\frac{c^2}{a^2}=\frac{c^2}{a}\left(1+\frac{a^2}{c^2}t^2\right)^{1/2}$$

Squaring

$$\Rightarrow\left(x+\frac{c^2}{a^2}\right)^2=\frac{c^4}{a^2}\left(1+\frac{a^2}{c^2}t^2\right)$$

$$\left(x+\frac{c^2}{a^2}\right)^2=\frac{c^4}{a^2}+c^2t^2$$

$$\left(x+\frac{c^2}{a^2}\right)^2-c^2t^2=\frac{c^4}{a^2}$$

$$\frac{\left(x+c^2/a^2\right)^2}{c^4/a^2}-\frac{c^2t^2}{c^4/a^2}=0\Rightarrow\boxed{\frac{\left(x+\frac{c^2}{a^2}\right)^2}{\left(c^2/a\right)^2}-\frac{(t-0)^2}{(c/a)^2}=1}$$

This equation represents a hyperbola with centre $\left(-\frac{c^2}{a^2},0\right)$ & transverse axis $\frac{c^2}{a},\frac{c}{a}$.

INTERVAL BETWEEN EVENTS

Consider interval of two events (ds) which is the distance between two points is given as; $(ds)^2 = dx^2 + dy^2 + dz^2 + (icdt)^2$. Naturally ds is an invariant quantity. It can be imaginary, zero or real depending on $(ds)^2$ being negative, zero or positive.

TIMELIKE INTERVALS

When $dx^2 + dy^2 + dz^2 < c^2dt^2$, i.e. $(ds)^2$ is negative. The interval ds is then imaginary. It is called **timelike interval**. In case of timelike interval, the two events are separated by such a length of time (dt), that the distance (cdt) which a ray would travel in time dt is greater than the space distance $\sqrt{dx^2 + dy^2 + dz^2}$ between the two events. It is therefore possible to find an inertial frame S' such as a train moving with a speed $v < c$ with respect to frame S such that the two events appear to occur at the same place to the observer in S'.

NULL INTERVALS

When $dx^2 + dy^2 + dz^2 = c^2dt^2$, i.e. $(ds)^2$ is zero. Interval ds is now zero or 'null' and hence ds is called **null interval**.

In case of null interval, the two events are so separated in space and time that a ray of light starting from the place and time of the first event can reach the second event. Null interval is therefore also called **lightlike**.

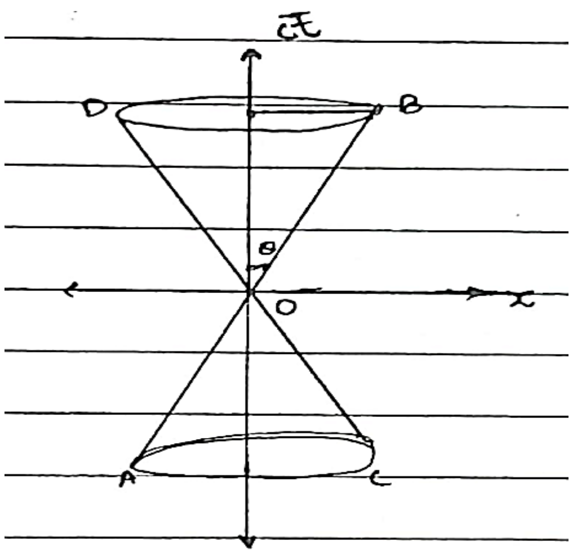
SPACELIKE INTERVALS

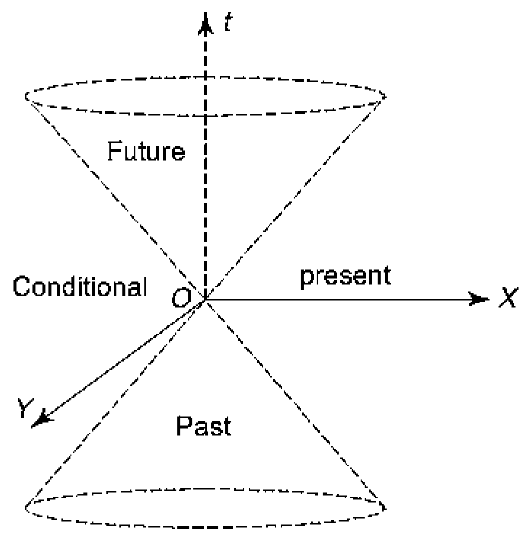
When $dx^2 + dy^2 + dz^2 > c^2dt^2$, i.e. $(ds)^2$is positive and ds is real. The interval ds is now called **spacelike interval.** In case of spacelike interval, the distance in space is so large that even a ray of light cannot cover it in the available time dt. Only an observer or a particle or a signal travelling faster than light would have covered the distance in the available time. Any pair of events which are simultaneous $(dt = 0)$ in an inertial frame such as S are separated by a spacelike interval. The interval is then spacelike for all inertial observers. Conversely if a pair of events is separated by a spacelike interval in one inertial frame, it is possible to find another inertial frame in which the events appear simultaneous.

LIGHT CONE / NULL CONE

A light cone is a surface describing the temporal evaluation of a flash of light in Minkowski Spacetime. It is a path that a flash of light, emanating from a single event and travelling in all directions, would take through spacetime.

Explanation: Consider an event at origin O, horizontal axis is x – axis and vertical axis is $c\tau$ – axis. Consider a light pulse at origin that expands in all directions with speed 'c' makes an angle θ with $c\tau$ −axis.





We have $tan\theta = \frac{x}{c\tau}$

$\Rightarrow tan\theta = \frac{c\tau}{c\tau} \Rightarrow tan\theta = 1 \Rightarrow \theta = 45°$ using $s = vt, x = c\tau$

Here $\theta = 45°$ is maximum angle of divergent of light. The lines AOB and COD are known as World lines.

All the events in sector ADC are called past events and in sector BOD are called future events.

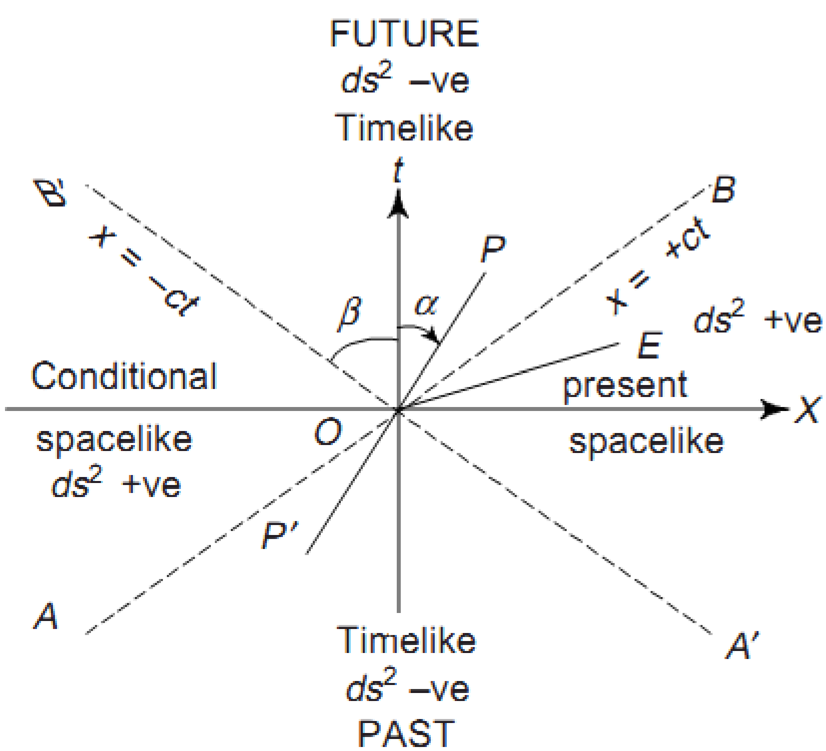
As $x^2 + y^2 + z^2 - c^2\tau^2 = 0$ represent hyper cone. So we use it as equation of light cone. Surface of the cone is made up of events which are lightlike $(ds = 0)$ with respect to the event O. Hence the name light cone or null cone, because a light signal transmitted from origin, will be along world line AOB and DOC. In which spacelike, timelike null vectors satisfied.

