Exercise: 1

★ Express each of the following complex number in polar form:-

★ **Q # 1:-** $Z = -\sqrt{3} + i$

$r = |Z| = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = 2$

$\because \cos\theta = \frac{x}{|Z|} = \frac{-\sqrt{3}}{2}$, $\sin\theta = \frac{y}{|Z|} = \frac{1}{2}$

$\Rightarrow \theta = Arg.Z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore Z = r(\cos\theta + i\sin\theta)$

$Z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

★ **Q # 2:-** $Z = -i$

$Z = 0 - i$

$r = |Z| = \sqrt{(0)^2 + (-1)^2} = 1$

$\because \cos\theta = \frac{x}{|Z|} = \frac{0}{1} = 0$, $\sin\theta = \frac{y}{|Z|} = \frac{-1}{1} = -1$

$\Rightarrow \theta = Arg\, Z = -\pi/2$

Hence $Z = 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

★ **Q # 3:-** $Z = -1 - \sqrt{3}i$

$r = |Z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$

$\because \cos\theta = \frac{x}{|Z|} = \frac{-1}{2}$, $\sin\theta = \frac{y}{|Z|} = \frac{-\sqrt{3}}{2}$

$\Rightarrow \theta = Arg.Z = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

Hence

$$Z = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

★ **Q # 4:-** $Z = -1 + i$

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$r = |Z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\because \cos\theta = \frac{-1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = Arg\, Z = \pi - \frac{\pi}{4} \Rightarrow \frac{3\pi}{4}$

★ **Q# 5:-** $Z = (-2+2i)(1-i)$

$$Z = -2+2i+2i-2i^2$$

$$= -2+4i-2(-1) \quad \therefore i^2=-1$$

$$= -\cancel{2}+4i+\cancel{2}$$

$$= 0+4i$$

$$r = |Z| = \sqrt{0^2+4^2} = 4$$

$$\because \cos\theta = \frac{0}{4} = 0 \quad , \quad \sin\theta = \frac{4}{4} \Rightarrow \sin\theta = 1$$

$$\theta = \text{Arg } Z = \frac{\pi}{2}$$

Hence

$$Z = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

★ **Q# 6:-** $Z = \dfrac{-34i}{5-3i}$

$$= \frac{-34i}{5-3i} \times \frac{5+3i}{5+3i} \Rightarrow \frac{(-34i)(5+3i)}{(5)^2-(3)^2 i^2}$$

$$= \frac{-170i+102}{25+9} \Rightarrow \frac{-170}{34}i + \frac{102}{34}$$

$$Z = 3-5i$$

$$\therefore r = |Z| = \sqrt{(3)^2+(-5)^2} = \sqrt{9+25} = \sqrt{34}$$

$$\theta = \text{Arg } Z = \tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left(\frac{-5}{3}\right)$$

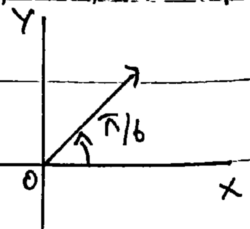
Hence

$$Z = \sqrt{34}\left(\cos\left(\tan^{-1}\frac{-5}{3}\right) + i\sin\left(\tan^{-1}\frac{-5}{3}\right)\right)$$

★ Express the given complex numbers in Cartesian form and plot argand diagram:-

★ **Q.# 7:-** $Z = 2\,\text{cis}\left(\frac{\pi}{6}\right)$

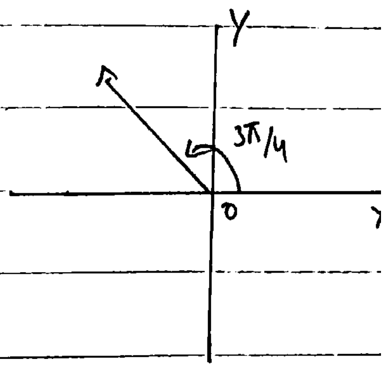
$$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$



$$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$Z = \sqrt{3} + i$$

Q # 8:- $Z = 5\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$

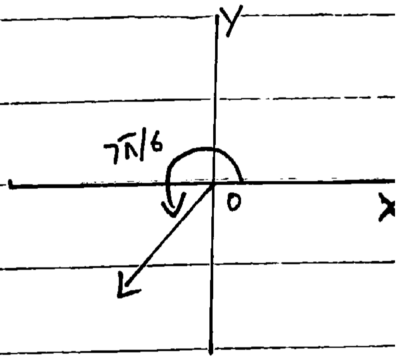


$$= 5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 5\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{-5}{\sqrt{2}} + i\frac{5}{\sqrt{2}}$$