CAUSALITY

Two physical events are causally related if the earlier event causes the later event. If two events are causally related in one frame, they must be causally related in all inertial frames. Interval between two causally related events must be timelike in all inertial frames. Causes always precede their effects in every inertial frame. This is the **principle of Causality.**

Explanation

Two physical events are said to be causally related if one event is the effect of another preceding event which causes it. The necessary condition for two events to be causally related is that the event being caused must occur at a later time than the event which causes it. (This does not mean that if one event is later in time than another, it is necessarily caused by the earlier event). Now if two events are causally related in one inertial frame, then by the Principle of Relativity they must be causally related in all inertial frames. Otherwise we could distinguish between inertial frames by whether two events are causally related or not.



Suppose events $E_1(0,0,0,0)$ and $E_2(0,0,0,t>0)$ in an inertial frame are causally related. Then $(ds)^2 = -c^2t^2 < 0$ and the interval between then is timelike. Since ds has the same value in all inertial frames, it follows that the interval between two causally related events must be timelike in all inertial frames.

If we suppress the y and z coordinates we may refer to the $x - t$ diagram of Figure. We see that event O can be the cause of event P with a velocity $v < c$. Similarly P' can be the cause of event O. Note that event O can only be the cause of events which are in its future. Also O can only be the effect of events which are in its past. Thus in all inertial frames causes always precede their effects. This is the principle of causality.

The event E in Figure is spacelike with respect to event O and the two events cannot be causally related. In fact it is impossible to make a frame-independent statement such as 'earlier' or 'later' about the pair of events O and E.

If a particle or a signal could travel faster than light, then it would be possible to reach E from O. Such a particle or signal could start from O and cause the effect E (event) at a later time. This is impossible because c is the limit for signal or particle velocity.

Hence the principle of causality is sometimes stated in the form:

"**Information or signal cannot travel faster than light**."

Binding Energy

The energy required to separate a particle from a system of particles or to disperse all the particles of the system. It is expressed in kJ/mol or MeV. For example binding energy of the Helium nucleus ${}^4_2He \rightarrow {}^2_1H + {}^2_1H$

Nuclear Binding Energy

The energy required to separate an atomic nucleus completely into its constituent protons and neutrons. Or, equivalently, then energy that would be liberated by combining individual protons and neutrons into a single nucleus.

Binding Energy of Nucleus

Mass of a nucleus is always less than the sum of the masses of its constituent nucleons i.e., protons and neutrons. Neutrons and protons within the nucleus are held together by attractive forces. Suppose W is the work done to break a nucleus into its constituent nucleons devoid of any kinetic or potential energy. Then from the conservation of mass energy;

(mass of nucleus M) c^2 + W = (sum of the masses of the constituent nucleons) c^2. Here W the work done or energy supplied in breaking the nucleus is expressed in Joules and the masses should be expressed in kilograms. However nuclear masses are usually expressed in atomic mass units. As an example we take a deuteron, the nucleus of deuterium ($^1H_2$) an isotope of hydrogen. Deuteron is made up of one proton and one neutron.

Mass of proton $\quad m_p = 1.00728$ u

Mass of neutron $\quad m_n = 1.00866$ u

$$m_p + m_n = 2.01594 \text{ u}$$

Mass of deuteron $\quad m_d = 2.01355$ u

Mass which has "disappeared" in the formation of deuteron is

$$\Delta m = (2.01594 \text{ u}) - (2.01355 \text{ u}) = 0.00239 \text{ u}$$

Since 1u is equivalent to 931.5 MeV, Δm is equivalent to (0.00239) (931.5) = 2.23 MeV. Experiments show that 2.23 MeV is indeed the energy required to break the deuteron into a proton and a neutron. W= 2.23 MeV. W is the binding energy (B.E.) of deuteron. B.E. of a nucleus is the energy which must be supplied to it in order to break it into its constituent nucleons. Another way of looking at it, is to say that the nucleons in the nucleus have a potential energy (P.E.) which is negative because of mutual attraction between nucleons. This negative potential energy reduces the mass of the nucleus by W/c^2. Note that (W/c^2) will be in kg only when W is expressed in J.

In the above case, PE of the nucleons is –2.23 MeV.

Vibrational Principle: According to this principle: nothing is at rest, everything moves, everything vibrates. It explain that matter, energy and even spirit are simply varying rates of vibration.

MASS DEFECT:

The difference between the mass of a nucles and the sum of masses of Neutron and proton (N+P) is called mass Defect.

it is denoted by Δm.

Calculate the binding energy of helium Nucleus.

Solution:

For this we have

$${}^{4}_{1}He \longrightarrow {}^{2}_{1}H + {}^{2}_{1}H$$

$\Rightarrow$ ${}^{4}_{1}He$ consist 2 protons + 2 neutrons.

Mass of helium nucleus = Mass (2 protons + 2 neutrons)

$= 2(1.007646) + 2(1.009034)$

$= 4.03336$ amu

Actual Mass of helium neucles = 4.003874 amu

Now

$$\text{Mass defect} = \Delta m = \begin{pmatrix}\text{Mass of} \\ \text{helium neclens}\end{pmatrix} - \begin{pmatrix}\text{Actual Mass of} \\ \text{Helium necleus}\end{pmatrix}$$

$= 4.03336 - 4.003874$

$\Delta m = 0.0295$ amu

$\because$ 1 amu = 1.6606×10^{-27} kg

Then

$$\Delta m = 0.0295 \times 1.6606 \times 10^{-27} \text{ kg}$$

$$\Delta m = 0.49 \times 10^{-28} \text{ kg}$$

For Binding Energy we have

$$\Delta E = \Delta m c^2$$

$$= 0.49 \times 10^{-28} \text{ kg} (3 \times 10^{8} \text{ ms}^{-1})^2$$

$$= 4.41 \times 10^{-2} \text{ kg m}^2 \text{s}^{-2}$$

$$\Delta E = 4.41 \times 10^{-2} \text{ J}$$

$$\because 1 \text{ Mev} = 1.602 \times 10^{-12} \text{ J}$$

$$\Rightarrow 1 \text{ J} = \frac{1}{1.602 \times 10^{-12}} \text{ Mev}$$

&

$$\Delta E = 4.41 \times 10^{-2} \times \frac{1}{1.602 \times 10^{-12}} \text{ Mev}$$

$$\Delta E = 27.53 \text{ Mev}$$

is a binding energy of ${}^{4}_{2}He$.

Particle Scattering

A change in the direction of motion of a particle because of a collision with another particle.

Particle Decay

Particle decay is the spontaneous process of one unstable subatomic particle transforming into multiple other particles. The particle created in this process must each be less massive than the original, although the total invariant mass of the system must be conserved.

Because of their intrinsic instability a number of particles are known to break up or decay into two or more particles. The simplest example of such a decay is that of a particle at rest decaying into two particles.

When a unstable particle of rest mass M decays into two particles of rest mass M_1 and M_2 the two particles emitted must carry equal and opposite momenta if the unstable particle is stationary. The total energies carried by the decay products are then given by

$$E_1 = \frac{\left(M^2 + M_1^2 - M_2^2\right)c^2}{2M} \text{ and } E_2 = \frac{\left(M^2 + M_2^2 - M_1^2\right)c^2}{2M}$$

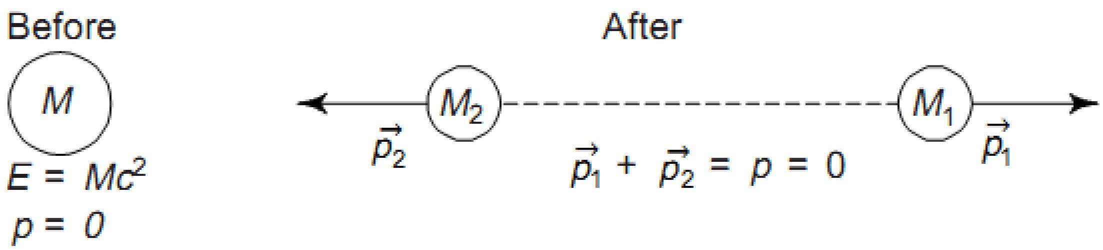
In this decay, a part of the original rest mass energy (Mc^2) is converted into kinetic energy of the decay products.

When an atom (rest mass M_0) absorbs a photon of energy Q, in order to conserve linear momentum, the atom recoils in the direction of the incident photon with a velocity

$$v = \frac{Q}{c(M_0 + Q/c^2)}$$

Decay of a Particle at Rest

Suppose a particle (rest mass M) at rest (momentum $P = 0$) decays into two particles of rest masses M_1 and M_2 as shown in Figure.



Since the initial momentum is zero, the two particles M_1 and M_2 must fly apart with equal and opposite momenta $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$ respectively. Thus in order to conserve momentum, P_1 and P_2 must be equal in magnitude i.e. $P_1 = P_2$ and opposite in direction. If E_1 and E_2 represent the total energies of particles of rest masses M_1 and M_2 respectively, then we must have

$$p_1^2 c^2 = p_2^2 c^2 \quad \text{or} \quad E_1^2 - M_1^2 c^4 = E_2^2 - M_2^2 c^4$$

$$\therefore \quad E_1^2 - E_2^2 = M_1^2 c^4 - M_2^2 c^4 \qquad \dots (1)$$

From conservation of total energy, we have

$$E_1 + E_2 = Mc^2 = \text{initial total energy} \qquad \dots (2)$$

Dividing Eqn. (1) by Eqn. (2) we get

$$E_1 - E_2 = \frac{M_1^2 c^4 - M_2^2 c^4}{Mc^2} \qquad \dots (3)$$

Addition and subtraction of Eqns. (2) and (3) leads respectively to

$$E_1 = \frac{(M^2 + M_1^2 - M_2^2)c^2}{2M} \qquad \dots (4)$$

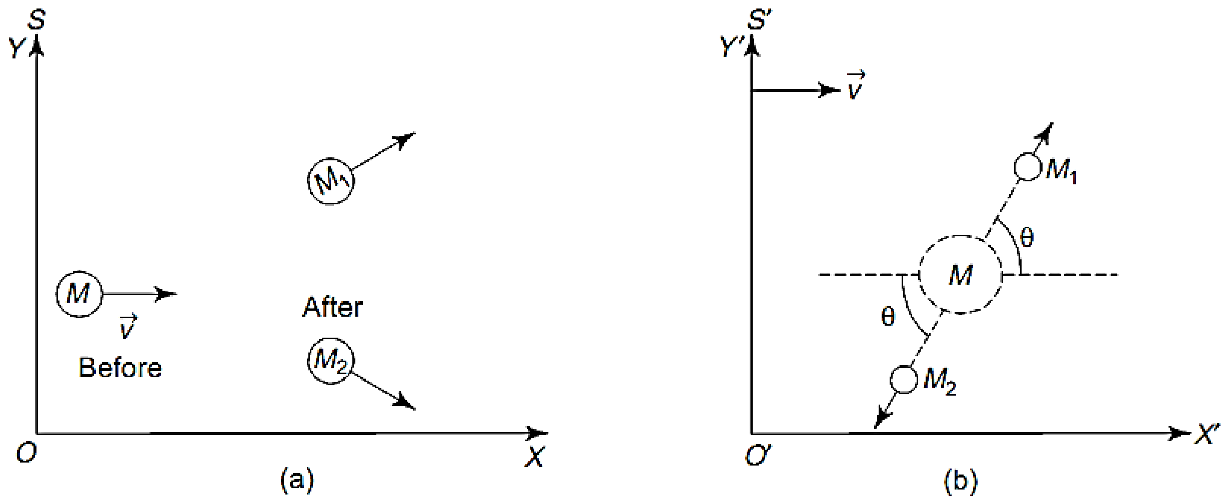
And

$$E_2 = \frac{(M^2 + M_2^2 - M_1^2)c^2}{2M} \qquad \dots (5)$$

In this decay process, part of the original rest mass energy Mc^2 has been converted into sum of rest mass energies $(M_1 + M_2)c^2$ of decay products. The remaining energy $[M - (M_1 + M_2)]c^2$ appears as the kinetic energy of the decay products.

Decay of a Particle in Flight

Suppose a particle of rest mass M moving with velocity $\vec{v}$ relative to frame S decays into two particles of rest mass M_1 and M_2. See Figure (a).



In frame S' moving to the right with velocity $\vec{v}$ as shown in Figure (b), the particle of rest mass M is at rest. It decays into two particles of rest mass M_1 and M_2 as shown.

Using Eqns. (4) and (5) above, the total energies of particles emitted are (note that these quantities are now measured in S') given by

$$E'_1 = \frac{\left(M^2 + M_1^2 - M_2^2\right)c^2}{2M} \qquad \text{... (6)}$$

$$E'_2 = \frac{\left(M^2 + M_2^2 - M_1^2\right)c^2}{2M} \qquad \text{... (7)}$$

Their momenta are given by

$$p_1'c = p_2'c = \sqrt{E_1^2 - M_1^2c^4} = \sqrt{E_2^2 - M_2^2c^4} \qquad \text{... (8)}$$

Pair Production

Simultaneous production of a pair consisting of a particle and it's antiparticle (electron e^- and position e^+ pair for example) at the expense of the entire energy $(h\upsilon)$ of a photon is called pair production. It is a very convincing example of conversion of electromagnetic energy into rest mass energy and kinetic energy of particles.

Pair Annihilation

The annihilation of a particle-antiparticle pair and the concomitant creation of photons is the inverse of pair production. An electron and a positron which are essentially at rest near one another unite and are annihilated.

Steller Energy

The internal energy of a star or the energy radiated by a star or the energy of the stars.

In 1904 Rutherford discovered that radioactive $\propto -$ decay releases energy. Radioactive decays are responsible for the energy of the stars but very heavy radioactive elements are seen in the spectra of Steller Atmosphere.

Natural Radioactivity

The spontaneous emission of radiations from an unstable nuclei is called Natural Radioactivity. For example natural radioactivity includes isotopes of potassium, uranium and thorium.

COMPTON EFFECT

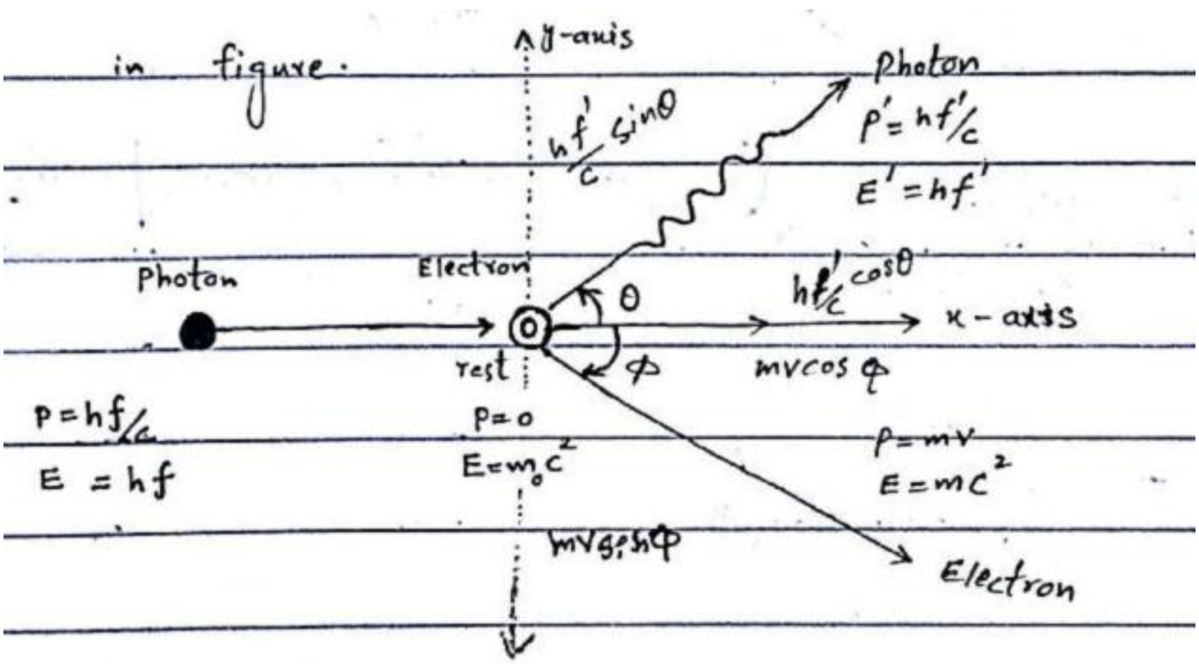
A.H. Compton found that

"When a radiation (Photon) scatters from a stationary particle its frequency decreases, i.e. wave length increases"