Q # 9:- $Z = \sqrt{3}\,\mathrm{CIS}\left(\frac{7\pi}{6}\right)$

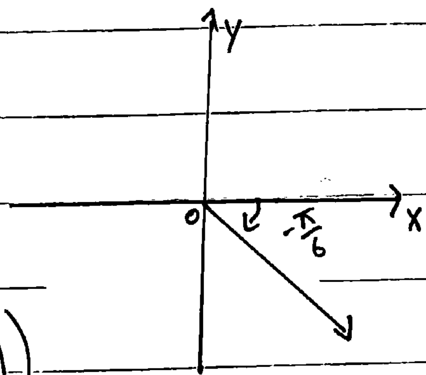


$$= \sqrt{3}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$= \sqrt{3}\left(\frac{-\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$$

$$= \frac{-3}{2} - i\frac{\sqrt{3}}{2}$$

Q # 10:- $Z = \dfrac{5\,\mathrm{CIS}\frac{\pi}{3}}{2\,\mathrm{CIS}\frac{\pi}{2}}$



$$= \frac{5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$$

$$= \frac{5}{2}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{2}\right)\right)$$

$$= \frac{5}{2}\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= \frac{5}{2}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$= \frac{5\sqrt{3}}{4} - \frac{5}{4}i$$

Q # 11:- Find the $|Z|$.

$$Z = -2i(1+i)(2+4i)(3+i)$$

$$= |-2i|\,|1+i|\,|2+4i|\,|3+i|$$

$= \sqrt{4}\sqrt{2}\sqrt{20}\sqrt{10}$

$= \sqrt{1600}$

$|z| = 40$

ii) $z = \dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$

$|z| = \left|\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right| \Rightarrow \dfrac{|3+4i||-1+2i|}{|-1-i||3-i|}$

$= \dfrac{\sqrt{3^2+4^2}\sqrt{-1^2+2^2}}{\sqrt{(-1)^2+(-1)^2}\sqrt{3^2+(-1)^2}} = \dfrac{\sqrt{9+16}\sqrt{1+4}}{\sqrt{1+1}\sqrt{9+1}} = \dfrac{\sqrt{25}\sqrt{5}}{\sqrt{2}\sqrt{10}}$

$= \dfrac{5\sqrt{5}}{\sqrt{20}} \Rightarrow \dfrac{5\cancel{\sqrt{5}}}{2\cancel{\sqrt{5}}} \Rightarrow \dfrac{5}{2}$

★ **Q # 12:- Show that z is real if and only if $z = \bar{z}$**

Suppose $z = \bar{z}$ $\quad \therefore z = a+ib$

$\cancel{a}+ib = \cancel{a}-ib$

$ib+ib = 0$

$2ib = 0 \quad \therefore 2 \neq 0,\ i \neq 0$

$b = 0$

$\therefore z = a+i0 = a$, z is real.

Conversely Suppose z is real

$z = a$

$= a+i0$ and $\bar{z} = a-i0$

Thus $z = \bar{z}$

★ **Q # 13:- Show that $|z_1 - z_2| \leq |z_1| + |z_2|$**

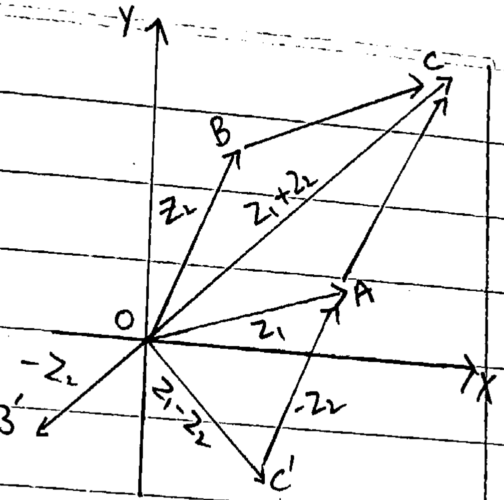
Draw the clear diagram showing

$|z_1 - z_2| = |z_1| + |z_2|$

Let z_1 and z_2 be represented by

$\overrightarrow{OA} = z_1$, $\overrightarrow{OB} = z_2$

and $z_1 + z_2 = \overrightarrow{OC}$



Now $|z_1| = OA$

$|z_2| = OB = AC$

$\Rightarrow |z_1 + z_2| = OC$

Also $\overrightarrow{OB'} = -z_2 = \overrightarrow{AC'}$, $\overrightarrow{OC'} = z_1 - z_2$

thus $OB' = |-z_2| = |z_2| = AC$ $\quad OC' = |z_1 - z_2|$

By Triangle OAC'

$$OC' < OA + AC'$$

$$\Rightarrow |z_1 - z_2| < |z_1| + |z_2| \text{ ——— (I)}$$

If OA and OB are parallel then parallelogram OACB becomes straight line.

$$OC = OA + AC$$

i.e. $|z_1 + z_2| = |z_1| + |z_2|$

replace z_2 by $-z_2$

$$\therefore |z_1 - z_2| = |z_1| + |-z_2|$$

$$|z_1 - z_2| = |z_1| + |z_2| \text{ ——— (II)}$$

By (I) and (II)

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

- Prove analytically:-

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

Since $|z|^2 = z\bar{z}$

$$\therefore |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$$

$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + |z_2|^2 + 2(Re\, z_1\bar{z}_2)$$

$$\Rightarrow |z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| \qquad \because |z| \geq Re.z$$

$$|z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$$

$$|z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \qquad \therefore |\bar{z}_2| = |z_2|$$

$$|z_1+z_2|^2 \leq (|z_1| + |z_2|)^2$$

$$|z_1+z_2| \leq |z_1| + |z_2| \quad \text{——— I}$$

Since

$$|z_1| = |(z_1+z_2) - z_2|$$

$$|z_1| \leq |z_1+z_2| + |-z_2|$$

$$|z_1| \leq |z_1+z_2| + |z_2| \qquad \because |-z_2| = |z_2|$$

$$|z_1 - z_2| \leq |z_1+z_2| \quad \text{——— II}$$

By I and II

$$|z_1| - |z_2| \leq |z_1+z_2| \leq |z_1| + |z_2|$$

★ **Q # 14:-** Let $z_1 = 24+7i$ and $|z_2| = 6$, Find the greatest and least value of $|z_1+z_2|$.

$$z_1 = 24+7i$$

$$|z_1| = \sqrt{(24)^2+(7)^2} = \sqrt{625} = 25$$

Also $|z_2| = 6$

Greatest value of $|z_1+z_2| = |z_1| + |z_2|$

$$= 25+6 = 31$$

Least value of $|z_1+z_2| = ||z_1| - |z_2||$

$$= |25-9| = 19$$

★ **Q # 15:-** If z_1, z_2 are complex no. Show that $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

$$L.H.S = |z_1+z_2|^2 + |z_1-z_2|^2$$

$$= (z_1+z_2)(\overline{z_1+z_2}) + (z_1-z_2)(\overline{z_1-z_2}) \qquad \therefore |z|$$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$= z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{z_2\bar{z}_1} + z_2\bar{z}_2 + z_1\bar{z}_1 - \cancel{z_1\bar{z}_2} - \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2$

$= 2|z_1|^2 + 2|z_2|^2$

$= 2(|z_1|^2 + |z_2|^2) \Rightarrow$ R.H.S

★ **Q # 16:–** Prove that $\left|\frac{az+b}{bz+a}\right| = 1$ for $|z| = 1$

L.H.S $\left|\frac{az+b}{bz+a}\right|$

$= \left|\frac{az+b}{a\bar{z}+a}\right|$

$= \frac{|az+b|}{|b\bar{z}+a|} \Rightarrow \frac{|z||az+b|}{|z||b\bar{z}+a|}$ ∴ x and ÷ by $|z|$

$= \frac{|z||az+b|}{|bz\bar{z}+za|} \Rightarrow \frac{|z|\cancel{|az+b|}}{\cancel{|b+az|}}$ ∴ $z\bar{z} = |z|^2 = 1$

$= |z|$ ∴ $|z| = 1$

$= 1$ R.H.S

★ **Q # 17:–** Find the locus of points in plane satisfy each of given.

i- $|z-5| = 6$

$|x+iy-5| = 6$ ∴ $z = x+iy$

$|(x-5)+iy| = 6$

$\sqrt{(x-5)^2+y^2} = 6$ Taking square on both sides.

$(x-5)^2+y^2 = 36$

Which is Circle.

ii- $|z-2i| \geq 1$

$|x+iy-2i| \geq 1$ ∴ Let $z = x+iy$

$|x+i(y-2)| \geq 1$

$\sqrt{x^2+(y-2)^2} \geq 1$ Taking square on both sides.