This phenomenon is known as Compton Effect. i.e. $\Delta\lambda = \frac{h}{m_0 c}(1 - Cos\theta)$

Explanation

Consider a photon particle scattering with the electron particle which is at rest. After scattering both particles moves in different directions. Photon makes direction θ with horizontal and electrons makes φ with horizontal as shown



Before collision

Momentum of Photon $= \frac{hf}{c}$

Momentum of Electron $= 0$

Energy of Photon $= hf$

Momentum of Electron $= m_0 c^2$

After collision

Momentum of Photon $= \frac{hf\prime}{c}$

Momentum of Electron $= mv$

Energy of Photon $= hf'$

Momentum of Electron $= mc^2$

According to law of conservation of energy

Total energy before collision = Total energy after collision

$$hf + m_0c^2 = hf' + mc^2$$

$$hf - hf' + m_0c^2 = mc^2$$

$$h\frac{c}{\lambda} - h\frac{c}{\lambda\prime} + m_0c^2 = mc^2$$

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{c}{\lambda}, f' = \frac{c}{\lambda'}$$

$$mc = \frac{h}{\lambda} - \frac{h}{\lambda\prime} + m_0c$$

$$(mc)^2 = \left(\frac{h}{\lambda} - \frac{h}{\lambda\prime} + m_0c\right)^2$$

$$m^2c^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} + m_0^2c^2 - 2\frac{h^2}{\lambda\lambda'} - 2\frac{m_0ch}{\lambda'} + 2\frac{m_0ch}{\lambda}$$

$$m^2c^2 - m_0^2c^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'} + 2m_0ch\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$(m^2 - m_0^2)c^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'} + 2m_0ch\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) \qquad \ldots\ldots\ldots\ldots(1)$$

According to law of conservation of momentum

Total momentum before collision = Total momentum after collision

$$\frac{hf}{c} + 0 = \frac{hf'}{c}Cos\theta + mvCos(-\varphi) \qquad \text{x – axis}$$

$$0 + 0 = \frac{hf'}{c}Sin\theta + mvSin(-\varphi) \qquad \text{y – axis}$$

This implies that

$\frac{hf}{c} = \frac{hf'}{c}Cos\theta + mvCos\varphi$ and $0 = \frac{hf'}{c}Sin\theta - mvSin\varphi$

$mvCos\varphi = \frac{hf}{c} - \frac{hf'}{c}Cos\theta$ …………..(2)

$mvSin\varphi = \frac{hf'}{c}Sin\theta$ …………..(3)

Squaring and adding (2) and (3)

$$m^2v^2Cos^2\varphi + m^2v^2Sin^2\varphi = \frac{h^2f^2}{c^2} + \frac{h^2f'^2}{c^2}Cos^2\theta - 2\frac{h^2ff'}{c^2}Cos\theta + \frac{h^2f'^2}{c^2}Sin^2\theta$$

$$m^2v^2 = \frac{h^2f^2}{c^2} + \frac{h^2f'^2}{c^2}Cos^2\theta - 2\frac{h^2ff'}{c^2}Cos\theta + \frac{h^2f'^2}{c^2}Sin^2\theta$$

$$v = f\lambda$$
$$\Rightarrow c = f\lambda, \quad \frac{1}{\lambda} = \frac{f}{c}$$

$$m^2v^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2}Cos^2\theta - 2\frac{h^2}{\lambda\lambda'}Cos\theta + \frac{h^2}{\lambda'^2}Sin^2\theta$$

$m^2v^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}Cos\theta$ …………..(4)

Now using $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\Rightarrow m^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}} \Rightarrow m_0^2 = m^2\left(1 - \frac{v^2}{c^2}\right) \Rightarrow m_0^2 = m^2\left(\frac{c^2-v^2}{c^2}\right) \Rightarrow m^2v^2 = (m_0^2 - m^2)c^2$$

$(4) \Rightarrow (m_0^2 - m^2)c^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}Cos\theta$ …………..(5)

$\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'} + 2m_0ch\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}Cos\theta$ Comparing (1),(5)

$$\Rightarrow -2\frac{h^2}{\lambda\lambda'} + 2m_0ch\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = -2\frac{h^2}{\lambda\lambda'}Cos\theta$$

$$\Rightarrow 2m_0ch\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right) = 2\frac{h^2}{\lambda\lambda'} - 2\frac{h^2}{\lambda\lambda'}Cos\theta \Rightarrow \lambda' - \lambda = \frac{\lambda\lambda'}{2m_0ch}\left(2\frac{h^2}{\lambda\lambda'} - 2\frac{h^2}{\lambda\lambda'}Cos\theta\right)$$

$$\Rightarrow \lambda' - \lambda = \frac{\lambda\lambda'}{2m_0ch} \times 2\frac{h^2}{\lambda\lambda'}(1 - Cos\theta) \Rightarrow \Delta\boldsymbol{\lambda} = \frac{\boldsymbol{h}}{\boldsymbol{m_0c}}(\mathbf{1} - \boldsymbol{Cos\theta})$$

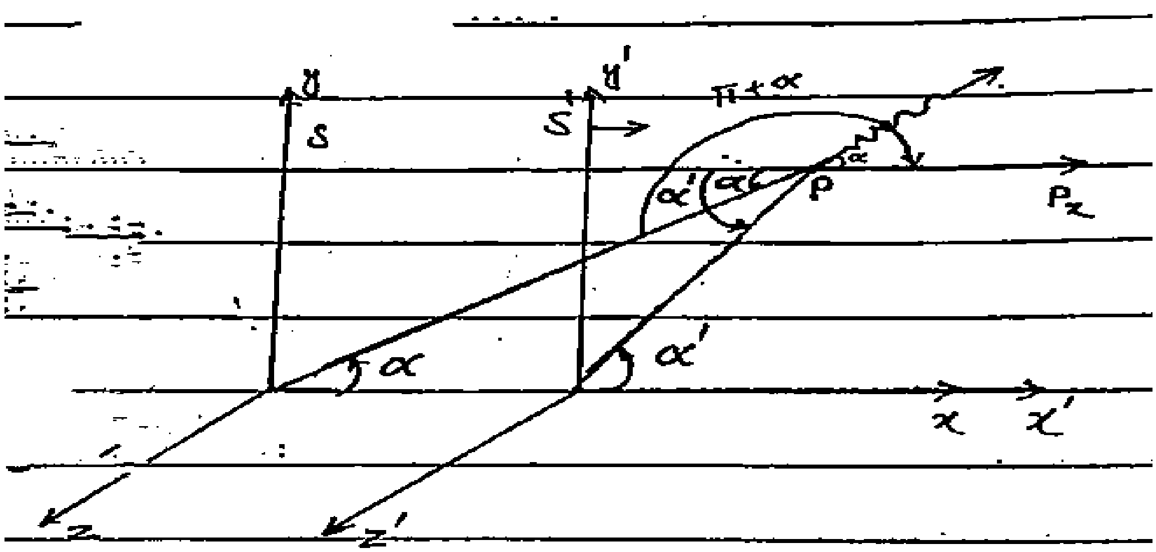
Here $\Delta\lambda$ is called Compton shift representing change of wavelength of the photon particles due to θ.

DOPPLER EFFECT / DOPPLER SHIFT

The change in frequency of light radiation emitted by a source of light due to relativistic motion of source and observer is called Doppler Effect /Shift.

The expression for Doppler's Effect can be obtained by using the transformation law of energy and momentum.

Consider two frames S and S'. S' $-$ frame moving with velocity v relative to frame S' along the direction of common x – axis.



Let source of light P is emitting the radiations. Then energy and momentum of photon of light in S is given by

$E = hf$ …………(1) and $P = \frac{hf}{c}$ …………(2)

Similarly in frame S' is given by

$E' = hf'$ …………(3) and $P' = \frac{hf'}{c}$ …………(4)

Where h is Planks constant, f is frequency in S frame and f' is proper frequency in S' frame.

Now by using law of transformation of Energy

$E' = \gamma(E - vP_x)$

Then inverse transformation law becomes

$E = \gamma(E' + vP_x)$ …………(A)

Since $P_x' = \frac{hf\prime}{c} Cos(\propto ' + 180°) = -\frac{hf\prime}{c} Cos \propto '$ …………(5)

Using (1),(3) and (5) in (A)

$\Rightarrow E = \gamma \left(E' - v\frac{hf\prime}{c} Cos \propto '\right)$

$\Rightarrow hf = \gamma \left(hf' - v\frac{hf\prime}{c} Cos \propto '\right)$ Since $E = hf\ , E' = hf'$

$\Rightarrow hf = \gamma hf' \left(1 - \frac{v}{c} Cos \propto '\right)$

$\Rightarrow \boldsymbol{f = \gamma f' \left(1 - \frac{v}{c} Cos \propto '\right)}$

Inverse relation is

$\Rightarrow \boldsymbol{f' = \gamma f \left(1 + \frac{v}{c} Cos \propto '\right)}$

This change in frequency due to the relative motion of the observer and the source is known as Doppler Effect/Shift.

Significance of Doppler Effect

It has great significance in the field of astronomy. Observation of stellar spectra determines the rate of moving motion of stars. i.e. at what rate the stars are moving towards or away from us. While the observation of red shift in the spectra of distance in galaxies indicate that the universe is continuously expanding.

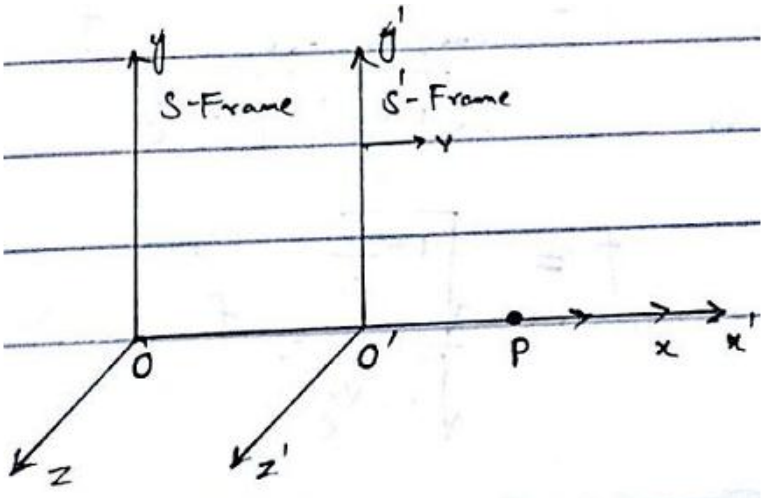
It is useful in

- Radar system
- Motorway (Speed)
- Blood flow
- Sonar system

Here are few cases for Doppler Effect

Horizontal Case (Longitudinal Doppler Effect)

(a) When source P moving away from the observer O or O′ then $\propto' = \propto = 0°$



Then using the equation

$$f = \gamma f'\left(1 - \frac{v}{c} Cos \propto '\right)$$

$$\Rightarrow f = \gamma f'\left(1 - \frac{v}{c} Cos 0°\right)$$

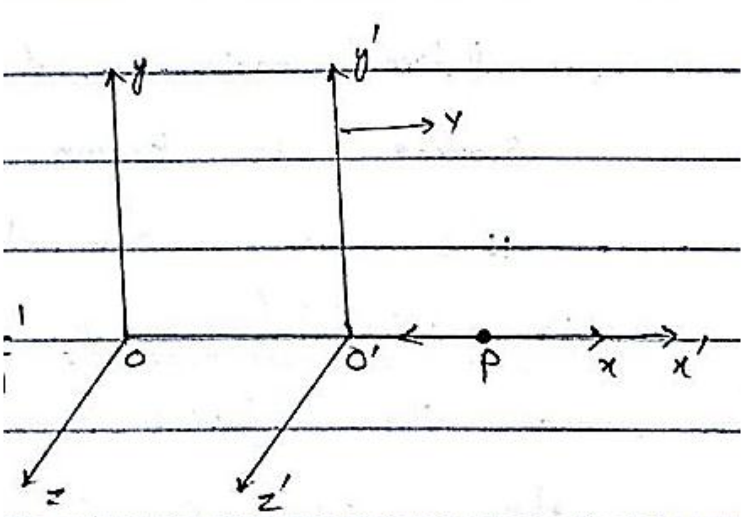
$$\Rightarrow f = \gamma f'\left(1 - \frac{v}{c}\right) = \frac{f'}{\sqrt{1-\frac{v^2}{c^2}}}\left(1 - \frac{v}{c}\right) = \frac{f'}{\sqrt{1-\frac{v}{c}}\sqrt{1+\frac{v}{c}}}\left(1 - \frac{v}{c}\right)$$

$$\Rightarrow f = \frac{\sqrt{1-\frac{v}{c}}}{\sqrt{1+\frac{v}{c}}} f' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} f'$$

$\Rightarrow \boldsymbol{f < f'}$ since $\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} < 1$

Thus frequency f with respect to $S -$ frame is less than the frequency f' with respect to $S' -$ frame

(b) When source P moving towards the observer O or O′ then $\propto' = \propto = 180°$



Then using the equation

$$f = \gamma f'\left(1 - \frac{v}{c} Cos \propto '\right)$$

$$\Rightarrow f = \gamma f'\left(1 - \frac{v}{c} Cos 180°\right)$$

$$\Rightarrow f = \gamma f'\left(1 + \frac{v}{c}\right) = \frac{f'}{\sqrt{1 - \frac{v^2}{c^2}}}\left(1 + \frac{v}{c}\right) = \frac{f'}{\sqrt{1 - \frac{v}{c}}\sqrt{1 + \frac{v}{c}}}\left(1 + \frac{v}{c}\right)$$

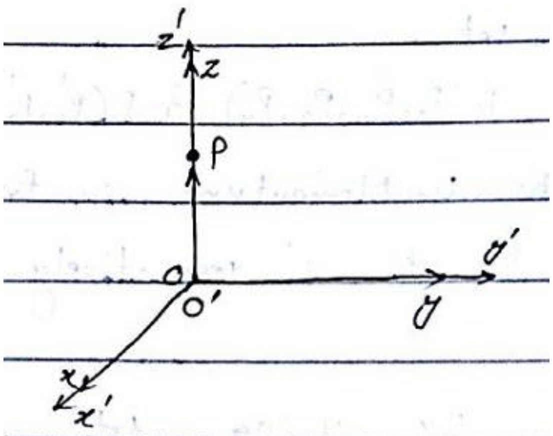
$$\Rightarrow f = \frac{\sqrt{1 + \frac{v}{c}}}{\sqrt{1 - \frac{v}{c}}} f' = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} f'$$

$\Rightarrow \boldsymbol{f} > \boldsymbol{f'}$ since $\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} > 1$

Thus frequency f with respect to $S -$ frame is grater than the frequency f' with respect to $S' -$ frame

Vertical Case (Transverse Doppler Effect)

In this case both frames coincide and source P moving perpendicular to x – direction then $\propto' = \propto = 90°$



Then using the equation

$$f = \gamma f' \left(1 - \frac{v}{c} Cos \propto '\right)$$

$$\Rightarrow f = \gamma f' \left(1 - \frac{v}{c} Cos 90°\right)$$

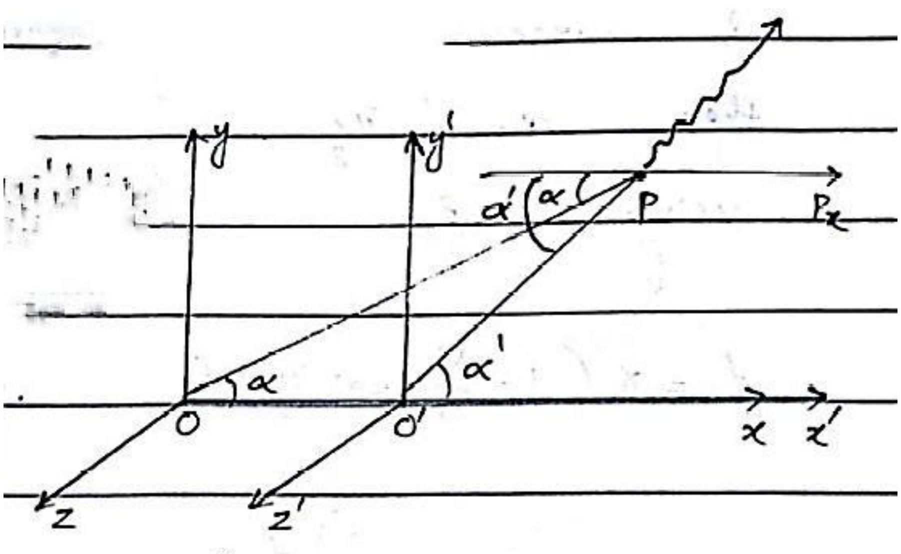
$$\Rightarrow \boldsymbol{f = \gamma f'}$$

Thus frequency of light changed when the source moving perpendicular to the direction of motion of light.