$x^2+(y-2)^2 \geq 1$

Loucs is set of points on and out side circle.

iii- $Re(z+2) = -1$

$Re(x+iy+2) = -1$ ∴ $z = x+iy$

$Re((x+2)+iy) = -1$

$x+2 = -1$ consider only Real part.

$x = -3$

Which is line parallel to Y-axis.

iv- $Re(iz) = 3$

$Re(i(x+iy)) = 3$ ∴ $z = x+iy$

$Re(xi + i^2y) = 3$

$Re(ix - y) = 3$ ∴ $i^2 = -1$

$-y = 3$ consider Real part only.

$y = -3$

Locus is a line parallel to X-axis.

v- $|z+i| = |z-i|$

$|x+iy+i| = |x+iy-i|$ ∴ $z = x+iy$

$|x+i(y+1)| = |x+i(y-1)|$

$\sqrt{x^2+(y+1)^2} = \sqrt{x^2+(y-1)^2}$ Taking square on b...

$x^2+(y+1)^2 = x^2+(y-1)^2$

$$\cancel{x^2}+\cancel{y^2}+\cancel{1}+2y = \cancel{x^2}+\cancel{y^2}+\cancel{1}-2y$$

$$2y=-2y$$

$$4y=0$$

$$y=0$$

Locus is X-axis.

vi- $|z+3|+|z+1|=4$

$$|x+iy+3|+|x+iy+1|=4$$

$$|(x+3)+iy|+|(x+1)+iy|=4$$

$$\sqrt{(x+3)^2+y^2}+\sqrt{(x+1)^2+y^2}=4$$

$$\sqrt{(x+3)^2+y^2}=4-\sqrt{(x+1)^2+y^2}$$ Taking square on both sides.

$$\left(\sqrt{(x+3)^2+y^2}\right)^2=\left(4-\sqrt{(x+1)^2+y^2}\right)^2$$

$$(x+3)^2+y^2=16+(x+1)^2+y^2-8\sqrt{(x+1)^2+y^2}$$

$$\cancel{x^2}+9+6x+\cancel{y^2}=16+\cancel{x^2}+1+2x+\cancel{y^2}-8\sqrt{(x+1)^2+y^2}$$

$$4x-8=-8\sqrt{(x+1)^2+y^2}$$

$$4(x-2)=-8\sqrt{(x+1)^2+y^2}$$

$$x-2=\frac{-8}{4}\sqrt{(x+1)^2+y^2}$$

$$x-2=-2\sqrt{(x+1)^2+y^2}$$ Again square.

$$x^2+4-4x=4(x+1)^2+y^2$$

$$x^2+4-4x=4x^2+4+8x+4y^2$$

$$3x^2+4y^2+12x=0$$

$$\{(x,y)\mid 3x^2+4y^2+12x.$$

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ii- $-1\le \operatorname{Re} z<1$ $\quad\therefore z=x+iy$

$\therefore -1\le x\le 1$ $\quad \operatorname{Re} z=x$

Locus is interval $[-1,1[$

viii- $\operatorname{Im} z < 0$

$\operatorname{Im}(x+iy) < 0$ $\quad\because z = x+iy$

$y < 0$ $\quad \operatorname{Im} z = y$

Locus is -ve y-axis.

ix- $\operatorname{Arg} z = \pi/3$

$\operatorname{Arg}(x+iy) = \pi/3$ $\quad\because z = x+iy$

$\tan^{-1}\frac{y}{x} = \pi/3$

$\frac{y}{x} = \tan\frac{\pi}{3}$

$\frac{y}{x} = \sqrt{3}$

$y = \sqrt{3}\,x$

$\therefore \{(x,y) \mid y = \sqrt{3}\,x\}$

x- $\operatorname{Arg}(z-1) = -\frac{3\pi}{4}$

$\operatorname{Arg}(x+iy-1) = -\frac{3\pi}{4}$ $\quad\because z = x+iy$

$\operatorname{Arg}((x-1)+iy) = -3\pi/4$

$\tan^{-1}\frac{y}{x-1} = -3\pi/4$

$\frac{y}{x-1} = \tan -3\pi/4$

$= -1$

$y = -x+1$

$x+y = 1$ Locus is Line.

i.e. $\{(x,y) \mid x+y = 1\}$