Also note that in classical mechanics $\gamma = 1$ then $\boldsymbol{f = f'}$ that is no such effect has been seen in classical mechanics.

ABERRATION OF LIGHT

Aberration is the variation in the apparent position of a heavenly body such as a star, due to the motion of the observer with the earth. The change in measurement of angles is called Aberration of light.



Consider two observers at origin O and O' of two frames S and S' respectively. Let the frame S' is moving with velocity v relative to S. Also suppose that a source of light is placed at point P in frame S', and also consider a light beam makes different angles $\propto, \propto'$ to the observer at O and O'.

Let $P(P_1, P_2, P_3, P_4)$ and $P'(P_1', P_2', P_3', P_4')$ be $4-$ momentum in S and S' respectively. Then by using first component of law of transformation of $4-$ momentum;

$$P_x' = \gamma\left(P_x + \frac{iv}{c}P_4\right) \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

Using $P_x = \frac{hf}{c}Cos(\propto + 180°) = -\frac{hf}{c}Cos \propto, P_4 = imc$

And $P_x' = \frac{hf'}{c}Cos(\propto' + 180°) = -\frac{hf'}{c}Cos \propto'$

$$(1) \Rightarrow -\frac{hf'}{c}Cos \propto' = \gamma\left(-\frac{hf}{c}Cos \propto + \frac{iv}{c}.imc\right)$$

$\Rightarrow -\frac{hf'}{c} Cos \propto' = \gamma \left(-\frac{hf}{c} Cos \propto - mv \right)$

$\Rightarrow -\frac{hf'}{c} Cos \propto' = \gamma \left(-\frac{hf}{c} Cos \propto - \frac{hf}{c^2} v \right) \qquad \because E = mc^2 \Rightarrow m = \frac{E}{c^2} = \frac{hf}{c^2}$

$\Rightarrow \frac{hf'}{c} Cos \propto' = \gamma \frac{hf}{c} \left(Cos \propto + \frac{v}{c} \right)$

$\Rightarrow f' Cos \propto' = \gamma f \left(Cos \propto + \frac{v}{c} \right)$ ....(2)

Using Doppler's Effect

$f' = \gamma \left(1 + \frac{v}{c} Cos \propto \right) f$

$(2) \Rightarrow \gamma f \left(1 + \frac{v}{c} Cos \propto \right) Cos \propto' = \gamma f \left(Cos \propto + \frac{v}{c} \right)$

$\Rightarrow Cos \propto' = \frac{Cos\propto + \frac{v}{c}}{1 + \frac{v}{c} Cos\propto} = \frac{cCos\propto + v}{c + vCos\propto}$

$\Rightarrow \boldsymbol{Cos} \propto' = \frac{\boldsymbol{v + cCos\propto}}{\boldsymbol{c + vCos\propto}}$ This equation gives us the relation between $\propto$ and $\propto'$

Significance of Aberration of light

It has great significance in the field of astronomy. Observation of stellar spectra determines the rate of moving motion of stars. i.e. at what rate the stars are moving towards or away from us. While the observation of red shift in the spectra of distance in galaxies indicate that the universe is continuously expanding.

The formula for aberration of light relate the true position of stars with the observers position. The displacement being caused by the motion of the earth relative to the velocity of light

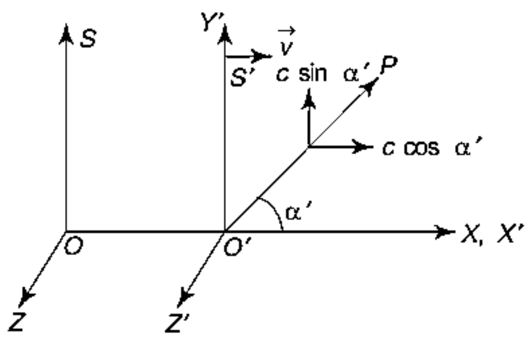
if $c \to \infty$ then $Cos \propto' = \frac{v + cCos\propto}{c + vCos\propto}$ becomes $Cos \propto' = Cos \propto$

implies $\propto = \propto'$

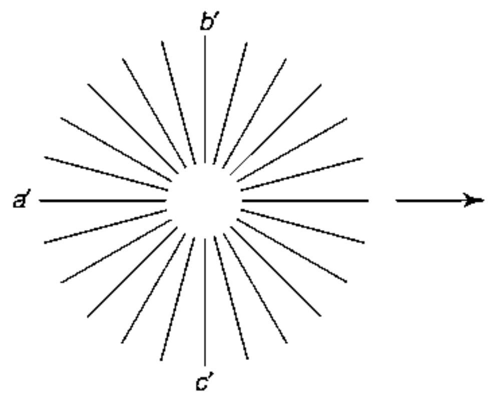
HEADLIGHT EFFECT

A moving source of radiation radiating uniformly in all directions in its rest frame, appears to radiate predominantly along its direction of motion. This observed bunching of radiation along the forward direction is due to aberration and is called as headlight effect.

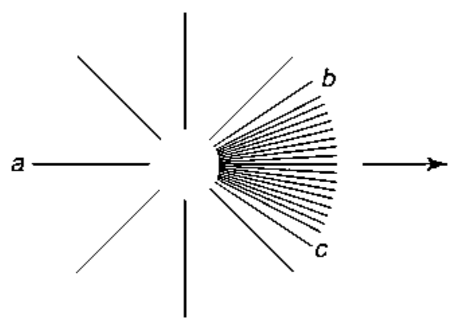
Consider a source in frame S' radiating uniformly in all directions. Frame S' is moving along positive $X, X' -$ axis with velocity v. See Figure;



(a)



(b)



(c)

In frame S' the ray $O'P$ (in plane $X'Y'$) has velocity components

$u'_x = cCos \propto'$ and $u'_y = cSin \propto'$

According to the relativistic law of composition of velocities, in S frame the x component of the velocity is

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2} u'_x}$$

$$u_x = \frac{cCos \propto ' + v}{1 + \frac{v}{c^2}.cCos \propto '}$$

$$u_x = \frac{cCos \propto ' + v}{1 + \frac{v}{c}Cos \propto '}$$

$$u_x = \frac{c\left(Cos\propto\prime+\frac{v}{c}\right)}{1+\frac{v}{c}Cos\propto\prime} = \frac{c(Cos\propto\prime+\beta)}{1+\beta Cos\propto\prime}$$ where $\frac{v}{c} = \beta$

Since $u_x = cCos \propto$, the angle made by the ray with the X −axis as observed in frame S is given by

$Cos \propto = \frac{u_x}{c}$ or $Cos \propto = \frac{Cos\propto\prime+\beta}{1+\beta Cos\propto\prime}$

From this equation we see that $\propto = \propto '$ when $\propto = 0$ or π but $Cos \propto = \beta$ when $\propto' = \pm\frac{\pi}{2}$.

Then as β approaches 1, the angle approaches zero. Thus most of the radiation appears to be strongly concentrated in the forward direction with very little radiation coming off in the backward direction. This is shown in Fig.(c) qualitatively for a source radiating uniformly in its rest frame as in Fig.(b).

The headlight effect can be observed as visible light in case of radiation emitted (synchrotron radiation) by circulating charged particles accelerated to extremely high energies in modern accelerators. A similar phenomenon is observed in nature when high energy cosmic ray protons decelerate on entering the earth's atmosphere.

Example

The speed of light in still water is $\frac{c}{n}$ where n is the refractive index of water. Experiments show that the speed of light in running water can be expressed as $v = \frac{c}{n} + kv$ where $k \approx 0.4$ is called the drag coefficient and v is the velocity of water. Determine the value of k using the law of addition of velocities. $n = \frac{4}{3}$.

Solution

According to the law of addition of velocities,

$U_x = \dfrac{U_x' + v}{1 + \dfrac{vU_x'}{c^2}}$ where $U_x' = \frac{c}{n}$ is the speed of light in a frame in which the water is at rest and v is the velocity of water in a frame S in which the velocity of light is U_x.

$$U_x = \frac{\frac{c}{n} + v}{1 + \frac{v}{c^2}\frac{c}{n}} = \left(\frac{c}{n} + v\right)\left(1 + \frac{v}{cn}\right)^{-1}$$

$$= \left(\frac{c}{n} + v\right)\left(1 - \frac{v}{cn}\right) \qquad \because \quad v/cn \ll 1$$

$$= \frac{c}{n} - \frac{cv}{cn^2} + v - \frac{v^2}{cn} = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) \text{ neglecting } v^2 \text{ terms.}$$

$$k = \left(1 - \frac{1}{n^2}\right) = \left(1 - \frac{9}{16}\right) = \frac{7}{16} = 0.438.$$

Example

How fast must you be driving your car to see a red light signal as green? Take the wavelengths of red and green lights as 6300 Å and 5400 Å respectively.

Solution In relativity only the relative motion between the source and the observer is of importance—there is no distinction between source approaching the observer and the observer approaching the source. For an observer in a car travelling directly towards a source of light with a velocity v, the source of light appears to approach directly towards him with velocity v. The observer in the car would measure a higher frequency or a lower wavelength.

Let f' be the frequency of light emitted (red signal) and f the frequency of light measured by the observer in the car. Then

$$\frac{f}{f'} = \sqrt{\frac{1+\beta}{1-\beta}} \text{ or } \frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} \qquad \because \; f'\lambda' = f\lambda = c$$

$$\therefore \qquad \frac{6300}{5400} = \sqrt{\frac{1+\beta}{1-\beta}} = \frac{7}{6}$$

$$\frac{49}{36} = \frac{1+\beta}{1-\beta}$$

$$\therefore \qquad 49 - 49\beta = 36 + 36\beta \text{ or } 85\beta = 13$$

$$\therefore \qquad \beta = \frac{13}{85} = 0.153$$

$$\therefore \qquad \frac{v}{c} = 0.153 \text{ or } v = 0.153\ c = 0.153 \times 3 \times 10^8 \text{ m/s}$$

$$\therefore \qquad v = 4.59 \times 10^7 \text{ m/s.}$$

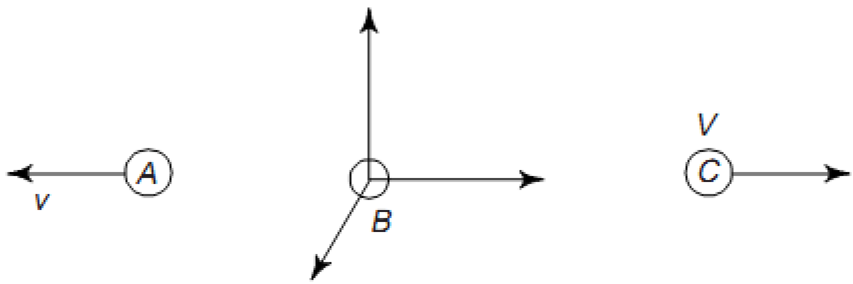
An observer in a car approaching a red signal with above velocity would see the red signal as green!

Example

Three identical radio transmitters A, B and C each transmitting at the frequency f_0 in its own rest frame are in motion as shown

(a) What is the frequency of B's signal as received by C?

(b) What is the frequency of A's signal as received by C?



Solution

(a) For an observer on C, the source B appears to recede directly away with a velocity v. Hence the frequency observed by C is

$$f = f_0\left(\frac{1-\beta}{1+\beta}\right)^{1/2} \quad \text{where } \beta = v/c.$$

(b) For an observer on C, the source A appears to recede directly away with a velocity

$$v' = \frac{v+v}{1+\frac{v}{c^2}\cdot v} = \frac{2v}{1+v^2/c^2}$$

Hence the frequency of transmitter A as observed by C is given by

$$f = f_0\left(\frac{1-v'/c}{1+v'/c}\right)^{1/2} = f_0\left(\frac{1-\frac{2v/c}{1+v^2/c^2}}{1+\frac{2v/c}{1+v^2/c^2}}\right)^{1/2}$$

$$= f_0\left(\frac{1+v^2/c^2-2v/c}{1+v^2/c^2+2v/c}\right)^{1/2} = f_0\left[\frac{(1-v/c)^2}{(1+v/c)^2}\right]^{1/2} = f_0\left(\frac{1-\beta}{1+\beta}\right)$$

Example

What is the Doppler shift in the wavelength of H∝(6561 Å) line from a star which is moving away from the earth with a velocity of 300 Km/s.

Solution Observed frequency

$$f = f'\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{or} \quad \frac{f}{f'} = \sqrt{\frac{1-\beta}{1+\beta}}$$

$$\therefore \quad \frac{\lambda'}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}} \qquad (\because f\lambda = f'\lambda' = c)$$

$$\text{or} \quad \lambda = \lambda'\sqrt{\frac{1+\beta}{1-\beta}} \quad \text{where} \quad \beta = v/c = \frac{3\times10^5}{3\times10^8} = 10^{-3} \ll 1$$

$$\therefore \quad \sqrt{\frac{1+\beta}{1-\beta}} = (1+\beta)^{1/2}(1-\beta)^{-\frac{1}{2}} \approx (1+\beta/2)(1+\beta/2)$$

$$\approx 1 + \beta/2 + \beta/2 = 1 + \beta \text{ neglecting } \beta^2.$$

$$\therefore \quad \lambda = \lambda'(1+\beta)$$

$\therefore$ Doppler shift $\quad \Delta\lambda = \lambda - \lambda' = \lambda'\beta = (6561\ \text{Å})(10^{-3})$

$$= 6.561\ \text{Å (increase)}.$$

Example

The sun rotates once in about 24.7 days. The radius of the sun is about $7.0 \times 10^8 kms$. Calculate the Doppler shift that we should observe for light of wavelength 6560 Å from the edge of the sun's disc near the equator. Is this shift towards the red end or the blue end of the spectrum?

Solution

The source of light at the edge A is instantaneously approaching the earth with a velocity $v = \omega r$ where ω is the angular velocity of the rotating sun and r its radius. The observed frequency will therefore increase, hence the observed wavelength will be less than 6560 Å. Such a decrease is called a shift towards the blue.

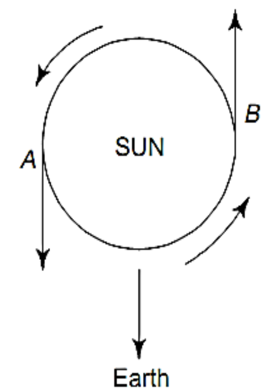


Fig. 5.6 For Illustrative Example 5

Since observed frequency $f = f'\sqrt{\dfrac{1+\beta}{1-\beta}}$ and $f\lambda = f\lambda' = c$,

$$\therefore \quad \frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{or} \quad \frac{\lambda}{\lambda'} = \sqrt{\frac{1-\beta}{1+\beta}} = (1-\beta)^{1/2}(1+\beta)^{-\frac{1}{2}}$$

$$\approx \left(1-\frac{\beta}{2}\right)\left(1-\frac{\beta}{2}\right) \approx 1-\beta \text{ neglecting } \beta^2 \text{ terms.}$$

$$\therefore \quad \lambda = \lambda'(1-\beta)$$

$$\therefore \quad \text{Doppler shift } \Delta\lambda = \lambda' - \lambda = \lambda'\beta$$

$$\text{Now} \quad \beta = \frac{v}{c} = \frac{\omega r}{c} = \frac{2\pi}{T}\frac{r}{c} \quad \text{where} \quad T = 24.7 \text{ days period}$$

$$\therefore \quad \Delta\lambda = \beta\lambda' = \frac{2\pi r}{Tc}\lambda' = \frac{2\pi \times 7.0 \times 10^8 \times 6560}{24.7 \times 24 \times 60 \times 60 \times 3 \times 10^8}$$

$$= 0.0455 \text{ Å}$$

$$\therefore \quad \Delta\lambda = 0.0455 \text{ Å (towards blue)}$$

The source of light at edge B is instantaneously receding away from the earth. The observed frequency is now reduced and hence observed wavelength will be more than 6560 Å. Such an increase in wave length is called a shift towards the red.

$$\text{In the present case} \quad \frac{\lambda}{\lambda'} = \sqrt{\frac{1+\beta}{1-\beta}} \approx (1+\beta)$$

$$\therefore \quad \lambda = \lambda'(1+\beta)$$

$$\therefore \quad \text{Increase in wavelength } \Delta\lambda = \lambda - \lambda' = \lambda'\beta$$

$$= 0.0455 \text{ Å as before (towards red)}$$

Example

Light of wavelength 6000 Å is incident normally on a mirror which is receding with a velocity $3 \times 10^4 ms^{-1}$ in a direction away from the incident light. Calculate the change in wavelength on reflection.

Solution

Consider an observer O moving with the mirror. To him the source of light appears to recede away directly from him. Hence the frequency of light incident on the mirror is $f = f'\sqrt{\frac{1-\beta}{1+\beta}}$.

This is also the frequency of light reflected from the mirror. The mirror is therefore a source of light of frequency f and this source (mirror) s receding away at 3×10^4 m/s. Hence the frequency of reflected light as observed by an observer with respect to whom the source is receding away at $v = 3 \times 10^4$ m/s is

$$f'' = f\sqrt{\frac{1-\beta}{1+\beta}} = f'\sqrt{\frac{1-\beta}{1+\beta}}\sqrt{\frac{1-\beta}{1+\beta}} = f'\left(\frac{1-\beta}{1+\beta}\right)$$

If λ'' is the observed wavelength of the reflected light then since $f''\lambda'' = c = f'\lambda'$ we find that

$$\lambda'' = \lambda'\left(\frac{1+\beta}{1-\beta}\right) = \lambda'(1+\beta)(1-\beta)^{-1} = (1+\beta)(1+\beta)$$

$$\approx \lambda'(1+2\beta)$$

$\therefore$ Change in wavelength is $\lambda'' - \lambda' = 2\beta\lambda'$

$$= 2 \times \frac{3 \times 10^4}{3 \times 10^8}\ 6000 = 2 \times 10^{-4} \times 6000 = 1.2\ \text{Å}.$$

Question

A stationary shell of rest mass M explodes into two fragments of rest masses M_1 and M_2. Show that their respective energies are given by

$$E_1 = \frac{M^2 + M_1^2 - M_2^2}{2M}\, c^2;\; E_2 = \frac{M^2 - M_1^2 + M_2^2}{2M}\, c^2.$$

Solution

Consider a stationary shell (of rest mass M) at rest momentum $P = 0$ exploded into two fragments of rest masses M_1 and M_2. Since the initial momentum is zero, in order to conserve momentum P_1, P_2 must be equal in magnitude. i.e. $P_1 = P_2$ and opposite in direction. If E_1, E_2 represent the total energies of particles of rest masses M_1 and M_2, then we must have

$$P_1^2 c^2 = P_2^2 c^2$$

Or $\quad E_1^2 - M_1^2 c^4 = E_2^2 - M_2^2 c^4$

$$E_1^2 - E_2^2 = M_1^2 c^4 - M_2^2 c^4 \qquad \ldots\ldots\ldots\ldots(1)$$

From conservation of total energy we have

$$E_1 + E_2 = M^2 c^4 = \text{initial total energy} \qquad \ldots\ldots\ldots\ldots(2)$$

Dividing (1) by (2) we get

$$\frac{E_1^2 - E_2^2}{E_1 + E_2} = \frac{M_1^2 c^4 - M_2^2 c^4}{Mc^2}$$

$$\frac{(E_1 + E_2)(E_1 - E_2)}{E_1 + E_2} = \frac{M_1^2 c^4 - M_2^2 c^4}{Mc^2}$$

$$E_1 - E_2 = \frac{M_1^2 c^4 - M_2^2 c^4}{Mc^2} \qquad \ldots\ldots\ldots\ldots(3)$$

$$\Rightarrow 2E_1 = \frac{(M^2 + M_1^2 - M_2^2)c^2}{M} \Rightarrow \boldsymbol{E_1 = \frac{(M^2 + M_1^2 - M_2^2)c^2}{2M}} \qquad \text{adding (2) and (3)}$$

$$\Rightarrow 2E_2 = \frac{(M^2 - M_1^2 + M_2^2)c^2}{M} \Rightarrow \boldsymbol{E_2 = \frac{(M^2 - M_1^2 + M_2^2)c^2}{2M}} \qquad \text{subtracting (2) and (3)}$$

Poincare Group

The **Poincaré group**, named after Henri Poincaré (1906),[1] was first defined by Hermann Minkowski (1908) as the group of Minkowski spacetime isometries.[2][3] It is a ten-dimensional non-abelian Lie group that is of importance as a model in our understanding of the most basic fundamentals of physics.

Lorentz Group

In physics and mathematics, the **Lorentz group** is the group of all Lorentz transformations of Minkowski spacetime, the classical and quantum setting for all (non-gravitational) physical phenomena. The Lorentz group is named for the Dutch physicist Hendrik Lorentz.

There are two differences. First, the difference between a *transformation* and a *group*. A transformation, in this context, is a change of the spacetime coordinates. A group, in this context, is a collection of all the possible transformations. So the difference is that a transformation is an element of a group.

Second, the difference between a *Lorentz* transformation and a *Poincare* transformation. The Lorentz group consists of the transformations which preserve the magnitude of 4-vectors and also transform the origin to itself. These transformations are the rotations around the origin and the boosts. (A boost is a change to a frame of reference moving with some relative velocity with respect to the original frame.)

The Poincare group consists of the transformations which preserve the magnitude of 4-vectors, but does not require the origin to be preserved. Therefore it includes, in addition to rotations and boosts, also translations in space and time.

So the Lorentz group has 6 parameters: 3 rotations (one around each direction of space) and 3 boosts (one in each direction of space). The Poincare group adds 4 more parameters for translations (one in the time direction and one in each space direction).

In short, Poincare = Lorentz + translations.

Question

Prove that Kronecker delta is unaltered (invariant) by coordinate transformation.

Solution

We have to prove $\delta_j^i = \delta_j^{i'}$

$$\delta_j^{i'} = \delta_v^u \frac{\partial x^{i'}}{\partial x^u} . \frac{\partial x^v}{\partial x^{j'}}$$

$$\delta_j^{i'} = \left(\delta_v^u \frac{\partial x^{i'}}{\partial x^u}\right) . \frac{\partial x^v}{\partial x^{j'}} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

Then

$$\delta_v^u \frac{\partial x^{i'}}{\partial x^u} = \delta_v^1 \frac{\partial x^{i'}}{\partial x^1} + \delta_v^2 \frac{\partial x^{i'}}{\partial x^2} + \cdots + \delta_v^v \frac{\partial x^{i'}}{\partial x^v} + \cdots + \delta_v^N \frac{\partial x^{i'}}{\partial x^N}$$

$$\delta_v^u \frac{\partial x^{i'}}{\partial x^u} = \frac{\partial x^{i'}}{\partial x^v} \qquad \because \delta_v^u = \begin{cases} 0 & ; u \neq v \\ 0 & ; u = v \end{cases}$$

$$\Rightarrow \delta_j^{i'} = \frac{\partial x^{i'}}{\partial x^v} . \frac{\partial x^v}{\partial x^{j'}}$$

$$\Rightarrow \delta_j^{i'} = \frac{\partial x^{i'}}{\partial x^{j'}}$$

$$\Rightarrow \delta_j^{i'} = \delta_j^i$$

Hence it is proved that Kronecker delta is unaltered by coordinate transformation.

حرفِ آخر (25-07-2022)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہو سکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالٰی آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔ (اٰمین)

محمد عثمان حامد

چک نمبر 105 شمالی (گودھے والا) سرگودھا

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